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Anisotropic Variable Exponent $(p(\cdot), q(\cdot))$-Laplacian with Large Exponents

Mayte Pérez-Llanos*

Department of Mathematics

Universidad Autónoma de Madrid, Cantoblanco, Madrid 28049, Spain

e-mail: mayte.perez@uam.es

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Abstract

In this work we consider the following Dirichlet problem corresponding to the anisotropic $(p(\cdot), q(\cdot))$−Laplacian operator with variable exponents

$$\begin{cases} -\mathrm{div}_x(|\nabla_x u|^{p(x,y)-2}\nabla_x u) - \mathrm{div}_y(|\nabla_y u|^{q(x,y)-2}\nabla_y u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and study the behaviour of the solutions as the exponents $p(x, y), q(x, y) \to \infty$ uniformly in $\Omega \subset \mathbb{R}^{N+K}$. We denote by $\nabla_x u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N})$ and similarly $\nabla_y u = (\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \ldots, \frac{\partial u}{\partial y_K})$ the gradients of u with respect to the first N variables (x variables) and with respect to the last K variables (y variables), respectively. We consider sequences of exponents $p_n(x, y), q_n(x, y)$ going to infinity uniformly in Ω and verifying suitable assumptions. We prove that u_n, the solutions with $p(x, y) = p_n(x, y)$, $q(x, y) = q_n(x, y)$, converge uniformly in $\overline{\Omega}$ to some nontrivial limit, u_∞, as $n \to \infty$. We determine the limit equation verified by u_∞ as well as some properties of this limit. Finally, we focus on the case $f = 0$ (with nontrivial Dirichlet boundary conditions, $u = g \not\equiv 0$ on $\partial\Omega$) and show uniqueness for the limit problem.

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1 Introduction

In this work we are interested in describing the behaviour of the solutions to the Dirichlet problem for the anisotropic $(p(\cdot), q(\cdot))$−Laplacian operator, as both functions $p, q \to \infty$ uniformly in Ω. More precisely, we consider the following sequence of problems,

$$\begin{cases} -\mathrm{div}_x(|\nabla_x u_n|^{p_n(x,y)-2}\nabla_x u_n) - \mathrm{div}_y(|\nabla_y u_n|^{q_n(x,y)-2}\nabla_y u_n) = f, & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $(x, y) \in \Omega \subset \mathbb{R}^{N+K}$ and Ω being a bounded and smooth domain ($\Omega \in C^1$). By $\nabla_x u$ and $\nabla_y u$ we understand the derivatives of u with respect to the first N variables and with respect to the last K variables, respectively, namely, $\nabla_x u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$ and $\nabla_y u = (\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \dots, \frac{\partial u}{\partial y_K})$. Thus the complete gradient reads as $\nabla u_n = (\nabla_x u_n, \nabla_y u_n)$. As f we take a continuous fixed function. In what follows we will denote as $z = (x, y) \in \mathbb{R}^{N+K}$.

Concerning the exponents, let us state our assumptions on the sequences p_n, q_n. We will assume that p_n, q_n are sequences of C^1 functions in $\overline{\Omega}$ such that

$$p_n(z), q_n(z) \to +\infty, \quad \text{uniformly in } \Omega. \tag{1.2}$$

The following quotients are bounded

$$\limsup_{n\to\infty} \frac{p_n^+}{p_n^-} \le k_1, \quad \limsup_{n\to\infty} \frac{q_n^+}{q_n^-}, \le k_2, ; \tag{1.3}$$

where for some function g we denote

$$g^- = \min_{z\in\overline{\Omega}} g(z), \qquad g^+ = \max_{z\in\overline{\Omega}} g(z). \tag{1.4}$$

We assume that both $p^-,\ q^- > 1$. Since we are interested in the limit as $p, q \to \infty$, without loss of generality, we suppose that $p_n^-, q_n^- > N + K$ for every n in the last sections. We also assume that there exists a positive and continuous function $0 < \theta < \infty$ such that

$$\lim_{n\to\infty} \frac{q_n(z)}{p_n(z)} = \theta(z), \ \text{ uniformly in } \ \Omega. \tag{1.5}$$

Moreover, we suppose that the following limits exist

$$\lim_{n\to\infty} \nabla_x(\ln p_n(z)) = \xi_p(z), \ \text{and} \ \lim_{n\to\infty} \nabla_y(\ln q_n(z)) = \xi_q(z), \ \text{uniformly in } \Omega, \tag{1.6}$$

being $\xi_p \in C(\Omega; \mathbb{R}^N)$ and $\xi_q \in C(\Omega; \mathbb{R}^K)$.

Let us now present some examples of possible sequences $p_n(z), q_n(z)$. We give special attention to hypothesis (1.5), which concerns to the anisotropy of the problem and is not present in previous works, such as [28, 30].

i) Our analysis extends the (constant) pseudo p-laplacian case studied in [2], by taking $p_n(z) = q_n(z) = n$; we have $\xi_p(z) = \xi_q(z) = 0$, $k_1 = k_2 = 1$ and $\theta(z) = 1$.

ii) If we choose $p_n(z) = p(z) + n$ and $q_n(z) = q(z) + n$ for some fixed positive functions p, q in $C^1(\Omega) \cap C(\overline{\Omega})$, we get a limit problem similar to the previous one, with $\xi_p(z) = \xi_q(z) = 0$, $k_1 = k_2 = 1$ and $\theta(z) = 1$.

iii) To obtain nontrivial vector fields ξ_p, ξ_q and a real anisotropic example, that is $\theta(z) \not\equiv 1$, let us consider the following exponents: $p_n(z) = np(z)$ and $q_n(z) = nq(z)$, for some fixed positive functions p, q in $C^1(\Omega) \cap C(\overline{\Omega})$. In this case

$$\xi_p(z) = \nabla_x(\ln(p(z))), \quad \xi_q(z) = \nabla_y(\ln(q(z))),$$

and

$$k_1 = \frac{p^+}{p^-}, \quad k_2 = \frac{q^+}{q^-}, \quad \theta(z) = \frac{q(z)}{p(z)}.$$

iv) We construct another example of an anisotropic limit by taking $p_n(z) = n^a p(x/n, y)$ [scaling in x] and $q_n(z) = n^a q(x, y/n)$ [scaling in y] with p, q being as in the previous examples; (the power a should be the same for both p_n and q_n to satisfy (1.5)). For this choice, we have that

$$\nabla_x(\ln p_n(x/n, y)) = \frac{\nabla_x p}{p}(x/n, y)\frac{1}{n} \to 0,$$
$$\nabla_y(\ln q_n(x, y/n)) = \frac{\nabla_y q}{q}(x, y/n)\frac{1}{n} \to 0$$

and thus $\xi_p(z) = \xi_q(z) = 0$. Moreover,

$$k_1 = \frac{\max_{(0,y)\in\overline{\Omega}} p}{\min_{(0,y)\in\overline{\Omega}} p}, \quad k_2 = \frac{\max_{(x,0)\in\overline{\Omega}} q}{\min_{(x,0)\in\overline{\Omega}} q} \quad \text{and} \quad \theta(z) = \frac{q(x,0)}{p(0,y)}.$$

The same conclusion holds for $p_n(z) = n^a + p(x/n, y)$, and $q_n(z) = n^a + q(x, y/n)$.

v) We note that taking $p_n(z) = n^a p(nx, y)$ and $q_n(z) = n^a q(x, ny)$ with p, q in $C^1(\Omega) \cap C(\overline{\Omega})$; we get

$$\nabla_x(\ln p_n(x, y)) = \frac{n\nabla_x p}{p}(nx),$$

which does not have a limit as $n \to \infty$. We either do not have limit for the gradient $\nabla_y(\ln q_n(x, ny))$.

The same happens with $p_n(z) = n + p(nx, y)$ and $q_n(z) = n + q(x, ny)$, for which

$$\nabla_x(\ln p_n(nx, y)) = \frac{n\nabla_x p(nx, y)}{n + p(nx, y)},$$

which does not have a uniform limit (although it is bounded). The analogous could be said about $\nabla_y(\ln q_n(x, ny))$.

vi) We can modify the previous example to get a nontrivial limit. Suppose that $r_i = r_i(\alpha)$ for $i = 1, 2$ are C^1 functions of the angular variable and that $0 \notin \Omega$. Choose $p_n(z) = n + r_1(nz)$ and $q_n(z) = n + r_2(nz)$ to obtain

$$\nabla_x(\ln p_n(nx, ny)) = \frac{n\nabla_x r_1(nx, ny)}{n + r_1(nx, ny)} \to \nabla_x r_1(\alpha),$$

and identically

$$\nabla_y(\ln q_n(nx, ny)) = \frac{n\nabla_y r_2(nx, ny)}{n + r_2(nx, ny)} \to \nabla_y r_2(\alpha),$$

In this case we get $k_1 = k_2 = \theta(z) = 1$.

vii) Finally, we can combine examples (iii) and (vi) to get an anisotropic nontrivial limit. Let $p_n(z) = np(z) + r_1(nz)$ and $q_n(z) = nq(z) + r_2(nz)$, with $0 \notin \Omega$ and $r_i = r_i(\alpha)$ for $i = 1, 2$ be angular functions as in (vi). We get

$$\nabla_x(\ln p_n(x, y)) = \frac{n\nabla_x p(x, y) + n\nabla_x r_1(nx, ny)}{np(x, y) + r_1(nx, ny)} \to \frac{\nabla_x p(x, y) + \nabla_x r_1(\alpha)}{p(x, y)}$$

and similarly

$$\nabla_y(\ln q_n(x, y)) \to \frac{\nabla_y q(x, y) + \nabla_y r_2(\alpha)}{q(x, y)}.$$

In this case we obtain the same conditions (1.3) and (1.5) as in iii), that is,

$$k_1 = \frac{p^+}{p^-}, \quad k_2 = \frac{q^+}{q^-}, \quad \theta(z) = \frac{q(z)}{p(z)}.$$

Our main result could be summarized as follows:

Theorem 1.1 *Let u_n be the sequence of solutions to* (1.1) *with $f \in C(\overline{\Omega})$ fixed and $p_n, q_n \to \infty$ uniformly in Ω, verifying* (1.5)*. Then, up to a subsequence, $u_n \to u_\infty$ uniformly in Ω. Moreover, the limit u_∞ verifies*

$$\max\{\|\nabla_x u_\infty\|_{L^\infty(\Omega)}, \|\nabla_y u_\infty\|_{L^\infty(\Omega)}\} \leq 1,$$

and is a maximizer of the following problem

$$\max_{\mathcal{K}} \int_\Omega f v \, dx,$$

with

$$\mathcal{K} = \{v \in W_0^{1,\infty}(\Omega), \max\{\|\nabla_x v\|_{L^\infty(\Omega)}, \|\nabla_y v\|_{L^\infty(\Omega)}\} \leq 1\}.$$

In addition, u_∞ verifies in the viscosity sense the following,

$$
\begin{aligned}
&H_\infty(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) = 0, && \textit{in } \Omega \setminus \textit{supp } f,\\
&\max\{|\nabla_x u_\infty|, |\nabla_y u_\infty|\} = 1, && \textit{in } \{f > 0\},\\
&-\max\{|\nabla_x u_\infty|, |\nabla_y u_\infty|\} = -1, && \textit{in } \{f < 0\},\\
&H_\infty(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) \geq 0, && \textit{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\},\\
&H_\infty(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) \leq 0, && \textit{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\},
\end{aligned}
$$

with the function H_∞ defined as follows

- *if* $|\nabla_y u_\infty|^{\theta(z)} < |\nabla_x u_\infty|$,

$$H_\infty = -\Delta_{\infty,x} u_\infty - |\nabla_x u_\infty|^2 \log(|\nabla_x u_\infty|)\langle \xi_p, \nabla_x u_\infty\rangle := -\Delta_{\infty(x)} u_\infty,$$

- *if* $|\nabla_y u_\infty|^{\theta(z)} > |\nabla_x u_\infty|$

$$H_\infty = -\theta\Delta_{\infty,y} u_\infty - \theta|\nabla_y u_\infty|^2 \log(|\nabla_y u_\infty|)\langle \xi_q, \nabla_y u_\infty\rangle := -\theta\Delta_{\infty(y)} u_\infty,$$

- *if* $|\nabla_y u_\infty|^{\theta(z)} = |\nabla_x u_\infty|$

$$
\begin{aligned}
H_\infty &= -\Delta_{\infty,x} u_\infty - |\nabla_x u_\infty|^2 \log(|\nabla_x u_\infty|)\langle \xi_p, \nabla_x u_\infty\rangle\\
&\quad -\theta\Delta_{\infty,y} u_\infty - \theta|\nabla_y u_\infty|^2 \log(|\nabla_y u_\infty|)\langle \xi_q, \nabla_y u_\infty\rangle\\
&= -\Delta_{\infty(x)} u_\infty - \theta\Delta_{\infty(y)} u_\infty.
\end{aligned}
$$

Here

$$\theta(z) = \lim_{n\to+\infty} \frac{q_n(z)}{p_n(z)} \in (0, +\infty),$$

the vectorial functions ξ_p, ξ_q are defined in (1.6) *and* $\Delta_{\infty,x} u = \langle D_x^2 u \nabla_x u, \nabla_x u\rangle$, $\Delta_{\infty,y} u = \langle D_y^2 u \nabla_y u, \nabla_y u\rangle$ *are the infinity Laplacian in x variables and in y variables, respectively.*

Remark 1.1 We observe that the limit problem depends strongly on the function θ appearing in the limit, which measures the anisotropy of the problem, see (1.5). This has to be contrasted with the limit problem related to the (constant) pseudo p-Laplacian, in [2].

Moreover, even in the case we take $p(z) = q(z)$ for all $z \in \Omega$, (pseudo $p(\cdot)$-Laplacian), the problem differs from the problem in [2] through the additional logarithmic terms involved in the limit function H_∞,

$$-|\nabla_x u_\infty|^2 \log(|\nabla_x u_\infty|)\langle \xi_p(z), \nabla_x u_\infty\rangle \text{ and } -\theta|\nabla_y u_\infty|^2 \log(|\nabla_y u_\infty|)\langle \xi_q(z), \nabla_y u_\infty\rangle.$$

Remark 1.2 We also want to emphasize an important difference between the problem in consideration and other previous problems related to the $p(\cdot)$-Laplacian (without anisotropy), see for instance [28, 30]. And it is the fact that for our problem, as well as for anisotropic problems treated in [2, 7, 17, 29], it happens that the limit function H_∞ is not continuous. Therefore, the concept of viscosity solutions has to be redefined considering a relaxation of those discontinuos equations, see Definitions 3.1 and 3.2 below.

We will conclude this work by focusing in the limit problem corresponding to $f = 0$ with nontrivial boundary conditions, namely

$$\begin{cases} H_\infty(D_x^2 u, D_y^2 u, \nabla_x u, \nabla_y u) = 0 & \text{in } \Omega, \\ u = g \not\equiv 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with the function H_∞ given in Theorem 1.1. We show that there is uniqueness of solutions to those problems. Thus in case we take limit in the problem $\Delta_{(p_n(\cdot),q_n(\cdot))} u_n = 0$ as $n \to \infty$ the whole sequence of solutions converges uniformly to the unique limit u_∞. The optimal regularity in this case is $u_\infty \in W^{1,\infty}(\Omega)$, since for the constant case, see [29], the function

$$u(x, y) = x + \frac{1}{2}|y|,$$

is a viscosity solution to (1.7), that has no further regularity than Lipschitz.

Let us end the introduction with a brief description of previous results. The infinity Laplacian operator $\Delta_\infty u = \langle D^2 u \nabla u, \nabla u\rangle$, appears naturally when possing the Lipschitz extension problem, closely related to the concept of AMLE (*absolutely minimizing Lipschitz extensions*), see [1, 25, 38]. In [1] the minimization problem of finding an AMLE is translated to a variational problem in $L^p(\Omega)$ spaces. Namely, solutions to

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ u_p = f & \text{on } \partial\Omega, \end{cases}$$

verify that $u_p \to u_\infty$ in uniformly in $C(\Omega)$ as $p \to \infty$, where the limit u_∞ is the desired AMLE and satisfies

$$\begin{cases} -\Delta_\infty u_\infty = 0 & \text{in } \Omega, \\ u_\infty = f & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

The issue of uniqueness of solutions to problem (1.8) is solved in [18], where it is shown that the appropriate framework for this matter consists of considering viscosity solutions to (1.8).

Recently a great number of contributions in this topic have appeared, due to the multitude of applications that this kind of problems offers, such as in image processing, [4, 34]; mass transfer and optimal transportation, [10, 13, 14]; Tug-of-War games, [26, 27] and the survey [31], etc.

Another important feature here is the anisotropic geometry of the problem in consideration, fact that involves the development of new techniques, for instance, for comparison or regularity matters, see [32, 36, 37] and references therein.

The rest of the paper is organized as follows. In the next section we introduce some notation and preliminary results of variable exponents spaces. We also show the existence of weak solutions to (1.1) using variational methods. In Section 3 we state the precise definition of viscosity solutions. Section 4 is devoted to study some a priori estimates in the variable Sobolev spaces, which ensure that u_n converge to some function as the exponents go to infinity. Moreover, we show that this limit turns out to be a maximizer of a variational problem. In the following section we determine the equation satisfied by this limit and give some explicit examples in special cases. Finally, in Section 6 we focus on the case $f = 0$ and prove our uniqueness result.

2 Anisotropic variable exponent Sobolev spaces and weak solutions

In order to establish a natural frame for our problem, let us give some brief introduction to variable exponent Sobolev and Lebesgue spaces and some of their main properties. Then we will introduce the natural anisotropic variable exponent Sobolev space related to our problem. See for example [6, 8, 11, 23], the survey [15] and the book [9] for more details. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as follows

$$L^{p(\cdot)}(\Omega) = \left\{ u \ \text{ such that } u : \Omega \to \mathbb{R} \text{ is measureable and } \int_\Omega |u(z)|^{p(z)}\, dz < +\infty \right\},$$

and is endowed with the norm

$$|u|_{p(\cdot)} = \inf\left\{ \tau > 0 \ \text{ such that } \int_\Omega \left| \frac{u(z)}{\tau} \right|^{p(z)} dz \leq 1 \right\}.$$

The anisotropic variable exponent Sobolev space $W^{1,p(\cdot),q(\cdot)}(\Omega)$ is given by

$$W^{1,p(\cdot),q(\cdot)}(\Omega) = \left\{ u \in L^\sigma(\Omega) \ : \ |\nabla_x u| \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla_y u| \in L^{q(\cdot)}(\Omega) \right\},$$

for some $1 \leq \sigma \leq \max\{p^-, q^-\}$ with the norm

$$\|u\| = \|u\|_{L^\sigma(\Omega)} + \inf\left\{ \tau > 0 \ : \ \int_\Omega \left(\left| \frac{\nabla_x u(z)}{\tau} \right|^{p(z)} + \left| \frac{\nabla_y u(z)}{\tau} \right|^{q(z)} \right) dz \leq 1 \right\}.$$

We denote by $W_0^{1,p(\cdot),q(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ with the norm above.

For any $u \in L^{p(\cdot)}(\Omega)$ the modular is defined as

$$\rho_{p(\cdot)}(u) = \int_\Omega |\nabla u|^{p(z)} dz.$$

We enclose in the following proposition some well known properties of the variable exponent Sobolev and Lebesgue spaces, that will be used in the future. See for instance the book [9].

Proposition 2.1

i) *The modular $\rho_{p(\cdot)}$ and the norm $|\cdot|_{p(\cdot)}$ are lower semicontinuous with respect to (sequential) weak convergence and almost everywhere convergence.*

ii) *If $|u|_{p(\cdot)} > 1$ then $|u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+}$. If conversely $|u|_{p(\cdot)} < 1$, it holds that $|u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}$.*

iii) *The spaces $\left(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)}\right)$, $\left(W^{1,p(\cdot),q(\cdot)}(\Omega), \|\cdot\|\right)$ and $\left(W_0^{1,p(\cdot),q(\cdot)}(\Omega), \|\cdot\|\right)$ are separable, reflexive and uniformly convex Banach spaces.*

iv) Hölder's inequality holds, namely

$$\int_\Omega |uv|\,dz \le 2|u|_{r(\cdot)}|v|_{r'(\cdot)}, \quad \forall u \in L^{r(\cdot)}(\Omega),\ \forall v \in L^{r'(\cdot)}(\Omega),$$

with $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$.

v) If $p, q \in C(\overline{\Omega})$ *verify that* $p^-, q^- > N$*, then, the imbedding from* $W^{1,p(\cdot),q(\cdot)}(\Omega)$ *to* $C^\beta(\Omega)$*, for some* $\beta < 1$*, is compact and continuous.*

We notice that property iii) above is performed in [9] for the variable Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$. Then, the anisotropic spaces $W^{1,p(\cdot),q(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot),q(\cdot)}(\Omega)$ can be considered as direct sum for the gradient, $\nabla u = (\nabla_x u, \nabla_y u) \in L^{p(\cdot)}(\Omega) \oplus L^{q(\cdot)}(\Omega)$, with the norm defined above, namely $\|u\| = |u|_{p(\cdot)} + |\nabla_x u|_{p(\cdot)} + |\nabla_y u|_{q(\cdot)}$. By the results in [3, 33] these anisotropic spaces are separable, reflexive and uniformly convex Banach spaces.

In the following result we establish a Poincare type inequality for anisotropic spaces with constant exponents, see Theorem 1 in [12].

Proposition 2.2 *For any* $u \in C_0^1(\Omega)$ *and* $r \ge 1$ *there exists some positive* $C = C(|\Omega|)$ *such that*

$$\|u\|_r \le Cr\|\nabla_x u\|_r,$$

where $\nabla_x u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$*, denotes the gradient with respect to the first N variables.*

We conclude these preliminary results on variable Sobolev and Lebesgue spaces with the following result obtained in [28]. We will refer to it several times throughout this work and include the brief proof for convenience of the reader.

Lemma 2.1 *Let* $v \in L^{p(\cdot)}(\Omega)$*. Then, for any* $1 < r < p^-$ *it holds that there exists a positive constant,* $C = C(\Omega)$ *such that*

$$\|v\|^r_{L^r(\Omega)} \le C(|v|_{p(\cdot)})^r.$$

Proof. The result follows just observing that

$$\|v\|^r_{L^r(\Omega)} \le 2|1|_{a'(\cdot)}\||v|^r|_{a(\cdot)} \le 2\max\{1, \mu(\Omega)\}(|v|_{p(\cdot)})^r,$$

where $ra(z) = p(z)$ and $\frac{1}{a(z)} + \frac{1}{a'(z)} = 1$, from applying Hölder inequality for variable exponent Sobolev spaces, see Proposition 2.1.

Now we are ready to show the existence result of solutions to problem (1.1) for fixed n. Namely, we consider the problem

$$\begin{cases} -\mathrm{div}_x(|\nabla_x u|^{p(z)-2}\nabla u) - \mathrm{div}_y(|\nabla_y u|^{q(z)-2}\nabla u) = f(z), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The Dirichlet problem corresponding to the $p(\cdot)$–Laplacian (without the anisotropy) is treated in [11]. We define solutions to (2.1) as follows.

Definition 2.1 *We say that $u \in W_0^{1,p(\cdot),q(\cdot)}(\Omega)$ is a weak solution to problem (2.1) if $\forall v \in C_0^1(\Omega)$ it holds that*

$$\int_\Omega \left(|\nabla_x u|^{p(z)-2}\nabla_x u \nabla_x v + |\nabla_y u|^{q(z)-2}\nabla_y u \nabla_y v\right) dz = \int_\Omega f v\, dz.$$

Theorem 2.1 *Let $f, p, q \in C(\overline{\Omega})$ such that $p^-, q^- > 1$. Then, problem (2.1) has a unique weak solution, which is the unique minimizer of the functional*

$$J(v) = \int_\Omega \left(\frac{1}{p(z)}|\nabla_x v|^{p(z)} + \frac{1}{q(z)}|\nabla_y v|^{q(z)}\right) dz - \int_\Omega f v\, dz. \tag{2.2}$$

in the set $W_0^{1,p(\cdot),q(\cdot)}(\Omega)$.

Proof. First we observe that the assumptions on f ensure that

$$J(v) \geq -\|f\|_{L^r(\Omega)}\|v\|_{L^{p^-}(\Omega)} \geq -C,$$

with r such that $\frac{1}{p^-} + \frac{1}{r} = 1$, i.e. J is bounded from below. Moreover, since the second term is linear and $p, q \in C(\overline{\Omega})$, J inherits the lower semicontinuity of the modular.

Next we show that if u_n is a sequence such that $\|u_n\| \to \infty$, with the norm defined on $W_0^{1,p(\cdot),q(\cdot)}(\Omega)$ it holds that

$$\lim_{\|u_n\|\to\infty} J(u_n) = \infty.$$

Since $p^-, q^- > 1$, there exists some δ verifying $p^-, q^- > \delta > 1$, hence it holds that $W_0^{1,p(\cdot),q(\cdot)}(\Omega) \subset W_0^{1,\delta}(\Omega)$. Set u_n a sequence such that $\|u_n\|_{W_0^{1,\delta}(\Omega)} \to \infty$ as $n \to \infty$. By Proposition 2.2 it implies that

$$\|\nabla_x u_n\|_{L^\delta(\Omega)} \to \infty \quad \text{and} \quad \|\nabla_y u_n\|_{L^\delta(\Omega)} \to \infty.$$

Lemma 2.1 yields that, in fact

$$|\nabla_x u_n|_{p(\cdot)} \to \infty \quad \text{and} \quad |\nabla_y u_n|_{q(\cdot)} \to \infty,$$

thus $\|u_n\| \to \infty$. Therefore, using Propositions 2.1, 2.2, Lemma 2.1 and the fact that $p^-, q^- > 1$, we deduce that

$$\begin{aligned} J(u_n) \quad &\geq \frac{1}{p^+}\rho_{p(\cdot)}(|\nabla_x u_n|) + \frac{1}{q^+}\rho_{q(\cdot)}(|\nabla_y u_n|) - C(f)\|u_n\|_{L^{p^-}(\Omega)} \\ &\geq \frac{1}{p^+}\Big(|\nabla_x u_n|_{p(\cdot)}\Big)^{p^-} + \frac{1}{q^+}\Big(|\nabla_y u_n|_{q(\cdot)}\Big)^{q^-} - C(f,\Omega)p^-\|\nabla_x u_n\|_{L^{p^-}(\Omega)} \\ &\geq \frac{C(\Omega)}{p^+}\Big(\|\nabla_x u_n\|_{L^{p^-}(\Omega)}\Big)^{p^-} + \frac{C(\Omega)}{q^+}\Big(\|\nabla_y u_n\|_{L^{q^-}(\Omega)}\Big)^{q^-} \\ &-C(f,\Omega)p^-\|\nabla_x u_n\|_{L^{p^-}(\Omega)} \to \infty. \end{aligned}$$

Therefore we have shown conditions that ensure the existence of a minimizer of J. The uniqueness follows from the convexity of the functional. Note that

$$(L(u), v) = \int_\Omega \left(|\nabla_x u|^{p(z)-2}\nabla_x u \nabla_x v + |\nabla_y u|^{q(z)-2}\nabla_y u \nabla_y v\right) dz$$

is the derivative of the first term in J. The well known p-Laplacian type inequalities (see [11] and references therein),

$$\left\langle |v|^{r-2}v - |w|^{r-2}w, v - w\right\rangle \geq 2^{2-r}|v - w|^r, \quad r \geq 2, \tag{2.3}$$

$$\left\langle |v|^{r-2}v - |w|^{r-2}w, v - w\right\rangle (|v|^r + |w|^r)^{\frac{2-r}{2}} \geq (r-1)|v - w|^r, \quad 1 < r < 2,$$

for any $v, w \in \mathbb{R}^M$, applied to each operator $p(\cdot)$ and $q(\cdot)$-Laplacian, prove the strict monotonicity of L. It is known, see for instance [35], that it yields the convexity for J, since the second term is linear.

Finally, if $f \not\equiv 0$, let us take some $u \in W_0^{1,p(\cdot),q(\cdot)}(\Omega)$ such that $\int_\Omega f u\, dz < 0$. Then, $J(\lambda u) < 0$ for some $\lambda > 0$ small enough, since $p^-, q^- > 1$. Therefore, the minimizer is not trivial.

3 Viscosity solutions.

This section is devoted to introduce the concept of viscosity solutions to problem (1.1). Suppose that u_n are smooth enough to differentiate equation (1.1). We get then

$$\begin{aligned}
&-|\nabla_x u_n|^{p_n(z)-2} \left(\Delta_x u_n + \log(|\nabla_x u_n|)\langle \nabla_x u, \nabla_x p_n\rangle\right)\\
&\qquad -(p_n(z) - 2)|\nabla_x u_n|^{p_n(z)-4}\langle D_x^2 u_n \nabla_x u_n, \nabla_x u_n\rangle\\
&-|\nabla_y u_n|^{q_n(z)-2} \left(\Delta_y u_n + \log(|\nabla_y u_n|)\langle \nabla_y u, \nabla_y q_n\rangle\right)\\
&\qquad -(q_n(z) - 2)|\nabla_y u_n|^{q_n(z)-4}\langle D_y^2 u_n \nabla_y u_n, \nabla_y u_n\rangle = f(z).
\end{aligned} \tag{3.1}$$

This equation is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it.

We recall now the definition of viscosity sub and supersolution to a nonlinear PDE problem of the form

$$\begin{cases} H(D^2u, \nabla u, z) = 0, & \text{in } \Omega,\\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

In general the function H can be discontinuous. Then we denote by H^* and H_* the upper and lower semicontinuous envelopes of H, respectively, defined as

$$\begin{aligned} H^*(S, w, z) = \lim_{\varepsilon\to 0} \sup \{H(S', w', z') \ :\\ |S - S'| + |w - w'| + |z - z'| < \varepsilon\} \end{aligned}$$

for $z, w \in \mathbb{R}^{N+K}$, and $S \in \mathbb{S}^{N+k}$ (we denote by $\mathbb{S}^L$ the set of symmetric matrices in $\mathbb{R}^{L\times L}$) and

$$H_*(S, w, z) = -(-H)^*(S, w, z).$$

Definition 3.1 *An upper semicontinuous function u defined in Ω is a viscosity subsolution of* (3.2) *if, $u|_{\partial\Omega} \leq g$ and, whenever $z_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u - \psi$ has a maximum at z_0, then*

$$H_*(D^2\psi(z_0), \nabla\psi(z_0), z_0) \leq 0.$$

Definition 3.2 *A lower semicontinuous function u defined in Ω is a viscosity supersolution of* (3.2) *if, $u|_{\partial\Omega} \geq g$ and, whenever $z_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u - \phi$ has a minimum at z_0, then*

$$H^*(D^2\phi(z_0), \nabla\phi(z_0), z_0) \geq 0.$$

In the sequel we will keep the notation used in the above definitions. Namely, by ϕ we will denote the test functions such that $u - \phi$ has a minimum in Ω and by ψ the test functions for which $u - \psi$ has a maximum at some point in Ω.

We refer to [5] for more details about general theory of viscosity solutions, and [19], [22] for viscosity solutions related to the ∞−Laplacian and the p−Laplacian operators. Related to viscosity solutions when H is discontinuos are for example the works [17, 20], in which operators of type normalized ∞−Laplacian are treated.

Now, let $w, z \in \mathbb{R}^{N+K}$, and $S \in \mathbb{S}^{N+K}$. To simplify the notation we will call

$$w_1 = (w^1, ..., w^N), \qquad \text{and} \qquad w_2 = (w^{N+1}, ..., w^{N+K}),$$

so w_1 stands for the first N components of w and w_2 for the last K components. As before, $z = (x, y) = (x_1, \cdots, x_N, y_1 \cdots, y_K)$. Also we denote

$$S_1 = (s_{ij})_{1\leq i,j\leq N}$$

the first $N \times N$ minor of the matrix S and

$$S_2 = (s_{ij})_{N+1\leq i,j\leq N+K}$$

the last $K \times K$ minor of S.

Solutions to (1.1) are to be considered as solutions to

$$\begin{cases} H_n(D_x^2 u_n, D_y^2 u_n, \nabla_x u_n, \nabla_y u_n, z) = 0, & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

in the sense of Definitions 3.1 and 3.2, where H_n is given by

$$\begin{aligned} H_n(S_1, S_2, w_1, w_2, z) = \quad & -|w_1|^{p_n(z)-2}\Big(\mathrm{trace}(S_1) + \log(|w_1|)\langle w_1, \nu_p\rangle\Big) \\ & -(p_n(z)-2)|w_1|^{p_n(z)-4}\langle S_1 w_1, w_1\rangle \\ & -|w_2|^{q_n(z)-2}\Big(\mathrm{trace}(S_2) + \log(|w_2|)\langle w_2, \nu_q\rangle\Big) \\ & -(q_n(z)-2)|w_2|^{q_n(z)-4}\langle S_2 w_2, w_2\rangle - f(z) \end{aligned} \tag{3.4}$$

From now on, our interest will be to study the limit problem as the functions $p, q \to \infty$. Therefore, there is no loss of generality if we assume that $p_n, q_n \geq 2$. In this case H_n is continuous in Ω, and then we have that $H_n = (H_n)_* = (H_n)^*$.

The following result can be shown as in [21]. See also [28] and [29].

Lemma 3.1 *A continuous weak solution to equation* (1.1) *is a viscosity solution to* (3.3).

4 Existence of a limit as $p_n(\cdot), q_n(\cdot)$ go to infinity.

In this Section we find some a priori estimates for the solutions to our problem that are uniform in n. These estimates ensure that, at least for a subsequence of solutions, they converge to some nontrivial limit, that is a maximizer of certain functional. We enclose these results in the following theorem.

Theorem 4.1 *Let $f \in L^{r'}(\Omega)$ for some $r' > 1$. There exists a subsequence $\{u_{n_i}\}$ of solutions to our problem converging to some nontrivial limit u_∞ in $C^\beta(\Omega)$, for some $0 < \beta < 1$. Moreover, u_∞ satisfies*

$$\max\{\|\nabla_x u_\infty\|_{L^\infty(\Omega)}, \|\nabla_y u_\infty\|_{L^\infty(\Omega)}\} \leq 1, \tag{4.1}$$

and it maximizes the following functional

$$\max_{\mathcal{K}} \int_\Omega f v \, dx, \tag{4.2}$$

with $\mathcal{K} = \{v \in W_0^{1,\infty}(\Omega), \max\{\|\nabla_x v\|_{L^\infty(\Omega)}, \|\nabla_y v\|_{L^\infty(\Omega)}\} \leq 1\}$.

Remark 4.1 We observe that properties (4.1) and (4.2) are the same properties verified by the limit of solutions to problem (1.1) for the constant anisotropic (p, q)-Laplacian, see [7, 29]. However here we use different arguments, since we work within variable Sobolev spaces, see [28, 30].

Proof. We first find the following bounds for the solutions to problem (1.1)

$$|\nabla_x u_n|_{p_n(\cdot)} \leq C_1(n) \text{ and } |\nabla_y u_n|_{q_n(\cdot)} \leq C_2(n), \text{ with } C_i(n) \to 1 \text{ as } n \to \infty \tag{4.3}$$

We perform in detail the proof of the boundedness of the norm of the first gradient, since the second term could be treated similarly. If we consider in (2.2) the trivial function, we get

$$\int_\Omega \frac{1}{p_n(z)} |\nabla u_n|^{p_n(z)} dz + \int_\Omega \frac{1}{q_n(z)} |\nabla u_n|^{q_n(z)} dz - \int_\Omega f u_n \, dz \leq 0.$$

In particular, if we take some $r < p_n^-, q_n^-$ (defined in (1.4)) and suppose that $f \in L^{r'}(\Omega)$ (this is not restrictive, since we have assumed (1.2)) the following inequalities hold (as we said we just deal with the first integral)

$$\begin{aligned} \int_\Omega \frac{1}{p_n(z)} |\nabla_x u_n|^{p_n(z)} dz &\leq \int_\Omega f u_n \, dz \\ &\leq \|f\|_{L^{r'}(\Omega)} \|u_n\|_{L^r(\Omega)} \leq C(\Omega, f, r) \|\nabla_x u_n\|_{L^r(\Omega)}, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ to apply Hölder inequality. By Lemma 2.1 we know that

$$\|\nabla_x u_n\|_{L^r(\Omega)} \leq C(\Omega, r) |\nabla_x u_n|_{p_n(\cdot)}, \tag{4.4}$$

hence, we obtain that

$$\int_\Omega \frac{1}{p_n(z)} |\nabla_x u_n|^{p_n(z)} dz \leq C(\Omega, f, r) |\nabla_x u_n|_{p_n(\cdot)}. \tag{4.5}$$

Let us assume that $|\nabla_x u_n|_{p_n(\cdot)} > 1$ (if $|\nabla_x u_n|_{p_n(\cdot)} \leq 1$, then (4.3) trivially holds). Hence, we can take some $\tau_0 > |\nabla_x u_n|_{p_n(\cdot)} > 1$ verifying that

$$\frac{1}{2} \leq \int_\Omega \left|\frac{\nabla_x u_n}{\tau_0}\right|^{p_n(z)} dz \leq 1.$$

Thus taking into account (4.5) we deduce that

$$\frac{\tau_0^{p_n^-}}{2p_n^+} \leq \int_\Omega \frac{1}{p_n(z)} |\nabla_x u_n|^{p_n(z)} dz \leq C(\Omega, f, r)|\nabla_x u_n|_{p_n(\cdot)} \leq C(f, \Omega, r)\tau_0. \tag{4.6}$$

Now, thanks to (1.3) it holds that

$$\limsup_{n\to\infty} \frac{\log(p_n^+)}{p_n^- - 1} = 0. \tag{4.7}$$

Therefore, by (4.6) and (4.7) we obtain

$$\tau_0 \leq (C(f, \Omega, r)p_n^+)^{\frac{1}{p_n^- - 1}} \to 1, \quad \text{as } n \to \infty,$$

which proves the first estimate of (4.3). As we pointed out before, we can get analogously $|\nabla_y u_n|_{q_n(\cdot)} \leq C(n)$, with $C(n) \to 1$ as $n \to \infty$. Notice that (4.3) and (4.4), (and the analogous inequality for the y-gradient) imply that u_n are uniformly bounded in $W_0^{1,r}(\Omega)$ for any $p_n^-, q_n^- > r > N$. Therefore, for a subsequence, still indexed by n, it holds that

$$u_n \rightharpoonup u_\infty, \ \text{ in } \ W_0^{1,r}(\Omega), \tag{4.8}$$

for some $u_\infty \in W_0^{1,r}(\Omega)$. By the compact Sobolev embedding $W^{1,r}(\Omega)$ into $C^\beta(\Omega)$ for some $0 < \beta < 1$ if $r > N$, we deduce also the strong convergence, up to subsequences,

$$u_n \to u_\infty, \ \text{strongly in } C^\beta(\Omega). \tag{4.9}$$

The lower semicontinuity of the norm and (4.8) yield

$$\begin{aligned} \|\nabla_x u_\infty\|_{L^r(\Omega)} &\leq \liminf_{n\to\infty} \|\nabla_x u_n\|_{L^r(\Omega)} \leq \liminf_{n\to\infty} \Big(2\max\{1, \mu(\Omega)\}\Big)^{\frac{1}{r}} |\nabla_x u_n|_{p_n(\cdot)} \\ &\leq \liminf_{n\to\infty} \Big(2\max\{1, \mu(\Omega)\}\Big)^{\frac{1}{r}} C(n) = \Big(2\max\{1, \mu(\Omega)\}\Big)^{\frac{1}{r}}, \end{aligned} \tag{4.10}$$

with the analogous estimate for the gradient with respect to the $y-$variables. These estimates hold for any sequence $\{r_h\}_{h\in\mathbb{N}} \nearrow \infty$. In fact, such a sequence being fixed, by diagonalization we can select a subsequence of (u_n) such that

$$u_n \rightharpoonup u_\infty, \ \text{ in } \ W_0^{1,r_h}(\Omega), \ \forall h \in \mathbb{N}.$$

Passing to the limit as $r \to \infty$ in (4.10) we get

$$\|\nabla_x u_\infty\|_{L^\infty(\Omega)} \leq 1.$$

Similar arguments prove an equivalent estimate for the gradient with respect to the y-variables and (4.1) follows.

Finally we see that u_∞ maximizes (4.2), (hence the limit u_∞ is nontrivial when $f \not\equiv 0$). Using the characterization (2.2) for the solutions to (1.1) with n fixed, we have that

$$\int_\Omega \left(\frac{1}{p_n(z)} |\nabla_x u_n|^{p_n(z)} + \frac{1}{q_n(z)} |\nabla_y u_n|^{q_n(z)} \right) dz - \int_\Omega f u_n \, dz \\ \leq \int_\Omega \left(\frac{1}{p_n(z)} + \frac{1}{q_n(z)} \right) dz - \int_\Omega f v \, dz,$$

for any $v \in \mathcal{K}$. After neglecting the first positive term on the left hand side we deduce

$$\int_\Omega f v \, dz \leq \int_\Omega f u_n \, dz + \int_\Omega \left(\frac{1}{p_n(z)} + \frac{1}{q_n(z)} \right) dz.$$

We take limit as $n \to \infty$ in the previous expression and thanks to the uniform convergence in (1.2) and (4.9), we obtain that

$$\int_\Omega f v \, dx \leq \int_\Omega f u_\infty \, dx,$$

for any function $v \in \mathcal{K}$, which proves (4.2).

5 The limit problem as $p(\cdot), q(\cdot) \to \infty$.

In the previous section we have shown that if we take the limit in (1.1) as the functions $p_n, q_n \to \infty$ uniformly in Ω, the solutions converge uniformly in $C(\overline{\Omega})$ to some function u_∞. We devote this section to identify the equation satisfied by u_∞.

To this aim, we first give some properties verified by the limit u_∞, that will be useful to determine the limit equation. Those properties are based on the property (4.1) and the characterization of the limit (4.2), which both are also verified by the limit of the constant anisotropic problem, see Remark 4.1. The following three lemmas and the corresponding proofs can be found in [7].

Lemma 5.1 *If u_∞ is a maximizer of* (4.2)*, then if $D \subset \Omega$ is open and smooth, u_∞ also maximizes $\int_D f v$, in*

$$\widetilde{\mathcal{K}} = \left\{ v \in W^{1,\infty}(D) \ : \ \max\{\|\nabla_x v\|_{L^\infty(D)}, \|\nabla_y v\|_{L^\infty(D)}\} \leq 1, \ v|_{\partial D} = u_\infty|_{\partial D} \right\}.$$

Now we consider the distance function to the boundary of $\Omega \in \mathbb{R}^{N+K}$ in the anisotropic ∞-norm

$$\mathrm{dist}_{\infty,a}(z, \partial\Omega) = \inf_{\overline{z} \in \partial\Omega} |z - \overline{z}|_{\infty,a}, \quad z \in \Omega, \tag{5.1}$$

where

$$|z - \overline{z}|_{\infty,a} = |(x, y) - (\overline{x}, \overline{y})|_{\infty,a} = \max\{|x - \overline{x}|, |y - \overline{y}|\}.$$

With the help of Lemma 5.1, we can find an expression for the limit u_∞ at the points where $\{f > 0\}$.

Lemma 5.2 *Let $D \subset \{f > 0\}$ be a convex set. For every $z \in D$, it holds that*

$$u_\infty(z) = \inf_{\overline{z}\in\partial D}\{u_\infty(\overline{z}) + |z - \overline{z}|_{\infty,a}\}.$$

In the same way, it is possible to prove a similar property of u_∞ in the set $\{f < 0\}$.

Lemma 5.3 *Let $D \subset \{f < 0\}$ be a convex set. For every $z \in D$, it holds that*

$$u_\infty(z) = \sup_{\overline{z}\in\partial D}\{u_\infty(\overline{z}) - |z - \overline{z}|_{\infty,a}\}.$$

We are ready to finish the proof of Theorem 1.1 identifying the limit problem. We divide the proof in several parts, corresponding to the different sets of points that determine the equation for u_∞ to solve.

5.1 The limit problem in $\{f > 0\}$

Let $z_0 \in \{f > 0\}$ such that $u_\infty - \phi$ reaches a minimum at z_0. By the convergence in (4.9), there exists a sequence $z_n \to z_0$, $z_n \in \{f > 0\}$ such that $u_n - \phi$ reaches a minimum at z_n, for any $n \in \mathbb{N}$. Since u_n are viscosity solutions of (1.1), (cf. Lemma 3.1) it holds that

$$\begin{aligned}
&-|\nabla_x\phi(zn)|^{p_n(z_n)-2}\Big(\Delta_x\phi(z_n) + \log(|\nabla_x\phi(z_n)|)\langle\nabla_x\phi(z_n), \nabla_x p_n(z_n)\rangle\Big)\\
&\quad-(p_n(z_n) - 2)|\nabla_x\phi(z_n)|^{p_n(z_n)-4}\langle D_x^2\phi(z_n)\nabla_x\phi(z_n), \nabla_x\phi(z_n)\rangle\\
&\quad-|\nabla_y\phi(z_n)|^{q_n(z_n)-2}\Big(\Delta_y\phi(z_n) + \log(|\nabla_y\phi(z_n)|)\langle\nabla_y\phi(z_n), \nabla_y q_n(z_n)\rangle\Big)\\
&\quad-(q_n(z_n) - 2)|\nabla_y\phi(z_n)|^{q_n(z_n)-4}\langle D_y^2\phi(z_n)\nabla_y\phi(z_n), \nabla_y\phi(z_n)\rangle \geq f(z_n).
\end{aligned} \tag{5.2}$$

We observe that in this case $\max\{|\nabla_x\phi(z_0)|, |\nabla_y\phi(z_0)|\} > 0$.

Taking the limit as $n \to \infty$, we wish to show that u_∞ is a viscosity supersolution to $\max\{|\nabla_x u_\infty|, |\nabla_y u_\infty|\} = 1$. If this is not the case and $|\nabla_x\phi(z_0)|,\ |\nabla_y\phi(z_0)| \leq 1$ we will find a contradiction.

Suppose first that $|\nabla_y\phi(z_0)|^{\theta(z_0)} \leq |\nabla_x\phi(z_0)|$. Then, $|\nabla_x\phi(z_0)| > 0$ and we can divide the equation by $(p_n(z_n) - 2)|\nabla_x\phi(z_n)|^{p_n(z_n)-4}$ to obtain

$$\begin{aligned}
&-\frac{|\nabla_x\phi(z_n)|^2}{p_n(z_n) - 2}\Big(\Delta_x\phi(z_n) + \log(|\nabla_x\phi(z_n)|)\langle\nabla_x\phi(z_n), \nabla_x p_n(z_n)\rangle\Big) - \Delta_{\infty,x}\phi(z_n)\\
&-\left(\frac{|\nabla_y\phi(z_n)|^{\frac{q_n(z_n)-4}{p_n(z_n)-4}}}{|\nabla_x\phi(z_n)|}\right)^{p_n(z_n)-4}\frac{q_n(z_n) - 2}{p_n(z_n) - 2}\\
&\quad\times\left[\frac{|\nabla_y\phi(z_n)|^2}{q_n(z_n) - 2}\Big(\Delta_y\phi(z_y) + \log(|\nabla_y\phi(z_n)|)\langle\nabla_y\phi(z_n), \nabla_y q_n(z_n)\rangle\Big) + \Delta_{\infty,y}\phi(z_n)\right]\\
&\quad\geq \frac{f(z_n)}{(p(z_n) - 2)|\nabla_x\phi(z_n)|^{p_n(z_n)-4}}.
\end{aligned}$$

The contradiction follows when taking limits as $n \to \infty$, since the right hand side term diverges, while the left hand side is bounded. In fact,

$$\frac{|\nabla_x\phi(z_n)|^2}{p_n(z_n)-2}\Delta_x\phi(z_n)\to 0, \qquad \Delta_{\infty,x}\phi(z_n)\to\Delta_{\infty,x}\phi(z_0) \qquad \text{and}$$
$$\frac{|\nabla_x\phi(z_n)|^2}{p_n(z_n)-2}\log(|\nabla_x\phi(z_n)|)\langle\nabla_x\phi(z_n),\nabla_x p_n(z_n)\rangle\to$$
$$|\nabla_x\phi(z_0)|^2\log(|\nabla_x\phi(z_0)|)\langle\nabla_x\phi(z_0),\xi_p(z_0)\rangle.$$

Analogously,

$$\frac{|\nabla_y\phi(z_n)|^2}{q_n(z_n)-2}\Delta_y\phi(z_n)\to 0, \qquad \Delta_{\infty,y}\phi(z_n)\to\Delta_{\infty,y}\phi(z_0) \qquad \text{and}$$
$$\frac{|\nabla_y\phi(z_n)|^2}{q_n(z_n)-2}\log(|\nabla_y\phi(z_n)|)\langle\nabla_y\phi(z_n),\nabla_y q_n(z_n)\rangle\to$$
$$|\nabla_y\phi(z_0)|^2\log(|\nabla_y\phi(z_0)|)\langle\nabla_y\phi(z_0),\xi_q(z_0)\rangle.$$

Finally,

$$\frac{q_n(z_n)-2}{p_n(z_n)-2}\to\theta(z_0), \qquad \frac{|\nabla_y\phi(z_n)|^{\frac{q_n(z_n)-4}{p_n(z_n)-4}}}{|\nabla_x\phi(z_n)|}\to\frac{|\nabla_y\phi(z_0)|^{\theta(z_0)}}{|\nabla_x\phi(z_0)|},$$

and since we are assuming that $|\nabla_y\phi(z_0)|^{\theta(z_0)}\leq|\nabla_x\phi(z_0)|$, we have that

$$\left(\frac{|\nabla_y\phi(z_n)|^{\frac{q_n(z_n)-4}{p_n(z_n)-4}}}{|\nabla_x\phi(z_n)|}\right)^{p_n(z_n)-4}\to 0.$$

Thus summing up we arrive to

$$-|\nabla_x\phi(z_0)|^2\log(|\nabla_x\phi(z_0)|)\langle\nabla_x\phi(z_0),\xi_p(z_0)\rangle-\Delta_{\infty,x}\phi(z_0)\geq+\infty,$$

a contradiction.

If we assume that $|\nabla_x\phi(z_0)|,\ |\nabla_y\phi(z_0)|\leq 1$ and also that $|\nabla_y\phi(z_0)|^{\theta(z_0)}\geq|\nabla_x\phi(z_0)|$. Then, $|\nabla_y\phi(z_0)|>0$ and we divide the equation by the term $(p_n(z_n)-2)|\nabla_y\phi(z_n)|^{q_n(z_n)-4}$. Arguing as before, we reach a similar contradiction

$$-\theta|\nabla_y\phi(z_0)|^2\log(|\nabla_y\phi(z_0)|)\langle\nabla_y\phi(z_0),\xi_q(z_0)\rangle-\theta\Delta_{\infty,y}\phi(z_0)\geq+\infty.$$

This shows that u_∞ is a viscosity supersolution to $\max\{|\nabla_x u_\infty|,|\nabla_y u_\infty|\}=1$.

To show that u_∞ is a viscosity subsolution, we consider $z_0=(x_0,y_0)\in\{f>0\}$ a point such that $u_\infty-\psi$ attains a maximum. Take D the anisotropic ball $\{z\in\mathbb{R}^{N+K}\ :\ \mathrm{dist}_{\infty,a}(z_0,z)\leq\varepsilon\}$, which is contained in $\{f>0\}$ for ε sufficiently small. Let us take $\overline{z}_0$ the point on ∂D such that $\overline{z}_0=(\overline{x}_0,y_0)$, with $\overline{x}_0=x_0-\varepsilon\frac{\nabla_x\psi(z_0)}{|\nabla_x\psi(z_0)|}$. By Lemma 5.2 and the definition of ψ, we know that

$$u_\infty(z_0)=\inf_{\overline{z}\in\partial D}\{u_\infty(\overline{z})+|z_0-\overline{z}|_{\infty,a}\}\leq u_\infty(\overline{z}_0)+\varepsilon\leq\psi(\overline{z}_0)+\varepsilon.$$

Taking into account that $u_\infty(z_0) = \psi(z_0)$, if we rearrange the previous expression, we obtain

$$1 \geq \frac{\psi(z_0) - \psi(\overline{z}_0)}{\varepsilon} = \frac{\psi(x_0, y_0) - \psi(x_0 - \varepsilon v, y_0)}{\varepsilon}, \qquad \text{with } v = \frac{\nabla_x \psi(z_0)}{|\nabla_x \psi(z_0)|}.$$

Taking limit as $\varepsilon \to 0$ we get that

$$|\nabla_x \psi(z_0)| \leq 1.$$

We can argue the same with the point $\hat{z}_0 \in \partial D$ such that $\hat{z}_0 = (x_0, \hat{y}_0)$ with $\hat{y}_0 = y_0 - \varepsilon \frac{\nabla_y \psi(z_0)}{|\nabla_y \psi(z_0)|}$, to show that

$$|\nabla_y \psi(z_0)| \leq 1,$$

which proves that u_∞ is also a viscosity subsolution.

5.2 The limit problem in $\{f < 0\}$

This case can be handled in a similar way and we omit the details.

Remark 5.1 We stress that in the sets $\{f > 0\}$ and $\{f < 0\}$ we have obtained the same limit problem as for the problem related to the constant anisotropic (p, q)-Laplacian, see [7]. Therefore, the following conclusions are also valid for our problem. We include them for completeness of the paper.

1. If $f > 0$ then $u_\infty = \mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$ in Ω, given in (5.1), thanks to Lemma 5.2 with $D = \Omega$. Moreover, this is the unique possible limit. Observe that $\mathrm{dist}_{\infty,a}(\cdot, \partial\Omega) \in \mathcal{K}$ and also that u_∞ is a maximizer in (4.2). Then

$$\int_\Omega f \mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)\, dx \leq \int_\Omega f u_\infty, \ \forall v \in \mathcal{K}$$

   being this inequality strict unless $u_\infty = \mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$. In addition, we have that $\mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$ is the unique solution to equation

$$\max\{|\nabla_x u_\infty|, |\nabla_y u_\infty|\} = 1$$

   in the viscosity sense, see [16].

2. If $f \geq 0$, $u_\infty = \mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$ in $\operatorname{supp} f$, while it solves

$$H_\infty(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) = 0 \qquad (5.3)$$

   in the interior of the set $\{f = 0\}$. We have also uniqueness of the limit u_∞ in this case. If, in addition, the points in which $\mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$ is not differentiable lay in $\operatorname{supp} f$, then $u_\infty = \mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$ in Ω, since this function verifies (5.3), which has a unique solution, see next section.

3. If $f < 0$ or $f \leq 0$ we obtain analogous results with $-\mathrm{dist}_{\infty,a}(\cdot, \partial\Omega)$.

From now on, to treat the rest of the cases, we deal with the function H_∞ given in Theorem 1.1. We recall its expression using the notation introduced in (3.4). Let $w \in \mathbb{R}^{N+K}$, and denote $w_1 = (w^1, ..., w^N)$, and $w_2 = (w^{N+1}, ..., w^{N+K})$, namely, w_1 stands for the first N components of w and w_2 for the last K components. Consider $\xi_p \in \mathbb{R}^N$, $\xi_q \in \mathbb{R}^K$, $\theta \in \mathbb{R}^+$ and $S \in \mathbb{S}^{N+K}$. As before we call $S_1 = (s_{ij})_{1\le i,j\le N}$ the first $N \times N$ minor of the matrix S and $S_2 = (s_{ij})_{N+1\le i,j\le N+K}$ the last $K \times K$ minor of S. Using this notation, the function H_∞ of Theorem 1.1 reads as follows

- if $|w_2|^\theta < |w_1|$,
$$H_\infty = -\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle := -\Delta_{\infty(x)} w_1,$$
- if $|w_2|^\theta > |w_1|$
$$H_\infty = -\theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle := -\theta\Delta_{\infty(y)} w_2, \tag{5.4}$$
- if $|w_2|^\theta = |w_1|$
$$\begin{aligned} H_\infty = &-\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle \\ &-\theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle := -\Delta_{\infty(x)} w_1 - \theta\Delta_{\infty(y)} w_2. \end{aligned}$$

Since this function H_∞ is discontinuous in the w variable, our first task is to characterize its upper and lower semicontinuous envelopes, $(H_\infty)^*$ and $(H_\infty)_*$.

Lemma 5.4 *The upper semicontinuous envelope of H_∞ is given by*

$$(H_\infty)^*(S, w) =$$
$$\begin{cases} -\Delta_{\infty(x)} w_1 & \text{for } |w_2|^\theta < |w_1|, \\ -\theta\Delta_{\infty(y)} w_2 & \text{for } |w_2|^\theta > |w_1|, \\ \max\Big\{ - \Delta_{\infty(x)} w_1,\ -\theta\Delta_{\infty(y)} w_2,\ -\Delta_{\infty(x)} w_1 - \theta\Delta_{\infty(y)} w_2\Big\} & \text{for } |w_2|^\theta = |w_1|. \end{cases}$$

The lower semicontinuous envelope has the same expression except for the last case in which the max *is replaced by*

$$\min\Big\{ - \Delta_{\infty(x)} w_1,\ -\theta\Delta_{\infty(y)} w_2,\ -\Delta_{\infty(x)} w_1 - \theta\Delta_{\infty(y)} w_2\Big\}.$$

Proof. If $w = 0$ the statement is trivial, hence we may assume that $w \neq 0$.

Let us treat first the case $|w_2|^\theta < |w_1|$. Observe that for ε small enough we also have $|w_2'|^\theta < |w_1'|$, for every w' such that $|w_1' - w_1| < \varepsilon$. Hence,

$$H_\infty(S', w') = -\langle S_1' w_1', w_1'\rangle - |w_1'|^2 \log(|w_1'|)\langle \xi_p, w_1'\rangle$$

for every $|S_1' - S_1| < \varepsilon$ and we conclude that

$$\begin{aligned} &(H_\infty)^*(S, w) = \lim_{\varepsilon\to 0} \sup \{H_\infty(S'w')\ :\ |w - w'| + |S - S'| < \varepsilon\} \\ &= \lim_{\varepsilon\to 0} \sup \Big\{-\langle S_1' w_1', w_1'\rangle - |w_1'|^2 \log(|w_1'|)\langle \xi_p, w_1'\rangle\ :\ |w - w'| + |S - S'| < \varepsilon\Big\} \\ &= -\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle. \end{aligned}$$

The proof of the case $|w_2|^\theta > |w_1|$ runs analogously.

We are left with the case $|w_2|^\theta = |w_1|$. We first see that

$$(H_\infty)^*(S, w) \geq -\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle.$$

Indeed, defining $w_k = (w_1, kw_2)$ with $k < 1$ and $k \nearrow 1$ we have that

$$\begin{aligned}(H_\infty)^*(S, w) &= \lim_{\varepsilon\to 0} \sup\{H_\infty(S', w') \ : \ |w - w'| + |S - S'| < \varepsilon\}\\ &\geq \lim_{k\nearrow 1} \sup_{w_k} \left\{-\langle S'_1 w'_1, w'_1\rangle - |w'_1|^2 \log(|w'_1|)\langle \xi_p, w'_1\rangle\right\}\\ &= -\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle.\end{aligned}$$

Similarly it can be shown that

$$(H_\infty)^*(S, w) \geq -\theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle.$$

Finally, observe that if $|w_2|^\theta = |w_1|$

$$\begin{aligned}(H_\infty)^*(S, w) &= \lim_{\varepsilon\to 0} \sup\{H_\infty(S', w') \ : \ |w - w'| + |S - S'| < \varepsilon\}\\ &\geq H_\infty(S, w) = -\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle\\ &\qquad -\theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle.\end{aligned}$$

Now, since the possible limit of $H_\infty(S', w')$ as $w' \to w$ and $S' \to S$ is given by

$$-\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle - \theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle,$$

$$-\langle S_1 w_1, w_1\rangle - |w_1|^2 \log(|w_1|)\langle \xi_p, w_1\rangle, \quad \text{or} \quad -\theta\langle S_2 w_2, w_2\rangle - \theta|w_2|^2 \log(|w_2|)\langle \xi_q, w_2\rangle,$$

the result follows.

The corresponding result for the lower envelope $(H_\infty)_*$ can be shown analogously and thus we leave it to the reader.

5.3 The limit problem in $\Omega \setminus \operatorname{supp}(f)$

Our task is to show that u_∞ is a viscosity solution to

$$H_\infty(D^2_x u_\infty, D^2_y u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) = 0$$

in the open set $\Omega \setminus \operatorname{supp}(f)$, with H_∞ given in (5.4), in the sense of Definitions 3.1 and 3.2. We see first that u_∞ is a supersolution. Let ϕ be such that $u_\infty - \phi$ has a strict local minimum at $z_0 \in \Omega$, with $\phi(z_0) = u_\infty(z_0)$. We want to prove that

$$(H_\infty)^*(D^2_x\phi(z_0), D^2_y\phi(z_0), \nabla_x\phi(z_0), \nabla_y\phi(z_0)) \geq 0. \tag{5.5}$$

By the uniform convergence (4.9), there exists a subsequence, still indexed by n, z_n, such that $z_n \to z_0$ and $u_n - \phi$ has a local minimum at z_n. Since by Lemma 3.1 u_n is a viscosity solution of (1.1), we have

$$\begin{aligned}
&-|\nabla_x\phi(z_n)|^{p_n(z_n)-2}\Big(\Delta_x\phi(z_n)+\log(|\nabla_x\phi(z_n)|)\langle\nabla_x\phi(z_n),\nabla_x p_n(z_n)\rangle\Big)\\
&\quad-(p_n(z_n)-2)|\nabla_x\phi(z_n)|^{p_n(z_n)-4}\langle D_x^2\phi(z_n)\nabla_x\phi(z_n),\nabla_x\phi(z_n)\rangle\\
&\quad\quad-|\nabla_y\phi(z_n)|^{q_n(z_n)-2}\Big(\Delta_y\phi(z_n)+\log(|\nabla_y\phi(z_n)|)\langle\nabla_y\phi(z_n),\nabla_y q_n(z_n)\rangle\Big)\\
&\quad\quad\quad-(q_n(z_n)-2)|\nabla_y\phi(z_n)|^{q_n(z_n)-4}\langle D_y^2\phi(z_n)\nabla_y\phi(z_n),\nabla_y\phi(z_n)\rangle\geq 0.
\end{aligned}$$

Notice that (5.5) trivially holds if $|\nabla_x\phi(z_0)|=|\nabla_y\phi(z_0)|=0$. Let us assume that $|\nabla_x\phi(z_0)|>|\nabla_y\phi(z_0)|^{\theta(z_0)}$. Then $\nabla_x\phi(z_0)\neq 0$ and we obtain,

$$\begin{aligned}
&\frac{|\nabla_x\phi(z_n)|^2}{p_n(z_n)-2}\Big(\Delta_x\phi(z_n)+\log(|\nabla_x\phi(z_n)|)\langle\nabla_x\phi(z_n),\nabla_x p_n(z_n)\rangle\Big)+\Delta_{\infty,x}\phi(z_n)\\
&\leq-\left(\frac{|\nabla_y\phi(z_n)|^{\frac{q_n(z_n)-4}{p_n(z_n)-4}}}{|\nabla_x\phi(z_n)|}\right)^{p_n(z_n)-4}\frac{(q_n(z_n)-2)}{(p_n(z_n)-2)}\\
&\times\left(\frac{|\nabla_y\phi(z_n)|^2}{q_n(z_n)-2}\Big(\Delta_y\phi(z_n)+\log(|\nabla_y\phi(z_n)|)\langle\nabla_y\phi(z_n),\nabla_y q_n(z_n)\rangle\Big)+\Delta_{\infty,y}\phi(z_n)\right).
\end{aligned}\tag{5.6}$$

From the limits we computed when treating the case $\{f>0\}$, we deduce that

$$-\Delta_{\infty,x}\phi(z_0)-|\nabla_x\phi(z_0)|^2\log(|\nabla_x\phi(z_0)|)\langle\nabla_x\phi(z_0),\xi_p(z_0)\rangle\geq 0.$$

On the other hand, if $|\nabla_x\phi(z_0)|<|\nabla_y\phi(z_0)|^{\theta(z_0)}$, then $\nabla_y\phi(z_0)\neq 0$ and we get

$$-\theta(z_0)\Delta_{\infty,y}\phi(z_0)-\theta(z_0)|\nabla_y\phi(z_0)|^2\log(|\nabla_y\phi(z_0)|)\langle\nabla_y\phi(z_0),\xi_q(z_0)\rangle\geq 0.$$

Finally, in case $|\nabla_x\phi(z_0)|=|\nabla_y\phi(z_0)|^{\theta(z_0)}$ let us argue by contradiction, and assume that

$$-\Delta_{\infty,x}\phi(z_0)-|\nabla_x\phi(z_0)|^2\log(|\nabla_x\phi(z_0)|)\langle\nabla_x\phi(z_0),\xi_p(z_0)\rangle<0,$$

$$-\theta(z_0)\Delta_{\infty,y}\phi(z_0)-\theta(z_0)|\nabla_y\phi(z_0)|^2\log(|\nabla_y\phi(z_0)|)\langle\nabla_y\phi(z_0),\xi_q(z_0)\rangle<0.$$

Notice that this assumption implies that both $\nabla_x\phi(z_0)\neq 0$ and $\nabla_y\phi(z_0)\neq 0$.

Suppose first that there exists a subsequence $n_i\in\mathbb{N}$ for which it holds that

$$\left(\frac{|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}{|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}\right)\geq 1.\tag{5.7}$$

Substituting this inequality in equation (5.6) and rearranging it as follows, along this subsequence $n_i\to\infty$, we obtain

$$\begin{aligned}
&\frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_x\phi(z_{n_i})+\frac{|\nabla_y\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_y\phi(z_{n_i})\leq-\Delta_{\infty,x}\phi(z_{n_i})\\
&\quad-\frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\log(|\nabla_x\phi(z_{n_i})|)\langle\nabla_x\phi(z_{n_i}),\nabla_x p_{n_i}(z_{n_i})\rangle\\
&\quad\quad-\frac{|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}{|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}\frac{(q_{n_i}(z_{n_i})-2)}{(p_{n_i}(z_{n_i})-2)}\\
&\quad\quad\quad\times\left(\Delta_{\infty,y}\phi(z_{n_i})+\frac{|\nabla_y\phi(z_{n_i})|^2}{q_{n_i}(z_{n_i})-2}\log(|\nabla_y\phi(z_{n_i})|)\langle\nabla_y\phi(z_{n_i}),\nabla_y q_{n_i}(z_{n_i})\rangle\right)<0,
\end{aligned}$$

for n_i large enough. Taking limit as $n_i \to \infty$, we get a contradiction. If we suppose now, that for a subsequence n_i it holds that

$$\left(\frac{|\nabla_y \phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}{|\nabla_x \phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}\right) < 1. \tag{5.8}$$

Then, using that

$$\left(\frac{|\nabla_x \phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}{|\nabla_y \phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}\right) > 1,$$

we argue as before (dividing the equation by $(p_{n_i}(z_{n_i}) - 2)|\nabla_y \phi(z_{n_i})|^{q_{n_i}(z_{n_i}-4)}$) to obtain

$$\begin{aligned}
&\frac{|\nabla_x \phi(z_{n_i})|^2}{p_{n_i}(z_{n_i}) - 2}\Delta_x \phi(z_{n_i}) + \frac{|\nabla_y \phi(z_{n_i})|^2}{p_{n_i}(z_{n_i}) - 2}\Delta_y \phi(z_{n_i}) \leq -\Delta_{\infty,x}\phi(z_{n_i}) \\
&\quad - \frac{|\nabla_x \phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}{|\nabla_y \phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}} \frac{|\nabla_x \phi(z_{n_i})|^2}{p_{n_i}(z_{n_i}) - 2} \log(|\nabla_x \phi(z_{n_i})|)\langle \nabla_x \phi(z_{n_i}), \nabla_x p_{n_i}(z_{n_i})\rangle \\
&\quad - \frac{(q_{n_i}(z_{n_i}) - 2)}{(p_{n_i}(z_{n_i}) - 2)} \\
&\quad\quad \times \left(\Delta_{\infty,y}\phi(z_{n_i}) + \frac{|\nabla_y \phi(z_{n_i})|^2}{q_{n_i}(z_{n_i}) - 2} \log(|\nabla_y \phi(z_{n_i})|)\langle \nabla_y \phi(z_{n_i}), \nabla_y q_{n_i}(z_{n_i})\rangle\right) < 0,
\end{aligned}$$

which leads to a contradiction after taking limit as $n_i \to \infty$.

The proof of the fact that u_∞ is a viscosity subsolution is analogous.

5.4 The limit problem in $\Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}$

In other words, if we consider $z_0 \in \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}$, we have that $f(z_0) = 0$ and this point z_0 can be approximated by points in the set $\{f > 0\}$ or in $\{f = 0\}$. Let us consider the sequence $z_n \to z_0$ where $u_n - \phi$ attains the minimum. At least for subsequences we have that $f(z_{n_i}) > 0$ or $f(z_{n_i}) = 0$. If $f(z_{n_i}) = 0$ we can argue as in the previous case to find that $(H_\infty)^*(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) \geq 0$. If $f(z_{n_i}) > 0$ we can deduce that $(H_\infty)^*(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) \geq 0$ proceeding as in the first case. However, if $f(z_{n_i}) = 0$ it is possible to show that $(H_\infty)^*(D_x^2 u_\infty, D_y^2 u_\infty, \nabla_x u_\infty, \nabla_y u_\infty) \leq 0$, but in case $f(z_{n_i}) > 0$ for subsolutions we just obtain the general estimates for the gradient (4.1).

5.5 The limit problem in $\Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}$

The proof runs analogously to the previous case.

5.6 The limit problem in $\Omega \cap \partial\{f < 0\} \cap \partial\{f > 0\}$

If $z_0 \in \Omega \cap \partial\{f < 0\} \cap \partial\{f > 0\}$ it means that it can be approximated by points in the sets $\{f > 0\}$, $\{f < 0\}$ or $\{f = 0\}$. Therefore, arguing as in the previous cases, we just deduce the general estimates for the gradient (4.1).

6 Uniqueness for the limit problem.

We end this work analyzing the limit problem when $f \equiv 0$, with a nontrivial boundary condition. Namely, we study

$$\begin{cases} H_\infty(D_x^2 u, D_y^2 u, \nabla_x u, \nabla_y u) = 0 & \text{in } \Omega, \\ u = g \not\equiv 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

with H_∞ given in (5.4) and u a solution in the sense of Definitions 3.1 and 3.2, and g some Lipschitz function defined on $\partial\Omega$. As we pointed out in the Introduction, the limit problem for the (constant) anisotropic p-Laplacian was treated in [29], where the authors found that Lipschitz regularity is the optimal that can be expected for the limit. They also proved the uniqueness of solutions of the limit problem.

As in the constant case, we will see that the limit problem (6.1) has a unique solution. For this reason in this section we take $g \neq 0$, to exclude the trivial solution. However, it is no possible to apply the arguments in [29], due to the logarithmic terms in the equations. We follow instead the ideas in [24], where the authors show the uniqueness of a single equation that arises from taking limit as $p_n(z) = np(z) \to \infty$ in $\Delta_{p_n(\cdot)}u = 0$, with the boundary condition $u = g$ on $\partial\Omega$.

6.1 Auxiliary equations

For our purposes we need to work with some auxiliary equations, that were introduced by Jensen in [18], see also [24].

- *Upper equation:*

$$\overline{H}_\infty = \begin{cases} \min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} = 0 & \text{if } |w_2|^\theta < |w_1|, \\ \min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} = 0 & \text{if } |w_2|^\theta > |w_1|, \\ \min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} & \\ \qquad + \min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} = 0 & \text{if } |w_2|^\theta = |w_1|. \end{cases} \tag{6.2}$$

- *Equation:*

$$H_\infty = \begin{cases} -\Delta_{\infty(x)} w_1 = 0 & \text{if } |w_2|^\theta < |w_1|, \\ -\theta\Delta_{\infty(y)} w_2 = 0 & \text{if } |w_2|^\theta > |w_1|, \\ -\Delta_{\infty(x)} w_1 - \theta\Delta_{\infty(y)} w_2 = 0 & \text{if } |w_2|^\theta = |w_1|. \end{cases} \tag{6.3}$$

- *Lower equation:*

$$\underline{H}_\infty = \begin{cases} \max\{\varepsilon - |w_1|, -\Delta_{\infty(x)} w_1\} = 0 & \text{if } |w_2|^\theta < |w_1|, \\ \max\{\varepsilon - |w_2|^\theta, -\theta\Delta_{\infty(y)} w_2\} = 0 & \text{if } |w_2|^\theta > |w_1|, \\ \max\{\varepsilon - |w_1|, -\Delta_{\infty(x)} w_1\} & \\ \qquad + \max\{\varepsilon - |w_2|^\theta, -\theta\Delta_{\infty(y)} w_2\} = 0 & \text{if } |w_2|^\theta = |w_1|. \end{cases} \tag{6.4}$$

Existence of solutions to the *equation* follows from previous sections: The limit u_∞ computed in Section 4 is a viscosity solution to equation (6.3). We show now existence of viscosity solutions to the *upper equation*. Consider problem (2.1) for n fixed, with $f(z) = \varepsilon^{p_n(z)}$. By Theorem 2.1 we have existence of a unique solution, let us denote it as u_n^+, given by the minimizer of (2.2). Thus, u_n^+ are variational and at least for a subsequence we have that $u_n^+ \to u_\infty^+$ in $C(\Omega)$.

Let us show that u_∞^+ is a viscosity solution to the *upper equation*, in the sense of Definitions 3.1 and 3.2. As in Lemma 5.4 we can show that the upper semicontinuous envelope of $\overline{H}_\infty$ is given by

$$(\overline{H}_\infty)^* = \begin{cases} \min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} & \text{if } |w_2|^\theta < |w_1|, \\ \min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} & \text{if } |w_2|^\theta > |w_1|, \\ \max\Big\{\min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\}, & \\ \qquad \min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\}, & \\ \qquad \min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} & \\ \qquad + \min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\}\Big\} & \text{if } |w_2|^\theta = |w_1|, \end{cases}$$

with the analogous expression for $(\overline{H}_\infty)_*$ replacing the maximum by the minimum, when $|w_2|^\theta = |w_1|$.

We begin by proving that u_∞^+ is a viscosity supersolution. Let z_0 be a point such that $u_\infty^+ - \phi$ attains a local minimum. Let us assume first that $|\nabla_x u_\infty^+(z_0)| > |\nabla_y u_\infty^+(z_0)|^{\theta(z_0)}$. Using the uniform convergence, let $z_n \to z_0$ be the points such that $u_n^+ - \phi$ attains a local minimum. The fact that u_n^+ are viscosity supersolutions imply that (5.2) holds with $f(z_n) = \varepsilon^{p_n(z_n)}$. Observe that, for each n at least the gradient $\nabla_x \phi(z_n) \neq 0$. Consequently, we can divide the equation by $(p_n(z_n) - 2)|\nabla_x \phi(z_n)|^{p_n(z_n)-4}$ to obtain

$$\begin{aligned} &-\frac{|\nabla_x \phi(z_n)|^2}{p_n(z_n) - 2}\Big(\Delta_x \phi(z_n) + \log(|\nabla_x \phi(z_n)|)\langle \nabla_x \phi(z_n), \nabla_x p_n(z_n)\rangle\Big) - \Delta_{\infty,x}\phi(z_n) \\ &-\left(\frac{|\nabla_y \phi(z_n)|^{\frac{q_n(z_n)-4}{p_n(z_n)-4}}}{|\nabla_x \phi(z_n)|}\right)^{p_n(z_n)-4} \frac{q_n(z_n) - 2}{p_n(z_n) - 2} \\ &\quad\times\left[\frac{|\nabla_y \phi(z_n)|^2}{q_n(z_n) - 2}\Big(\Delta_y \phi(z_y) + \log(|\nabla_y \phi(z_n)|)\langle \nabla_y \phi(z_n), \nabla_y q_n(z_n)\rangle\Big) + \Delta_{\infty,y}\phi(z_n)\right] \\ &\qquad \geq \frac{\varepsilon^{p_n(z_n)}}{(p(z_n) - 2)|\nabla_x \phi(z_n)|^{p_n(z_n)-4}}. \end{aligned}$$

Since the left hand side term is bounded, then necessarily $|\nabla_x \phi(z_0)| - \varepsilon \geq 0$, and hence also $-\Delta_{\infty,x}\phi(z_0) - |\nabla_x \phi(z_0)|^2 \log(|\nabla_x \phi(z_0)|)\langle \nabla_x \phi(z_0), \xi_p(z_0)\rangle \geq 0$. The proof for subsolutions follows straight forward by reversing the inequalities and noticing that the quotient on the right hand side can diverge.

In case $|\nabla_y u_\infty^+(z_0)|^{\theta(z_0)} > |\nabla_x u_\infty^+(z_0)|$ we have that at least $\nabla_y \phi(z_n) \neq 0$ for each n. So this time we divide the equation by $(p_n(z_n) - 2)|\nabla_y \phi(z_n)|^{q_n(z_n)-4}$ and conclude with similar observations as above.

We are just left with the case $|\nabla_y u_\infty^+(z_0)|^{\theta(z_0)} = |\nabla_x u_\infty^+(z_0)|$. First we observe that, since for each n the right hand side term is positive, $\nabla_x\phi(z_n) \neq 0$ or $\nabla_y\phi(z_n) \neq 0$, which in this case implies for n large that neither of them vanish.

As before, we argue by contradiction, assuming that

$$\min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} < 0$$

and

$$\min\{|w_2|^\theta - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} = \min\{|w_1| - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} < 0.$$

Let us start by analyzing the limit assuming that (5.7) holds for a subsequence $n_i \to \infty$. We have

$$\begin{aligned}
&\frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_x\phi(z_{n_i}) + \frac{|\nabla_y\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_y\phi(z_{n_i}) \leq -\frac{\varepsilon^{p_{n_i}(z_{n_i})}}{(p_{n_i}(z_{n_i})-2)|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}\\
&\quad -\Delta_{\infty,x}\phi(z_{n_i}) - \frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\log(|\nabla_x\phi(z_{n_i})|)\langle\nabla_x\phi(z_{n_i}), \nabla_x p_{n_i}(z_{n_i})\rangle\\
&\quad -\frac{|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}{|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}\frac{(q_{n_i}(z_{n_i})-2)}{(p_{n_i}(z_{n_i})-2)}\\
&\quad \times\left(\Delta_{\infty,y}\phi(z_{n_i}) + \frac{|\nabla_y\phi(z_{n_i})|^2}{q_{n_i}(z_{n_i})-2}\log(|\nabla_y\phi(z_{n_i})|)\langle\nabla_y\phi(z_{n_i}), \nabla_y q_{n_i}(z_{n_i})\rangle\right),
\end{aligned}$$

for n_i large enough. Taking limit as $n_i \to \infty$, we get a contradiction in any of the possible cases:

1. If
$$\min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} = -\Delta_{\infty(x)} w_1 < 0$$
and
$$\min\{|w_1| - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} = -\theta\Delta_{\infty(y)} w_2 < 0,$$
the right hand side term is strictly negative whereas the left hand side tends to zero. Therefore, for n large we get a contradiction.

2. If either
$$\min\{|w_1| - \varepsilon, -\Delta_{\infty(x)} w_1\} = |w_1| - \varepsilon < 0$$
or
$$\min\{|w_1| - \varepsilon, -\theta\Delta_{\infty(y)} w_2\} = |w_1| - \varepsilon < 0,$$
or both hold, then the term
$$\frac{-\varepsilon^{p_{n_i}(z_{n_i})}}{(p_{n_i}(z_{n_i})-2)|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}} = \frac{-|\nabla_x\phi(z_{n_i})|^4}{(p_{n_i}(z_{n_i})-2)}\left(\frac{\varepsilon}{|\nabla_x\phi(z_{n_i})|}\right)^{p_n(z_{n_i})} \to -\infty,$$
while the rest of the right hand side is bounded. We obtain again a contradiction with the fact that the left hand side goes to zero.

For subsequences verifying the estimate (5.8), we again divide the equation by the term $(p_{n_i}-2)|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i}-4)}$, and get a similar inequality as before. Precisely

$$
\begin{aligned}
&\frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_x\phi(z_{n_i})+\frac{|\nabla_y\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\Delta_y\phi(z_{n_i})\\
&\quad\le -\Delta_{\infty,x}\phi(z_{n_i})-\frac{|\nabla_x\phi(z_{n_i})|^4}{p_{n_i}(z_{n_i})-2}\left(\frac{\varepsilon}{|\nabla_x\phi(z_{n_i})|}\right)^{p_{n_i}(z_{n_i})}\\
&\qquad-\frac{|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}{|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}\frac{|\nabla_x\phi(z_{n_i})|^2}{p_{n_i}(z_{n_i})-2}\log(|\nabla_x\phi(z_{n_i})|)\langle\nabla_x\phi(z_{n_i}),\nabla_x p_{n_i}(z_{n_i})\rangle\\
&\qquad-\frac{(q_{n_i}(z_{n_i})-2)}{(p_{n_i}(z_{n_i})-2)}\\
&\qquad\times\left(\Delta_{\infty,y}\phi(z_{n_i})+\frac{|\nabla_y\phi(z_{n_i})|^2}{q_{n_i}(z_{n_i})-2}\log(|\nabla_y\phi(z_{n_i})|)\langle\nabla_y\phi(z_{n_i}),\nabla_y q_{n_i}(z_{n_i})\rangle\right).
\end{aligned}
$$

Arguing as before we get a contradiction. If 1 holds we just observe that the right hand side is strictly negative for n_i large, while the term on the left goes to zero. In case 2 just notice that the term

$$
\begin{aligned}
&\frac{-\varepsilon^{p_{n_i}(z_{n_i})}}{(p_{n_i}(z_{n_i})-2)|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}\\
&\qquad=-\frac{|\nabla_x\phi(z_{n_i})|^4}{p_{n_i}(z_{n_i})-2}\left(\frac{\varepsilon}{|\nabla_x\phi(z_{n_i})|}\right)^{p_{n_i}(z_{n_i})}\frac{|\nabla_x\phi(z_{n_i})|^{p_{n_i}(z_{n_i})-4}}{|\nabla_y\phi(z_{n_i})|^{q_{n_i}(z_{n_i})-4}}\\
&\qquad\le-\frac{|\nabla_x\phi(z_{n_i})|^4}{p_{n_i}(z_{n_i})-2}\left(\frac{\varepsilon}{|\nabla_x\phi(z_{n_i})|}\right)^{p_{n_i}(z_{n_i})}\to-\infty,
\end{aligned}
$$

and the rest of the terms on the right hand side are bounded. This is a contradiction, since the left hand side tends to zero.

The proof of the fact that u_∞^+ is a viscosity subsolution runs similarly.

The existence of solutions to the *lower equation* can be shown analogously. Theorem 2.1 ensures existence of (variational) solutions to problem (2.1) for n fixed, with $f(z)=-\varepsilon^{p_n(z)}$, denoted as u_n^-. Passing to the limit as $p_n,q_n\to\infty$ we can show that $u_n^-\to u_\infty^-$ uniformly in $C(\Omega)$, being u_∞^- a viscosity solution to (6.4).

The uniqueness of the limit u_∞**.** Let u_n^+,u_n,u_n^- be the variational weak solutions to (2.1) with $f(z)=\varepsilon^{p_n(z)}$, $f=0$ and $f(z)=-\varepsilon^{p_n(z)}$ respectively, and the same boundary data g. By comparison for the weak formulation we have that $u_n^+\ge u_n\ge u_n^-$, so by the uniform convergence in $C(\Omega)$, we deduce that $u_\infty^+\ge u_\infty\ge u_\infty^-$. Moreover, the following lemma holds

Lemma 6.1 *Let u_∞^+,u_∞^- be viscosity solutions to* (6.2) *and* (6.4)*, as limit of variational solutions. Then, for some $\kappa>0$ it holds that*

$$\|u_\infty^+-u_\infty^-\|\le C\varepsilon^\kappa$$

$$\max\{\left\|\nabla_x u_\infty^+-\nabla_x u_\infty^-\|_\infty,\|\nabla_y u_\infty^+-\nabla_y u_\infty^-\|_\infty\right\}\le C\varepsilon^\kappa.$$

Proof. Let us take $u_n^+ - u_n^-$ as test function in the weak formulation of problems (2.1) for $f(z) = \varepsilon^{p_n(z)}$ and $f(z) = -\varepsilon^{p_n(z)}$ and subtract the equations (notice that it is an admisible test function thanks to (4.3)). It gives us

$$\begin{aligned}&\int_\Omega \langle |\nabla_x u_n^+|^{p_n(z)-2}\nabla_x u_n^+ - |\nabla_x u_n^-|^{p_n(z)-2}\nabla_x u_n^-, \nabla_x u_n^+ - \nabla_x u_n^-\rangle\, dz\\ &+ \int_\Omega \langle |\nabla_y u_n^+|^{q_n(z)-2}\nabla_y u_n^+ - |\nabla_y u_n^-|^{q_n(z)-2}\nabla_y u_n^-, \nabla_y u_n^+ - \nabla_y u_n^-\rangle\, dz\\ &= 2\int_\Omega \varepsilon^{p_n(z)}(u_n^+ - u_n^-)\, dz.\end{aligned}$$

Thanks to the inequality (2.3) we deduce that

$$2\int_\Omega \left|\frac{\nabla_x u_n^+ - \nabla_x u_n^-}{2}\right|^{p_n(z)} dz + \int_\Omega \left|\frac{\nabla_y u_n^+ - \nabla_y u_n^-}{2}\right|^{q_n(z)} dz \le \int_\Omega \varepsilon^{p_n(z)}(u_n^+ - u_n^-)\, dz.$$

Obviously,

$$2^{\frac{1}{n}}\left(\int_\Omega \left(\left|\frac{\nabla_x u_n^+ - \nabla_x u_n^-}{2}\right|^{\frac{p_n(z)}{n}}\right)^n dz\right)^{\frac{1}{n}} \le \left(\int_\Omega \left(\varepsilon^{\frac{p_n(z)}{n}}(u_n^+ - u_n^-)^{\frac{1}{n}}\right)^n dz\right)^{\frac{1}{n}}. \tag{6.5}$$

Now we observe that, without loss of generality, we can assume that

$$\lim_{n\to\infty}\frac{p_n(z)}{n} = \ell_1(z), \qquad \lim_{n\to\infty}\frac{q_n(z)}{n} = \ell_2(z). \tag{6.6}$$

Note that if

$$\lim_{n\to\infty}\frac{p_n(z)}{\sigma(n)} = \ell_1(z), \quad \text{with} \quad \sigma(n)\ne n \quad \text{and} \quad \sigma(n)\to\infty \quad \text{as} \quad n\to\infty,$$

then, we can rearrange the index of the sequence such that $\frac{p_{n_k}(z)}{n_k} \to \ell_1(z)$, with $n_k = \sigma(n)$ (with similar conclusion for q_n). Taking this observation in mind, if we pass to the limit as $n\to\infty$ in (6.5) we get

$$\left\|\left|\frac{\nabla_x u_\infty^+ - \nabla_x u_\infty^-}{2}\right|^{\ell_1(z)}\right\|_\infty \le \|\varepsilon^{\ell_1(z)}\|_\infty.$$

Similarly,

$$\left\|\left|\frac{\nabla_y u_\infty^+ - \nabla_y u_\infty^-}{2}\right|^{\ell_2(z)}\right\|_\infty \le \|\varepsilon^{\ell_1(z)}\|_\infty,$$

hence

$$\|\nabla u_\infty^+ - \nabla u_\infty^-\|_\infty \le C\varepsilon^\kappa,$$

being κ some positive constant depending on the bounds on ℓ_1. This estimate together with the fact that $u_\infty^+ - u_\infty^- = 0$ on $\partial\Omega$ implies that

$$\|u_\infty^+ - u_\infty^-\|_\infty \le C\varepsilon^\kappa,$$

and the proof is complete.

Corollary 6.1 *u_∞, as limit of variational solutions to* (2.1) *with $f = 0$ and nontrivial boundary condition, is unique.*

Moreover, we go further and see that problem (6.1) has uniqueness of viscosity solutions that are not necessarily variational. This is the core of the last subsection of this paper.

6.2 Uniqueness of solutions to problem (6.1)

The following result will be the key to show the uniqueness of solutions to (6.1).

Lemma 6.2 *Let $u \in C(\overline{\Omega})$ be an arbitrary solution to* (6.1)*. It holds that*

$$u_\infty^- \le u \le u_\infty^+,$$

where u_∞^+, u_∞^- are the variational solutions to (6.2) *and* (6.4) *constructed in the previous subsection, with g as boundary data.*

Proof. We only perform the proof of the fact that, if u is a subsolution to (6.1) such that $u \le u_\infty^+$ on $\partial\Omega$, being u_∞^+ a supersolution to (6.2), then $u \le u_\infty^+$ in Ω, since the remaining inequality can be handled similarly. Assume, *ad contrarium* that this is not the case, namely that

$$\max_{\Omega}(u - u_\infty^+) > \max_{\partial\Omega}(u - u_\infty^+), \tag{6.7}$$

and we will reach a contradiction.

We make the following observations:

i) To simplify the notation we will denote as $v = u_\infty^+$ and we will write the proof formally: at each point $z_0 \in \Omega$ where $u - \phi$ or $v - \phi$ attains a minimum, instead of working with the test function ϕ in the equation, we will just write u or v.

ii) As in the proof of (4.1), see also Theorem 2.2 in [29], we can show that $\|\nabla_x v\|_\infty, \|\nabla_y v\|_\infty \le C$, with $C = C(L_g)$ (L_g the Lipschitz constant for g), but independent of p_n, q_n.

iii) We can asume that $v > 0$, otherwise we add a constant.

We need to get a contradiction assuming (6.7) with the equations for u and v, for which we need a strict inequality. Therefore, as first step we construct a strict supersolution, w, to (6.2), that also verifies (6.7), that is, such that

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w). \tag{6.8}$$

To ensure this inequality holds, as it is usual in this kind of problems, we take $w = h(v)$, being h the following approximation of the identity

$$h(t) = \frac{1}{\alpha}\log(1 + A(e^{\alpha t} - 1)), \quad \alpha > 1,\ A > 1.$$

that was examined in [22]. In this work the following properties for h can be found:

$$0 < h(t) - t < \frac{A-1}{\alpha} \quad \text{for } t \geq 0,$$

which ensures (6.8) for A close to one and α large. In addition, $0 < h'(t) - 1 < A - 1$ for $t \geq 0$,

$$\frac{h''(t)}{h'(t)} = -\alpha(h'(t) - 1) = -\frac{\alpha(A-1)}{1 + A(e^{\alpha t} - 1)}, \tag{6.9}$$

and, if $A < 2$,

$$0 \leq \ln(h'(t)) = \ln(1 + (h'(t) - 1)) \leq h'(t) - 1. \tag{6.10}$$

Let us show that $w = h(v)$ is the desired strict supersolution to (6.2). We compute the terms involved in the equations

$$\left\{\begin{array}{ll} \nabla_x w = h'(v)\nabla_x v & \nabla_y w = h'(v)\nabla_y v \\ w_{x_i x_j} = h''(v) v_{x_i} v_{x_j} + h'(v) v_{x_i x_j} & w_{y_i y_j} = h''(v) v_{y_i} v_{y_j} + h'(v) v_{y_i y_j} \\ \Delta_{\infty,x} w = h(v)^3 \Delta_{\infty,x} v & \Delta_{\infty,y} w = h(v)^3 \Delta_{\infty,y} v \\ \qquad + h'(v)^2 h''(v)|\nabla_x v|^4 & \qquad + h'(v)^2 h''(v)|\nabla_y v|^4. \end{array}\right. \tag{6.11}$$

We start by showing that w is a strict viscosity supersolution to the first equation in (6.2). We notice that this equation is similar to the single equation treated in [24], except for the fact that here the derivatives with respect to x are partial derivatives. We include the proof for convenience of the reader.

After multiplying the first equation by $[h'(v)]^3$ and using (6.11) we get

$$-\Delta_{\infty,x} w + h'(v)^2 h''(v)|\nabla_x v|^4 - |\nabla_x w|^2 \ln(|\nabla_x v|)\langle \nabla_x w, \xi_p\rangle \geq 0,$$

which yields that

$$-\Delta_{\infty,x} w - |\nabla_x w|^2 \ln(|\nabla_x w|)\langle \nabla_x w, \xi_p\rangle \geq -|\nabla_x w|^3 \left(\frac{h''(v)}{h'(v)}|\nabla_x v| + \ln(h'(v))|\xi_p|\right),$$

if we write $\ln(|\nabla_x v|) = \ln(|\nabla_x w|) - \ln(h'(v))$. Taking into account (6.9), (6.10) and the fact that $|\nabla_x v| \geq \varepsilon$, we deduce that

$$\begin{aligned} -\Delta_{\infty(x)} w &\geq -|\nabla_x w|^3(-\alpha\varepsilon + \|\xi_p\|_\infty)\frac{A-1}{1 + A(e^{\alpha v} - 1)} \\ &\geq -\varepsilon^3 h'(v)^4(-\alpha\varepsilon + \|\xi_p\|_\infty)(A-1)A^{-1}e^{-\alpha v}. \end{aligned}$$

Fix some $\varepsilon > 0$, then we choose $\alpha = \alpha(\varepsilon)$ large so that $-\alpha\varepsilon + \|\xi_p\|_\infty \leq -2$ (we had chosen α large to ensure (6.8)). On the other hand, since $|\nabla w| = |h'(v)||\nabla v|$, z_0 is a point such that $|\nabla_x v(z_0)| > |\nabla_y v(z_0)|^{\theta(z_0)}$, if and only if $|\nabla_x w(z_0)| > |\nabla_y w(z_0)|^{\theta(z_0)}$. Moreover, at that point z_0 we get

$$-\Delta_{\infty(x_0)} w \geq \nu_1 > 0,$$

with $\nu_1 = \varepsilon^3(A-1)A^{-1}e^{-\alpha\|v\|_\infty}$. Analogously, if $z_0 \in \Omega$ is a point such that $|\nabla_x v(z_0)| < |\nabla_y v(z_0)|^{\theta(z_0)}$, and hence $|\nabla_x w(z_0)| < |\nabla_y w(z_0)|^{\theta(z_0)}$ we obtain

$$-\theta(z_0)\Delta_{\infty(y_0)} w \geq \nu_2 > 0,$$

with $\nu_2 = \theta(z_0)\varepsilon^{\frac{3}{\theta(z_0)}}(A-1)A^{-1}e^{-\alpha\|v\|_\infty}$. Finally, at points such that $|\nabla_x w(z_0)| = |\nabla_y w(z_0)|^{\theta(z_0)}$ obviously

$$\max\Big\{-\Delta_{\infty(x_0)}w,\ -\theta(z_0)\Delta_{\infty(y_0)}w,\ -\Delta_{\infty(x_0)}w-\theta(z_0)\Delta_{\infty(y_0)}w\Big\} \geq \nu_1 > 0.$$

Turning back to our assumption (6.8) and applying the maximum principle for semicontinuos functions, see [5], doubling the variables $z=(x,y)$ and $\overline{z}=(\overline{x},\overline{y})$, we define

$$\Psi(z,\overline{z}) = u(z) - w(\overline{z}) - \frac{j}{2}|z-\overline{z}|, \quad j\in\mathbb{N}.$$

This function attains its maximum at interior points $(z_j,\overline{z}_j)$ for each j, (the maximum cannot be achieved on the boundary due to (6.8)). It is known that

$$z_j \to \hat{z} \qquad \text{and} \qquad \overline{z}_j \to \hat{z},$$

with $\hat{z}\in\Omega$, and

$$j|z-\overline{z}|^2 \to 0, \quad \text{as} \quad j\to\infty,$$

hence

$$j|x-\overline{x}|^2 \to 0, \quad \text{and} \quad j|y-\overline{y}|^2 \to 0 \quad \text{as} \quad j\to\infty.$$

The maximum principle also guarantees the existence of symmetric matrices $S_j, \overline{S}_j \in \mathbb{S}_{N+K}$ such that $S_j \leq \overline{S}_j$ and

$$(j|z_j-\overline{z}_j|, S_j) \in \overline{J^{2,+}}u(z_j) \quad \text{and} \quad (j|z_j-\overline{z}_j|, \overline{S}_j) \in \overline{J^{2,+}}w(\overline{z}_j).$$

As before we call

$$S_j^1 = (s_{kl})_{1\leq k,l\leq N}$$

the first $N\times N$ minor of the matrix S_j and

$$S_j^2 = (s_{kl})_{N+1\leq k,l\leq N+K}$$

the last $K\times K$ minor of S_j, and analogous indexes for the matrix $\overline{S}$. With this notation we rewrite the first equations as follows

$$-j^2\langle S_j^1(x_j-\overline{x}_j), x_j-\overline{x}_j\rangle - j^3|x_j-\overline{x}_j|^2\ln(j|x_j-\overline{x}_j|)\langle x_j-\overline{x}_j, \xi_p(z_j)\rangle \geq 0, \tag{6.12}$$

and the equations for w,

$$-j^2\langle \overline{S}_j^1(x_j-\overline{x}_j), x_j-\overline{x}_j\rangle - j^3|x_j-\overline{x}_j|^2\ln(j|x_j-\overline{x}_j|)\langle x_j-\overline{x}_j, \xi_p(\overline{z}_j)\rangle \geq \nu_1, \tag{6.13}$$

with $j|x_j-\overline{x}_j|\geq\varepsilon$.

Moreover, let us see that

$$j|x_j-\overline{x}_j| \leq C.$$

Indeed, just observe that

$$\begin{aligned} u(z_j)-w(\overline{z}_j)-\frac{j}{2}|x_j-\overline{x}_j|^2 &\geq u(z_j)-w(\overline{z}_j)-\frac{j}{2}|z_j-\overline{z}_j|^2 \\ &\geq u(z_j)-w(z_j)-\frac{j}{2}|z_j-z_j|^2 = u(z_j)-w(z_j), \end{aligned}$$

thus

$$\frac{j}{2}|x_j - \overline{x}_j|^2 \le w(z_j) - w(\overline{z}_j) \quad \le ||\nabla_x w||_\infty |x_j - \overline{x}_j|$$
$$= ||h'(v)\nabla_x v||_\infty |x_j - \overline{x}_j| \le CA|x_j - \overline{x}_j|,$$

according to observation *ii*), hence $C = 2CA$. Subtracting the equations (6.12) and (6.13) we get

$$-j^2\langle(\overline{S}^1_j - S^1_j)(x_j - \overline{x}_j), x_j - \overline{x}_j\rangle$$
$$\ge \nu_1 + j^3|x_j - \overline{x}_j|^2 \ln(j|x_j - \overline{x}_j|)\langle x_j - \overline{x}_j, \xi_p(\overline{z}_j) - \xi_p(z_j)\rangle$$
$$\ge \nu_1 - C^3 \ln\left(\frac{\widetilde{C}}{\varepsilon}\right)|\xi_p(x_j) - \xi_p(\overline{x}_j)| \ge \frac{1}{2}\nu_1 > 0,$$

for j large, by the continuity of ξ_p. But this is a contradiction, since the left hand side term is non positive, due to the fact that $S_j \le \overline{S}_j$. The contradiction with the remaining two equations can be handled similarly.

Corollary 6.2 *The unique solution to problem* (6.1) *is* u_∞.

Proof. Let u_1, u_2 be arbitrary solutions to (6.1). Then

$$|u_1 - u_2| \le |u_\infty - u_1| + |u_\infty - u_2| \le \varepsilon^\kappa,$$

by Lemma 6.1. Therefore, the solution to (6.1) is unique and hence coincides with the variational limit u_∞.

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