Advanced Nonlinear Studies **13** (2013), 391–410

Global Attractors for p-Laplacian Equations with Dynamic Flux Boundary Conditions

Bo You*

School of Mathematics and Statistics
Xi'an Jiaotong University, P. R. China, Xi'an 710049
e-mail: youb03@126.com

Chengkui Zhong

Department of Mathematics
Nanjing University, P. R. China, Nanjing 210093
e-mail: ckzhong@nju.edu.cn

Received 14 May 2012
Communicated by Peter Kloeden

Abstract

This paper studies the long-time behavior of solutions for p-Laplacian equations with a polynomial growth nonlinearity of arbitrary order and dynamic flux boundary conditions in d-dimensional bounded smooth domains. We first prove the existence of global attractors in $L^2(\Omega)\times L^2(\Gamma)$ and $L^q(\Omega)\times L^q(\Gamma)$ for the p-Laplacian evolution equations subject to dynamic nonlinear boundary conditions by using the Sobolev compactness embedding theory, and then the existence of $(L^2(\Omega)\times L^2(\Gamma), L^{2(q-1)}(\Omega)\times L^{2(q-1)}(\Gamma))$-global attractor is obtained by asymptotical regularity method. Finally, we prove the existence of global attractor in $\left(W^{1,p}(\Omega)\cap L^q(\Omega)\right)\times L^q(\Gamma)$ by asymptotic a priori estimate.

1991 Mathematics Subject Classification. 35K60.

Key words. Global attractor, p-Laplacian, norm-to-weak continuous semigroup, asymptotic a priori estimate, nonlinear flux boundary conditions, asymptotic regularity

1 Introduction

In this paper, we are concerned with the existence of global attractors in $\left(W^{1,p}(\Omega)\cap L^q(\Omega)\right)\times L^q(\Gamma)$ and $L^{2(q-1)}(\Omega)\times L^{2(q-1)}(\Gamma)$ for the semigroup associated with solutions of the following p-

*The authors acknowledge the support NSFC (11031003)

Laplacian equation

$$u_t - \Delta_p u + |u|^{p-2}u + f(u) = g(x), \qquad (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.1}$$

Equation (1.1) is subject to the dynamic flux boundary condition

$$u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + f(u) = 0, \qquad (x,t) \in \Gamma \times \mathbb{R}^+ \tag{1.2}$$

and the initial conditions

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{1.3}$$

$$u(x,0) = \theta_0(x),\ x \in \Gamma, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^d (d \geq 3)$ is a bounded domain with smooth boundary Γ, ν denotes the outer unit normal on Γ, $d > p \geq 2$.

In addition, we assume that $g : \Omega \to \mathbb{R}$ is locally Lipschitz continuous, the function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

$$f'(u) \geq -l \tag{1.5}$$

for some $l \geq 0$ and

$$c_1|u|^q - k \leq f(u)u \leq c_2|u|^q + k, \tag{1.6}$$

where $c_i > 0$, $i = 1, 2$, $q \geq 2$, $k > 0$ and $g \in L^2(\Omega)$.

Dynamic boundary conditions appear very naturally in many mathematical models such as heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, heat transfer in two medium, problems in fluid dynamics(see [13, 14, 15, 17, 24, 25, 30, 31, 33, 34, 37] etc.).

During the past several decades, many authors have extensively studied the well-posedness, the blow-up phenomenon and the long-time behavior of solutions for some quasilinear parabolic equations with many types of different boundary conditions and the multiplicity of solutions for some quasilinear elliptic equations with many types of different boundary conditions, for example, Dirichlet conditions, Neumann boundary conditions, nonlinear boundary conditions and dynamic boundary conditions and so on. The authors have studied the existence, the uniqueness and the regularity of solutions for the heat equations with nonlinear boundary conditions under properly dissipative assumptions on the nonlinear terms for which the initial data is in $L^q(\Omega)$, $W^{1,q}(\Omega)(1 < q < \infty)$ or measure space respectively in [3]. The existence of global attractors for parabolic problems with nonlinear boundary conditions was established in [4]. The authors have proved the existence of global attractors for reaction-diffusion equations with dynamic boundary conditions under some assumptions on the nonlinear terms in [13]. The existence of attractors for reaction-diffusions with many types of different boundary conditions was obtained in [5], [16], [33]-[35], [38], respectively. The authors have proved the existence of solutions for parabolic equations with generalized Wentzell boundary conditions in [18, 32]. The eigenvalue problems and the Fučik spectrum for p-Laplacian equations have been studied in [19, 22, 23]. The authors were concerned with the existence of solutions for p- Laplacian elliptic equation in [7, 12, 21]. The existence of global attractors for p-Laplacian equations was established in [36]. The authors have proved the existence of uniform attractors for the non-autonomous p-Laplacian equations in [11]. The long-time behavior of solutions for a class of quasi-linear parabolic equations involving weighted p- Laplacian operators in an arbitrary domain was considered via the concept of global attractor for multi-valued semi-flows in [2].

The authors have studied the existence of solutions, global in time, for some quasi-linear parabolic equations in [8]-[10]. The existence of solutions, global in time, for elliptic equations with dynamic boundary conditions was proved in [37]. The authors have considered the blow-up phenomenon of solutions for some quasi-linear equations with dynamic boundary conditions of parabolic type was studied in [30]. The well-posedness of weak solutions and the existence of global attractors for quasi-linear parabolic equations with nonlinear dynamic boundary conditions for which the nonlinearities are subcritical to a given exponent were studied in [15]. The authors have considered the well-posedness of weak solutions and the existence of global attractors for a class of degenerate parabolic equations with dynamic boundary conditions and a trajectory attractor of weak solutions for the associated parabolic equation was constructed under some assumptions on the nonlinear terms in [14], but the authors impose an additional restriction on the order of the nonlinear terms.

The main purpose of this paper is to study the long-time behavior of solutions for p-Laplacian evolutionary equation (1.1)-(1.4) under quite general assumptions (1.5)-(1.6) on the nonlinear term, however, there is no other restriction on the order q (≥ 2) of the nonlinear term. In this paper, we will extend the results for Laplacian parabolic equations in [13] to the ones for p- ($2 < p < d$) Laplacian parabolic equations for which the order of the nonlinear term is arbitrary. We will first prove the existence and the uniqueness of solutions for the problem (1.1)-(1.6), and then we obtain the existence of absorbing sets associated with the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6) in $\left(L^2(\Omega)\cap W^{1,p}(\Omega)\cap L^q(\Omega)\right)\times\left(L^2(\Gamma)\cap L^q(\Gamma)\right)$. Next, we will prove the existence of global attractors in $L^2(\Omega)\times L^2(\Gamma)$ and $L^q(\Omega)\times L^q(\Gamma)$ associated with the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6) by use of the Sobolev compactness embedding theory and interpolation inequality, and the existence of global attractor in $L^{2(q-1)}(\Omega)\times L^{2(q-1)}(\Gamma)$ of the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to (1.1)-(1.6) will be obtained by the asymptotic regularity method. Finally, we will establish the existence of global attractor in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$ associated with the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6) by the use of asymptotic a priori estimate.

This paper is organized as follows: In section 2, we will give some notations and lemmas used in the sequel. The existence and the uniqueness of solutions for (1.1)-(1.6) will be obtained in section 3. Section 4 is devoted to prove the existence of absorbing sets and the existence of global attractors in $\left(L^2(\Omega)\cap L^q(\Omega)\cap L^{2(q-1)}(\Omega)\cap W^{1,p}(\Omega)\right)\times\left(L^2(\Gamma)\cap L^q(\Gamma)\cap L^{2(q-1)}(\Gamma)\right)$ of the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to (1.1)-(1.6).

Throughout this paper, let X be a Banach space endowed with the norm $\|\cdot\|_X$, $(\cdot,\cdot)$ be the inner product in $L^2(\Omega)$(or $L^2(\Gamma)$) and C be a positive constant, which may be different from line to line (and even in the same line), the symbol $C(s)$ denote a positive constants dependent of s, we denote the trace of u by v and let γ be the trace operator.

2 Preliminaries

In order to study the problem (1.1)-(1.6), we will recall the Sobolev space $W^{1,p}(\Omega)$ defined as the closure of $C^\infty(\Omega)\cap W^{1,p}(\Omega)$ with respect to the norm

$$\|u\|_{1,p}=\left(\int_\Omega |\nabla u|^p+|u|^p\,dx\right)^{\frac{1}{p}}$$

and denote the dual space of Banach space X by X^*.

In the following, for the sake of brevity, we will first introduce some notations used in this paper.

Denote

$$
\begin{aligned}
&p^{'} = \frac{p}{p-1},\\
&\Omega_T = \Omega \times (0,T),\\
&\Gamma_T = \Gamma \times (0,T),\\
&V = \Big(L^p(0,T;W^{1,p}(\Omega)) \cap L^2(\Omega_T) \cap L^q(\Omega_T)\Big) \times\\
&\Big(L^p(0,T;W^{1-\frac{1}{p},p}(\Gamma)) \times L^2(\Gamma_T) \cap L^q(\Gamma_T)\Big),\\
&V^* = (L^{p^{'}}(0,T;(W^{1,p}(\Omega))^*) + L^2(\Omega_T) + L^{q^{'}}(\Omega_T)) \times\\
&\Big(L^{p^{'}}(0,T;(W^{1-\frac{1}{p},p}(\Gamma))^*) + L^2(\Gamma_T) + L^{q^{'}}(\Gamma_T)\Big)
\end{aligned}
$$

and define the operator $A : L^p(0,T;W^{1,p}(\Omega)) \to (L^p(0,T;W^{1,p}(\Omega)))^*$ given by

$$
\langle A(u), v\rangle = \int_{\Omega_T} |\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv \tag{2.1}
$$

for any $u, v \in L^p(0,T;W^{1,p}(\Omega))$.

Next, we will recall briefly some lemmas used to prove the well-posedness of solutions and the existence of global attractors of the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to (1.1)-(1.4) under some assumptions on the nonlinear term f.

Lemma 2.1 *([27])* Let O be a bounded domain in $\mathbb{R}^d$ and $\{g_n\} \subset L^q(O)$, where $1 < q < \infty$ is given. Assume that there exist a positive constant C independent of n and $g \in L^q(O)$ such that $\|g_n\|_{L^q(O)} \leq C$ for any natural number n and $g_n \to g$ almost everywhere in O, as $n \to \infty$. Then $g_n \to g$ weakly in $L^q(O)$, as $n \to \infty$.

Lemma 2.2 *([7]) Let $x, y \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ be the standard scalar product in $\mathbb{R}^d$. Then for any $p \geq 2$, there exist two positive constants C_1, C_2 depend only on p such that*

$$
\begin{aligned}
&\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq C_1|x-y|^p,\\
&\Big||x|^{p-2}x - |y|^{p-2}y\Big| \leq C_2(|x|+|y|)^{p-2}|x-y|.
\end{aligned}
$$

Lemma 2.3 *([1]) Let Ω be a bounded subset of $\mathbb{R}^d$ with smooth boundary Γ and $1 \leq p < \infty$. Then the following inclusion*

$$
W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Gamma)
$$

is compact for any $r \in [1, p*)$, where

$$
p* = \begin{cases} \frac{(d-1)p}{d-p}, & p < d;\\ \infty, & p = d. \end{cases}
$$

Lemma 2.4 *([19]) Let A be defined by* (2.1) *and $X = L^p(0,T;W^{1,p}(\Omega))$. Then for any $u, v \in X$, one has*

$$
\langle A(u) - A(v), u - v\rangle \geq (\|u\|_X^{p-1} - \|v\|_X^{p-1})(\|u\|_X - \|v\|_X).
$$

Furthermore, $\langle A(u) - A(v), u - v\rangle = 0$ if and only if $u = v$ a.e. in Ω_T.

3 The well-posedness of solutions

In the following, we assume that $(u_0, \theta_0) \in L^2(\Omega) \times L^2(\Gamma)$ is given and $T > 0$.

Definition 3.1 *A function $u(x,t)$ is called a weak solution of* (1.1)-(1.6) *on* $(0,T)$, *if*

$$\begin{aligned} &(u,v) \in V, \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) \in V^*, \\ &u|_{t=0} = u_0, \ \ a.e.\ in\ \Omega, \\ &v|_{t=0} = \theta_0, \ \ a.e.\ in\ \Gamma \end{aligned}$$

and

$$\int_{\Omega_T} \left(u_t\xi + |\nabla u|^{p-2}\nabla u \cdot \nabla \xi + |u|^{p-2}u\xi + f(u)\xi\right) + \int_{\Gamma_T} (v_t\xi + f(v)\xi) = \int_{\Omega_T} g(x)\xi$$

for all test functions $\xi \in V$.

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^d (d \geq 3)$, f satisfies* (1.5)-(1.6)*, g is locally Lipschitz continuous and $g \in L^2(\Omega)$. Then for any $(u_0, \theta_0) \in L^2(\Omega) \times L^2(\Gamma)$ and any $T \geq 0$, there exists a unique weak solution $u(x,t)$ of* (1.1)-(1.4) *and the mapping*

$$(u_0, \theta_0) \to (u(t), v(t))$$

is continuous on $L^2(\Omega) \times L^2(\Gamma)$.

Proof. We will first prove the existence of solutions for (1.1)-(1.6) by the Faedo-Galerkin method (See [29]).

Consider the approximate solution $u_n(t)$ in the form

$$u_n(t) = \sum_{i=1}^{n} u_{n_i}(t) e_i,$$

where $\{(e_j, \gamma e_j)\}_{j=1}^{\infty}$ is an orthogonal basis of $L^2(\Omega) \times L^2(\Gamma)$, which is included in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$. We get u_n from solving the following problem

$$\langle \frac{du_n}{dt}, e_k\rangle + \langle \frac{dv_n}{dt}, e_k\rangle + \langle A(u_n) + |u_n|^{p-2}u_n, e_k\rangle + \langle f(u_n), e_k\rangle + \langle f(v_n), e_k\rangle = \langle g(x), e_k\rangle, \quad (3.1)$$

$$(u_n(0), e_k) = (u_0, e_k),\ k = 1, \cdots, n, \quad (3.2)$$

$$(v_n(0), e_k) = (\theta_0, e_k),\ k = 1, \cdots, n. \quad (3.3)$$

Since f is continuous and g is locally Lipschitz continuous, using the Peano theorem, we get the existence of the solutions (u_n, v_n), local in time, for (3.1)-(3.3). Next, we will establish some a priori estimates for (u_n, v_n). We have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|v_n(t)\|^2_{L^2(\Gamma)} + \|u_n\|^p_{1,p} + \int_\Omega f(u_n)u_n + \int_\Gamma f(v_n)v_n \\ = &\int_\Omega g(x)u_n. \end{aligned}$$

Thanks to (1.5)-(1.6), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|v_n(t)\|^2_{L^2(\Gamma)} + \|u_n\|^p_{1,p} + c_1\|u_n\|^q_{L^q(\Omega)} + c_1\|v_n\|^q_{L^q(\Gamma)} \\ \leq \quad & \frac{1}{2}\|g(x)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} + k|\Omega| + k|\Gamma|. \end{aligned} \tag{3.4}$$

By virtue of the following inequality (see Theorem 2.3.1 in [6]):

$$-\mu z^q + \lambda z^2 \leq C\mu^{\frac{-2}{q-2}}\lambda^{\frac{q}{q-2}} \tag{3.5}$$

for any $q \geq 2$, $z \geq 0$, $\mu > 0$ and $\lambda \geq 0$.

Let $\mu = c_1$ and $\lambda = 1$. We deduce from (3.4) and (3.5) that

$$\begin{aligned} & \frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\|v_n(t)\|^2_{L^2(\Gamma)} + 2\|u_n\|^p_{1,p} + c_1\|u_n\|^q_{L^q(\Omega)} + 2c_1\|v_n\|^q_{L^q(\Gamma)} \\ \leq \quad & \|g(x)\|^2_{L^2(\Omega)} + C. \end{aligned} \tag{3.6}$$

Integrating (3.6) from 0 to t, we obtain

$$\begin{aligned} & \|u_n(t)\|^2_{L^2(\Omega)} + \|v_n(t)\|^2_{L^2(\Gamma)} + 2\int_0^t \|u_n\|^p_{1,p} + c_1\int_0^t \|u_n\|^q_{L^q(\Omega)} + 2c_1\int_0^t \|v_n\|^q_{L^q(\Gamma)} \\ \leq \quad & CT + \|u_n(0)\|^2_{L^2(\Omega)} + \|v_n(0)\|^2_{L^2(\Gamma)} \end{aligned} \tag{3.7}$$

for any $t \in (0, T]$.

Due to (3.7), we get

$$\begin{aligned} &\{u_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^{\infty}(0,T;L^2(\Omega)), \\ &\{v_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^{\infty}(0,T;L^2(\Gamma)), \\ &\{u_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^{p}(0,T;W^{1,p}(\Omega)), \\ &\{u_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^{q}(\Omega_T), \\ &\{v_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^{q}(\Gamma_T), \end{aligned}$$

which imply $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded with respect to n in $L^p(0,T;W^{1,p}(\Omega))$, $L^q(\Omega_T)$, respectively, and $\{v_n\}_{n=1}^{\infty}$ is uniformly bounded with respect to n in $L^q(\Gamma_T)$. Therefore, we can extract a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned} &u_{n_j} \rightharpoonup u \textit{ weakly in } L^p(0,T;W^{1,p}(\Omega)), \textit{ as } j \to \infty, \\ &u_{n_j} \rightharpoonup u \textit{ weakly in } L^q(\Omega_T), \textit{ as } j \to \infty, \\ &v_{n_j} \rightharpoonup v \textit{ weakly in } L^q(\Gamma_T), \textit{ as } j \to \infty. \end{aligned}$$

Let $P_n : V \to span\{(e_j, \gamma e_j)\}_{j=1}^n$ be a projection. For any $\phi \in V$, set $\phi_n = P_n\phi$, we have

$$\langle \frac{du_n}{dt}, \phi_n\rangle + \langle \frac{dv_n}{dt}, \phi_n\rangle + \langle A(u_n), \phi_n\rangle + \langle f(u_n), \phi_n\rangle + \langle f(v_n), \phi_n\rangle = \langle g(x), \phi_n\rangle. \tag{3.8}$$

From Hölder inequality and Young inequality, we deduce

$$\begin{aligned} |\langle A(u_n), \phi_n\rangle| \quad &= \quad \left|\int_{\Omega_T} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi_n + |u_n|^{p-2}u_n\phi_n\right| \\ &\leq \quad \|\nabla u_n\|^{p-1}_{L^p(\Omega_T)}\|\nabla\phi_n\|_{L^p(\Omega_T)} + \|u_n\|^{p-1}_{L^p(\Omega_T)}\|\phi_n\|_{L^p(\Omega_T)} \\ &\leq \quad \|u_n\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))}\|\phi_n\|_{L^p(0,T;W^{1,p}(\Omega))}. \end{aligned}$$

Using the boundedness of $\{u_n\}_{n=1}^{\infty}$ with respect to n in $L^p(0,T;W^{1,p}(\Omega))$ again, we infer that

$$\{A(u_n)\}_{n=1}^{\infty} \text{ is uniformly bounded in } L^{p'}(0,T;(W^{1,p}(\Omega))^*).$$

Since $g \in L^2(\Omega_T)$, $f(u_n) \in L^{q'}(\Omega_T)$, $f(v_n) \in L^{q'}(\Gamma_T)$, we find

$$(u_n', v_n') \in V^*.$$

Therefore, we can extract a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned}
&(u_{n_j}', v_{n_j}') \rightharpoonup (u', v') \text{ weakly in } V^*, \text{ as } j \to \infty,\\
&A(u_{n_j}) \rightharpoonup \xi \text{ weakly in } L^{p'}(0,T;(W^{1,p}(\Omega))^*), \text{ as } j \to \infty.
\end{aligned}$$

By virtue of the Aubin-Lions compactness theorem, we can extract a further subsequence (still denote by $\{u_{n_j}\}_{j=1}^{\infty}$) such that additionally

$$u_{n_j} \to u \text{ strongly in } L^p(\Omega_T), \text{ as } j \to \infty, \tag{3.9}$$

$$v_{n_j} \to v \text{ strongly in } L^p(\Gamma_T), \text{ as } j \to \infty. \tag{3.10}$$

Due to the boundedness of $\{u_n\}_{n=1}^{\infty}$ in $L^q(\Omega_T)$ and (1.6), we obtain that $\{f(u_n)\}_{n=1}^{\infty}$ is uniformly bounded with respect to n in $L^{q'}(\Omega_T)$ and hence $f(u_n) \rightharpoonup \chi$ in $L^{q'}(\Omega_T)$. Similarly, $f(v_n) \rightharpoonup \eta$ in $L^{q'}(\Gamma_T)$, and by virtue of (3.9)-(3.10), we see that $u_{n_j} \to u$ *a.e.* in Ω_T and $v_{n_j} \to v$ *a.e.*in Γ_T. Then $f(u_{n_j}) \to f(u)$ *a.e.* in Ω_T and $f(v_{n_j}) \to f(v)$ *a.e.* in Γ_T. Thanks to Lemma 2.1, we know that

$$\chi = f(u),\ \eta = f(v).$$

Therefore, we have

$$\langle u', \phi\rangle + \langle v', \phi\rangle + \langle \xi, \phi\rangle + \langle f(u), \phi\rangle + \langle f(v), \phi\rangle = \langle g(x), \phi\rangle \tag{3.11}$$

for any $\phi \in V$.

In order to prove that u is a weak solution of (1.1)-(1.4), it remains to show that $\xi = A(u)$. Noticing that

$$\begin{aligned}
\langle A(u_n), u_n\rangle &= \int_0^T \|u_n\|_{1,p}^p\\
&= \frac{1}{2}\|u_n(0)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_n(T)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v_n(0)\|_{L^2(\Gamma)}^2 - \frac{1}{2}\|v_n(T)\|_{L^2(\Gamma)}^2\\
&\quad - \int_0^T\int_\Omega f(u_n)u_n - \int_0^T\int_\Gamma f(v_n)v_n + \int_0^T\int_\Omega g(x)u_n.
\end{aligned} \tag{3.12}$$

It follows from the formulation of $u_n(0)$ and $v_n(0)$ that $u_n(0) \to u_0$ in $L^2(\Omega)$ and $v_n(0) \to \theta_0$ in $L^2(\Gamma)$. Moreover, by the lower semi-continuity of $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma)}$, we obtain

$$\|u(T)\|_{L^2(\Omega)}^2 \le \liminf_{n\to\infty}\|u_n(T)\|_{L^2(\Omega)}^2, \|v(T)\|_{L^2(\Omega)}^2 \le \liminf_{n\to\infty}\|v_n(T)\|_{L^2(\Omega)}^2. \tag{3.13}$$

Meanwhile, by the Lebesgue dominated theorem, one can check that

$$\begin{aligned}
&\int_0^T\int_\Omega f(u)u + \int_0^T\int_\Gamma f(v)v = \lim_{n\to\infty}\int_0^T\int_\Omega f(u_n)u_n + \lim_{n\to\infty}\int_0^T\int_\Gamma f(v_n)v_n.\\
&\lim_{n\to\infty}\int_0^T\int_\Omega g(x)u_n = \int_0^T\int_\Omega g(x)u.
\end{aligned}$$

This fact and (3.12)-(3.13) imply

$$\begin{aligned}
&\limsup_{n\to\infty}\langle A(u_n),u_n\rangle\\
\le\ & \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v(0)\|^2_{L^2(\Gamma)}-\frac{1}{2}\|v(T)\|^2_{L^2(\Gamma)}\\
&-\int_0^T\int_\Omega f(u)u-\int_0^T\int_\Gamma f(v)v+\int_0^T\int_\Omega g(x)u. \qquad (3.14)
\end{aligned}$$

In view of (3.11), we have

$$\begin{aligned}
\langle\xi,u\rangle\ =\ & \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v(0)\|^2_{L^2(\Gamma)}-\frac{1}{2}\|v(T)\|^2_{L^2(\Gamma)}\\
&-\int_0^T\int_\Omega f(u)u-\int_0^T\int_\Gamma f(v)v+\int_0^T\int_\Omega g(x)u.
\end{aligned}$$

From this and (3.14), we deduce

$$\limsup_{n\to\infty}\langle A(u_n),u_n\rangle\le\langle\xi,u\rangle. \qquad (3.15)$$

On one hand, we observe that

$$\begin{aligned}
&\lim_{n\to\infty}\langle A(u_n)-A(u),u_n-u\rangle\\
=\ &\lim_{n\to\infty}(\langle A(u_n),u_n\rangle-\langle A(u_n),u\rangle-\langle A(u),u_n-u\rangle)\\
\le\ &\langle\xi,u\rangle-\langle\xi,u\rangle=0.
\end{aligned}$$

On the other hand, it follows from Lemma 2.4 that

$$\begin{aligned}
&\langle A(u_n)-A(u),u_n-u\rangle\\
\ge\ &(\|u_n\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))}-\|u\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))})(\|u_n\|_{L^p(0,T;W^{1,p}(\Omega))}\\
&-\|u\|_{L^p(0,T;W^{1,p}(\Omega))})\ge 0.
\end{aligned}$$

Hence

$$\|u_n\|_{L^p(0,T;W^{1,p}(\Omega))}\to\|u\|_{L^p(0,T;W^{1,p}(\Omega))},\ \ as\ n\to\infty. \qquad (3.16)$$

Combining (3.16) with $u_n\rightharpoonup u$ in $L^p(0,T;W^{1,p}(\Omega))$, we obtain

$$u_n\to u\ strongly\ in\ L^p(0,T;W^{1,p}(\Omega)),\ \ as\ n\to\infty.$$

Therefore, from Lemma 2.2, Hölder inequality and Young inequality, we deduce that for any $\phi\in L^p(0,T;W^{1,p}(\Omega))$,

$$\begin{aligned}
&|\langle A(u_n)-A(u),\phi\rangle|\\
=\ &|\int_{\Omega_T}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\cdot\nabla\phi+(|u_n|^{p-2}u_n-|u|^{p-2}u)\phi|\\
\le\ &C_2\int_{\Omega_T}(|\nabla u_n|+|\nabla u|)^{p-2}|\nabla u_n-\nabla u||\nabla\phi|+C_2\int_{\Omega_T}(|u_n|+|u|)^{p-2}|u_n-u||\phi|\\
\le\ &C(\|u_n\|^{p-2}_{L^p(0,T;W^{1,p}(\Omega))}+\|u\|^{p-2}_{L^p(0,T;W^{1,p}(\Omega))})\times\\
&\|u_n-u\|_{L^p(0,T;W^{1,p}(\Omega))}\|\phi\|_{L^p(0,T;W^{1,p}(\Omega))},
\end{aligned}$$

which implies that $A(u_n) \rightharpoonup A(u)$ weakly in $(L^p(0,T;W^{1,p}(\Omega)))^*$, as $n \to \infty$. Hence $\xi = A(u)$.

Finally, we will prove the uniqueness and the continuous dependence on the initial data of solutions for (1.1)-(1.6).

Let u_1, u_2 be two solutions of (1.1)-(1.6) with the initial data (u_{1_0}, θ_{1_0}), (u_{2_0}, θ_{2_0}), respectively. Denote $w = u_1 - u_2$, therefore w satisfies the following equation

$$\begin{cases} w_t - \Delta_p u_1 + \Delta_p u_2 + |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2 + f(u_1) - f(u_2) = 0, \\ w_t + |\nabla u_1|^{p-2}\frac{\partial u_1}{\partial \nu} - |\nabla u_2|^{p-2}\frac{\partial u_2}{\partial \nu} + f(u_1) - f(u_2) = 0, \\ w(x,0) = u_{1_0} - u_{2_0}, \\ w(x,0) = \theta_{1_0} - \theta_{2_0}. \end{cases} \tag{3.17}$$

Taking the inner product of (3.17) with w, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Gamma)} + \int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u_1 - u_2) \\ &+ \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2) \\ &+ \int_\Omega (f(u_1) - f(u_2), u_1 - u_2) + \int_\Gamma (f(v_1) - f(v_2), v_1 - v_2) = 0. \end{aligned} \tag{3.18}$$

By virtue of (1.5) and Lemma 2.2, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Gamma)} \\ \leq\ & l\|w(t)\|^2_{L^2(\Omega)} + l\|w(t)\|^2_{L^2(\Gamma)}, \end{aligned}$$

which implies that

$$\begin{aligned} &\|w(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \\ \leq\ & \exp(2lt)(\|w(0)\|^2_{L^2(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)}). \end{aligned}$$

Therefore, $u_1(x,t) = u_2(x,t)$ a.e. in $\overline{\Omega_T}$, if $u_{1_0}(x) = u_{2_0}(x)$ in Ω and $\theta_{1_0}(x) = \theta_{2_0}(x)$ in Γ, and $(u(x,t), v(x,t))$ is continuously dependent on the initial datum. This is the end of the proof.

By Theorem 3.1, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ on $L^2(\Omega) \times L^2(\Gamma)$ as follows

$$S(.)(.,.) : \mathbb{R}^+ \times L^2(\Omega) \times L^2(\Gamma) \to L^2(\Omega) \times L^2(\Gamma),$$

which is continuous in $L^2(\Omega) \times L^2(\Gamma)$.

4 The existence of global attractors

In this section, we will prove the existence of global attractors of the semigroup $\{S(t)\}_{t\geq 0}$ associated with (1.1)-(1.6).

4.1 The existence of absorbing sets

In this subsection, we will first recall some definitions and lemmas used in the sequel.

Definition 4.1 *([38]) Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We say that $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous (or C^0) semigroup on X, if $\{S(t)\}_{t\geq 0}$ satisfies*

(i) $S(0) = Id$ *(the identity),*

(ii) $S(t)S(s) = S(t+s)$, $\forall t, s \geq 0$,

(iii) $S(t_n)x_n \rightharpoonup S(t)x$ *(or* $S(t_n)x_n \to S(t)x$ *), if* $t_n \to t$ *and* $x_n \to x$ *in* X.

Definition 4.2 *([29]) Let X be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X. A subset $\mathcal{A}$ in X is called a global attractor if and only if*

(i) $\mathcal{A}$ *is invariant, i.e.,* $S(t)\mathcal{A} = \mathcal{A}$ *for all* $t \geq 0$,

(ii) $\mathcal{A}$ *is compact in* X,

(iii) $\mathcal{A}$ *attracts all bounded sets of* X. *That is, for any bounded subset* B *of* X,

$$dist(S(t)B, \mathcal{A}) \to 0, \quad as \quad t \to \infty,$$

where $dist(A, B)$ *is the Hausdorff semi-distance of two sets* A *and* B *defined by*

$$dist(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_X.$$

Definition 4.3 *([29, 36]) Let $\{S(t)\}_{t\geq 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\geq 0}$ is called (X, Z)-asymptotically compact, if for any bounded (in X) sequence $\{x_n\}_{n=1}^{\infty} \subset X$ and $t_n \geq 0$ with $t_n \to \infty$, as $n \to \infty$, $\{S(t_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence with respect to the topology of Z.*

Definition 4.4 *([6, 20, 27, 29]) Let $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in a complete metric space X. A subset B_0 of X is called an absorbing set in X, if for any bounded subset B of X, there exists some time $T = T_B \geq 0$ such that $S(t)B \subset B_0$ for any $t \geq T$.*

Lemma 4.1 *([38]) Let X, Y be two Banach spaces, X^*, Y^* be the dual spaces of X, Y, respectively, where X is dense in Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is dense. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X and Y, respectively, and assume furthermore that $\{S(t)\}_{t\geq 0}$ is continuous or weak continuous on Y. Then $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on X if and only if $\{S(t)\}_{t\geq 0}$ maps compact subsets of $X \times \mathbb{R}^+$ into bounded sets of X.*

Definition 4.5 *([29]) Let $\{S(t)\}_{t\geq 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\geq 0}$ is called (X, Z)-uniformly compact, if for any bounded (in X) subset $B \subset X$, there exists $t_0 = t_0(B) \geq 0$ such that $\bigcup_{t\geq t_0} S(t)B$ is relatively compact in Z.*

Lemma 4.2 *([29],[38]) Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup on X. Then $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}$ in X provided that the following conditions hold true:*

(i) $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set* B_0 *in* X,

(ii) $\{S(t)\}_{t\geq 0}$ *is uniformly compact or asymptotically compact.*

Lemma 4.3 *([6], [29]) Let $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup and $\{S(t)\}_{t\geq 0}$ is uniformly compact. If there exists an absorbing set B, then $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}$ and*

$$\mathcal{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}.$$

Next, we will prove the existence of absorbing sets for the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6).

Theorem 4.1 *Assume that $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6). *Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by* (1.1)-(1.4) *has $\left(L^2(\Omega)\times L^2(\Gamma), (W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)\right)$- bounded absorbing sets. That is, for any bounded subset $B \subset L^2(\Omega)\times L^2(\Gamma)$ and any $(u_0,\theta_0)\in B$, there exists a positive constant T such that for any $t \geq T$,*

$$\|u(t)\|_{L^q(\Omega)} + \|v(t)\|_{L^q(\Gamma)} + \|u(t)\|_{W^{1,p}(\Omega)} \leq \rho,$$

where ρ is a positive constant independent of B, $(u(t), v(t)) = S(t)(u_0,\theta_0)$, and T is dependent on the $L^2(\Omega)\times L^2(\Gamma)$-norm of B.

Proof. Taking the inner product of (1.1) with u and integrating by parts, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Gamma)} + \|u\|^p_{1,p} + \int_\Omega f(u)u + \int_\Gamma f(v)v \\ = \;& \int_\Omega g(x)u. \end{aligned} \tag{4.1}$$

As a result of (1.6), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Gamma)} + \|u\|^p_{1,p} + c_1\|u\|^q_{L^q(\Omega)} + c_1\|v\|^q_{L^q(\Gamma)} \\ \leq\;& \frac{1}{2}\|g(x)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + k|\Omega| + k|\Gamma| \\ \leq\;& \frac{1}{2}\|g(x)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Gamma)} + k|\Omega| + k|\Gamma|. \end{aligned} \tag{4.2}$$

By virtue of the following inequality (see Theorem 2.3.1 in [6]):

$$-\mu z^q + \lambda z^2 \leq C\mu^{\frac{-2}{q-2}}\lambda^{\frac{q}{q-2}}, \tag{4.3}$$

for any $q \geq 2$, $z \geq 0$, $\mu > 0$ and $\lambda \geq 0$.

Let $\mu = \frac{1}{2}c_1$ and $\lambda = 1$, we deduce from (4.2) and (4.3) that

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\|v(t)\|^2_{L^2(\Gamma)} + 2\|u\|^p_{1,p} + c_1\|u\|^q_{L^q(\Omega)} \\ &+c_1\|v\|^q_{L^q(\Gamma)} + \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)} \\ \leq\;& \|g(x)\|^2_{L^2\Omega)} + C \end{aligned} \tag{4.4}$$

(when $q = 2$, (4.4) holds). That is,

$$\begin{aligned} &\frac{d}{dt}\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}\right) + \|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} \\ \leq\;& \|g(x)\|^2_{L^2(\Omega)} + C. \end{aligned}$$

By the Classical Gronwall Lemma, we obtain

$$\begin{aligned} &\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} \\ \leq\;& \exp(-t)(\|u_0\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Gamma)}) + (\|g(x)\|^2_{L^2(\Omega)} + C)(1-\exp(-t)), \end{aligned}$$

which implies

$$\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} \le \rho_0 \tag{4.5}$$

for any $(u_0, \theta_0) \in B$ and $t \ge T_0$, where

$$\rho_0 = 2(\|g(x)\|^2_{L^2(\Omega)} + C),$$
$$T_0 = \ln\left\{\frac{\|u_0\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Gamma)}}{\|g(x)\|^2_{L^2(\Omega)} + C}\right\}.$$

Denote $F(s) = \int_0^s f(\tau)d\tau$. By virtue of (1.6), we deduce

$$\alpha_1|u|^q - \beta \le F(u) \le \alpha_2|u|^q + \beta \tag{4.6}$$

and

$$\alpha_1\|u\|^q_{L^q(\Omega)} - \beta|\Omega| \le \int_\Omega F(u) \le \alpha_2\|u\|^q_{L^q(\Omega)} + \beta|\Omega|, \tag{4.7}$$

$$\alpha_1\|v\|^q_{L^q(\Gamma)} - \beta|\Gamma| \le \int_\Gamma F(v) \le \alpha_2\|v\|^q_{L^q(\Gamma)} + \beta|\Gamma|. \tag{4.8}$$

It follows (4.4), (4.7)-(4.8) that

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\|v(t)\|^2_{L^2(\Gamma)} + 2\|u\|^p_{1,p} + \frac{c_1}{\alpha_2}\int_\Omega F(u) + \frac{c_1}{\alpha_2}\int_\Gamma F(v) \\ \le\ & \|g(x)\|^2_{L^2(\Omega)} + C. \end{aligned} \tag{4.9}$$

Integrating (4.9) from t to $t+1$, we deduce that for any $t \ge T_0$,

$$\begin{aligned} &2\int_t^{t+1}\|u\|^p_{1,p} + \frac{c_1}{\alpha_2}\int_t^{t+1}\int_\Omega F(u) + \frac{c_1}{\alpha_2}\int_t^{t+1}\int_\Gamma F(v) \\ \le\ & \|g(x)\|^2_{L^2(\Omega)} + C + \|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}. \end{aligned} \tag{4.10}$$

On the other hand, taking the inner product of (1.1) with u_t, we obtain

$$\begin{aligned} &\|u_t\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(\Gamma)} + \frac{d}{dt}\left(\frac{1}{p}\|u\|^p_{1,p} + \int_\Omega F(u) + \int_\Gamma F(v)\right) \\ \le\ & \frac{1}{2}\|g(x)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.11}$$

It follows from (4.11) that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{p}\|u\|^p_{1,p} + \int_\Omega F(u) + \int_\Gamma F(v)\right) \\ \le\ & \frac{1}{2}\|g(x)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.12}$$

Using the Uniformly Gronwall Lemma, we deduce from (4.5), (4.10) and (4.12) that

$$\begin{aligned} &\frac{1}{p}\|u\|^p_{1,p} + \int_\Omega F(u) + \int_\Gamma F(v) \\ \le\ & 2\max\{\frac{1}{2p}, \frac{\alpha_2}{c_1}\}\rho_0, \end{aligned} \tag{4.13}$$

which implies that for any $(u_0, \theta_0) \in B$ and $t \geq T_0 + 1$, there exists a positive constant ρ such that

$$\|u(t)\|_{L^q(\Omega)} + \|v(t)\|_{L^q(\Gamma)} + \|u(t)\|_{W^{1,p}(\Omega)} \leq \rho. \tag{4.14}$$

This is the end of the proof.

Since $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Gamma)$ is compact, we obtain the following result.

Theorem 4.2 *Assume that $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6). *Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by* (1.1)-(1.4) *has a $(L^2(\Omega)\times L^2(\Gamma), L^2(\Omega)\times L^2(\Gamma))$-global attractor $\mathcal{A}_2$ which is compact, connected and invariant.*

4.2 The existence of $(L^2\Omega) \times L^2(\Gamma), L^q(\Omega) \times L^q(\Gamma))$- global attractor

Since $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Gamma)$ is compact, it follows from Theorem 4.1 that the semigroup generated by (1.1)-(1.6) is compact in $L^2(\Omega)\times L^2(\Gamma)$. From Theorem 4.1 and Lemma 4.1, we deduce that the semigroup generated by (1.1)-(1.6)) maps a compact set of $L^q(\Omega) \times L^q(\Gamma)$ into a bounded set of $L^q(\Omega) \times L^q(\Gamma)$, i.e.,the semigroup generated by (1.1)-(1.6) is norm-to-weak continuous in $L^q(\Omega)\times L^q(\Gamma)$. In this subsection, we will prove the existence of $(L^2(\Omega)\times L^2(\Gamma), L^q(\Omega)\times L^q(\Gamma))$-global attractor of the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6).

Theorem 4.3 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^d (d \geq 3)$, f satisfies* (1.5)-(1.6) *and $g \in L^2(\Omega)$. Then for any bounded subset B in $L^2(\Omega) \times L^2(\Gamma)$, there exist two positive constants $T = T(B)$ and ρ_1 such that*

$$\|u_t(s)\|^2_{L^2(\Omega)} + \|v_t(s)\|^2_{L^2(\Gamma)} \leq \rho_1$$

for any $(u_0, \theta_0) \in B$ and $s \geq T$, where $(u_t(s), v_t(s)) = \frac{d}{dt}\left(S(t)(u_0, \theta_0)\right)|_{t=s}$ and ρ_1 is a positive constant which is independent of B.

Proof. First of all, we will differentiate (1.1) and (1.2) with respect to t and denote $\zeta = u_t$, $\eta = v_t$, we get

$$\begin{aligned} &\zeta_t - div(|\nabla u|^{p-2}\nabla\zeta) - (p-2)div\left(|\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)\nabla u\right) \\ &+(p-1)|u|^{p-2}\zeta + f'(u)\zeta = 0, \end{aligned} \tag{4.15}$$

$$\eta_t + (p-2)|\nabla v|^{p-4}(\nabla v\cdot\nabla\eta)\frac{\partial v}{\partial \nu} + |\nabla v|^{p-2}\frac{\partial \eta}{\partial \nu} + f'(v)\eta = 0, \tag{4.16}$$

where "$\cdot$" denotes the dot product in $\mathbb{R}^d$.

Multiplying (4.15) by ζ, integrating over Ω and combining (1.5) with (4.16), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Gamma)} + \int_\Omega |\nabla u|^{p-2}|\nabla\zeta|^2 \\ &+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)^2 + (p-1)\int_\Omega |u|^{p-2}|\zeta|^2 \\ \leq\ & l\left(\|\zeta\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}\right), \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Gamma)} \\ \leq\ & l\left(\|\zeta\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}\right). \end{aligned}$$

On the other hand, integrating (4.11) from t to $t+1$ and using (4.14), we find

$$\begin{aligned} & \int_t^{t+1} \|\zeta\|^2_{L^2(\Omega)} + \int_t^{t+1} \|\eta\|^2_{L^2(\Gamma)} \\ \leq \quad & C(\rho_0, \|g\|_{L^2(\Omega)}) \end{aligned}$$

as t large enough, where

$$\rho_0 = 2(\|g(x)\|^2_{L^2(\Omega)} + C).$$

Therefore, we deduce from the Uniformly Gronwall Inequality that there exists two positive constants T and ρ_1 such that

$$\|u_t(s)\|^2_{L^2(\Omega)} + \|v_t(s)\|^2_{L^2(\Gamma)} \leq \rho_1$$

for any $(u_0, \theta_0) \in B$ and $s \geq T$. This is the end of the proof.

Next, we will prove the existence of absorbing sets in $L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)$ of the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.6) by using Theorem 4.3.

Theorem 4.4 *Assume that $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6). *Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by* (1.1)-(1.4) *has a $\left(L^2(\Omega) \times L^2(\Gamma), L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)\right)$-positively invariant bounded absorbing sets. That is, for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$ and any $(u_0, \theta_0) \in B$, there exists two positive constants T and ρ_2 such that for any $t \geq T$, we have*

$$\|u(t)\|^{2(q-1)}_{L^{2(q-1)}(\Omega)} + \|v(t)\|^{2(q-1)}_{L^{2(q-1)}(\Gamma)} \leq \rho_2,$$

where ρ_2 is a positive constant independent of B, $(u(t), v(t)) = S(t)(u_0, \theta_0)$, and T is dependent on B.

Proof. Taking the inner product of (1.1) with $|u|^{q-2}u$ and integrating by parts, we get

$$\begin{aligned} & \int_\Omega u_t|u|^{q-2}u + \int_\Gamma v_t|v|^{q-2}v + (q-1)\int_\Omega |\nabla u|^p|u|^{q-2} \\ & + \int_\Omega |u|^{p+q-2} + \int_\Omega f(u)|u|^{q-2}u + \int_\Gamma f(v)|v|^{q-2}v \\ = \quad & \int_\Omega g|u|^{q-2}u. \end{aligned}$$

From the Hölder inequality and (1.6), we deduce that

$$\begin{aligned} & c_1\int_\Omega |u|^{2q-2} + c_1\int_\Gamma |v|^{2q-2} - k\int_\Omega |u|^{q-2} - k\int_\Gamma |v|^{q-2} \\ \leq \quad & \int_\Omega g|u|^{q-2}u - \int_\Omega u_t|u|^{q-2}u - \int_\Gamma v_t|v|^{q-2}v, \end{aligned}$$

which implies that

$$\begin{aligned} & c_1\|u\|^{2(q-1)}_{L^{2(q-1)}(\Omega)} + c_1\|v\|^{2(q-1)}_{L^{2(q-1)}(\Gamma)} \\ \leq \quad & C\|g\|^2_{L^2(\Omega)} + C\|u_t\|^2_{L^2(\Omega)} + C\|v_t\|^2_{L^2(\Gamma)} + C. \end{aligned}$$

Thanks to Theorem 4.3, we know that there exists two positive constants ρ_2 and T such that for any $t \geq T$,

$$\|u\|^{2(q-1)}_{L^{2(q-1)}(\Omega)} + \|v\|^{2(q-1)}_{L^{2(q-1)}(\Gamma)} \leq \rho_2 \tag{4.17}$$

for any bounded subset $B \subset L^2(\Omega) \times L^2(\Gamma)$ and any $(u_0, \theta_0) \in B$. This is the end of the proof.

From Theorem 4.3 and Theorem 4.4, we deduce that $-\Delta_p u \in L^2(\Omega)$. In the case of Dirichlet or Neumann boundary condition, we can obtain that $|\nabla u|^{p-2}\nabla u \in H_0^1(\Omega)$, where $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_\Gamma = 0\}$ or $H_0^1(\Omega) = \{u \in H^1(\Omega) : \frac{\partial u}{\partial \nu}|_\Gamma = 0\}$. Thanks to the Sobolev embedding theory $H_0^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$, yield $|\nabla u| \in L^{(p-1)\frac{2d}{d-2}}(\Omega)$.

Since $\|u\|^q_{L^q(\Omega)} \le \|u\|_{L^2(\Omega)}\|u\|^{q-1}_{L^{2(q-1)}(\Omega)}$ and $\|u\|^q_{L^q(\Gamma)} \le \|u\|_{L^2(\Gamma)}\|u\|^{q-1}_{L^{2(q-1)}(\Gamma)}$, the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is compact in $L^2(\Omega) \times L^2(\Gamma)$ and bounded in $L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)$. Therefore, the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is compact in $L^q(\Omega) \times L^q(\Gamma)$. From Lemma 4.2 and Theorem 4.1, we have the following result.

Theorem 4.5 *Assume that $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6). *Then the semigroup $\{S(t)\}_{t\ge 0}$ generated by* (1.1)-(1.4) *possesses a $(L^2(\Omega) \times L^2(\Gamma), L^q(\Omega) \times L^q(\Gamma))$-global attractor $\mathcal{A}_q$, which is compact, invariant in $L^q(\Omega) \times L^q(\Gamma)$ and attracts every bounded subset in $L^2(\Omega) \times L^2(\Gamma)$ with respect to the topology of $L^q(\Omega) \times L^q(\Gamma)$.*

4.3 The existence of $(L^2(\Omega) \times L^2(\Gamma), L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma))$- global attractor

Since the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is compact in $L^q(\Omega) \times L^q(\Gamma)$, from Theorem 4.4 and Lemma 4.1, we deduce that the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) maps a compact subset of $L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)$ into a bounded subset of $L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)$, i.e.,the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is norm-to-weak continuous in $L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma)$. In this subsection, we will prove the existence of $(L^2(\Omega) \times L^2(\Gamma), L^{2(q-1)}(\Omega) \times L^{2(q-1)}(\Gamma))$-global attractor associated with the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6). From Theorems 4.1, 4.4, we know that the semigroup $\{S(t)\}_{t\ge 0}$ has a positively invariant absorbing set in $\left(W^{1,p}(\Omega) \cap L^{2(q-1)}(\Omega)\right) \times L^{2(q-1)}(\Gamma)$ denoted by B_0, which satisfies

$$B_0 \subset \Big\{(u, v) \in \left(W^{1,p}(\Omega) \cap L^{2(q-1)}(\Omega)\right) \times L^{2(q-1)}(\Gamma) : \\ \|u\|_{\left(W^{1,p}(\Omega)\cap L^{2(q-1)}(\Omega)\right)} + \|v\|_{L^{2(q-1)}(\Gamma)} \le \varrho\Big\}$$

for some constant ϱ. Note that here positively invariant means $S(t)B_0 \subset B_0$ for any $t \ge 0$.

Next, we will consider the asymptotic regularity for (1.1)-(1.6), which is similar to the one in [26, 28]. Consider the following elliptic equation

$$\begin{cases} -\Delta_p \phi + |\phi|^{p-2}\phi + f(\phi) = g(x), & x \in \Omega, \\ |\nabla \phi|^{p-2}\frac{\partial \phi}{\partial \nu} + f(\phi) = 0, & x \in \Gamma. \end{cases} \tag{4.18}$$

Due to the assumptions (1.5)-(1.6), from the classical results about elliptic equation, we know that (4.18) has at least one solution $\phi(x)$ with

$$(\phi, \gamma\phi) \in \left(W^{1,p}(\Omega) \cap L^{2(q-1)}(\Omega)\right) \times L^{2(q-1)}(\Gamma). \tag{4.19}$$

For the solution $u(x,t)$ of (1.1)-(1.6), we now decompose $u(x,t)$ as follows

$$u(x,t) = \phi(x) + w(x,t),$$

where $\phi(x)$ is a fixed solution of (4.18) and $w(x,t)$ satisfies the following equation:

$$\begin{cases} w_t - \Delta_p u + \Delta_p \phi + |u|^{p-2}u - |\phi|^{p-2}\phi + f(u) - f(\phi) = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ w_t + |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} - |\nabla \phi|^{p-2}\frac{\partial \phi}{\partial \nu} + f(v) - f(\phi) = 0, & (x,t) \in \Gamma \times \mathbb{R}^+, \\ w(x,0) = u_0(x) - \phi(x), & x \in \Omega, \\ w(x,0) = \theta_0(x) - \phi(x), & x \in \Gamma. \end{cases} \tag{4.20}$$

Then, (4.20) is also globally well-posed. From Theorem 4.1, Theorem 4.4 and (4.19), we have that

$$\|w(x,t)\|_{\left(W^{1,p}(\Omega)\cap L^{2(q-1)}(\Omega)\right)}+\|\gamma w(x,t)\|_{L^{2(q-1)}(\Gamma)}\leq \varrho_1, \tag{4.21}$$

where ϱ_1 is a positive constant. Without loss of generality, hereafter we assume $(u_0,\theta_0)\in B_0$ and $(w(x,0),\gamma w(x,0))\in \left(W^{1,p}(\Omega)\cap L^{2(q-1)}(\Omega)\right)\times L^{2(q-1)}(\Gamma)$.

Multiplying (4.20) by $|w|^{2(q-2)}w$, integrating over Ω and using Lemma 2.2 and (1.5), we obtain

$$\begin{aligned}&\frac{1}{2(q-1)}\frac{d}{dt}\|w\|^{2(q-1)}_{L^{2(q-1)}(\Omega)}+\frac{1}{2(q-1)}\frac{d}{dt}\|w\|^{2(q-1)}_{L^{2(q-1)}(\Gamma)}\\ &+C_1\int_\Omega|\nabla w|^p|w|^{2(q-2)}+C_1\int_\Omega|w|^{2q+p-4}\\ \leq\ & l\left(\|w\|^{2(q-1)}_{L^{2(q-1)}(\Omega)}+\|w\|^{2(q-1)}_{L^{2(q-1)}(\Gamma)}\right).\end{aligned} \tag{4.22}$$

Integrating (4.22) from t to $t+1$, we deduce from (4.21) that

$$\int_t^{t+1}\int_\Omega|\nabla w|^p|w|^{2(q-2)}+\int_t^{t+1}\int_\Omega|w|^{2q+p-4}\leq C(\varrho,\varrho_1,p,q),$$

which implies that

$$\int_t^{t+1}\|w^{\frac{2q+p-4}{p}}\|^p_{W^{1,p}(\Omega)}\leq C(\varrho,\varrho_1,p,q) \tag{4.23}$$

for all $t\geq 0$. Therefore, by virtue of the Sobolev embedding theory and the Sobolev trace embedding theory, we obtain

$$W^{1,p}(\Omega)\subset L^{\frac{pn}{n-p}}(\Omega),\ W^{1,p}(\Omega)\subset L^{\frac{p(n-1)}{n-p}}(\Gamma).$$

It follows from (4.23) that for any $(u_0,\theta_0)\in B_0$,

$$\begin{aligned}&\exists t_0\in[0,1]\ \textit{such that}\ w(x,t_0)\in L^{(2q+p-4)\frac{n}{n-p}}(\Omega)\ \textit{and}\ w(x,t_0)\in L^{(2q+p-4)\frac{(n-1)}{n-p}}(\Gamma),\\ &\textit{with}\ \|w(t_0)\|_{L^{(2q+p-4)\frac{n}{n-p}}(\Omega)}+\|w(t_0)\|_{L^{(2q+p-4)\frac{(n-1)}{n-p}}(\Gamma)}\leq C(\varrho,\varrho_1,p,q).\end{aligned} \tag{4.24}$$

Thanks to (4.23)-(4.24), we can perform a bootstrap estimate, which is the completely same as that in [38], to deduce the following estimates:

For each $k=1,2,\cdots$, there exists C_k, depending only on k and ϱ (recall ϱ is the bound of the bounded absorbing set B_0) such that

$$\|w\|_{L^{p_k}(\Omega)}+\|w\|_{L^{q_k}(\Gamma)}\leq C_k$$

for any $(u_0,\theta_0)\in B_0$ and $t\geq 1$, where $(w(t),\gamma w(t))=S(t)(u_0-\phi(x),\theta_0-\gamma\phi(x))$, $p_k=(2q-2)(\frac{n}{n-p})^k+(p-2)\sum_{i=1}^k(\frac{n}{n-p})^k$, $q_k=(2q-2)(\frac{n-1}{n-p})^k+(p-2)\sum_{i=1}^k(\frac{n-1}{n-p})^k$.

Since $\frac{n-1}{n-p}>1$ and $\frac{n}{n-p}>1$, by taking k large enough, (4.24) implies that for any $\delta\in[0,\infty)$,

$$\|w(t)\|_{L^{2q-2+\delta}(\Omega)}+\|w(t)\|_{L^{2q-2+\delta}(\Gamma)}\leq C(\varrho,\delta,\varrho_1,p,q) \tag{4.25}$$

for all $t\geq 1$.

Theorem 4.6 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^d$, $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6)*. Then for any $\delta \in [0,\infty)$, there is a bounded subset B_δ satisfying the following property*

$$B_\delta = \left\{(u,v) : \|u\|_{\left(W^{1,p}(\Omega)\cap L^{2q-2+\delta}(\Omega)\right)} + \|v\|_{L^{2q-2+\delta}(\Gamma)} \le \varrho_1 + C(\varrho,\delta,\varrho_1,p,q)\right\}$$

and for any bounded (in $L^2(\Omega)\times L^2(\Gamma)$) subset $B \subset L^2(\Omega)\times L^2(\Gamma)$, there exists a $T = T(\|B\|,\delta)$ such that

$$S(t)B \subset \phi + B_\delta \tag{4.26}$$

for all $t \ge T$, where ϕ is a fixed solution of (4.18)*, the constants ϱ_1 and $C(\varrho,\delta,\varrho_1,p,q)$ are as specified in* (4.21)*,* (4.25) *respectively, and all are independent of B.*

Using the interpolation inequality and Theorem 4.4, we obtain the following result.

Theorem 4.7 *Assume that $g \in L^2(\Omega)$ and f satisfies* (1.5)-(1.6)*. Then for any $\delta \in [0,\infty)$, the semigroup $\{S(t)\}_{t\ge 0}$ generated by* (1.1)-(1.4) *possesses a $(L^2(\Omega)\times L^2(\Gamma), L^{2(q-1)}(\Omega)\times L^{2(q-1)}(\Gamma))$-global attractor $\mathcal{A}_{2(q-1)}$, which attracts every bounded subset of $L^2(\Omega)\times L^2(\Gamma)$ with respect to the topology of $L^{2(q-1)+\delta}(\Omega)\times L^{2(q-1)+\delta}(\Gamma)$. Moreover, $\mathcal{A}_{2(q-1)} = \phi + \mathcal{A}_0$, where $\mathcal{A}_0$ is compact in $L^{2(q-1)}(\Omega)\times L^{2(q-1)}(\Gamma)$.*

4.4 $(L^2(\Omega)\times L^2(\Gamma), (W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma))$-global attractor

In this subsection, we will prove the existence of global attractor in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$ associated with the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6). First, from Lemma 4.1 and Theorem 4.1, we know that the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is norm-to-weak continuous in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$. Next, we first prove the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)-(1.6) is asymptotically compact in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$.

Theorem 4.8 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^d (d \ge 3)$, f satisfies* (1.5)-(1.6) *and $g \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t\ge 0}$ generated by* (1.1)-(1.4) *is $(L^2(\Omega)\times L^2(\Gamma), (W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma))$- asymptotically compact.*

Proof. Let B_0 be a $(L^2(\Omega)\times L^2(\Gamma), (W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma))$-bounded absorbing set obtained in Theorem 4.1. Then we need only to show that for any $\{(u_{0n}, v_{0n})\} \subset B_0$ and $t_n \to \infty$, $\{(u_n(t_n), v_n(t_n))\}_{n=0}^\infty$ is precompact in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$, where $(u_n(t_n), v_n(t_n)) = S(t_n)(u_{0n}, v_{0n})$.

In view of Theorem 4.5, it remains to prove that for any $\{(u_{0n}, v_{0n})\} \subset B_0$ and $t_n \to \infty$, $\{u_n(t_n)\}_{n=0}^\infty$ is precompact in $W^{1,p}(\Omega)$.

In fact, from Theorem 4.2 and Theorem 4.5, we know that $\{(u_n(t_n), v_n(t_n))\}_{n=0}^\infty$ is precompact in $L^2(\Omega)\times L^2(\Gamma)$ and $L^q(\Omega)\times L^q(\Gamma)$. Without loss of generality, we assume that $\{(u_n(t_n), v_n(t_n))\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\Omega)\times L^2(\Gamma)$ and $L^q(\Omega)\times L^q(\Gamma)$.

In the following, we will prove that $\{u_n(t_n)\}_{n=0}^\infty$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Then, by simply calculations, we deduce from Lemma 2.2 that

$$\begin{aligned}
& C_1\|u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})\|^p_{W^{1,p}(\Omega)} \\
\le\ & (-\frac{d}{dt}u_{n_k}(t_{n_k}) - f(u_{n_k}(t_{n_k})) + \frac{d}{dt}u_{n_j}(t_{n_j}) + f(u_{n_j}(t_{n_j})), u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})) \\
& +(-\frac{d}{dt}v_{n_k}(t_{n_k}) - f(v_{n_k}(t_{n_k})) + \frac{d}{dt}v_{n_j}(t_{n_j}) + f(v_{n_j}(t_{n_j})), v_{n_k}(t_{n_k}) - v_{n_j}(t_{n_j})) \\
=\ & I_1 + I_2.
\end{aligned} \tag{4.27}$$

Now, we estimate separately the two terms I_1 and I_2. By simple calculations and Hölder inequality, we deduce that

$$\begin{aligned} I_1 \leq\ & \|\frac{d}{dt}u_{n_k}(t_{n_k}) - \frac{d}{dt}u_{n_j}(t_{n_j})\|_{L^2(\Omega)}\|u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})\|_{L^2(\Omega)} + C(1 + \\ & \|u_{n_k}(t_{n_k})\|^{q-1}_{L^q(\Omega)} + \|u_{n_j}(t_{n_j})\|^{q-1}_{L^q(\Omega)})\|u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})\|_{L^q(\Omega)} \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} I_2 \leq\ & \|\frac{d}{dt}u_{n_k}(t_{n_k}) - \frac{d}{dt}u_{n_j}(t_{n_j})\|_{L^2(\Gamma)}\|u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})\|_{L^2(\Gamma)} + C(1 + \\ & \|u_{n_k}(t_{n_k})\|^{q-1}_{L^q(\Gamma)} + \|u_{n_j}(t_{n_j})\|^{q-1}_{L^q(\Gamma)})\|u_{n_k}(t_{n_k}) - u_{n_j}(t_{n_j})\|_{L^q(\Gamma)}, \end{aligned} \tag{4.29}$$

which combines with Theorems 4.2-4.3, 4.5, yields Theorem 4.8 immediately. This is the end of the proof.

Combining Lemma 4.3 with Theorems 4.1, 4.8, we immediately obtain the following result:

Theorem 4.9 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^d (d \geq 3)$, f satisfies* (1.5)-(1.6) *and $g \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by* (1.1)-(1.4) *has a $(L^2(\Omega) \times L^2(\Gamma), (W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma))$-global attractor $\mathcal{A}$.*

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] C. T. Anh, and T. D. Ke, *On quasilinear parabolic equations involving weighted p-Laplacian operators*, Nonlinear Differential Equations and Applications **17** (2010), 195–212.

[3] J. M. Arrieta, A. N. Carvalho, and A. R. Bernal,*Parabolic Problems with Nonlinear Boundary Conditions and Critical Noninearities*, Journal of Differential Equations **156** (1999), 376–406.

[4] J. M. Arrieta, A. N. Carvalho, and Aníbal Rodríguez-Bernal, *Attractors of parabolic problems with nonlinear boundary conditions uniform bounds*, Partial Differential Equations **25** (2000), 1–37.

[5] A. R. Bernal, *Attractors for Parabolic Equations with Nonlinear Boundary Conditions, Critical Exponents and Singular Initial Data*, Journal of Differential Equations **181** (2002), 165–196.

[6] A. V. Babin, and M. I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.

[7] T. Bartsch, and Z. Liu, *On a superlinear elliptic p-Laplacian equation*, Journal of Differential Equations **198** (2004), 149–175.

[8] A. Constantin, and J. Escher, *Global existence for fully parabolic boundary value problems*, Nonlinear Differential Equations and Applications **13** (2006), 91–118.

[9] A. Constantin, J. Escher, and Z. Yin, *Global solutions for quasilinear parabolic systems*, Journal of Differential Equations **197** (2004), 73–84.

[10] A. Constantin, and J. Escher, *Global solutions for quasilinear parabolic problems*, Journal of Evolution Equations **2** (2002), 97–111.

[11] G. X. Chen, and C. K. Zhong, *Uniform attractors for non-autonomous p-Laplacian equations*, Nonlinear Analysis **68** (2008), 3349–3363.

[12] J. Fernandez Bonder, and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions*, Journal Mathematical Analysis and Applications **263** (2001), 195–223.

[13] Z. H. Fan, and C. K. Zhong,*Attractors for parabolic equations with dynamic boundary conditions*, Nonlinear Analysis **68** (2008), 1723–1732.

[14] C. G. Gal, *On a class of degenerate parabolic equations with dynamic boundary conditions*, Journal of Differential Equations **253** (2012), 126–166.

[15] C. G. Gal, and M. Warma, *Well-posedness and long term behavior of quasilinear parabolic equations with nonlinear dynamic boundary conditions*, Differential and Integral Equations **23** (2010), 327–358.

[16] C. G. Gal, *Sharp estimates for the global attractor of scalar reaction-diffusion equations with a Wentzell boundary condition*, Journal of Nonlinear Science **22** (2012), 85–106.

[17] G. R. Goldstein, *Derivation of dynamical boundary conditions*, Advances in Differential Equations **11** (2006), 457–480.

[18] T.A. Kwembe, and Z. Zhang, *A semilinear equation with generalized Wentzell boundary condition*, Nonlinear Analysis **73** (2010), 3162–3170.

[19] A. Lê,*Eigenvalue problems for the p-Laplacian*, Nonlinear Analysis **64** (2006), 1057–1099.

[20] Q. F. Ma, S. H. Wang, and C. K. Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications*, Indiana University Mathematics **51** (2002), 1541–1549.

[21] S. Martinez, and J. D. Rossi,*Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance*, Electronic Journal of Differential Equations **27** (2003), 1–14.

[22] S. Martinez,and J. D. Rossi, *Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary conditions*, Abstract and Applied Analysis **7** (2002), 287–293.

[23] S. R. Martinez, and J. D. Rossi, *On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition*, Nonlinear Analysis **59** (2004), 813–848.

[24] J. Petersson, *A note on quenching for parabolic equations with dynamic boundary conditions*, Nonlinear Analysis **58** (2004), 417–423.

[25] L. Popescu, and A. Rodriguez-Bernal, *On a singularly perturbed wave equation with dynamical boundary conditions*, Proceedings of the Royal Society of Edinburgh **134** (2004), 389–413.

[26] V. Pata, and S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity **19** (2006), 1495–1506.

[27] J. C. Robinson, *Infinite-dimensional Dynamical Systems*, Cambridge University Press, 2001.

[28] C. Y. Sun, *Asymptotic regularity for some dissipative equations*, Journal of Differential Equations **248** (2010), 342–362.

[29] R. Temam, *Infinite-dimensional systems in mechanics and physics*, Springer-Verlag, New York, 1997.

[30] L. G. Vulkov, *Blow up for some quasilinear equations with dynamical boundary conditions of parabolic type*, Applied Mathematics and Computation **191** (2007), 89–99.

[31] H. Wu, *Convergence to equilibrium for the semilinear parabolic equation with dynamic boundary condition*, Advances in Mathematical Sciences and Applications **17** (2007), 67–88.

[32] M. Warma, *Quasilinear parabolic equations with nonlinear Wentzell-Robin type boundary conditions*, Journal of Mathematical Analysis and Applications **336** (2007), 1132–1148.

[33] L. Yang, *Uniform attractors for the closed process and applications to the reaction-diffusion with dynamical boundary condition*, Nonlinear Analysis **71** (2009), 4012–4025.

[34] L. Yang, and M. Yang,*Long-time behavior of reaction-diffusion equations with dynamical boundary condition*, Nonlinear Analysis **74** (2011), 3876–3883.

[35] L. Yang, and M. Yang, *Attractors of the non-autonomous reaction-diffusion equation with nonlinear boundary condition*, Nonlinear Analysis: Real World Applications **11** (2010), 3946–3954.

[36] M. H. Yang, C. Y. Sun, and C. K. Zhong,*Global attractors for p-Laplacian equation*, Journal of Mathematical Analysis and Applications **327** (2007), 1130–1142.

[37] Z. Yin, *Global existence for elliptic equations with dynamic boundary conditions*, Archiv der Mathematik **81** (2003), 567–574.

[38] C. K. Zhong, M. H. Yang, and C. Y. Sun, *The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations*, Journal of Differential Equations **223** (2006), 367–399.