Advanced Nonlinear Studies **12** (2012), 49–66

Radial Solutions of a Supercritical Elliptic Equation with Hardy Potential

Zuji Guo, Zhaoli Liu[*]

School of Mathematical Sciences
Capital Normal University, Beijing 100048, People's Republic of China
e-mail: zliu@mail.cnu.edu.cn

Received 20 January 2011
Communicated by Yiming Long

Abstract

Various properties of radial solutions of the supercritical elliptic equation with Hardy Potential

$$\begin{cases} -\Delta u + \mu \dfrac{u}{|x|^2} = |u|^{p-2}u & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are studied, where $\Omega = \text{int}\{x \in \mathcal{R}^N \mid a \leq |x| < b\}$, which is a ball if $0 = a < b < +\infty$, an annulus if $0 < a < b < +\infty$, an exterior domain if $0 < a < b = +\infty$, and the whole space $\mathcal{R}^N$ if $a = 0, b = +\infty$. We assume p is supercritical, that is, $p > 2^*$ with $2^* = \frac{2N}{N-2}$ being the critical Sobolev exponent, and $N \geq 3$.

2010 Mathematics Subject Classification. 35J20, 35J75, 35J92, 70K05.
Key words. Elliptic equation with Hardy potential, supercritical nonlinearity, radial solution

1 Introduction

This paper concerns radial solutions of the supercritical elliptic equation with Hardy potential

$$\begin{cases} -\Delta u + \mu \dfrac{u}{|x|^2} = |u|^{p-2}u & \text{in } \Omega \setminus \{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

---

[*] The author acknowledges the support by NSFC (10825106, 10831005).

where $\Omega = \text{int}\{x \in \mathcal{R}^N \mid a \le |x| < b\}$, $p > 2^*$ with $2^* = \frac{2N}{N-2}$ being the critical Sobolev exponent, and $N \ge 3$. Note that Ω is a ball if $0 = a < b < +\infty$, an annulus if $0 < a < b < +\infty$, an exterior domain if $0 < a < b = +\infty$, and the whole space $\mathcal{R}^N$ if $a = 0, b = +\infty$. By a solution of (1.1) we mean a classical solution which satisfies (1.1) pointwise. Since a solution u may be singular at $x = 0$, we drop 0 from Ω in (1.1). Existence, uniqueness, and multiplicity of radial solutions with prescribed properties of (1.1) are studied in this paper. For radial solutions, (1.1) is rewritten as

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + \mu\dfrac{u}{r^2} = |u|^{p-2}u, & a < r < b, \\ u(a) = 0 \ \text{ if } a > 0, \quad u(b) = 0 \ \text{ if } b < +\infty. \end{cases} \tag{1.2}$$

Here, if $b = +\infty$ then we do not impose any boundary condition at $+\infty$ since, from phase plane analysis, any solution of the first equation in (1.2) automatically satisfies $\lim_{r\to+\infty} u(r) = 0$. Note also that there is no boundary condition at 0 if $a = 0$. Equation (1.1) with $\mu = 0$ is the famous Lane-Emden-Fowler equation. In this case, solutions of this equation have been studied extensively in the last century.

If $2 < p < 2^*$ and Ω is a bounded domain, then it is well known that (1.1) has infinitely many solutions ([1]).

If $p = 2^*$ and Ω is a bounded star-shaped domain then $u = 0$ is the only solution of (1.1), as seen from Pohozaev's identity, but nontrivial solutions do exist if Ω has a nontrivial topology or if the equation is suitably perturbed ([2, 5]).

For $p > 2^*$, (1.1) is more involved since in this case no Sobolev imbedding is available and variational methods can not be directly applied to (1.1).

While in various cases solutions of (1.1) with supercritical exponent have been constructed (see for example [10, 11, 13, 25]) and uniqueness of positive and sign-changing radial solutions has been proved (see for example [9, 12, 22, 23]), the question concerning properties of solutions of (1.1) remains largely open. See [16, 19] for more results in this direction. When $\mu \ne 0$, the singularity at origin will increase essential difficulties in solving (1.1), as indicated in [30]. There have also been many results concerning existence of solutions of (1.1) in this case; see, for instance, [6, 7, 17, 27], where variational methods were applied to (1.1) with either a subcritical exponent or a critical exponent for some ranges of μ. If the exponent is supercritical then additional difficulties arise, since not only is compactness no longer in effect but also variational methods are not applicable. In the case where Ω is the whole space and $-\frac{(N-2)^2}{4} < \mu < 0$, positive solutions of (1.2) were studied in [31].

The present paper is focused on nodal and asymptotic properties of radial solutions. It turns out that these properties depend heavily on ranges of the parameter μ. We now state the main results which extend some known results in the literature; see Remarks 1.1, 1.2, and 1.3. We state the first theorem for the more general m-Laplacian equation

$$\begin{cases} -\Delta_m u + \dfrac{\mu|u|^{m-2}u}{|x|^m} = |u|^{p-2}u & \text{in } B_R(0) \setminus \{0\}, \\ u = 0 & \text{on } \partial B_R(0), \end{cases} \tag{1.3}$$

where $\Delta_m u = \text{div}(|\nabla u|^{m-2}\nabla u)$, $B_R(0)$ is the ball centered at 0 with radius R, and $\mu \in \mathcal{R}$. We assume that $N > m > 1$ and $p > \frac{mN}{N-m}$. If $m \ge N$ then any p is subcritical, which is not the case considered here. As before, we study only radial solutions and rewrite (1.3) as

$$\begin{cases} -(r^{N-1}|u'|^{m-2}u')' + \mu r^{N-1-m}|u|^{m-2}u = r^{N-1}|u|^{p-2}u, \quad r \in (0, R), \\ u(R) = 0. \end{cases} \tag{1.4}$$

By a solution of (1.2) or (1.4) we always mean a nontrivial solution.

Theorem 1.1 *Any solution of* (1.4) *changes sign infinitely many times.*

Remark 1.1 In the case $\mu = 0$, the result of Theorem 1.1 is due to Ni and Serrin [24, Theorem 5.2], who proved that (1.4) has no positive solution, and hence has no solution which changes sign only finitely many times. Our result generalizes Ni and Serrin's result to the case $\mu \neq 0$. We use an idea from [24] together with some new techniques to prove Theorem 1.1 in the case $\mu < 0$. The case $\mu > 0$ is more involved and we have to use a more detailed analysis in this case.

In the annulus case, we have the following theorem which also covers the range $2 < p \leq 2^*$.

Theorem 1.2 *Assume* $0 < a < b < +\infty$ *and* $p > 2$. *Denote*

$$\mu_n = -\frac{(N-2)^2}{4} - \left(\frac{n\pi}{\ln b - \ln a}\right)^2, \quad n = 1, 2, 3, \cdots.$$

(i) *If* $\mu > \mu_1$ *then, for any nonnegative integer* n, (1.2) *has a unique solution* u, $u'(a) > 0$, *which has exactly* n *zeros in* (a, b).

(ii) *If* $\mu_{k+1} < \mu \leq \mu_k$ *for some* $k \geq 1$ *then any solution of* (1.2) *has at least* k *zeros in* (a, b), *and for any integer* $n \geq k$, (1.2) *has a unique solution* u, $u'(a) > 0$, *which has exactly* n *zeros in* (a, b).

Remark 1.2 a) Assuming $0 < a < b < +\infty$, it was proved by Ni and Nussbaum [23, Theorem 3.1] that for any nonnegative integer n, the equation (i.e. (1.2) with $\mu = 0$)

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' = r^l|u|^{p-2}u, \quad a < r < b, \\ u(a) = 0, \quad u'(a) > 0, \quad u(b) = 0 \end{cases}$$

with $l \in \mathcal{R}$ and $p > 2$ has at most one solution having exactly n zeros in (a, b). Using a transformation, we will see that the uniqueness part of Theorem 1.2 is a consequence of Ni and Nussbaum's result. For $\mu > 0$, uniqueness of the positive solution is also a consequence of [14, Theorem 1.2], where nondegeneracy of the unique positive solution was also given. The existence part of Theorem 1.2 in the case $\mu = 0$ is a result in [3]. Combining [23] and [3] yields a result stating that if $\mu = 0$ then for any nonnegative integer n, (1.2) has a unique solution u, $u'(a) > 0$, which has exactly n zeros in (a, b). Here in Theorem 1.2 we extend this result to the case $\mu \neq 0$. We give exact information on the number of solutions with exactly prescribed number of zeros in (a, b) for all $\mu \in \mathcal{R}$, in accordance with the location of μ with respect to μ_n. We will use Rabinowitz's global bifurcation theorem to prove Theorem 1.2.

b) A more general theorem than Theorem 1.2 will be stated in Section 3.

Assume $a \geq 0$ and let u_1 and u_2 be two solutions of the equation

$$-u'' - \frac{N-1}{r}u' + \mu\frac{u}{r^2} = |u|^{p-2}u, \quad a < r < +\infty.$$

If there exists $\alpha > 1$ such that $u_2(r) = \alpha^{2/(p-2)}u_1(\alpha r)$ for all $a < r < +\infty$ then u_1 and u_2 are said to be equivalent, otherwise they are said to be nonequivalent. We now turn to consider exterior domains, that is, we consider the case where $0 < a < b = +\infty$. Set

$$\beta = \frac{2}{p-2}(N-2-\frac{2}{p-2}) + \mu.$$

For $\mu \geq -\frac{(N-2)^2}{4}$, denote

$$\lambda^{\pm} = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 4\mu}}{2}.$$

Let $u(r)$ be a function defined for large r (small r, respectively) and λ be a real number. We say that $u(r)$ has order r^{λ} near $r = +\infty$ ($r = 0$, respectively) if the limit $\lim_{r\to+\infty} u(r)/r^{\lambda}$ ($\lim_{r\to 0} u(r)/r^{\lambda}$, respectively) exists and is a nonzero real number.

Theorem 1.3 *Assume* $0 < a < b = +\infty$. (i) *If* $\mu < -\frac{(N-2)^2}{4}$, *then any solution of* (1.2) *changes sign infinitely many times.*

(ii) *If* $\mu \geq -\frac{(N-2)^2}{4}$, *then any solution of* (1.2) *has only a finite number of zeroes in* $(a, +\infty)$, *and, for any nonnegative integer* n, (1.2) *has infinitely many nonequivalent solutions with exactly* n *zeros in* $(a, +\infty)$, *among which only one pair, denoted by* $\pm u_n$, *have order* r^{λ^-} *near* $+\infty$. *Moreover,* $u_0, u_1, \cdots, u_n, \cdots$ *are equivalent if we specify* u_n *so that* $(-1)^n u_n'(a) > 0$.

Remark 1.3 a) In the case $\mu = 0$, the results concerning positive solutions in Theorem 1.3 are due to [21, 23], see [21, Theorem 2] and the remark following it. Theorem 1.3 extend the related results in [21, 23] to all $\mu \in \mathcal{R}$ and to sign-changing solutions. For $p \neq p_i$, where $\{p_i\}$ is an increasing sequence tending to infinity, non-degeneracy of the unique positive solution with fast decay rate $r^{-(N-2)}$ was proved in [12] and non-degeneracy of the unique pair of solutions with n zeros and with fast decay rate $r^{-(N-2)}$ was proved in [9]. These solutions were used in [9, 12] as building blocks to construct solutions of elliptic equations with supercritical exponent on a bounded domain with a very small hole.

b) Theorem 1.3 also improves obviously a result in [20]. It was proved in [20] that if $\mu = 0$, $a > 0$ and $b = +\infty$ then (1.2) has a radially symmetric solution which satisfies $|\nabla u| \in L^2(\mathcal{R}^N \setminus B_a(0))$ and $u \in L^p(\mathcal{R}^N \setminus B_a(0))$. This is just the positive solution with order r^{λ^-} near $+\infty$ stated in Theorem 1.3(ii). Theorem 1.3 gives not only existence but also uniqueness of the positive solution, and, in addition, it asserts existence and uniqueness of the pair of solutions with arbitrary prescribed number of zeros in the class of functions which satisfy $|\nabla u| \in L^2(\mathcal{R}^N \setminus B_a(0))$ and $u \in L^p(\mathcal{R}^N \setminus B_a(0))$.

Our last result is for the case of entire $\mathcal{R}^N$.

Theorem 1.4 *Assume* $a = 0$ *and* $b = +\infty$.

(i) *If* $\mu < -\frac{(N-2)^2}{4}$, *then each solution* $u(r)$ *of* (1.2) *changes sign infinitely many times near* 0 *and also near* $+\infty$.

(ii) *If* $-\frac{(N-2)^2}{4} \le \mu \le -\frac{2}{p-2}(N-2-\frac{2}{p-2})$*, then each solution of* (1.2) *changes sign infinitely many times near* 0*, and does not change sign and satisfies* $u(r) = O((r^2 \ln r)^{-1/(p-2)})$ *near* $+\infty$*. Among all the solutions, only one pair have order* r^{λ^-} *near* $+\infty$*.*

(iii) *If* $\mu > -\frac{2}{p-2}(N-2-\frac{2}{p-2})$*, then* (1.2) *has two pairs of solutions* $\pm u_1, \pm u_2$ *which do not change sign and all the other solutions change sign infinitely many times near* 0 *and do not change sign near* $+\infty$*. Moreover,* $u_1(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}$*,* $u_2(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1+o(1))$ *near* $+\infty$*,* u_2 *has order* r^{λ^+} *near* 0*,* (1.2) *has one pair of sign changing solutions with order* r^{λ^-} *near* $+\infty$*, and all the other sign changing solutions satisfy* $|u(r)| = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1+o(1))$ *near* $+\infty$*.*

The pair of solutions with order r^{λ^-} near $+\infty$ in Theorem 1.3 or in Theorem 1.4 are called the fast decay solutions. The paper is organized as follows. In Section 2 we prove Theorem 1.1 and prove at the end of this section two lemmas as byproduct which will be used at a later stage. We prove Theorem 1.2 and state a more general result in Section 3. Section 4 is devoted to the study of the fast decay solutions. In Section 5 we complete the proofs of Theorems 1.3 and 1.4, using phase plane analysis.

2 Ω being a ball

We denote $\phi(s) = |s|^{m-2}s$ for $s \in (-\infty, +\infty)$. The inverse function of ϕ is given by $\phi^{-1}(s) = |s|^{\frac{1}{m-1}-1}s$. Using ϕ, we write (1.4) as

$$\begin{cases} -(r^{N-1}\phi(u'))' + \mu r^{N-1-m}\phi(u) = r^{N-1}|u|^{p-2}u, \quad r \in (0,R), \\ u(R) = 0. \end{cases} \tag{2.1}$$

The following lemma is Pohozaev's identity [26], whose proof is standard and thus is omitted.

Lemma 2.1 *If* u *is a solution of* (2.1)*, then for any* $0 < \epsilon < r < R$*,*

$$\begin{aligned}(\frac{N}{p} - \frac{N-m}{m})\int_\epsilon^r s^{N-1}|u|^p = \frac{s^N|u|^p}{p}\Big|_\epsilon^r + \frac{m-1}{m}s^N|u'|^m\Big|_\epsilon^r - \frac{\mu}{m}s^{N-m}|u|^m\Big|_\epsilon^r \\ + \frac{N-m}{m}s^{N-1}|u'|^{m-2}u'u\Big|_\epsilon^r.\end{aligned} \tag{2.2}$$

The following lemma is proved following the arguments in [24]. In the sequel, we shall use C as a positive constant which may be variant in the context if not specified.

Lemma 2.2 *Assume* $\mu \le 0$*. Then there exists a positive constant* $C = C(m, N, p)$ *such that, for any positive solution* $u(r)$ *of* (2.1)*,*

$$u(r) \le C r^{-\frac{m}{p-m}}, \quad 0 < r < R.$$

Proof. We first claim that $u' < 0$ in $(0,R)$. In fact, since $u(r)$ is a positive solution of (2.1) and $\mu \le 0$, from (2.1) we see that $-(r^{N-1}\phi(u'))' > 0$ which implies that $r^{N-1}\phi(u')$ is strictly decreasing in $(0,R)$. If $u'(r_0) \ge 0$ for some $r_0 \in (0,R)$ then there would exist a constant $C > 0$ such that $r^{N-1}\phi(u'(r)) \ge C$ for $r \in (0, r_0/2)$. As a consequence, for $0 < r_1 < r_2 < r_0/2$, we have

$$u(r_2) \ge \int_{r_1}^{r_2} u'(r)dr \ge \frac{C(m-1)}{N-m}(r_1^{-\frac{N-m}{m-1}} - r_2^{-\frac{N-m}{m-1}}).$$

For fixed r_2, letting $r_1 \to 0$, we obtain $u(r_2) = +\infty$, a contradiction. Therefore, $u'(r) < 0$ for $0 < r < R$. For $0 < \epsilon < r < R$, integrating (2.1) from ϵ to r, since $u'(r) < 0$ for $0 < r < R$, we have

$$-r^{N-1}\phi(u'(r)) \geq -s^{N-1}\phi(u'(s))\Big|_{\epsilon}^{r} \geq \int_{\epsilon}^{r} s^{N-1}|u|^{p-2}u \geq u^{p-1}(r)\frac{r^N - \epsilon^N}{N}.$$

Sending ϵ to 0, we obtain

$$-r^{N-1}\phi(u'(r)) \geq \frac{r^N}{N}u^{p-1}(r), \quad 0 < r < R,$$

which can be rewritten as

$$-u'(r)u^{-\frac{p-1}{m-1}}(r) \geq N^{-\frac{1}{m-1}}r^{\frac{1}{m-1}}, \quad 0 < r < R.$$

Integrating the last inequality from ϵ to r, we see that

$$\frac{m-1}{p-m}u^{-\frac{p-m}{m-1}}(r) \geq \frac{m-1}{m}N^{-\frac{1}{m-1}}(r^{\frac{m}{m-1}} - \epsilon^{\frac{m}{m-1}}).$$

Letting $\varepsilon \to 0$, we arrive at

$$u^{-\frac{p-m}{m-1}}(r) \geq \frac{p-m}{m}N^{-\frac{1}{m-1}}r^{\frac{m}{m-1}}, \quad 0 < r < R,$$

which implies the result. This is the end of the proof.

In order to estimate $u'(r)$, we need the following lemma.

Lemma 2.3 *Assume $\mu \leq 0$. If $u(r)$ is a positive solution of* (2.1) *then*

$$\lim_{\epsilon \to 0} \epsilon^{N-1}\phi(u'(\epsilon)) = 0.$$

Proof. According to the proof of Lemma 2.2, $u'(r) < 0$ and $(r^{N-1}\phi(u'(r)))' < 0$ for $0 < r < R$, and therefore the limit $\lim_{\epsilon\to 0}\epsilon^{N-1}\phi(u'(\epsilon))$ exists and is nonpositive. Assume by contradiction that $\lim_{\epsilon\to 0}\epsilon^{N-1}\phi(u'(\epsilon)) := -a_0 < 0$, then

$$r^{N-1}\phi(u'(r)) \leq -a_0, \quad 0 < r < R.$$

That is

$$u'(r) \leq \phi^{-1}(-a_0 r^{-(N-1)}), \quad 0 < r < R.$$

For any $0 < r < R/2$, integrating the last inequality from r to $2r$, we see that

$$u(2r) - u(r) \leq -\frac{m-1}{N-m}a_0^{\frac{1}{m-1}}(1 - 2^{-\frac{N-m}{m-1}})r^{-\frac{N-m}{m-1}}.$$

As a consequence, we obtain

$$u(r) \geq \frac{m-1}{N-m}a_0^{\frac{1}{m-1}}(1 - 2^{-\frac{N-m}{m-1}})r^{\frac{m-N}{m-1}},$$

which combined with Lemma 2.2 yields, for some positive constant C_1 and for all $0 < r < R/2$,

$$r^{-\frac{N-m}{m-1}} \leq C_1 r^{-\frac{m}{p-m}}.$$

Note that $-\frac{m}{p-m} > -\frac{N-m}{m-1}$ since p is supercritical. We obtain a contradiction by letting $r \to 0$. This is the end of the proof.

Lemma 2.4 *Assume $\mu \le 0$. If $u(r)$ is a positive solution of* (2.1) *then*

$$0 < -u'(r) \le Cr^{-\frac{p}{p-m}}, \quad 0 < r < R,$$

where $C = C(\mu, m, N, p)$ is a positive constant.

Proof. Integrating (2.1) from ϵ to r , we have

$$-s^{N-1}\phi(u'(s))\Big|_{\epsilon}^{r} = \int_{\epsilon}^{r} -\mu s^{N-1-m}\phi(u) + s^{N-1}|u|^{p-2}u.$$

Letting $\epsilon \to 0$ and using Lemma 2.3, we obtain

$$-r^{N-1}\phi(u'(r)) = \int_{0}^{r} (-\mu s^{N-1-m}\phi(u) + s^{N-1}|u|^{p-2}u).$$

Then Lemma 2.2 can be used to deduce that

$$-r^{N-1}\phi(u'(r)) \le C\int_{0}^{r} s^{N-1-\frac{m(p-1)}{p-m}},$$

from which we easily obtain the result. This is the end of the proof.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume by contradiction that (2.1) has a solution u which changes sign only finitely many times. Then, choosing a smaller ball if necessary, we may assume that u is a positive solution of (2.1). To come to a contradiction, we first consider the case $\mu \le 0$. In Pohozaev's identity (2.2), letting $\epsilon \to 0$ and $r \to R$ and using the estimates obtained in Lemmas 2.2 and 2.4, we see that

$$(\frac{N}{p} - \frac{N-m}{m})\int_{0}^{R} s^{N-1}|u|^{p} = \frac{m-1}{m}R^{N}|u'(R)|^{m}.$$

But this is a contradiction since the left side is negative and the right side is positive. Now we assume $\mu > 0$, and divide the remaining of the proof into three cases.

Case I: There exists $r_0 \in (0, R)$ such that $u(r) \ge (2\mu)^{\frac{1}{p-m}}r^{-\frac{m}{p-m}}$ for $0 < r < r_0$. Then we have

$$-(r^{N-1}\phi(u'))' = r^{N-1}u^{p-1} - \mu r^{N-m-1}u^{m-1} \ge \frac{1}{2}r^{N-1}u^{p-1}.$$

Repeating the arguments in Lemmas 2.2-2.4, we see that, for some positive constant C and for $0 < r < r_0$,

$$u(r) \le Cr^{-\frac{m}{p-m}}, \quad 0 < -u'(r) \le Cr^{-\frac{p}{p-m}}.$$

This yields a contradiction as above.

Case II: There exists $r_0 \in (0, R)$ such that $u(r) \le (2\mu)^{\frac{1}{p-m}}r^{-\frac{m}{p-m}}$ for $0 < r < r_0$. Let $\{r_n\}$ be a decreasing sequence such that $r_1 \le r_0/2$ and $r_n \to 0$. Set $f(r) = r^{\frac{m}{p-m}}u(r)$ and then choose $r_n^* \in (r_n, 2r_n)$ such that $f(2r_n) - f(r_n) = f'(r_n^*)r_n$ for $n \in \mathcal{N}$. Then it is easy to see that, for some positive constant C_1 and for all n,

$$|u'(r_n^*)| \le C_1(r_n^*)^{-\frac{p}{p-m}}.$$

In Pohozaev's identity (2.2), letting $\epsilon = r_n^* \to 0$ and $r \to R$, we come to a contradiction.

Case III: There exist sequences $\{r_n\}$ and $\{\tilde{r}_n\}$ satisfying $0 < r_n < \tilde{r}_n$ and $\tilde{r}_n \to 0$ such that

$$u(r_n) = (2\mu)^{\frac{1}{p-m}}(r_n)^{-\frac{m}{p-m}}, \quad u(\tilde{r}_n) = (2\mu)^{\frac{1}{p-m}}(\tilde{r}_n)^{-\frac{m}{p-m}},$$

$$u(r) < (2\mu)^{\frac{1}{p-m}} r^{-\frac{m}{p-m}}, \quad \text{for } r_n < r < \tilde{r}_n.$$

Choose $r_n^* \in (r_n, \tilde{r}_n)$ such that $f'(r_n^*) = 0$. Then, for some positive constant C_2 and for all n,

$$u(r_n^*) < (2\mu)^{\frac{1}{p-m}}(r_n^*)^{-\frac{m}{p-m}}, \quad |u'(r_n^*)| \le C_2(r_n^*)^{-\frac{p}{p-m}}.$$

We then arrive at a contradiction as in Case II. This is the end of the proof.

The argument for proving Theorem 1.1 can be used to prove the following lemma, which provides the order at $r = 0$ of the solution of (1.2) which corresponds to the heteroclinic orbit of (5.3).

Lemma 2.5 *Assume $a = 0$, $b = +\infty$ and $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$. Then the equation* (1.2) *has a positive solution u with order r^{λ^+} near $r = 0$.*

Proof. Recall that $\lambda^+ = \frac{-(N-2)+\sqrt{\Delta}}{2}$, where $\Delta = (N-2)^2 + 4\mu$. Set $w(r) = r^{-\lambda^+}u(r)$. Then u is a positive solution of (1.2) if and only if w is a positive solution of the equation

$$-(r^{N-1+2\lambda^+}w')' = r^{p\lambda^+ + N-1}|w|^{p-2}w, \quad r \in (0, +\infty). \tag{2.3}$$

Let $\theta^+ = p\lambda^+ - 2\lambda^+$. Then $\theta^+ > -2$ since $\mu > -\frac{2}{p-2}(N-2-\frac{2}{p-2})$. As in [8, 18], we take the coordinate transformation

$$t = \frac{2}{2+\theta^+} r^{\frac{\theta^+ + 2}{2}}, \quad v(t) = w(r).$$

Then w is a positive solution of (2.3) if and only if v is a positive solution of the equation

$$-v'' - \frac{\widetilde{N} - 1}{t}v' = |v|^{p-2}v, \quad t \in (0, +\infty),$$

where $\widetilde{N} = \frac{2}{2+\theta^+}\sqrt{\Delta} + 2$. Consider the initial value problem

$$\begin{cases} -v'' - \dfrac{\widetilde{N} - 1}{t}v' = |v|^{p-2}v, & t > 0, \\ v(0) = 1, \\ v'(0) = 0, \end{cases} \tag{2.4}$$

which is equivalent to the integral equation

$$v(t) = -\int_0^t \int_0^s \frac{\tau^{\widetilde{N}-1}}{s^{\widetilde{N}-1}} |v(\tau)|^{p-2} v(\tau) d\tau ds + 1.$$

By the Banach fixed point theorem, the integral equation and therefore (2.4) has a unique continuous solution v near $t = 0$. Since the Hamiltonian function $H(t) := \frac{1}{2}|v'(t)|^2 + \frac{1}{p}|v(t)|^p$ satisfies

$$\frac{dH}{dt} = -\frac{\widetilde{N} - 1}{t}|v'|^2 \le 0,$$

v exists for all $t > 0$. We claim that $v(t)$ must be positive for all $t > 0$. Otherwise, $v(T) = 0$ for some $T > 0$ and we may assume T to be the smallest such number. As in Lemma 2.1, for any $0 < \epsilon < t < T$, we have

$$\left(\frac{\widetilde{N}}{p} - \frac{\widetilde{N}-2}{2}\right)\int_\epsilon^t s^{\widetilde{N}-1}|v|^p = \frac{s^{\widetilde{N}}|v|^p}{p}\Big|_\epsilon^t + \frac{1}{2}s^{\widetilde{N}}|v'|^2\Big|_\epsilon^t + \frac{\widetilde{N}-2}{2}s^{\widetilde{N}-1}v'v\Big|_\epsilon^t, \tag{2.5}$$

Also, as in Lemmas 2.2 and 2.4, we have, for $0 < t < T$,

$$v(t) \le Ct^{-\frac{2}{p-2}}, \quad 0 < -v'(t) \le Ct^{-\frac{p}{p-2}}, \tag{2.6}$$

where $C = C(\mu, m, N, p)$ is a positive constant. Since

$$p > \frac{2N}{N-2},$$

we see that

$$\sqrt{\Delta} > -(N-2) + \sqrt{\Delta} + \frac{4}{p-2} = \frac{2}{p-2}((p-2)\lambda^+ + 2) = \frac{2}{p-2}(\theta^+ + 2).$$

Therefore,

$$p > \frac{2(\theta^+ + 2)}{\sqrt{\Delta}} + 2 = \frac{2\widetilde{N}}{\widetilde{N}-2}.$$

Letting $\epsilon \to 0$ and $t \to T$ in (2.5), we obtain

$$0 > \left(\frac{\widetilde{N}}{p} - \frac{\widetilde{N}-2}{2}\right)\int_0^T s^{\widetilde{N}-1}|v|^p = \frac{1}{2}T^{\widetilde{N}}|v'(T)|^2 > 0,$$

a contradiction. As a consequence, v is positive in $(0, +\infty)$. Setting $u(r) = r^{\lambda^+}v(\frac{2}{\theta^++2}r^{\frac{\theta^++2}{2}})$, then u is a positive solution of equation (1.2) with order r^{λ^+} near $r = 0$. This is the end of the proof.

Remark 2.1 It is clear that Lemmas 2.1-2.4 are valid if the ball $B_R(0)$ is replaced with the entire $\mathcal{R}^N$. More precisely, if $\mu \le 0$ and u is a positive solution of

$$-(r^{N-1}\phi(u'))' + \mu r^{N-1-m}\phi(u) = r^{N-1}|u|^{p-2}u, \quad r \in (0, +\infty),$$

then, for $0 < r < +\infty$,

$$u(r) \le Cr^{-\frac{m}{p-m}}, \quad 0 < -u'(r) \le Cr^{-\frac{p}{p-m}},$$

where $C = C(\mu, m, N, p)$ is a positive constant. The same argument also leads to the following lemma.

Lemma 2.6 *For any $\mu \in \mathcal{R}$, if u is a positive solution of the equation*

$$-u'' - \frac{N-1}{r}u' + \mu\frac{u}{r^2} = |u|^{p-2}u, \quad 0 < r < r_0,$$

then

$$u(r) \le Cr^{-\frac{2}{p-2}}, \quad |u'(r)| \le Cr^{-\frac{p}{p-2}}, \quad 0 < r < r_0.$$

Proof. If $\mu \le 0$ then the result follows from Lemmas 2.1-2.4. If $\mu > 0$, then the result is a consequence of (2.6). This is the end of the proof.

3 Ω being an annulus

Let $0 < a < b < +\infty$, $p > 2$, and $\mu, l \in \mathcal{R}$. Consider the boundary value problem

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + \mu\dfrac{u}{r^2} = r^l|u|^{p-2}u, \quad r \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{3.1}$$

We shall prove the following theorem which generalizes Theorem 1.2.

Theorem 3.1 *Assume* $0 < a < b < +\infty$, $p > 2$, *and* $l \in \mathcal{R}$. *Denote*

$$\mu_n = -\frac{(N-2)^2}{4} - \left(\frac{n\pi}{\ln b - \ln a}\right)^2, \quad n = 1, 2, 3, \cdots.$$

(i) *If* $\mu > \mu_1$ *then, for any nonnegative integer* n, (1.2) *has a unique solution* u, $u'(a) > 0$, *which has exactly* n *zeros in* (a,b).

(ii) *If* $\mu \le \mu_k$, $k \ge 1$, *then any solution of* (1.2) *has at least* k *zeros in* (a,b).

(iii) *If* $\mu_{k+1} < \mu \le \mu_k$, $k \ge 1$, *then for any integer* $n \ge k$, (1.2) *has a unique solution* u, $u'(a) > 0$, *which has exactly* n *zeros in* (a,b).

Proof. Set $u(r) = r^\epsilon v(s)$, $s = \ln r$. Then (3.1) is transformed into

$$\begin{cases} -v'' - (N+2\epsilon-2)v' - [\epsilon(N+\epsilon-2) - \mu]v = e^{\tau s}|v|^{p-2}v, \quad s \in (\alpha,\beta), \\ v(\alpha) = v(\beta) = 0, \end{cases} \tag{3.2}$$

where $\tau = l + 2 + (p-2)\epsilon$, $\alpha = \ln a$ and $\beta = \ln b$. According to [23, Theorem 3.1] and the proof therein, for any $\mu \in \mathcal{R}$ and any nonnegative integer n, (1.2) has at most one solution u, $u'(a) > 0$, which has exactly n zeros in (a,b). Therefore, the uniqueness part of (i) and (iii) holds. We choose $\epsilon = -\frac{N-2}{2}$ and rewrite (3.2) as

$$\begin{cases} -v'' + \left[\dfrac{(N-2)^2}{4} + \mu\right]v = e^{(l+2-\frac{(N-2)(p-2)}{2})s}|v|^{p-2}v, \quad s \in (\alpha,\beta), \\ v(\alpha) = v(\beta) = 0. \end{cases} \tag{3.3}$$

Since $\lambda_k = \left(\frac{k\pi}{\beta-\alpha}\right)^2$ is the kth eigenvalue of

$$\begin{cases} -v'' = \lambda v, \quad s \in (\alpha,\beta), \\ v(\alpha) = v(\beta) = 0, \end{cases}$$

and the associated eigenfunctions have exactly $k-1$ zeros in (α,β), the Sturm comparison theorem implies that if $\mu \le \mu_k$ for some $k \ge 1$ then any solution of (3.3) has at least k zeros in (α,β). This proves (ii). For existence part of (i) and (iii), we apply Rabinowitz' global bifurcation theorem from [28, 29]. Let $E = \{u \in C^1[\alpha,\beta] \mid u(\alpha) = u(\beta) = 0\}$ and S_k^+ denote the set of $u \in E$ such that u has exactly $k-1$ simple zeros in (α,β), $u > 0$ near $s = \beta$, and all zeros of u in $[\alpha,\beta]$ are simple. Set $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. Let $\mathcal{S}$ denote the closure of the set $\{(\mu, v) \mid v \neq 0,\ (\mu, v) \text{ satisfies } (3.3)\}$

in $\mathcal{R}\times E$. According to [28, 29], $\mathcal{S}$ has an unbounded connected component C_k such that $C_k \subset \mathcal{R}\times S_k$ and $C_k \cap \{(\mu, 0) \mid \mu \in \mathcal{R}\} = \{(\mu_k, 0)\}$. From (ii) we see that $C_k \setminus \{(\mu_k, 0)\} \subset (\mu_k, +\infty) \times S_k$. The same argument as the proof of [29, Lemma 2.22] shows that for any bounded interval I, $C_k \cap (I \times E)$ is bounded in $\mathcal{R}\times E$. Therefore, the image of the projection of C_k onto the parameter space is $[\mu_k, +\infty)$. The existence part of (i) and (iii) follows. This is the end of the proof.

4 The fast decay solution

Hereafter, we always assume $b = +\infty$ in (1.2). We assume $\mu \geq -(N-2)^2/4$ in this section. To study the fast decay solution, we take the transformation

$$w(s) = r^{-\lambda^-}u(r), \quad s = r^{-1},$$

where $\lambda^- = \frac{-(N-2)-\sqrt{\Delta}}{2}$ and $\Delta = (N-2)^2 + 4\mu$. Then this transformation converts (1.2) into

$$\begin{cases} -w'' - \dfrac{3-N-2\lambda^-}{s}w' = s^{-p\lambda^-+2\lambda^--4}|w|^{p-2}w, & 0 < s < a^{-1} \\ w(a^{-1}) = 0 \quad \text{if } a > 0. \end{cases} \tag{4.1}$$

Here $a^{-1} = +\infty$ if $a = 0$ and there is no boundary condition in this case. Let $\theta^- = -p\lambda^- + 2\lambda^- - 4$. Then $\theta^- > -2$. Using the coordinate transformation (see for instance [8, 18])

$$t = \frac{2}{2+\theta^-}s^{\frac{\theta^-+2}{2}}, \quad v(t) = w(s)$$

again, we see that w is a solution of (4.1) if and only if v is a solution of the equation

$$\begin{cases} -v'' - \dfrac{\widetilde{N}-1}{t}v' = |v|^{p-2}v, & t \in (0, T), \\ v(T) = 0 \quad \text{if } T < +\infty, \end{cases}$$

where $\widetilde{N} = \frac{2}{2+\theta^-}\sqrt{\Delta} + 2$, and

$$T = \begin{cases} \dfrac{2}{2+\theta^-}a^{-\frac{\theta^-+2}{2}} & \text{if } a > 0, \\ +\infty & \text{if } a = 0. \end{cases}$$

Now we consider the initial value problem

$$\begin{cases} -(t^{\widetilde{N}-1}v')' = t^{\widetilde{N}-1}|v|^{p-2}v, & t \geq 0, \\ v(0) = 1, \\ v'(0) = 0. \end{cases} \tag{4.2}$$

The same argument of the proof of Lemma 2.5 show that the equation (4.2) has a unique solution $\tilde{v}$ which exists globally. For $0 < r < +\infty$, define

$$t = \frac{2}{2+\theta^-}r^{-\frac{\theta^-+2}{2}}, \quad \tilde{v}(t) = r^{-\lambda^-}\tilde{u}(r).$$

Then $\tilde{u}$ is a solution of (1.2) with $a = 0$ and $b = +\infty$, and $\tilde{u}(r) = r^{\lambda^-}(1 + o(1))$ for r large.

Lemma 4.1 *$\tilde{u}$ changes sign infinitely many times near $r = 0$.*

Proof. Assume the result were false. That is, $\tilde{u}$ has only a finite number of zeros in $(0, +\infty)$. Then $\tilde{v}$ has only a finite number of zeros in $(0, +\infty)$. Without loss of generality, we assume that $t^* > 0$ is the largest zero of $\tilde{v}$ and $\tilde{v}(t) > 0$ for $t^* < t < +\infty$. Define $r^* := (\frac{2+\theta^-}{2} t^*)^{-\frac{2}{2+\theta^-}}$. Then $\tilde{u}(r)$ is a positive solution of

$$-u'' - \frac{N-1}{r} u' + \mu \frac{u}{r^2} = |u|^{p-2} u, \quad 0 < r < r^*.$$

According to Lemma 2.6, we have

$$\tilde{u}(r) \leq C r^{-\frac{2}{p-2}}, \quad |\tilde{u}'(r)| \leq C r^{-\frac{p}{p-2}}, \quad 0 < r < r^*.$$

This implies

$$\tilde{v}(t) \leq C t^{-\frac{2}{p-2}}, \quad |\tilde{v}'(t)| \leq C t^{-\frac{p}{p-2}}, \quad t^* < t < +\infty. \tag{4.3}$$

In the Pohozaev's identity

$$(\frac{\widetilde{N}}{p} - \frac{\widetilde{N} - 2}{2}) \int_{t^*}^{\tau} t^{\widetilde{N}-1} |\tilde{v}'|^2 = \frac{\widetilde{N}}{p} t^{\widetilde{N}-1} \tilde{v}' \tilde{v} \Big|_{t^*}^{\tau} + \frac{t^{\widetilde{N}} |\tilde{v}|^p}{p} \Big|_{t^*}^{\tau} + \frac{1}{2} t^{\widetilde{N}} |\tilde{v}'|^2 \Big|_{t^*}^{\tau},$$

letting $\tau \to +\infty$ and using (4.3), we obtain

$$(\frac{\widetilde{N}}{p} - \frac{\widetilde{N} - 2}{2}) \int_{t^*}^{+\infty} t^{\widetilde{N}-1} |\tilde{v}'|^2 = -\frac{1}{2} t^{\widetilde{N}} |\tilde{v}'(t^*)|^2. \tag{4.4}$$

It is easy to see that $\widetilde{N} = 2$ if $\Delta = 0$ and

$$\frac{2\widetilde{N}}{\widetilde{N} - 2} = p + \frac{p(N-2) - 2N}{\sqrt{\Delta}} > p$$

if $\Delta > 0$. Therefore, the left side of (4.4) is positive while the right side is negative, which is a contradiction. This is the end of the proof.

Lemma 4.2 *Assume $\mu \geq -\frac{(N-2)^2}{4}$.*

If $a > 0$ then for any nonnegative integer n, (1.2) *has exactly one pair of solutions $\pm u_n$ with exactly n zeros in $(a, +\infty)$ and having order r^{λ^-} near $+\infty$, and all $u_0, u_1, u_2, \cdots$ are equivalent.*

If $a = 0$ then (1.2) *has exactly one pair of solutions having order r^{λ^-} near $+\infty$ and these solutions have infinitely many zeros in* $(0, +\infty)$.

Proof. If u is a solution of (1.2) having order r^{λ^-} near $+\infty$ then u is equivalent to $\tilde{u}$. Therefore, if $a = 0$ then, up to equivalence, (1.2) has exactly one pair of solutions $\pm\tilde{u}$ having order r^{λ^-} near $+\infty$ and these solutions have infinitely many zeros in $(0, +\infty)$, according to Lemma 4.1. We arrange the zeros of $\tilde{u}$ in the decreasing sequence

$$r_0 > r_1 > r_2 > \cdots > r_n > \cdots.$$

In the case $a > 0$, define for any nonnegative integer n and $r \geq a$,

$$u_n(r) = (\frac{r_n}{a})^{\frac{2}{p-2}} \tilde{u}(\frac{r_n}{a} r).$$

Then $\pm u_n$ are the only solutions of (1.2) with order r^{λ^-} near $+\infty$ which have exactly n zeros in $(a, +\infty)$. These solutions are equivalent. This is the end of the proof.

5 Phase plane analysis

Taking the coordinate transformation introduced by Fowler [15]

$$v(s) = r^{\frac{2}{p-2}} u(r), \quad r = e^s, \tag{5.1}$$

we transform (1.2) with $a = 0$ into the equation

$$-\ddot{v} - \alpha \dot{v} + \beta v = |v|^{p-2} v, \quad s \in (-\infty, +\infty), \tag{5.2}$$

where $\alpha = N - 2 - \frac{4}{p-2} > 0$, $\beta = \frac{2}{p-2}(N - 2 - \frac{2}{p-2}) + \mu$. Translation invariance of the solutions of (5.2) corresponds to equivalence of the solutions of (1.2). Setting $x = v$ and $y = v'$, then (5.2) can be written as the 2-dimensional dynamical system

$$\begin{cases} x' = y, \\ y' = \beta x - |x|^{p-2} x - \alpha y. \end{cases} \tag{5.3}$$

Consider the Hamiltonian function

$$H(s) = \frac{1}{2} y^2(s) + \frac{1}{p} |x(s)|^p - \frac{\beta}{2} x^2(s). \tag{5.4}$$

Clearly, we have

$$\frac{dH}{ds} = -\alpha y^2(s) \leq 0, \tag{5.5}$$

which implies that $(x(s), y(s))$ exists for all positive s and that (5.3) has no homoclinic orbit. To see that $(x(s), y(s))$ exists for all negative s and to study the picture of the phase flow, we also consider

$$\begin{cases} \xi' = -\eta, \\ \eta' = -\beta \xi + |\xi|^{p-2} \xi + \alpha \eta. \end{cases} \tag{5.6}$$

Lemma 5.1 (a) *If* $\mu \leq -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$ *then for any* $\xi_0 > 0$ *the phase curve of* (5.6) *starting from* $(\xi_0, 0)$ *will turn counterclockwise around* $(0, 0)$ *and reach some point* $(\xi_1, 0)$ *with* $\xi_1 > \xi_0$ *in finite time.*

(b) *If* $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$ *then the result in* (a) *also holds provided that* ξ_0 *is large enough.*

Proof. (a) We first assume $\mu \leq -\frac{2}{p-2}(N-2-\frac{2}{p-2})$. Then $\beta \leq 0$. Suppose $(\xi(t), \eta(t))$ is the phase curve of (5.6) such that $(\xi(0), \eta(0)) = (\xi_0, 0)$. Since $\eta'(0) = -\beta\xi_0 + \xi_0^{p-1} > 0$, the phase curve enters the first quadrant at $t = 0$. From (5.6) it is easy to see that $\xi(t)$ is strictly decreasing and $\eta(t)$ is strictly increasing as long as $(\xi(t), \eta(t))$ is in the first quadrant. Let $t_0 > 0$ be such that $(\xi(t), \eta(t))$ is in the first quadrant for $0 < t \leq t_0$. Then $\xi'(t) \leq -\eta(t_0) < 0$ and $\eta'(t) \leq -\beta\xi_0 + \xi_0^{p-1} + \alpha\eta$ for $t \geq t_0$ before $(\xi(t), \eta(t))$ leaves the first quadrant. As a consequence, there is $t_1 > t_0$ such that $(\xi(t_1), \eta(t_1))$ is on the positive η-axis, that is,

$$\xi(t_1) = 0, \quad \eta(t_1) > 0$$

and $(\xi(t), \eta(t))$ stays in the first quadrant for $0 < t < t_1$. Now since

$$\xi'(t_1) = -\eta(t_1) < 0, \quad \eta'(t_1) = \alpha\eta(t_1) > 0,$$

there exists $\bar{t}_1 > t_1$ such that $(\xi(t), \eta(t))$ lies in the second quadrant and $\eta'(t) > 0$ for $t_1 < t \leq \bar{t}_1$. We claim that there exists $\tilde{t}_1 > \bar{t}_1$ such that $(\xi(t), \eta(t))$ lies in the second quadrant for $t_1 < t \leq \tilde{t}_1$ and $\eta'(\tilde{t}_1) < 0$. If this is not the case then in the maximal existence interval $[t_1, T)$ of $(\xi(t), \eta(t))$, we have $\xi'(t) < 0$ and $\eta'(t) \geq 0$. Hence $\xi(t)$ is strictly decreasing and $\eta(t)$ is increasing in $t \in [t_1, T)$. Therefore, for $t_1 \leq t < T$,

$$-\beta\xi(t) + |\xi(t)|^{p-2}\xi(t) + \alpha\eta(t) \geq 0, \tag{5.7}$$

which implies

$$\xi'(t) = -\eta(t) \leq -\frac{\beta}{\alpha}\xi(t) + \frac{1}{\alpha}|\xi(t)|^{p-2}\xi(t) \leq \frac{1}{\alpha}|\xi(t)|^{p-2}\xi(t). \tag{5.8}$$

For $t \in (\bar{t}_1, T)$, integrating this inequality from $\bar{t}_1$ to t, we obtain

$$\int_{\xi(\bar{t}_1)}^{\xi(t)} \frac{d\xi}{|\xi|^{p-2}\xi} \geq \frac{1}{\alpha}(t - \bar{t}_1),$$

which implies

$$\xi(t) \leq -(C_1 - C_2 t)^{-\frac{1}{p-2}}, \tag{5.9}$$

where $C_1 = |\xi(\bar{t}_1)|^{-(p-2)} + (p-2)\alpha^{-1}\bar{t}_1$ and $C_2 = (p-2)\alpha^{-1}$ are positive. Therefore, $T < +\infty$ and $\lim_{t \to T-0} \xi(t) = -\infty$. For $t \in (\bar{t}_1, T)$, we also have $\eta'(t) \leq \alpha\eta(t)$ and thus $\eta(t) \leq \eta(\bar{t}_1)e^{\alpha(t-\bar{t}_1)}$. Therefore,

$$\lim_{t \to T-0} (-\beta\xi(t) + |\xi(t)|^{p-2}\xi(t) + \alpha\eta(t)) = -\infty,$$

which contradicts (5.7) and shows the claim. From (5.6), we observe that $\xi'(t) \geq -\eta(\tilde{t}_1)$ and $\eta'(t) \leq \eta'(\tilde{t}_1) < 0$ as long as $(\xi(t), \eta(t))$ stays in the second quadrant for $t \geq \tilde{t}_1$. This implies there exists $t_2 > \tilde{t}_1$ such that $(\xi(t), \eta(t))$ stays in the second quadrant for $t_1 < t < t_2$ and $(\xi(t_2), \eta(t_2))$ is on the negative ξ-axis. Repeating the above discussions, we obtain $t_4 > t_3 > t_2$ such that $(\xi(t), \eta(t))$ stays in the third quadrant for $t_2 < t < t_3$, $(\xi(t_3), \eta(t_3))$ is on the negative η-axis, $(\xi(t), \eta(t))$ stays in the fourth quadrant for $t_3 < t < t_4$, and $(\xi(t_4), \eta(t_4))$ is on the positive ξ-axis. Then $\xi(t_4) > 0$ and $\eta(t_4) = 0$. Denote $\xi_1 = \xi(t_4)$. Then $\xi_1 > \xi_0$ since $\frac{1}{2}\eta^2(t) + \frac{1}{p}|\xi(t)|^p - \frac{\beta}{2}\xi^2(t)$ is strictly increasing when $\eta(t) \neq 0$.

(b) If $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$ then $\beta > 0$. In this case the argument is similar and we only indicate the main differences. Assume $\xi_0 \geq (\frac{p\beta}{2})^{1/(p-2)}$ and consider the the phase curve $(\xi(t), \eta(t))$ of (5.6) such that $(\xi(0), \eta(0)) = (\xi_0, 0)$. Noting that the energy function takes its maximal value 0 in the interval $[0, \beta^{1/(p-2)}]$ on ξ-axis, that it takes nonnegative value at $(\xi_0, 0)$, and that it is strictly increasing along phase curve when $\eta \neq 0$, we see that $(\xi(t), \eta(t))$ can not reach the ξ-axis before it reaches the positive η-axis. Therefore, similar to the first paragraph, there is $t_1 > 0$ such that $(\xi(t_1), \eta(t_1))$ is on the positive η-axis, and $(\xi(t), \eta(t))$ stays in the first quadrant for $0 < t < t_1$. To study the behavior of the phase curve in the second quadrant, we first observe that the curve must cross the line $\xi = -\beta^{1/(p-2)}$ from the right to the left. Therefore, we can assume that the time $\bar{t}_1$ selected above also satisfies $\xi(\bar{t}_1) < -\beta^{1/(p-2)}$. To prove the claim in the second paragraph, instead of (5.8), we have, for some $c > 0$,

$$\xi'(t) \leq c|\xi(t)|^{p-2}\xi(t), \quad \bar{t}_1 \leq t < T,$$

which also leads to a contradiction. The rest of the argument is the same as above. This is the end of the proof.

Remark 5.1 a) According to Lemma 5.1, any orbit of (5.3) exists for all $s \in \mathcal{R}^1$. b) If $a = 0$, $b = +\infty$ and $\mu \leq -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$ then the equation (1.2) has no positive solution.

Lemma 5.2 *Assume* $\mu < -\frac{(N-2)^2}{4}$. *Then any solution* u *of* (1.2) *changes sign infinitely many times near* $r = +\infty$.

Proof. We note that $\mu < -\frac{(N-2)^2}{4}$ implies $\beta < 0$. Therefore (5.3) only has one equilibrium point $O(0, 0)$. The linearized matrix at $O(0, 0)$ is

$$L = \begin{pmatrix} 0 & 1 \\ \beta & -\alpha \end{pmatrix}.$$

It is easy to see that $\mathrm{tr}(L) = -\alpha < 0$, $\det(L) = -\beta > 0$ and

$$\Delta = \alpha^2 + 4\beta = (N - 2)^2 + 4\mu < 0.$$

Thus $O(0, 0)$ is a spiral point of the linearization system of (5.3). By classical perturbation theory (see, for instance, [4]), $O(0, 0)$ is also a spiral point of the nonlinear system (5.3). Therefore, along any trajectory, x changes sign infinitely many times near $s = +\infty$. Then any solution u of (1.2) also changes sign infinitely many times near $r = +\infty$. This is the end of the proof.

Lemma 5.3 *If* $\mu \geq -\frac{(N-2)^2}{4}$, *then any solution* u *of* (1.2) *does not change sign near* $+\infty$. *If* $-\frac{(N-2)^2}{4} \leq \mu \leq -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$, *then any solution* u *of* (1.2) *satisfies* $u(r) = O((r^2 \ln r)^{-1/(p-2)})$ *as* $r \to +\infty$.

Proof. Assume $\mu \geq -\frac{(N-2)^2}{4}$ and let $\tilde{v}$ be the solution of (4.2) obtained in section 4 which is positive near $t = 0$. Then the solution $\tilde{u}$ of (1.2) defined by

$$t = \frac{2}{2 + \theta^-} r^{-\frac{\theta^- + 2}{2}}, \quad \tilde{v}(t) = r^{-\lambda^-} \tilde{u}(r).$$

is positive near $r = +\infty$, and along the trajectory of (5.3) corresponding to $\tilde{u}$, $x(s)$ is positive near $s = +\infty$. Therefore, along any trajectory of (5.3), $x(s)$ does not change sign near $s = +\infty$. Consequently, any solution of (1.2) does not change sign near $r = +\infty$. Assume $-\frac{(N-2)^2}{4} \leq \mu \leq -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$ and let u be any solution of (1.2). Since $v(s)$ defined as in (5.1) does not change sign near $s = +\infty$, we may assume that $v(s)$ is positive for s large enough. From (5.2) we see that if $v(s) > 0$ and $v'(s) = 0$ then $v''(s) < 0$. Therefore, there exists $s_0 > 0$ such that if $s \geq s_0$ then $v(s) > 0$ and $v'(s) < 0$. For $s \geq s_0$, we have

$$-v'' - \alpha v' \geq v^{p-1},$$

which implies

$$-e^{\alpha s} v'(s) > -e^{\alpha\tau} v'(\tau)\Big|_{\tau=s_0}^{\tau=s} \geq \int_{s_0}^{s} e^{\alpha\tau} v^{p-1}(\tau) d\tau > \frac{1}{\alpha} v^{p-1}(s)(e^{\alpha s} - e^{\alpha s_0}).$$

Therefore, there exists $s_1 > s_0$ such that for $s \geq s_1$,

$$-v' v^{1-p} \geq \frac{1}{\alpha}(1 - e^{\alpha(s_0 - s)}) \geq \frac{1}{2\alpha}.$$

For $s > s_1$, integrating the last inequality from s_1 to s, we obtain $v(s) = O(s^{-\frac{1}{p-2}})$ as $s \to +\infty$. According to the transformation (5.1), u satisfies

$$u(r) = O((r^2 \ln r)^{-1/(p-2)}), \quad \text{as } r \to +\infty,$$

which finishes the proof. This is the end of the proof.

If $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$, then $\beta > 0$ and (5.3) has three equilibrium points, $O(0,0)$, $P_+(\beta^{\frac{1}{p-2}}, 0)$ and $P_-(-\beta^{\frac{1}{p-2}}, 0)$. It is easy to see that $O(0,0)$ is a saddle point of the linearization system of (5.3) at $O(0,0)$. By classical perturbation theory (see [4]), $O(0,0)$ is a saddle point of (5.3). $P_\pm$ are the global minimizers of the energy function H in the phase plane.

Lemma 5.4 *Assume* $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$, $a = 0$ *and* $b = +\infty$. *Then* (1.2) *has exactly two positive solutions* u_1 *and* u_2 *and two negative solutions* $-u_1$ *and* $-u_2$. *Moreover,* $u_1(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}$, $u_2(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1 + o(1))$ *near* $r = +\infty$, *and* u_2 *has order* r^{λ^+} *near* $r = 0$. *Here* u_2 *stands for a class of solutions which are equivalent.*

Proof. Since $O(0,0)$ is a saddle point of (5.3) and $P_\pm$ are the only global minimizers of the energy function H in the phase plane, there is a unique heteroclinic orbit connecting $O(0,0)$ and $P_+(\beta^{\frac{1}{p-2}}, 0)$. This heteroclinic orbit lies in the half plane $\{(x, y) \in \mathcal{R}^2 | x > 0\}$ since the energy function is decreasing along the orbit and is nonnegative on the y-axis. Therefore, this heteroclinic orbit defines a family of positive solutions to problem (1.2), which, by the uniqueness of the heteroclinic, are equivalent to the solution u obtained in Lemma 2.5. Accordingly, (1.2) has exactly two positive solutions u_1 and u_2 and two negative solutions $-u_1$ and $-u_2$, u_1 corresponding to the equilibrium point $P_+(\beta^{\frac{1}{p-2}}, 0)$ and u_2 corresponding to the unique heteroclinic orbit connecting $O(0,0)$ and $P_+(\beta^{\frac{1}{p-2}}, 0)$. Thus, $u_1(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}$, $u_2(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1 + o(1))$ as $r \to +\infty$, and $u_2(r)$ has order r^{λ^+} near $r = 0$. This is the end of the proof.

Now we are ready to prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. In the case $\mu < -\frac{(N-2)^2}{4}$, the result has been proved in Lemma 5.2. Now we assume $\mu \geq -\frac{(N-2)^2}{4}$. Then according to Lemma 5.3, any solution of (1.2) changes sign only finitely many times. Let $\tilde{u}$ be the solution of (1.2) with $a = 0$ and $b = +\infty$ obtained in Section 4 and u_n be the solution of (1.2) with $a > 0$ and $b = +\infty$ defined in the proof of Lemma 4.2. For any n, according to the discussion in Section 4, $\pm u_n$ are the only solutions with exactly n zeros in $(a, +\infty)$ and with order r^{λ^-} near $+\infty$. Also, $(-1)^n u_n'(a) > 0$ and $u_0, u_1, \cdots, u_n, \cdots$ are equivalent. Denote by $(\tilde{x}(s), \tilde{y}(s))$ the orbit of (5.3) corresponding to $\tilde{u}$. This orbit lies in the right half plane for s large enough and there exists a decreasing sequence $\{s_n\}$ with $\lim_{n\to+\infty} s_n = -\infty$ such that $\{(\tilde{x}(s_n), \tilde{y}(s_n))\}$ are all the intersection points of this orbit with the y-axis. Denote $y_0 = 0$ and $y_n = (-1)^{n-1} y(s_n)$. Observe that $\{y_n\}$ is a strictly increasing sequence and any orbit $(x(s), y(s))$ of (5.3) with $y(0) \in [-y_{n+1}, -y_n) \cup (y_n, y_{n+1}]$ intersects with the y-axis exactly n times in $s \in (0, +\infty)$. Then $u(r) = r^{-\frac{2}{p-2}} x(\ln r)$ corresponding to $(x(s), y(s))$ with $y(0) \in [-y_{n+1}, -y_n) \cup (y_n, y_{n+1}]$ is a solution of (1.2) with $a = 1$ and $b = +\infty$. Therefore, using the transformation $u_1(r) = a^{-\frac{2}{p-2}} u(\frac{r}{a})$ one sees that (1.2) with $a > 0$ and $b = +\infty$ has infinitely many solutions with exactly n zeros in $(a, +\infty)$. This is the end of the proof.

Proof of Theorem 1.4. In the case $\mu < -\frac{(N-2)^2}{4}$, the result is a consequence of combination of Lemmas 5.1 and 5.2. If $-\frac{(N-2)^2}{4} \leq \mu \leq -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$, then according to Lemmas 5.1 and 5.3, the set of zeros of any solution of (1.2) is a decreasing sequence $\{r_n\}_1^{+\infty}$ which satisfies $\lim_{n\to+\infty} r_n = 0$, and any solution of (1.2) satisfies $u(r) = O((r^2 \ln r)^{-1/(p-2)})$ near $+\infty$. That (1.2) has exactly one pair of solutions with order r^{λ^-} near $+\infty$ has been obtained in Lemma 4.2. If $\mu > -\frac{2}{p-2}(N - 2 - \frac{2}{p-2})$, then, thanks to Lemma 5.4, (1.2) has exactly two positive solutions u_1, u_2 and two negative solutions $-u_1, -u_2$, $u_1(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}$, $u_2(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1 + o(1))$ near $r = +\infty$, and u_2 has order r^{λ^+} near $r = 0$. By lemmas 5.1 and 5.3, the set of zeros of any sign changing solution of (1.2) is a decreasing sequence $\{r_n\}_1^{+\infty}$ which satisfies $\lim_{n\to+\infty} r_n = 0$. According to the discussions in Section 4, (1.2) has exactly one pair of sign changing solutions with order r^{λ^-} near $+\infty$ which corresponds to the unique pair of orbits of $(x(s), y(s))$ of (5.3) satisfying $\lim_{s\to+\infty}(x(s), y(s)) = (0, 0)$. Any other orbit of (5.3) satisfies either $\lim_{s\to+\infty}(x(s), y(s)) = P_+$ or $\lim_{s\to+\infty}(x(s), y(s)) = P_-$. Therefore, any other sign changing solution of (1.2) satisfies either $u(r) = \beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1 + o(1))$ or $u(r) = -\beta^{\frac{1}{p-2}} r^{-\frac{2}{p-2}}(1 + o(1))$ near $r = +\infty$. This is the end of the proof.

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis, **14** (1973), 349–381.

[2] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math., **41** (1988), 255-294.

[3] T. Bartsch and M. Willem, *Infinitely many radial solutions of a semilinear elliptic problem on* $\mathcal{R}^N$, Arch. Rational Mech. Anal., **124** (1993), 261–276.

[4] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations*, 7th ed., John Wiley & Sons, Inc., 2001.

[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), 437–477.

[6] D. Cao and S. Peng, *A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms*, J. Differential Equations, **193** (2003), 424–434.

[7] D. Cao and S. Peng, *A global compactness result for singular elliptic problems involving critical Sobolev exponent*, Proc. Amer. Math. Soc., **131** (2003), 1857–1866.

[8] C. Cortázar, M. García-Huidobro, and C. S. Yarur, *On the uniqueness of the second bound state solution of a semilinear equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[9] E. N. Dancer and J. Wei, *Sign-changing solutions for supercritical elliptic problems in domains with small holes*, Manuscripta Math., **123** (2007), 493–511.

[10] J. Dávila, M. del Pino, and M. Musso, *The supercritical Lane-Emden-Fowler equation in exterior domains*, Comm. PDEs., **32** (2007), 1225–1243.

[11] M. del Pino, *Nonlinear elliptic problems above criticality*, Milan J. Math., **74** (2006), 313–338.

[12] M. del Pino and J. Wei, *Supercritical elliptic problems in domains with small holes*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **24** (2007), 507–520.

[13] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathcal{R}^N$, J. Math. Pures Appl., **87** (2007), 537–561.

[14] P. Felmer, S. Martinez, and K. Tanaka, *Uniqueness of radially sysmmetric positive solutions for* $-\Delta u + u = u^p$ *in an annulus*, J. Differential Equations, **245** (2008), 1198–1209.

[15] R. H. Fowler, *Further studies on Emden's and similar differential equations*, Quart. J. Math., **2** (1931), 259–288.

[16] Y. Ge, R. Jing, and F. Pacard, *Bubble towers for supercritical semilinear elliptic equations*, J. Funct. Anal., **221** (2005), 251–302.

[17] N. Ghoussoub and C. Yuan, *Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc., **352** (2000), 5703–5743.

[18] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., **34** (1981), 528–598.

[19] C. Gui, W.-M. Ni, and X. Wang, *On the stsbility and instability of positive steady states of a semilinear heat equation in* $\mathcal{R}^N$, Comm. Pure Appl. Math., **45** (1992), 1153–1181.

[20] S. Hashimoto and M. Ôtani, *Existence of nontrivial solutions for some elliptic equations with supercritical nonlinearity in exterior domains*, Disc. Cont. Dyn. Sys., **19** (2007), 323–333.

[21] R. Johnson, X.-B. Pan, and Y. Yi, *Positive solution of supercritical elliptic equations and asymptotics*, Comm. PDEs., **18** (1993), 1153–1181.

[22] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal., **49** (1973), 321–380.

[23] W.-M. Ni and R. Nussbaum, *Uniqueness and nonuniqueness for positive solutions of* $\Delta u + f(u,r) = 0$, Comm. Pure Appl. Math., **38** (1985), 67–108.

[24] W.-M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math., **39** (1986), 379–399.

[25] D. Passaseo, *Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains*, Duke Math. J., **92** (1998), 429–457.

[26] S. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., **6** (1965), 1408-1411.

[27] P. Pucci and R. Servadei, *Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25** (2008), 505–537.

[28] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., **7** (1971), 487–513.

[29] P.H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mount. J. Math., **3** (1973), 161–202.

[30] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities*, Trans. Amer. Math. Soc., **357** 2004, 2909–2938.

[31] S. Terracini, *On positive entire solutions to a class of equations with a singular coefficient and critical exponent*, Adv. Differential Equations, **1** (1996), 241–264.