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Positive Solutions for Quasilinear Systems with Critical Growth

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Abstract

In this paper, we consider the following system:

$$\begin{cases} \Delta u + u\Delta u^2 + \alpha u^{\alpha-1}v^{\beta} + \lambda u^{\lambda-1}v^{\mu} = 0 & \text{in } \Omega \\ \Delta v + v\Delta v^2 + \beta u^{\alpha}v^{\beta-1} + \mu u^{\lambda}v^{\mu-1} = 0 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain and $\alpha, \beta, \lambda, \mu > 1$ are parameters with $\alpha + \beta = \frac{4N}{N-2}, 4 < \lambda + \mu < \frac{4N}{N-2}$. By using the perturbation method, we prove the existence of positive solutions for the problem (P). Indeed, the method we are using in this paper can be applied to the more general quasilinear system (see (1.7)).

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1 Introduction

In this paper, we are concerned with the existence of positive solutions for the following quasilinear system with critical exponents:

$$\begin{cases} \Delta u + u\Delta u^2 + \alpha u^{\alpha-1}v^{\beta} + \lambda u^{\lambda-1}v^{\mu} = 0 & \text{in } \Omega \\ \Delta v + v\Delta v^2 + \beta u^{\alpha}v^{\beta-1} + \mu u^{\lambda}v^{\mu-1} = 0 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain and $\alpha, \beta, \lambda, \mu > 1$ are parameters satisfying $\alpha + \beta = \frac{4N}{N-2}, 4 < \lambda + \mu < \frac{4N}{N-2}$.

Note that the system (1.1) has variational structure, and the functional is defined by:

$$I(u,v) = \frac{1}{2}\int_{\Omega}(1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\Omega}(1+2v^2)|\nabla v|^2 dx - \int_{\Omega}|u|^{\alpha}|v|^{\beta}dx - \int_{\Omega}|u|^{\lambda}|v|^{\mu}dx. \tag{1.2}$$

The directional derivative of I at (u,v) in the direction $(\varphi,\psi) \in C_0^{\infty}(\Omega) \times C_0^{\infty}(\Omega)$ is defined by

$$\begin{aligned} &\langle I'(u,v),(\varphi,\psi)\rangle \\ &= \lim_{t\to 0}\frac{1}{t}(I(u+t\varphi, v+t\psi) - I(u,v)) \\ &= \int_{\Omega}\nabla u\nabla\varphi dx + \int_{\Omega}\nabla u^2\nabla(u\varphi)dx - \alpha\int_{\Omega}|u|^{\alpha-2}u|v|^{\beta}\varphi dx - \lambda\int_{\Omega}|u|^{\lambda-2}u|v|^{\mu}\varphi dx \\ &\quad + \int_{\Omega}\nabla v\nabla\psi dx + \int_{\Omega}\nabla v^2\nabla(v\psi)dx - \beta\int_{\Omega}|u|^{\alpha}|v|^{\beta-2}v\psi dx - \mu\int_{\Omega}|u|^{\lambda}|v|^{\mu-2}v\psi dx. \end{aligned}$$

The weak form of the problem (P) is as follows:

$$\begin{cases} \displaystyle\int_{\Omega}\nabla u\nabla\varphi dx + \int_{\Omega}\nabla u^2\nabla(u\varphi)dx - \alpha\int_{\Omega}|u|^{\alpha-2}u|v|^{\beta}\varphi dx \\ \quad \displaystyle -\lambda\int_{\Omega}|u|^{\lambda-2}u|v|^{\mu}\varphi dx = 0, \forall\varphi \in C_0^{\infty}(\Omega), \\ \displaystyle\int_{\Omega}\nabla v\nabla\psi dx + \int_{\Omega}\nabla v^2\nabla(v\psi)dx - \beta\int_{\Omega}|u|^{\alpha}|v|^{\beta-2}v\psi dx \\ \quad \displaystyle -\mu\int_{\Omega}|u|^{\lambda}v^{\mu-2}v\psi dx = 0, \forall\psi \in C_0^{\infty}(\Omega). \end{cases} \tag{1.3}$$

We say that (u,v) is a critical point of I if $(u,v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, $\int_{\Omega} u^2|\nabla u|^2 dx < \infty$, $\int_{\Omega} v^2|\nabla v|^2 dx < \infty$, and $\langle I'(u,v),(\varphi,\psi)\rangle = 0$ for all $(\varphi,\psi) \in C_0^{\infty}(\Omega) \times C_0^{\infty}(\Omega)$. Hence, formally one can see that to prove the existence of the solutions for the problem (P) is equivalent to look for the critical points of the functional I.

However, note that there is no suitable space such that the functional I is smooth and at the same time, has some compactness properties, for instance, the usual Palais-Smale ((PS) in short) condition. To overcome this difficulty, there are several approaches in the literatures, such as, by minimizations method [1, 2]; by Nehari manifold method [3]; by changes of variables [4, 5]. Later on, the method of changes of variable has been explored by several authors to deal with critical problems, such as do Ó, Miyagaki, Soares [6] and Silva, Vieira [7]. However, this approach does not work for the more general quasilinear problems (1.7). Very recently A Nehari manifold method combing with the subcritical approximation was used in [8] to study the quasilinear single equation, but this method ask some monotonicity conditions for the structure for the equations. In this paper, follow the idea of [9], we present a perturbation method to study the existence of positive solutions for a quasilinear system with critical exponents. More precisely, let $X := W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$, for $s \in (0, 1]$, we define a perturbed functional I_s by:

$$I_s(u, v) = \frac{1}{4} s \int_\Omega (|\nabla u|^4 + |\nabla v|^4)dx + I(u, v), \tag{1.4}$$

where I is defined as in (1.2). Then one can see, by the standard arguments, that for $\forall s \in (0, 1]$, I_s is a C^1 functional on X, and for all $(\varphi, \psi) \in X$,

$$\begin{aligned}
&\langle I_s'(u, v), (\varphi, \psi)\rangle \\
&= s \int_\Omega |\nabla u|^2 \nabla u \nabla \varphi dx + s \int_\Omega |\nabla v|^2 \nabla v \nabla \psi dx + \langle I'(u, v), (\varphi, \psi)\rangle \\
&= s \int_\Omega |\nabla u|^2 \nabla u \nabla \varphi dx + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla u^2 \nabla (u\varphi) dx \\
&\quad -\alpha \int_\Omega |u|^{\alpha-2} u |v|^\beta \varphi dx - \lambda \int_\Omega |u|^{\lambda-2} u |v|^\mu \varphi dx \\
&\quad +s \int_\Omega |\nabla v|^2 \nabla v \nabla \psi dx + \int_\Omega \nabla v \nabla \psi dx + \int_\Omega \nabla v^2 \nabla (v\psi) dx \\
&\quad -\beta \int_\Omega |u|^\alpha |v|^{\beta-2} v \psi dx - \mu \int_\Omega |u|^\lambda |v|^{\mu-2} v \psi dx.
\end{aligned} \tag{1.5}$$

Moreover the critical point (u, v) of I_s satisfies the following system:

$$\left\{\begin{aligned}
&s \int_\Omega |\nabla u|^2 \nabla u \nabla \varphi dx + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla u^2 \nabla (u\varphi) dx \\
&\quad -\alpha \int_\Omega |u|^{\alpha-2} u |v|^\beta \varphi dx - \lambda \int_\Omega |u|^{\lambda-2} u |v|^\mu \varphi dx = 0, \forall \varphi \in C_0^\infty(\Omega), \\
&s \int_\Omega |\nabla v|^2 \nabla v \nabla \psi dx + \int_\Omega \nabla v \nabla \psi dx + \int_\Omega \nabla v^2 \nabla (v\psi) dx \\
&\quad -\beta \int_\Omega |u|^\alpha |v|^{\beta-2} v \psi dx - \mu \int_\Omega |u|^\lambda |v|^{\mu-2} v \psi dx = 0, \forall \psi \in C_0^\infty(\Omega).
\end{aligned}\right. \tag{1.6}$$

In order to prove the existence of positive solutions for the system (P), the main idea of our method is as following: we first prove the existence of critical points (u_s, v_s) of the perturbed functional I_s for any $s > 0$ small. Then we establish some suitable estimates for the critical points set $\{(u_s, v_s)\}$. So that we may pass limit as $s \to 0$ to get the solutions for the original problem (P). Indeed, the method we are using in this paper can be applied to the more general quasilinear system of the form:

$$\begin{cases} \sum_{i,j=1}^{N} D_j(a_{ij}(x,u)D_iu) - \frac{1}{2}\sum_{i,j=1}^{N} D_sa_{ij}(x,u)D_iuD_ju + G_u(u,v) = 0 & \text{in } \Omega \\ \sum_{i,j=1}^{N} D_j(a_{ij}(x,v)D_iv) - \frac{1}{2}\sum_{i,j=1}^{N} D_sa_{ij}(x,v)D_ivD_jv + G_v(u,v) = 0 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $D_i = \frac{\partial}{\partial x_i}$ and $D_sa_{ij}(x,s) = \frac{\partial}{\partial s}a_{ij}(x,s)$. Note that the above system (1.7) includes the system (P) as special case, namely, $a_{ij}(x,s) = (1+2s^2)\delta_{ij}, G(u,v) = |u|^\alpha|v|^\beta + |u|^\lambda|v|^\mu$.

For the sake of simplicity of notations and of the clear presentation of the ideas of the methods, we first consider the simple case of the problem (P). Then we give the sketch proof for the general quasilinear system (1.7).

Our main result for the system (P) is :

Theorem 1.1 *Suppose that* $\alpha,\beta,\lambda,\mu > 1$ *satisfying* $\alpha+\beta = \frac{4N}{N-2}$, $P_N := \max(4, \frac{2(N+2)}{N-2}) < \lambda+\mu < \frac{4N}{N-2}$. *Then the system* (P) *admits at least one positive solution.*

2 Properties for perturbation functional

Recall that a functional I defined on a Banach space X is said to satisfy (Palais-Smale)$_c$ condition ($(PS)_c$ condition in short) if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c, I'(u_n) \to 0 \tag{2.1}$$

is relatively compact. And we say a sequence $\{u_n\}$ is a $(PS)_c$ sequence if (2.1) is satisfied. Let $X := W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ equipped with the norm:

$$\|(u,v)\|_X := \|u\|_{W_0^{1,4}(\Omega)} + \|v\|_{W_0^{1,4}(\Omega)}.$$

Lemma 2.1 *For* $s \in (0,1]$, *the perturbation functional* I_s *defined in* (1.4) *satisfies the* $(PS)_c$ *condition.*

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence for I_s. We first show that $\{(u_n,v_n)\}$ is bounded in X. Indeed, we have

$$\begin{aligned} & c + \|(u_n,v_n)\|_X o(1) \\ \geq\ & I_s(u_n,v_n) - \frac{1}{\lambda+\mu}\langle I_s'(u_n,v_n),(u_n,v_n)\rangle \\ =\ & (\frac{1}{4} - \frac{1}{\lambda+\mu})s\int_\Omega(|\nabla u_n|^4 + |\nabla v_n|^4)dx + (\frac{1}{2} - \frac{1}{\lambda+\mu})\int_\Omega|\nabla u_n|^2dx + \int_\Omega(1 - \frac{4}{\lambda+\mu})u_n^2|\nabla u_n|^2dx \\ & +(\frac{1}{2} - \frac{1}{\lambda+\mu})\int_\Omega|\nabla v_n|^2dx + \int_\Omega(1 - \frac{4}{\lambda+\mu})v_n^2|\nabla v_n|^2dx + (\frac{\alpha+\beta}{\lambda+\mu} - 1)\int_\Omega|u_n|^\alpha|v_n|^\beta dx. \end{aligned} \tag{2.2}$$

Note that $4 < \lambda + \mu < \alpha + \beta = \frac{4N}{N-2} < \frac{4N}{N-4}$ and the embedding $W_0^{1,4}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le \frac{4N}{N-4}$ and is compact for $1 \le p < \frac{4N}{N-4}$. It follows from (2.2) that

$$(\frac{1}{4} - \frac{1}{\lambda + \mu})s \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx \le c + \|(u_n, v_n)\|_X o(1),$$

which implies that $\{(u_n, v_n)\}$ is bounded in X. As a consequence, we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega),$$

$$(u_n, v_n) \to (u, v) \text{ in } L^q(\Omega) \times L^q(\Omega) \text{ for } 1 \le q < \frac{4N}{N-4}.$$

Take $(\varphi, \psi) = (u_n - u_m, v_n - v_m)$ in (1.5), we deduce that

$$\begin{aligned}
& o(1) \\
&= \langle I_s'(u_n, v_n) - I_s'(u_m, v_m), (u_n - u_m, v_n - v_m)\rangle \\
&= s \int_\Omega (|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m)\nabla(u_n - u_m)dx \\
&\quad + s \int_\Omega (|\nabla v_n|^2 \nabla v_n - |\nabla v_m|^2 \nabla v_m)\nabla(v_n - v_m)dx \\
&\quad + \int_\Omega |\nabla u_n - \nabla u_m|^2 dx + \int_\Omega |\nabla v_n - \nabla v_m|^2 dx \\
&\quad + 2\int_\Omega (u_n^2 \nabla u_n - u_m^2 \nabla u_m)(\nabla u_n - \nabla u_m)dx \\
&\quad + 2\int_\Omega (v_n^2 \nabla v_n^2 - v_m^2 \nabla v_m)(\nabla v_n - \nabla v_m)dx \\
&\quad + 2\int_\Omega (u_n|\nabla u_n|^2 - u_m^2|\nabla u_m|^2)(u_n - u_m)dx \\
&\quad + 2\int_\Omega (v_n|\nabla v_n|^2 - v_m^2|\nabla v_m|^2)(v_n - v_m)dx \\
&\quad - \alpha \int_\Omega (|u_n|^{\alpha-2}u_n|v_n|^\beta - |u_m|^{\alpha-2}u_m|v_m|^\beta)(u_n - u_m)dx \\
&\quad - \lambda \int_\Omega (|u_n|^{\lambda-2}u_n|v_n|^\mu - |u_m|^{\lambda-2}u_m|v_m|^\mu)(u_n - u_m)dx \\
&\quad - \beta \int_\Omega (|u_n|^\alpha|v_n|^{\beta-2}v_n - |u_m|^\alpha|v_m|^{\beta-2}v_m)(v_n - v_m)dx \\
&\quad - \mu \int_\Omega (|v_n|^{\mu-2}v_n|u_n|^\lambda - |u_m|^\lambda|v_m|^{\mu-2}v_m)(v_n - v_m)dx.
\end{aligned} \tag{2.3}$$

In the following, we will estimate the above terms one by one. First, we have

$$\begin{aligned}
& \int_\Omega (u_n^2 \nabla u_n - u_m^2 \nabla u_m)(\nabla u_n - \nabla u_m)dx \\
&= \int_\Omega u_n^2 |\nabla u_n - \nabla u_m|^2 dx + \int_\Omega (u_n^2 - u_m^2)\nabla u_m(\nabla u_n - \nabla u_m)dx \\
&\ge -|u_n - u_m|_4(|u_n|_4 + |u_m|_4)\|u_n\|_{W_0^{1,4}(\Omega)}(\|u_n\|_{W_0^{1,4}(\Omega)} + \|u_m\|_{W_0^{1,4}(\Omega)}) \\
&= o(1).
\end{aligned} \tag{2.4}$$

Similarly, we have

$$\int_\Omega (v_n^2\nabla v_n - v_m^2\nabla v_m)(\nabla v_n - \nabla v_m)dx \geq o(1). \tag{2.5}$$

And

$$\begin{aligned} &\Big|\int_\Omega (u_n|\nabla u_n|^2 - u_m|\nabla u_m|^2)(u_n - u_m)dx\Big| \\ &\leq (|u_n|_4\|u_n\|^2_{W_0^{1,4}(\Omega)} + |u_m|_4\|u_m\|^2_{W_0^{1,4}(\Omega)})|u_n - u_m|_4 = o(1). \end{aligned} \tag{2.6}$$

$$|\int_\Omega (v_n|\nabla v_n|^2 - v_m|\nabla v_m|^2)(v_n - v_m)dx| \leq o(1). \tag{2.7}$$

Moreover, we have

$$\begin{aligned} &\Big|\int_\Omega (|u_n|^{\alpha-2}u_n|v_n|^\beta - |u_m|^{\alpha-2}u_m|v_m|^\beta)(u_n - u_m)dx\Big| \\ &\leq (|u_n|^{\alpha-1}_{\alpha+\beta}|v_n|^\beta_{\alpha+\beta} + |u_m|^{\alpha-1}_{\alpha+\beta}|v_m|^\beta_{\alpha+\beta})|u_n - u_m|_{\alpha+\beta} = o(1). \end{aligned} \tag{2.8}$$

Similarly

$$\begin{aligned} &\int_\Omega (|u_n|^{\lambda-2}u_n|v_n|^\mu - |u_m|^{\lambda-2}u_m|v_m|^\mu)(u_n - u_m)dx = o(1), \\ &\int_\Omega (|u_n|^\alpha|v_n|^{\beta-2}v_n - |u_m|^\alpha|v_m|^{\beta-2}v_m)(v_n - v_m)dx = o(1), \\ &\int_\Omega (|v_n|^{\mu-2}v_n|u_n|^\lambda - |u_m|^\lambda|v_m|^{\mu-2}v_m)(v_n - v_m)dx = o(1). \end{aligned}$$

Finally

$$s\int_\Omega (|\nabla u_n|^2\nabla u_n - |\nabla u_m|^2\nabla u_m)(\nabla u_n - \nabla u_m)dx \geq cs\int_\Omega |\nabla u_n - \nabla u_m|^4dx. \tag{2.9}$$

$$s\int_\Omega (|\nabla v_n|^2\nabla v_n - |\nabla v_m|^2\nabla v_m)(\nabla v_n - \nabla v_m)dx \geq cs\int_\Omega |\nabla v_n - \nabla v_m|^4dx. \tag{2.10}$$

Combine with the above obtained results, we get

$$Cs\int_\Omega (|\nabla u_n - \nabla u_m|^4 + |\nabla v_n - \nabla v_m|^4)dx \leq o(1),$$

which implies that $\|(u_n - u_m, v_n - v_m)\|_X \to 0$, that is $(u_n, v_n) \to (u, v)$ strongly in X.

□

Lemma 2.2 *Let*

$$c_s = \inf_{\gamma\in\Gamma_s}\sup_{t\in[0,1]} I_s(\gamma(t))$$

$$\Gamma_s = \{\gamma|\gamma \in C([0,1], X), \gamma(0) = (0,0), I_s(\gamma(1)) < 0\}.$$

Then c_s is a critical value of I_s. Moreover, there exists a constant $m > 0$ such that $c_s \geq m$, for $\forall s \in (0,1]$.

Proof. For $\rho > 0$, we define

$$E_\rho = \{(u,v) \in X | \int_\Omega (1+2u^2)|\nabla u|^2 dx + \int_\Omega (1+2v^2)|\nabla v|^2 dx \le \rho^2\}. \tag{2.11}$$

Then for $(u,v) \in \partial E_\rho$, we have

$$\begin{aligned} I_s(u,v) &= \frac{1}{4}s\int_\Omega (|\nabla u|^4 + |\nabla v|^4)dx + I(u,v) \\ &= \frac{1}{4}s\int_\Omega (|\nabla u|^4 + |\nabla v|^4)dx + \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_\Omega (1+2v^2)|\nabla v|^2 dx \\ &\quad - \int_\Omega |u|^\alpha |v|^\beta dx - \int_\Omega |u|^\lambda |v|^\mu dx \\ &\ge \frac{1}{2}\rho^2 - |u|^\alpha_{\alpha+\beta}|v|^\beta_{\alpha+\beta} - |u|^\lambda_{\lambda+\mu}|v|^\mu_{\lambda+\mu} \\ &\ge \frac{1}{2}\rho^2 - c(\rho^{\frac{\alpha+\beta}{2}} + \rho^{\frac{\lambda+\mu}{2}}). \end{aligned} \tag{2.12}$$

In the last inequality, we have used the fact that $|u|_{\frac{4N}{N-2}} \le C(\int_\Omega u^2|\nabla u|^2 dx)^{\frac{1}{4}}$ and the embedding $W_0^{1,2} \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$. Now we take $\rho_0 > 0$ such that $c(\rho_0^{\frac{\alpha+\beta}{2}} + \rho_0^{\frac{\lambda+\mu}{2}}) = \frac{1}{4}\rho_0^2$, then we have

$$I_s(u,v) \ge \frac{1}{4}\rho_0^2, \qquad \text{for } (u,v) \in \partial E_{\rho_0}.$$

But for $\rho \le \rho_0$, $c(\rho^{\frac{\alpha+\beta}{2}} + \rho^{\frac{\lambda+\mu}{2}}) \le \frac{1}{4}\rho^2$,

$$\begin{aligned} I_s(u,v) &\ge \tfrac{1}{2}\rho^2 - c(\rho^{\frac{\alpha+\beta}{2}} + \rho^{\frac{\lambda+\mu}{2}}) \\ &= \tfrac{1}{2}\rho^2 - \tfrac{1}{4}\rho^2 \\ &\ge \tfrac{1}{4}\rho^2. \end{aligned}$$

Note that $\alpha + \beta > \lambda + \mu > 4$, we obtain

$$c_s = \inf_{\gamma \in \Gamma_s} \sup_{t \in [0,1]} I_s(\gamma(t)) \ge \inf_{(u,v) \in \partial E_{\rho_0}} I(\gamma(s)) \ge \frac{1}{4}\rho_0^2. \tag{2.13}$$

Since I_s satisfies the $(PS)_c$ condition. By using Mountain Pass Lemma, we obtain that c_s is the critical value of I_s and $m = \frac{1}{4}\rho_0^2$ is the desired lower bound.

□

Remark 2.1 If we consider the functional I_s^+ defined by:

$$\begin{aligned} I_s^+(u,v) &= \frac{1}{4}s\int_\Omega (|\nabla u|^4 + |\nabla v|^4)dx + \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_\Omega (1+2v^2)|\nabla v|^2 dx \\ &\quad - \int_\Omega u_+^\alpha v_+^\beta dx - \int_\Omega u_+^\lambda v_+^\mu dx, \end{aligned} \tag{2.14}$$

where $u_+ = \max(u,0), v_+ = \max(v,0)$. Let

$$c_s^+ = \inf_{\gamma \in \Gamma_s^+} \sup_{t \in [0,1]} I_s^+(\gamma(t)),$$

$$\Gamma_s^+ = \{\gamma | \gamma \in C([0,1], X), \gamma(0) = (0,0), I_s^+(\gamma(1) < 0)\}.$$

Then by using the similar arguments as in the proofs of Lemma 2.1 and Lemma 2.2, we can verify that I_s^+ satisfies the $(PS)_c$ condition and c_s^+ is a critical value of I_s^+. Moreover, $c_s^+ \geq m$ for $\forall s \in (0,1]$. On the other hand, we see that the critical points of I_s^+ are positive, it deduces that c_s^+ is a critical value of I_s too. As a result, I_s admits a positive critical point.

Note that the functional I_s (and I_s^+) is monotone with respect to $s \in (0,1]$, we have $c_s \leq c_1$ (and $c_s^+ \leq c_1^+$) $\forall s \in (0,1]$. On the other hand, by Lemma 2.2, we see that

$$c_s \geq \frac{1}{4}\rho_0^2 = m, \text{ for } \forall s \in (0,1].$$

The idea of the following of the paper is to prove that for $s \to 0$, the sequence $\{(u_s, v_s)\}$, the critical points of the perturbed functional I_s, is convergent and its limit (u,v) is the solution of the original problem (P). More precisely, we have:

Proposition 2.1 *Suppose that (u_n, v_n) is the critical point of I_{s_n} for $s_n \in (0,1]$ with $s_n \to 0$ as $n \to \infty$. Let $d = \lim_{n\to\infty} I_{s_n}(u_n, v_n)$. Then up to a subsequence, we have*

(1) (a) $u_n \rightharpoonup u, v_n \rightharpoonup v \in W_0^{1,2}(\Omega)$;
(b) $u_n \nabla u_n \rightharpoonup u\nabla u, v_n \nabla v_n \rightharpoonup v \nabla v$ in $L^2(\Omega)$;
(c) $u_n \to u, v_n \to v$ in $L^p(\Omega)$ for $1 \leq p < \frac{4N}{N-2}$.
(2) $u, v \in L^\infty(\Omega)$ and (u,v) is a solution of the problem (P).
(3) $I(u,v) \leq \lim_{n\to\infty} I_{s_n}(u_n, v_n)$.
(4) Let $S_{\alpha,\beta} = \frac{2S^{\frac{N}{2}}}{(N-2)(\alpha^\alpha \beta^\beta)^{\frac{N-2}{8}}}$, where S is the Sobolev constant. If $d \in (0, S_{\alpha,\beta})$, then $(u,v) \neq (0,0)$.

Proof. (1) Note that $I'_{s_n}(u_n, v_n) = 0$ for $\forall s_n \in (0,1]$. By using the similar arguments as in the proof of Lemma 2.1, we have

$$\begin{aligned} c &\geq I_{s_n}(u_n, v_n) - \frac{1}{\lambda+\mu}\langle I'_{s_n}(u_n, v_n), (u_n, v_n)\rangle \\ &= (\frac{1}{4} - \frac{1}{\lambda+\mu})s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + (\frac{1}{2} - \frac{1}{\lambda+\mu})\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx \\ &\quad + (1 - \frac{4}{\lambda+\mu})\int_\Omega (u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2)dx + (\frac{\alpha+\beta}{\lambda+\mu} - 1)\int_\Omega |u_n|^\alpha |v_n|^\beta dx, \end{aligned}$$

which implies that

$$\begin{aligned} & s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + (\frac{1}{2} - \frac{1}{\lambda+\mu})\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx \\ & + (1 - \frac{4}{\lambda+\mu})\int_\Omega (u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2)dx \leq C, \end{aligned} \tag{2.15}$$

for some constant C independent of n. Thus, up to a subsequence, we have:

(a) $u_n \rightharpoonup u, v_n \rightharpoonup v \in W_0^{1,2}(\Omega)$;
(b) $u_n \nabla u_n \rightharpoonup u \nabla u, v_n \nabla v_n \rightharpoonup v \nabla v$ in $L^2(\Omega)$;
(c) $u_n \to u, v_n \to v$ in $L^p(\Omega)$ for $1 \leq p < \frac{4N}{N-2}$.

(d) $u_n(x) \to u(x), v_n(x) \to v(x)\, a.e. x \in \Omega$.

(2) We prove that $u, v \in L^\infty(\Omega)$. Since $I'_{s_n}(u_n, v_n) = 0$, for $\forall \varphi \in W_0^{1,4}(\Omega)$, we have

$$\begin{aligned} & s_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \varphi dx + \int_\Omega (1 + 2u_n^2)\nabla u_n \nabla \varphi dx + 2\int_\Omega u_n |\nabla u_n|^2 \varphi dx \\ = {} & \alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta \varphi dx + \lambda \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu \varphi dx. \end{aligned} \tag{2.16}$$

For $T > 0$, we define the cut-off function $u^T(x)$ by

$$u^T(x) = \begin{cases} u(x), & |u(x)| \le T; \\ T, & u(x) \ge T; \\ -T, & u(x) \le -T. \end{cases}$$

Let $K > T > 0$, take $\varphi = |u_n^T|^{2r} u_n^K$ as the test function in (2.16), we have

$$\begin{aligned} & s_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla (|u_n^T|^{2r} u_n^K) dx \\ & \quad + \int_\Omega (1 + 2u_n^2) \nabla u_n \nabla (|u_n^T|^{2r} u_n^K) dx + 2\int_\Omega u_n |\nabla u_n|^2 (|u_n^T|^{2r} u_n^K) dx \\ & = \alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta (|u_n^T|^{2r} u_n^K) dx + \lambda \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu (|u_n^T|^{2r} u_n^K) dx. \end{aligned}$$

Note that

$$\begin{aligned} \nabla(|u_n^T|^{2r} u_n^K) & = u_n^K \nabla(|u_n^T|^{2r}) + |u_n^T|^{2r} \nabla u_n^K \\ & = 2r u_n^K |u_n^T|^{2r-2} u_n^T + |u_n^T|^{2r} \nabla u_n^K. \end{aligned}$$

We have

$$\begin{aligned} & \int_\Omega |\nabla u_n|^2 u_n u_n^K |u_n^T|^{2r} dx \\ & \le C\left(\int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta u_n^K |u_n^T|^{2r} dx + \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu u_n^K |u_n^T|^{2r} dx \right), \end{aligned} \tag{2.17}$$

for some constant $C > 0$ independent of n.

Since $u_n \nabla u_n \rightharpoonup u\nabla u$ in $L^2(\Omega), u_n \to u$ in $L^p(\Omega)$ for $1 \le p < \frac{4N}{N-2}$. The integral on the left side of (2.17) is lower semi-continuous, while the right side of the inequality is continuous due to the facts $2 < \alpha + \beta - 1 < \frac{4N}{N-2}, 2 < \lambda + \mu - 1 < \frac{4N}{N-2}$ ($u_n^K |u_n^T|^{2r}$ is bounded for fixed $K > T > 0$). Let $n \to \infty$ (then $k \to \infty$) in (2.17), we obtain

$$\int_\Omega |\nabla u|^2 u^2 |u^T|^{2r} dx \le C\left(\int_\Omega |u|^\alpha |v|^\beta |u^T|^{2r} dx + \int_\Omega |u|^\lambda |v|^\mu |u^T|^{2r} dx \right).$$

In a similar way, we have

$$\int_\Omega |\nabla v|^2 v^2 |v^T|^{2r} dx \le C\left(\int_\Omega |u|^\alpha |v|^\beta |v^T|^{2r} dx + \int_\Omega |u|^\lambda |v|^\mu |v^T|^{2r} dx \right).$$

It follows that

$$\begin{aligned}&\int_\Omega |\nabla u|^2u^2|u^T|^{2r}dx+\int_\Omega |\nabla v|^2v^2|v^T|^{2r}dx\\ &\leq C\left(\int_\Omega |u|^\alpha|v|^\beta(|u^T|^{2r}+|v^T|^{2r})dx+\int_\Omega |u|^\lambda|v|^\mu(|u^T|^{2r}+|v^T|^{2r})dx\right)\\ &\leq C\int_\Omega(|u|^{\frac{4N}{N-2}}+|v|^{\frac{4N}{N-2}})(|u^T|^{2r}+|v^T|^{2r})dx+C\int_\Omega(|u|^{\lambda+\mu}+|v|^{\lambda+\mu})(|u^T|^{2r}+|v^T|^{2r})dx\\ &\leq C(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r}dx+\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r}dx)+C(\int_\Omega |u|^{\lambda+\mu}|u^T|^{2r}dx+\int_\Omega |v|^{\lambda+\mu}|v^T|^{2r}dx).\end{aligned} \tag{2.18}$$

Take $r \geq \frac{4}{N-2} := r_0$. By the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ and the Poincare inequality respectively, we have

$$\int_\Omega |\nabla u|^2u^2|u^T|^{2r}dx \geq \frac{C}{r^2}\int_\Omega |\nabla(u^2|u^T|^r)|^2dx \geq \frac{C}{r^2}(\int_\Omega (u^2|u^T|^r)^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}}$$

and

$$\int_\Omega |\nabla u|^2u^2|u^T|^{2r}dx \geq \frac{C}{r^2}\int_\Omega |\nabla u^2|u^T|^r|^2dx \geq \frac{C}{r^2}\int_\Omega (u^2|u^T|^r)^2dx, \tag{2.19}$$

where C is a constant independent of n.

Similarly,

$$\int_\Omega |\nabla v|^2v^2|v^T|^{2r}dx \geq \frac{C}{r^2}\int_\Omega |\nabla(v^2|v^T|^r)|^2dx \geq \frac{C}{r^2}(\int_\Omega (v^2|v^T|^r)^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}},$$

and

$$\int_\Omega |\nabla v|^2v^2|v^T|^{2r}dx \geq \frac{C}{r^2}\int_\Omega |\nabla v^2|v^T|^r|^2dx \geq \frac{C}{r^2}\int_\Omega (v^2|v^T|^r)^2dx.$$

On the other hand, since $4 < \lambda+\mu < \frac{4N}{N-2}$, we have the following inequality, for some $0 < q := \frac{4}{\lambda+\mu-4}$,

$$|t|^{\lambda+\mu} \leq \frac{\epsilon}{r^2}t^4 + (\frac{r^2}{\epsilon})^q|t|^{\frac{4N}{N-2}}.$$

Indeed, if $|t| \leq (\frac{\epsilon}{r^2})^{\frac{1}{\lambda+\mu-4}}$, then $|t|^{\lambda+\mu-4} \leq \frac{\epsilon}{r^2}$, $|t|^{\lambda+\mu} \leq \frac{\epsilon}{r^2}t^4$;

If $|t| \geq (\frac{\epsilon}{r^2})^{\frac{1}{\lambda+\mu-4}}$, then $|t|^{\frac{4N}{N-2}-(\lambda+\mu)} \geq (\frac{\epsilon}{r^2})^{\frac{\frac{4N}{N-2}-(\lambda+\mu)}{\lambda+\mu-4}}$. Take $q := \frac{\frac{4N}{N-2}-(\lambda+\mu)}{\lambda+\mu-4}$, we obtain $|t|^{\lambda+\mu} \leq (\frac{r^2}{\epsilon})^q|t|^{\frac{4N}{N-2}}$.

Thus

$$\begin{aligned}&\int_\Omega |u|^{\lambda+\mu}|u^T|^{2r}dx+\int_\Omega |v|^{\lambda+\mu}|v^T|^{2r}dx\\ &\leq (\frac{r^2}{\epsilon})^q\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r}dx+\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r}dx\right)\\ &\quad+\frac{\epsilon}{r^2}\left(\int_\Omega |u|^4|u^T|^{2r}dx+\int_\Omega |v|^4|v^T|^{2r}dx\right).\end{aligned} \tag{2.20}$$

Combine the formulae (2.18)-(2.20) together, we obtain

$$\begin{aligned}&\left(\int_\Omega (u^2|u^T|^r)^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}} + \left(\int_\Omega (v^2|v^T|^r)^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}\\ &\le Cr^{\tilde q}\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r}dx + \int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r}dx\right)(\tilde q = 2q+2).\end{aligned} \tag{2.21}$$

Therefore

$$\begin{aligned}&\max\left\{\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{\frac{2Nr}{N-2}}dx\right)^{\frac{N-2}{2Nr}},\ \left(\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{\frac{2Nr}{N-2}}dx\right)^{\frac{N-2}{2Nr}}\right\}\\ &\le (Cr^{\tilde q})^{\frac{1}{2r}}\max\left\{(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r}dx)^{\frac{1}{2r}}, (\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r}dx)^{\frac{1}{2r}}\right\}.\end{aligned}$$

By iteration, we have

$$\begin{aligned}&\max\left\{\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0d^k}dx\right)^{\frac{1}{2r_0d^k}}, \left(\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0d^k}dx\right)^{\frac{1}{2r_0d^k}}\right\}\\ &\le \Pi_{i=1}^k(C(r_0d^{i-1})^{\tilde q})^{\frac{1}{2r_0d^{i-1}}}\max\left\{\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx\right)^{\frac{1}{2r_0}}, \left(\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0}dx\right)^{\frac{1}{2r_0}}\right\}\\ &\le C\max\left\{\left(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx\right)^{\frac{1}{2r_0}}, \left(\int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0}dx\right)^{\frac{1}{2r_0}}\right\},\end{aligned} \tag{2.22}$$

where $d = \frac{N}{N-2}$.

In the following, it is sufficient to prove that $u, v \in L^{\frac{4N}{N-2}+2r_0}(\Omega)$. Indeed, we have

$$\begin{aligned}\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx &\le K^{\frac{4N}{N-2}}\int_{|u|\le K}|u^T|^{2r_0}dx + \int_{|u|\ge K}|u|^{\frac{8}{N-2}}(u^2|u^T|^{r_0})^2dx\\ &\le C + (\int_{|u|\ge K}|u|^{\frac{4N}{N-2}}dx)^{\frac{2}{N}}\cdot(\int_{|u|\ge K}(u^2|u^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}}\\ &\le C + (\int_{|u|\ge K}|u|^{\frac{4N}{N-2}}dx)^{\frac{2}{N}}\cdot(\int_\Omega (u^2|u^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}},\end{aligned} \tag{2.23}$$

take K large enough such that $(\int_{|u|\ge K}|u|^{\frac{4N}{N-2}})^{\frac{2}{N}} \le \epsilon$, then we have

$$\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx \le C + \epsilon(\int_\Omega (u^2|u^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}}. \tag{2.24}$$

Similarly, we have the estimate for the function v and

$$\begin{aligned}&\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx + \int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0}dx\\ &\le C + \epsilon((\int_\Omega (u^2|u^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}} + (\int_\Omega (v^2|v^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}}).\end{aligned} \tag{2.25}$$

Return to (2.21) with $r = r_0$, we obtain

$$\begin{aligned}&(\int_\Omega (u^2|u^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}} + (\int_\Omega (v^2|v^T|^{r_0})^{\frac{2N}{N-2}}dx)^{\frac{N-2}{N}}\\ &\le C(\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx + \int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0}dx).\end{aligned} \tag{2.26}$$

Substitute (2.26) into (2.25), and take $C\epsilon < \frac{1}{2}$, we have

$$\int_\Omega |u|^{\frac{4N}{N-2}}|u^T|^{2r_0}dx + \int_\Omega |v|^{\frac{4N}{N-2}}|v^T|^{2r_0}dx \le C,$$

where C is a constant independent of T. Let $T \to \infty$, we get

$$\int_\Omega |u|^{\frac{4N}{N-2}+2r_0}dx + \int_\Omega |v|^{\frac{4N}{N-2}+2r_0}dx \le C.$$

Thus we proved that $u, v \in L^{\frac{4N}{N-2}+2r_0}(\Omega)$.

(3) Now we prove that (u, v) is the solution of (P). Since $I'_{s_n}(u_n, v_n) = 0$, we have, for $\forall \varphi, \psi \in W_0^{1,4}(\Omega)$,

$$\begin{aligned}&s_n \int_\Omega |\nabla u_n|^2\nabla u_n \nabla\varphi dx + \int_\Omega (1+2u_n^2)\nabla u_n\nabla\varphi dx + 2\int_\Omega u_n^2|\nabla u_n|^2\varphi dx\\ &= \alpha\int_\Omega |u_n|^{\alpha-2}u_n|v_n|^\beta\varphi dx + \lambda\int_\Omega |u_n|^{\lambda-2}u_n|v_n|^\mu\varphi dx\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&s_n \int_\Omega |\nabla v_n|^2\nabla v_n \nabla\psi dx + \int_\Omega (1+2v_n^2)\nabla v_n\nabla\psi dx + 2\int_\Omega v_n^2|\nabla v_n|^2\psi dx\\ &= \beta\int_\Omega |v_n|^{\beta-2}v_n|u_n|^\alpha\psi dx + \mu\int_\Omega |u_n|^{\lambda}v_n|v_n|^{\mu-2}\psi dx.\end{aligned} \tag{2.28}$$

Choose $\varphi = \phi e^{-u_n^+}$ as the test function in (2.27), where $\phi \in C_0^\infty(\Omega), \phi \ge 0$, we get

$$\begin{aligned}&s_n \int_\Omega (|\nabla u_n|^2\nabla u_n\nabla\phi e^{-u_n^+} - |\nabla u_n^+|^4\phi e^{-u_n^+})dx + \int_\Omega (1+2u_n^2)\nabla u_n\nabla\phi e^{-u_n^+}dx\\ &- \int_\Omega (1+2u_n^2)|\nabla u_n^+|^2\phi e^{-u_n^+}dx + 2\int_\Omega u_n|\nabla u_n|^2\phi e^{-u_n^+}dx\\ &= \alpha\int_\Omega |u_n|^{\alpha-2}u_n|v_n|^\beta\phi e^{-u_n^+}dx + \lambda\int_\Omega |u_n|^{\lambda-2}u_n|v_n|^\mu\phi e^{-u_n^+}dx.\end{aligned} \tag{2.29}$$

Note that the left side of (2.29) is upper semi-continuous, while the right side is continuous. We have

$$|s_n\int_\Omega (|\nabla u_n|^2\nabla u_n\nabla\phi e^{-u_n^+}dx| \le Cs_n(\int_\Omega |\nabla u_n|^4dx)^{\frac{3}{4}} \le Cs_n^{\frac{1}{4}} \to 0,$$

$$\int_\Omega (1+2u_n^2)\nabla u_n\nabla\phi e^{-u_n^+}dx \to \int_\Omega (1+2u^2)\nabla u\nabla\phi e^{-u^+}dx,$$

$$\begin{aligned}&\alpha\int_\Omega |u_n|^{\alpha-2}u_n|v_n|^\beta\phi e^{-u_n^+}dx + \lambda\int_\Omega |u_n|^{\lambda-2}u_n|v_n|^\mu\phi e^{-u_n^+}dx\\ &\to \alpha\int_\Omega |u|^{\alpha-2}u|v|^\beta\phi e^{-u^+}dx + \lambda\int_\Omega |u|^{\lambda-2}u|v|^\mu\phi e^{-u^+}dx.\end{aligned}$$

and

$$
\begin{aligned}
&\underline{\lim}_{n\to\infty}\{\int_\Omega(1+2u_n^2)|\nabla u_n^+|^2\phi e^{-u_n^+}dx-2\int_\Omega u_n|\nabla u_n|^2\phi e^{-u_n^+}dx\}\\
=\ &\underline{\lim}_{n\to\infty}\{\int_\Omega(1-2u_n+2u_n^2)|\nabla u_n^+|^2\phi e^{-u_n^+}dx-2\int_\Omega u_n|\nabla u_n^-|^2\phi e^{-u_n^+}dx\}\\
\ge\ &\int_\Omega(1-2u+2u^2)|\nabla u^+|^2\phi e^{-u^+}dx-2\int_\Omega u|\nabla u^-|^2\phi e^{-u^+}dx\\
=\ &\int_\Omega(1+2u^2)|\nabla u^+|^2\phi e^{-u^+}dx-2\int_\Omega u|\nabla u|^2\phi e^{-u^+}dx\,.
\end{aligned}
$$

Let $n\to\infty$ in (2.29) we obtain

$$
\begin{aligned}
&\int_\Omega(1+2u^2)\nabla u\nabla\phi e^{-u^+}dx-\int_\Omega(1+2u^2)|\nabla u^+|^2\phi e^{-u^+}dx+2\int_\Omega u|\nabla u|^2\phi e^{-u^+}dx\\
\ge\ &\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\phi e^{-u^+}dx+\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\phi e^{-u^+}dx.
\end{aligned}
\tag{2.30}
$$

Since $u\in L^\infty(\Omega),\phi e^{u^+}\in H_0^1(\Omega)\cap L^\infty(\Omega)$, we may choose a sequence of nonnegative function $\phi_n\in C_0^\infty(\Omega)$ such that $\phi_n\to\phi e^{u^+}$ a.e. $x\in\Omega$ and $|\phi_n|_\infty\le C$. Thus by (2.30), we get

$$
\int_\Omega(1+2u^2)\nabla u\nabla\varphi dx+2\int_\Omega u|\nabla u|^2\varphi dx\ge\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\varphi dx+\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\varphi dx.
$$

Choose $\phi e^{-u_n^-}$ as the test function in (2.27), then by repeating the same arguments as above, we obtain for $\forall\varphi\in C_0^\infty(\Omega),\varphi\ge 0$,

$$
\int_\Omega(1+2u^2)\nabla u\nabla\varphi dx+2\int_\Omega u|\nabla u|^2\varphi dx\le\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\varphi dx+\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\varphi dx.
\tag{2.31}
$$

Therefore we have, for $\varphi\in C_0^\infty(\Omega),\varphi\ge 0$,

$$
\int_\Omega(1+2u^2)\nabla u\nabla\varphi dx+2\int_\Omega u|\nabla u|^2\varphi dx=\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\varphi dx+\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\varphi dx.
\tag{2.32}
$$

By approximation, we see that (2.32) is true for all $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$. Similarly, it holds that for all $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$
\int_\Omega(1+2v^2)\nabla v\nabla\varphi dx+2\int_\Omega v|\nabla v|^2\varphi dx=\beta\int_\Omega|u|^\alpha v|v|^{\beta-2}\varphi dx+\mu\int_\Omega|u|^\lambda v|v|^{\mu-2}\varphi dx,
$$

hence (u,v) is a solution of the system (P).

(4) We prove that $I(u,v)\le\lim_{n\to\infty}I_{s_n}(u_n,v_n)$. Indeed, by (2.31) and (2.32), we have

$$
\begin{aligned}
&\int_\Omega(1+4u^2)|\nabla u|^2dx+\int_\Omega(1+4v^2)|\nabla v|^2dx\\
=\ &(\alpha+\beta)\int_\Omega|u|^\alpha|v|^\beta dx+(\lambda+\mu)\int_\Omega|u|^\lambda|v|^\mu dx.
\end{aligned}
$$

Since (u_n, v_n) is the critical point of I_{s_n}, we obtain

$$
\begin{aligned}
& s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + \int_\Omega (1 + 4u_n^2)|\nabla u_n|^2 dx + \int_\Omega (1 + 4v_n^2)|\nabla v_n|^2 dx \\
&= (\alpha + \beta) \int_\Omega |u_n|^\alpha |v_n|^\beta dx + (\lambda + \mu) \int_\Omega |u_n|^\lambda |v_n|^\mu dx.
\end{aligned} \tag{2.33}
$$

Thus

$$
\begin{aligned}
& I_{s_n}(u_n, v_n) \\
&= \frac{1}{4} s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + \frac{1}{2} \int_\Omega (1 + 2u_n^2)|\nabla u_n|^2 dx + \frac{1}{2} \int_\Omega (1 + 2v_n^2)|\nabla v_n|^2 dx \\
&\quad - \int_\Omega |u_n|^\alpha |v_n|^\beta dx - \int_\Omega |u_n|^\lambda |v_n|^\mu dx \\
&= (\frac{1}{4} - \frac{1}{\lambda + \mu}) s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx \\
&\quad + (\frac{1}{2} - \frac{1}{\lambda + \mu}) \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx + (1 - \frac{4}{\lambda + \mu}) \int_\Omega (u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2)dx \\
&\quad + (\frac{\alpha + \beta}{\lambda + \mu} - 1) \int_\Omega |u_n|^\alpha |v_n|^\beta dx.
\end{aligned} \tag{2.34}
$$

By lower semi-continuous and Fatou's lemma, we have

$$
\begin{aligned}
\lim_{n\to\infty} I_{s_n}(u_n, v_n) \quad & \geq (\frac{1}{2} - \frac{1}{\lambda + \mu}) \int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx \\
& \quad + (1 - \frac{4}{\lambda + \mu}) \int_\Omega (u^2 |\nabla u|^2 + v^2 |\nabla v|^2)dx + (\frac{\alpha + \beta}{\lambda + \mu} - 1) \int_\Omega |u|^\alpha |v|^\beta dx \\
& = I(u, v).
\end{aligned}
$$

(5) We prove that if $(u, v) = (0, 0)$, then $d \notin (0, S_{\alpha,\beta})$.

Suppose that $(u, v) = (0, 0)$, then $u_n \to 0, v_n \to 0$ in $L^p(\Omega)$ for $1 \leq p < \frac{4N}{N-2}$. In particular, note that $\lambda + \mu < \frac{4N}{N-2}$, we have

$$
\int_\Omega |u_n|^\lambda |v_n|^\mu dx = o(1), \text{ as } n \to \infty.
$$

It follows from (2.33) that

$$
\begin{aligned}
& s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx + 4 \int_\Omega (u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2)dx \\
&= (\alpha + \beta) \int_\Omega |u_n|^\alpha |v_n|^\beta dx + o(1).
\end{aligned} \tag{2.35}
$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega |u_n|^\alpha |v_n|^\beta dx\\
&\le |u_n^2|_{2^*}^{\frac{\alpha}{2}} |v_n^2|_{2^*}^{\frac{\beta}{2}}\\
&\le (\frac{1}{S}\|u_n^2\|^2_{H_0^1(\Omega)})^{\frac{\alpha}{4}} (\frac{1}{S}\|v_n^2\|^2_{H_0^1(\Omega)})^{\frac{\beta}{4}}\\
&= \frac{\alpha^{\frac{\alpha}{4}}\beta^{\frac{\beta}{4}}}{S^{\frac{\alpha+\beta}{4}}(\alpha+\beta)^{\frac{\alpha+\beta}{4}}}\{(\frac{\alpha+\beta}{\alpha}\|u_n^2\|^2_{H_0^1(\Omega)})^{\frac{\alpha}{\alpha+\beta}} \cdot (\frac{\alpha+\beta}{\beta}\|v_n^2\|^2_{H_0^1(\Omega)})^{\frac{\beta}{\alpha+\beta}}\}^{\frac{\alpha+\beta}{4}}\\
&\le \frac{\alpha^{\frac{\alpha}{4}}\beta^{\frac{\beta}{4}}}{S^{\frac{\alpha+\beta}{4}}(\alpha+\beta)^{\frac{\alpha+\beta}{4}}}(\|u_n^2\|^2_{H_0^1(\Omega)} + \|v_n^2\|^2_{H_0^1(\Omega)})^{\frac{\alpha+\beta}{4}}.
\end{aligned} \tag{2.36}$$

We have two cases:

(1) If $\|u_n^2\|^2_{H_0^1(\Omega)} + \|v_n^2\|^2_{H_0^1(\Omega)} \to 0$. Then it follows from (2.35) and (2.36) that

$$s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx \to 0, \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx \to 0,$$

and $d = \lim_{n\to\infty} I_{s_n}(u_n, v_n) = 0$;

(2) If $\|u_n^2\|^2_{H_0^1(\Omega)} + \|v_n^2\|^2_{H_0^1(\Omega)} \nrightarrow 0$. Then again it follows from (2.35) and (2.36)

$$\|u_n^2\|^2_{H_0^1(\Omega)} + \|v_n^2\|^2_{H_0^1(\Omega)} \ge \frac{(\alpha+\beta)S^{\frac{N}{2}}}{(\alpha^\alpha\beta^\beta)^{\frac{N-2}{8}}} + o(1).$$

It turns out that,

$$\begin{aligned}
&I_{s_n}(u_n, v_n)\\
&= (\frac{1}{4} - \frac{1}{\alpha+\beta})s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)dx + (\frac{1}{2} - \frac{1}{\alpha+\beta})\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)dx\\
&\quad +(1 - \frac{4}{\alpha+\beta})\int_\Omega (|\nabla u_n^2|^2 + |\nabla v_n^2|^2)dx + (\frac{\lambda+\mu}{\alpha+\beta} - 1)\int_\Omega |u_n|^\lambda |v_n|^\mu dx + o(1)\\
&\ge (\frac{1}{4} - \frac{1}{\alpha+\beta})\int_\Omega (|\nabla u_n^2|^2 + |\nabla v_n^2|^2)dx + o(1)\\
&\ge \frac{2}{N-2} \cdot \frac{S^{\frac{N}{2}}}{(\alpha^\alpha\beta^\beta)^{\frac{N-2}{8}}} + o(1)\\
&= S_{\alpha,\beta} + o(1).
\end{aligned}$$

Hence $d = \lim_{n\to\infty} I_{s_n}(u_n, v_n) \ge S_{\alpha,\beta}$. All in all, we have $d \notin (0, S_{\alpha,\beta})$.

□

3 The proof of the main results

Recall that c_s is defined by

$$c_s = \inf_{\gamma\in\Gamma_s} \sup_{t\in[0,1]} I_s(\gamma(t)),$$

where $\Gamma_s = \{\gamma | \gamma \in C([0,1], X), \gamma(0) = (0,0), I_s(\gamma(1)) < 0\}$. In the following we will prove that $d = \lim_{s\to 0} c_s \in (0, S_{\alpha,\beta})$. Set

$$w_s(x) = \frac{(N(N-2)\epsilon)^{\frac{N-2}{8}}}{(\epsilon + |x|^2)^{\frac{N-2}{4}}}. \tag{3.1}$$

Without loss of generality, we may assume that $0 \in \Omega, B_{2\rho}(0) \subset \Omega$. Let $\varphi \in C_0^\infty(\Omega)$ be the cut-off function such that $\varphi \geq 0, \varphi(x) = 1$ for $|x| \leq \rho; \varphi(x) = 0$ for $|x| \geq 2\rho; |\nabla\varphi| \leq \frac{2}{\rho}$. Let $Z_\epsilon = \varphi w_s, U_\epsilon = (\alpha^{\frac{1}{4}} Z_\epsilon, \beta^{\frac{1}{4}} Z_\epsilon)$. We have

Lemma 3.1 *$lim_{s\to 0}$* $\sup_{t\geq 0} I_s(tU_\epsilon) \leq S_{\alpha,\beta} - c\epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)}$.

Proof. By direct computation, we get

$$\begin{cases} \int_\Omega |\nabla Z_\epsilon^2|^2 dx = S^{\frac{N}{2}} + O(\epsilon^{\frac{N-2}{2}}) \\ \int_\Omega |\nabla Z_\epsilon|^2 dx = O(\epsilon^{\frac{N-2}{4}} |log\epsilon|) \\ \int_\Omega Z_\epsilon^{\frac{4N}{N-2}} dx = S^{\frac{N}{2}} + O(\epsilon^{\frac{N}{2}}) \\ \int_\Omega Z_\epsilon^r \, dx \sim \begin{cases} \epsilon^{\frac{N}{2}-\frac{1}{8}r(N-2)}, & \frac{2N}{N-2} < r < \frac{4N}{N-2} \\ \epsilon^{\frac{N}{4}} |log\epsilon|, & r = \frac{2N}{N-2} \\ \epsilon^{\frac{1}{8}r(N-2)}, & r < \frac{2N}{N-2}. \end{cases} \end{cases} \tag{3.2}$$

On the other hand, it is easy to see that there exist $0 < T_1 < T_2$ such that

$$\sup_{t\geq 0} I_s(tU_\epsilon) = \sup_{T_1\leq t\leq T_2} I_s(tU_\epsilon) \leq cs + \sup_{T_1\leq t\leq T_2} I(tU_\epsilon), \tag{3.3}$$

and

$$\begin{aligned} &\sup_{T_1\leq t\leq T_2} I(tU_\epsilon) \\ &\leq c_1 \int_\Omega |\nabla Z_\epsilon|^2 dx - c_2 \int_\Omega Z_\epsilon^{\lambda+\mu} dx + \sup_{T_1\leq t\leq T_2} \{(\alpha+\beta)t^4 \int_\Omega Z_\epsilon^2 |\nabla Z_\epsilon|^2 dx - \alpha^{\frac{\alpha}{4}}\beta^{\frac{\beta}{4}} t^{\frac{4N}{N-2}} \int_\Omega Z_\epsilon^{\frac{4N}{N-2}} dx\} \\ &\leq \frac{N}{N-2} t_0^4 (S^{\frac{N}{2}} + O(\epsilon^{\frac{N-2}{2}})) - \alpha^{\frac{\alpha}{4}}\beta^{\frac{\beta}{4}} t_0^{\frac{4N}{N-2}} (S^{\frac{N}{2}} + O(\epsilon^{\frac{N}{2}})) \\ &\quad + O(\epsilon^{\frac{N-2}{4}} |log\epsilon|) - 2c_0 \epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)} \\ &\leq (\frac{N}{N-2} t_0^4 - \alpha^{\frac{\alpha}{4}}\beta^{\frac{\beta}{4}} t_0^{\frac{2N}{N-2}}) S^{\frac{N}{2}} - c_0 \epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)} \\ &\leq S_{\alpha,\beta} - c_0 \epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)}. \end{aligned}$$

Thus

$$\lim_{s\to 0} \sup_{t\geq 0} I_s(tU_\epsilon) \leq S_{\alpha,\beta} - C_0 \epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)}.$$

□

Proof of Theorem 1.1

By Lemma 2.2, we see that c_s is a critical value of I_s and $\lim_{s\to 0} c_s = d \geq m > 0$. It follows from Lemma 3.1,

$$0 < m \leq d = \lim_{s\to 0} c_s \leq \lim_{s\to 0} \sup_{t\geq 0} I_s(tU_\epsilon) \leq S_{\alpha,\beta} - C_0\epsilon^{\frac{N}{2}-\frac{1}{8}(\lambda+\mu)(N-2)}.$$

Now we suppose $s_n \to 0$, $I'_{s_n}(u_n, v_n) = 0$ and $I_{s_n}(u_n, v_n) = c_{s_n}$. Then

$$d = \lim_{n\to\infty} c_{s_n} \in (0, S_{\alpha,\beta}).$$

By Lemma 2.2, without loss of generality, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u, \quad v_n \rightharpoonup v \text{ in } H_0^1(\Omega);\\ &u_n\nabla u_n \rightharpoonup u\nabla u, \quad v_n\nabla v_n \rightharpoonup v\nabla v \text{ in } L^2(\Omega). \end{aligned}$$

Moreover by Proposition 2.4 $(u, v) \neq (0, 0)$ is the solution of the problem (1.1) and it follows from the system (P) that $u \neq 0, v \neq 0$. If we use $I_s^+(c_s^+)$ instead of $I_s(c_s)$, then we can obtain the nonnegative critical point (u_s, v_s) of I_s^+, as a result we can get the nonnegative solution (u, v) of the problem (P). At last by using the standard elliptic estimate we can see that $u > 0, v > 0$ in Ω. This finishes the proof of Theorem 1.1.

□

4 General quasilinear systems

In this section, we will show that how the approach we used in the previous sections works for the more general quasilinear system of the form:

$$\begin{cases} \displaystyle\sum_{ij=1}^{N}(D_j(a_{ij}(x,u)D_iu) - \frac{1}{2}D_sa_{ij}(x,u)D_iuD_ju) + \alpha u^{\alpha-1}v^{\beta} + \lambda u^{\lambda-1}v^{\mu} = 0 & \text{in } \Omega\\ \displaystyle\sum_{ij=1}^{N}(D_j(a_{ij}(x,v)D_iv) - \frac{1}{2}D_sa_{ij}(x,v)D_ivD_jv) + \beta u^{\alpha}v^{\beta-1} + \mu u^{\lambda}v^{\mu-1} = 0 & \text{in } \Omega\\ u > 0, v > 0 & \text{in } \Omega\\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $D_i = \frac{\partial}{\partial x_i}$, $D_sa_{ij}(x,s) = \frac{\partial}{\partial s}a_{ij}(x,s)$, $\alpha,\beta,\mu,\lambda > 1$ satisfying $4 < \lambda+\mu < \alpha+\beta = \frac{4N}{N-2}$.

Assume:

(A_1) $a_{ij} \in C^1(\bar{\Omega}\times\mathbb{R},\mathbb{R})$, $a_{ij} = a_{ji}$ and there exist constants $a_1 \geq a_0 > 0$ such that

$$a_0(1+s^2)|\xi|^2 \leq \sum_{ij=1}^{N} a_{ij}(x,s)\xi_i\xi_j \leq a_1(1+s^2)|\xi|^2, \forall x \in \bar{\Omega}, s \in \mathbb{R}, \xi \in \mathbb{R}^N.$$

(A_2) There exists $\eta \in (0, 2)$ such that

$$(-2+\eta)\sum_{ij=1}^{N} a_{ij}(x,s)\xi_i\xi_j \leq \sum_{ij=1}^{N} sD_sa_{ij}(x,s)\xi_i\xi_j \leq (\lambda+\mu-2+\eta)\sum_{ij=1}^{N} a_{ij}(x,s)\xi_i\xi_j.$$

(A_3) There exists $\gamma \in [0,2)$ such that

$$\sum_{ij=1}^{N} a_{ij}(x,s)\xi_i\xi_j = 2s^2 + O(|s|^\gamma), \frac{1}{2}\sum_{ij=1}^{N} sD_s a_{ij}(x,s)\xi_i\xi_j = 2s^2 + O(|s|^\gamma),$$

as $|s| \to \infty$, uniformly in $x \in \bar{\Omega}, \xi \in S^{N-1}$, the unit sphere of R^N.

Our main result is:

Theorem 4.1 *Assume* $(A_1), (A_2), (A_3)$ *hold. Moreover assume* $\lambda + \mu > P_N + \alpha$. *Then the system* (3.1) *has at least one positive solution.*

We will use the similar notations as in the previous sections. The perturbation functional I_s now is defined as:

$$I_s(u,v) = \frac{1}{4}s\int_\Omega (|\nabla u|^4 + |\nabla v|^4)dx + I(u,v), \tag{3.2}$$

where

$$\begin{aligned} I(u,v) \quad &= \frac{1}{2}\int_\Omega \sum_{ij=1}^{N} a_{ij}(x,u)D_iuD_judx + \frac{1}{2}\int_\Omega \sum_{ij=1}^{N} a_{ij}(x,v)D_ivD_jvdx \\ &- \int_\Omega |u|^\alpha |v|^\beta dx - \int_\Omega |u|^\lambda |v|^\mu dx. \end{aligned} \tag{3.3}$$

The weak form of (3.3) is:

$$\begin{cases} s\displaystyle\int_\Omega |\nabla u|^2\nabla u\nabla\varphi dx + \int_\Omega \sum_{ij=1}^{N} a_{ij}(x,u)D_iuD_j\varphi dx + \frac{1}{2}\int_\Omega \sum_{ij=1}^{N} D_s a_{ij}(x,u)uD_iuD_ju\varphi dx \\ -\alpha\displaystyle\int_\Omega |u|^{\alpha-2}u|v|^\beta\varphi dx - \lambda\int_\Omega |u|^{\lambda-2}u|v|^\mu\varphi dx = 0, \\ s\displaystyle\int_\Omega |\nabla v|^2\nabla v\nabla\psi dx + \int_\Omega \sum_{ij=1}^{N} a_{ij}(x,u)D_ivD_j\psi\, dx + \frac{1}{2}\int_\Omega \sum_{ij=1}^{N} D_s a_{ij}(x,v)vD_ivD_jv\psi dx \\ -\beta\displaystyle\int_\Omega |u|^\alpha|v|^{\beta-2}v\psi dx - \mu\int_\Omega |u|^\lambda|v|^{\mu-2}v\psi dx = 0, \end{cases}$$

for all $\varphi, \psi \in C_0^\infty(\Omega)$.

Lemma 4.1 *Assume* $(A_1), (A_2)$. *Then for fixed* $s \in (0,1]$, *the perturbation functional* I_s *satisfies the* $(PS)_c$ *condition.*

Proof. The proof is similar as the proof of Lemma 2.1, we only give the sketch. Let $\{(u_n, v_n)\}$

be a $(PS)_c$ sequence of I_s. We have

$$\begin{aligned} & c+o(1)(\|(u_n,v_n)\|_X) \\ \geq\ & I_s(u_n,v_n)-\frac{1}{\lambda+\mu}\langle I_s'(u_n,v_n),(u_n,v_n)\rangle \\ =\ & (\frac{1}{4}-\frac{1}{\lambda+\mu})s\int_\Omega(|\nabla u_n|^4+|\nabla v_n|^4)dx \\ & +\int_\Omega\sum_{ij=1}^N[(\frac{1}{2}-\frac{1}{\lambda+\mu})a_{ij}(x,u_n)-\frac{1}{2(\lambda+\mu)}u_nD_sa_{ij}(x,u_n)]D_iu_nD_ju_ndx \\ & +\int_\Omega\sum_{ij=1}^N[(\frac{1}{2}-\frac{1}{\lambda+\mu})a_{ij}(x,v_n)-\frac{1}{2(\lambda+\mu)}v_nD_sa_{ij}(x,v_n)]D_iv_nD_jv_ndx \\ & +(\frac{\alpha+\beta}{\lambda+\mu}-1)\int_\Omega|u_n|^\alpha|v_n|^\beta dx. \end{aligned} \tag{3.4}$$

By $(A_1),(A_2)$ we have

$$s\int_\Omega(|\nabla u_n|^4+|\nabla v_n|^4)dx+\int_\Omega(1+u_n^2)|\nabla u_n|^2dx+\int_\Omega(1+v_n^2)|\nabla v_n|^2dx\leq c+o(1)\|(u_n,v_n)\|_X,$$

which implies that $\{(u_n,v_n)\}$ is bounded in $X=:\ W_0^{1,4}(\Omega)\times W_0^{1,4}(\Omega)$. Then up to a subsequence, we may assume that

$$(u_n,v_n)\rightharpoonup(u,v)\text{ in }H_0^1(\Omega)\times H_0^1(\Omega),$$

$$u_n\nabla u_n\rightharpoonup u\nabla u,\ v_n\nabla v_n\rightharpoonup v\nabla v\text{ in }L^2(\Omega),$$

$$(u_n,v_n)\to(u,v)\text{ in }L^q(\Omega)\times L^q(\Omega)\text{ for }1\leq q<\frac{4N}{N-2}.$$

Note that (u_n,v_n) satisfy for all $(\varphi,\psi)\in X$,

$$\begin{aligned} & s_n\int_\Omega|\nabla u_n|^2\nabla u_n\nabla\varphi dx+\int_\Omega\sum_{ij=1}^N a_{ij}(x,u_n)D_iu_nD_j\varphi dx \\ & \quad+\int_\Omega\frac{1}{2}\sum_{ij=1}^N D_sa_{ij}(x,u_n)D_iu_nD_ju_n\varphi dx \\ & \quad-\alpha\int_\Omega|u_n|^{\alpha-2}u_n|v_n|^\beta\varphi dx-\lambda\int_\Omega|u_n|^{\lambda-2}u_n|v_n|^\mu\varphi dx=o(1)\|\varphi\|, \\ & s_n\int_\Omega|\nabla v_n|^2\nabla v_n\nabla\psi dx+\int_\Omega\sum_{ij=1}^N a_{ij}(x,v_n)D_iv_nD_j\psi dx \\ & \quad+\int_\Omega\frac{1}{2}\sum_{ij=1}^N D_sa_{ij}(x,v_n)D_iv_nD_jv_n\psi dx \\ & \quad-\beta\int_\Omega|u_n|^\alpha|v_n|^{\beta-2}v_n\psi dx-\mu\int_\Omega|u_n|^\lambda|v_n|^{\mu-2}v_n\psi dx=o(1)\|\psi\|. \end{aligned} \tag{3.5}$$

Choose $\varphi = u_n - u_m, \psi = v_n - v_m$ as the test function in (3.5), we obtain

$$\begin{aligned}
&o(1)\\
&= \langle I_s'(u_n, v_n) - I_s'(u_m, v_m), (u_n - u_m, v_n - v_m)\rangle\\
&= s\int_\Omega (|\nabla u_n|^2\nabla u_n\nabla(u_n - u_m) - |\nabla u_m|^2\nabla u_m\nabla(u_n - u_m))dx\\
&\quad + s\int_\Omega (|\nabla v_n|^2\nabla v_n\nabla(v_n - v_m) - |\nabla v_m|^2\nabla v_m\nabla(v_n - v_m))dx\\
&\quad + s\int_\Omega \sum_{ij=1}^N (a_{ij}(x, u_n)D_iu_nD_j(u_n - u_m) - a_{ij}(x, u_m)D_iu_mD_j(u_n - u_m))dx\\
&\quad + s\int_\Omega \sum_{ij=1}^N (a_{ij}(x, v_n)D_iv_nD_j(v_n - v_m) - a_{ij}(x, v_m)D_iv_mD_j(v_n - v_m))dx\\
&\quad + \frac{1}{2}\int_\Omega \sum_{ij=1}^N (D_sa_{ij}(x, u_n)D_iu_nD_ju_n(u_n - u_m) - D_sa_{ij}(x, u_m)D_iu_mD_ju_m(u_n - u_m))dx\\
&\quad + \frac{1}{2}\int_\Omega \sum_{ij=1}^N (D_sa_{ij}(x, v_n)D_iv_nD_jv_n(v_n - v_m) - D_sa_{ij}(x, u_m)D_iv_mD_jv_m(v_n - v_m))dx\\
&\quad + \alpha\int_\Omega (|u_n|^{\alpha-2}u_n|v_n|^\beta - |u_m|^{\alpha-2}u_m|v_m|^\beta)(u_n - u_m)dx\\
&\quad + \lambda\int_\Omega (|u_n|^{\lambda-2}u_n|v_n|^\mu - |u_m|^{\lambda-2}u_m|v_m|^\mu)(u_n - u_m)dx\\
&\quad + \beta\int_\Omega (|u_n|^\alpha|v_n|^{\beta-2}v_n - |u_m|^\alpha|v_m|^{\beta-2}v_m)(v_n - v_m)dx\\
&\quad + \mu\int_\Omega (|u_n|^\lambda|v_n|^{\mu-2}v_n - |u_m|^\lambda|v_m|^{\mu-2}v_m)(v_n - v_m)dx.
\end{aligned}\tag{3.6}$$

Since $|Du_n|_4 \le C, |Du_m|_4 \le C$ and $|a_{ij}(x, s)| \le c(1 + s^2), u_n \to u$ in $L^4(\Omega), |a_{ij}(x, u_n) - a_{ij}(x, u_m)|_2 \to 0$, we have

$$\begin{aligned}
&\int_\Omega \sum_{ij=1}^N (a_{ij}(x, u_n)D_iu_n - a_{ij}(x, u_m)D_iu_m)D_j(u_n - u_m)dx\\
&= \int_\Omega \sum_{ij=1}^N a_{ij}(x, u_n)(D_iu_n - D_iu_m)(D_ju_n - D_ju_m)dx\\
&\quad + \int_\Omega \sum_{ij=1}^N (a_{ij}(x, u_n) - a_{ij}(x, u_m))D_iu_m(D_ju_n - D_ju_m)dx\\
&\ge -\sum_{ij=1}^N |a_{ij}(x, u_n) - a_{ij}(x, u_m)|_2|Du_m|_4|Du_n - Du_m|_4 = o(1),
\end{aligned}\tag{3.7}$$

Also, we have

$$\begin{aligned}
&\left|\int_\Omega \sum_{ij=1}^N (D_sa_{ij}(x, u_n)D_iu_nD_ju_n - D_sa_{ij}(x, u_m)D_iu_mD_ju_m)(u_n - u_m)dx\right|\\
&\le (|D_sa_{ij}(x, u_n)|_4|Du_n|_4^2 + |D_sa_{ij}(x, u_m)|_4|Du_m|_4^2)|u_n - u_m|_4\\
&\le C|u_n - u_m|_4 = o(1).
\end{aligned}\tag{3.8}$$

Similar to the proof of Lemma 2.1, we obtain

$$cs\int_\Omega(|\nabla u_n-\nabla u_m|^4+|\nabla v_n-\nabla v_m|^4)dx\le o(1),$$

which implies $\|(u_n-u_m,v_n-v_m)\|_X\to 0$, i. e. $\{(u_n,v_n)\}$ converges strongly in X.

□

In order to prove the main result, in the following, we will show that Proposition 2.1 is still true in this general situation. That is:

Proposition 4.1 *Assume* $(A_1),(A_2),(A_3)$ *hold. Suppose that* (u_n,v_n) *is a critical point of* I_{s_n} *for* $s_n\in(0,1]$ *with* $s_n\to 0$ *as* $n\to\infty$. *Let* $d=\lim_{n\to\infty}I_{s_n}(u_n,v_n)$. *Then up to a subsequence, we have*

(1) (a) $u_n\rightharpoonup u, v_n\rightharpoonup v\in W_0^{1,2}(\Omega)$;

  (b) $u_n\nabla u_n\rightharpoonup u\nabla u, v_n\nabla v_n\rightharpoonup v\nabla v$ *in* $L^2(\Omega)$;

  (c) $u_n\to u, v_n\to v$ *in* $L^p(\Omega)$ *for* $1\le p<\frac{4N}{N-2}$;

  (d) $u_n(x)\to u(x), v_n(x)\to v(x)$ *a.e.* $x\in\Omega$.

(2) $u,v\in L^\infty(\Omega)$, (u,v) *is a solution of the problem* (P).

(3) $I(u,v)\le\lim_{n\to\infty}I_{s_n}(u_n,v_n)$.

(4) If $d\in(0,S_{\alpha,\beta})$, *then* $(u,v)\ne(0,0)$.

Proof. We only give the sketch of the proof. Let $\{(u_n,v_n)\}$ be the critical point of I_{s_n} for $s_n\in(0,1]$ with $s_n\to 0$ as $n\to\infty$.

(1) Since $I'_{s_n}(u_n,v_n)=0$, for $\forall s_n\in(0,1]$, by $(A_1),(A_2)$, using the similar arguments as in the proof of Lemma 4.1, we obtain

$$s_n\int_\Omega(|\nabla u_n|^4+|\nabla v_n|^4)dx+\int_\Omega(1+u_n^2)|\nabla u_n|^2dx+\int_\Omega(1+v_n^2)|\nabla v_n|^2dx\le C,$$

for some constant C independent of n. Then

(a) $u_n\rightharpoonup u, v_n\rightharpoonup v$ in $W_0^{1,2}(\Omega)$;

(b) $u_n\nabla u_n\rightharpoonup u\nabla u, v_n\nabla v_n\rightharpoonup v\nabla v$ in $L^2(\Omega)$;

(c) $u_n\to u, v_n\to v$ in $L^p(\Omega)$ for $1\le p<\frac{4N}{N-2}$;

(d) $u_n(x)\to u(x), v_n(x)\to v(x)$ a.e. $x\in\Omega$.

(2) By similar arguments as that of Proposition 2.1 (2), that is by Moser's iteration, we can show that $u,v\in L^\infty(\Omega)$.

(3) We prove that (u,v) is the solution of (P). Take $T>0$ such that $\|u\|_{L^\infty(\Omega)}<T$ and $D_sa_{ij}(s)s\ge 0$ for $|s|\ge T$. Define

$$w_n=\begin{cases}u_n, & \text{if } u_n\ge -T\\ -T, & \text{if } u_n\le -T.\end{cases}$$

Choose $\varphi = \psi exp(-Mw_n)$ as the test function, where $\psi \geq 0, \psi \in C_0^\infty(\Omega), M > 0$ is a parameter, since $I'_{s_n}(u_n, v_n) = 0$, we have

$$
\begin{aligned}
0 \;&= s_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \psi \exp(-Mw_n)dx - s_n M \int_{u_n \geq -T} \psi |\nabla u_n|^4 \exp(-Mw_n))dx \\
&\quad + \int_\Omega \sum_{ij=1}^N a_{ij}(x, u_n) D_i u_n D_j \psi \exp(-Mw_n)dx \\
&\quad - \int_{u_n \geq -T} \sum_{ij=1}^N (M a_{ij}(x, u_n) - \frac{1}{2} D_s a_{ij}(x, u_n)) D_i u_n D_j u_n \psi \exp(-Mw_n)dx \\
&\quad + \int_{u_n \leq -T} \sum_{ij=1}^N \frac{1}{2} D_s a_{ij}(x, u_n) D_i u_n D_j u_n \psi \exp(-Mw_n)dx \\
&\quad - \alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta \psi \exp(-Mw_n)dx - \lambda \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu \psi \exp(-Mw_n)dx\,.
\end{aligned}
\tag{3.9}
$$

Note that

$$
-s_n M \int_{u_n \geq -T} |\nabla u_n|^4 \psi \exp(-Mw_n) \leq 0,
$$

$$
\int_{u_n \leq -T} \sum_{ij=1}^N D_s a_{ij}(x, u_n) D_i u_n D_j u_n \psi \exp(-Mw_n)dx \leq 0,
$$

$$
s_n |\int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \psi \exp(-Mw_n)dx| \leq C s_n^{\frac{1}{4}} \to 0.
$$

And since $\lambda + \mu - 1 < \alpha + \beta - 1 < \frac{4N}{N-2}$,

$$
\alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta \psi \exp(-Mw_n)dx \to \alpha \int_\Omega |u|^{\alpha-2} u |v|^\beta \psi \exp(-Mu)dx,
$$

$$
\int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu \psi \exp(-Mw_n)dx \to \lambda \int_\Omega |u|^{\lambda-2} u |v|^\mu \psi \exp(-Mu)dx.
$$

On the other hand, choose $M > 0$ such that $M a_{ij}(x, s) - \frac{1}{2} D_s a_{ij}(x, s) > 0$. Then by lower semi-continuous we have

$$
\begin{aligned}
&\liminf_{n \to \infty} \int_{u_n \geq -T} \sum_{ij=1}^N (M a_{ij}(x, u_n) - \frac{1}{2} D_s a_{ij}(x, u_n)) D_i u_n D_j u_n \psi \exp(-Mw_n)dx \\
&\geq \int_{u \geq -T} \sum_{ij=1}^N (M a_{ij}(x, u) - \frac{1}{2} D_s a_{ij}(x, u)) D_i u D_j u \psi \exp(-Mu)dx \\
&= \int_\Omega \sum_{ij=1}^N (M a_{ij}(x, u) - \frac{1}{2} D_s a_{ij}(x, u)) D_i u D_j u \psi \exp(-Mu)dx\,.
\end{aligned}
$$

Thus by Fatou's lemma, we obtain

$$\begin{aligned}&\int_\Omega a_{ij}(x,u)D_iuD_j(\psi\exp(-Mu))dx+\frac{1}{2}\int_\Omega D_sa_{ij}(x,u)D_iuD_ju(\psi\exp(-Mu))dx\\&\quad-\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\psi\exp(-Mu)dx-\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\psi\exp(-Mu)dx\geq 0.\end{aligned}\tag{3.10}$$

From here, by using the same arguments as we used in the proof of Proposition 2.1, we can obtain the opposite inequality by taking $\psi\exp(M\tilde{w}_n)$ such that $\tilde{w}_n=u_n$ if $u_n\leq T$ and $\tilde{w}_n=T$ if $u_n\geq T$. Thus, we have for $\forall\varphi,\psi\in C_0^\infty(\Omega)$,

$$\begin{aligned}&\int_\Omega\sum_{ij=1}^N a_{ij}(x,u)D_iuD_j\varphi dx+\frac{1}{2}\int_\Omega\sum_{ij=1}^N D_sa_{ij}(x,u)D_iuD_ju\varphi dx\\&\quad-\alpha\int_\Omega|u|^{\alpha-2}u|v|^\beta\varphi dx-\lambda\int_\Omega|u|^{\lambda-2}u|v|^\mu\varphi dx=0\end{aligned}$$

and similarly

$$\begin{aligned}&\int_\Omega\sum_{ij=1}^N a_{ij}(x,v)D_ivD_j\psi dx+\frac{1}{2}\int_\Omega\sum_{ij=1}^N D_sa_{ij}(x,v)D_ivD_jv\psi dx\\&\quad-\beta\int_\Omega|u|^{\alpha}|v|^{\beta-2}v\psi dx-\mu\int_\Omega|u|^{\lambda}|v|^{\mu-2}v\psi dx=0,\end{aligned}$$

which implies that (u,v) is the critical point of I_0 and hence a solution of the problem (P).

(4) We prove that $I(u,v)\leq\lim_{n\to\infty}I_{s_n}(u_n,v_n)$. Indeed, since (u,v) is a critical point of $I(u,v)$, we have

$$\begin{aligned}&\int_\Omega\sum_{ij=1}^N a_{ij}(x,u)D_iuD_judx+\int_\Omega\sum_{ij=1}^N a_{ij}(x,v)D_ivD_jvdx\\&\quad+\frac{1}{2}\int_\Omega\sum_{ij=1}^N(D_sa_{ij}(x,u)uD_iuD_ju+D_sa_{ij}(x,v)vD_ivD_jv)dx\\&=(\alpha+\beta)\int_\Omega|u|^\alpha|v|^\beta dx+(\lambda+\mu)\int_\Omega|u|^\lambda|v|^\mu dx.\end{aligned}$$

On the other hand, since (u_n,v_n) is the critical point of I_{s_n}, we obtain

$$\begin{aligned}&s_n\int_\Omega(|\nabla u_n|^4+|\nabla v_n|^4)dx\\&\quad+\int_\Omega\sum_{ij=1}^N a_{ij}(x,u_n)D_iu_nD_ju_ndx+\int_\Omega\sum_{ij=1}^N a_{ij}(x,v_n)D_iv_nD_jv_ndx\\&\quad+\frac{1}{2}\int_\Omega\sum_{ij=1}^N(D_sa_{ij}(x,u_n)u_nD_iu_nD_ju_n+D_sa_{ij}(x,v_n)v_nD_iv_nD_jv_n)dx\\&=(\alpha+\beta)\int_\Omega|u_n|^\alpha|v_n|^\beta dx+(\lambda+\mu)\int_\Omega|u_n|^\lambda|v_n|^\mu dx.\end{aligned}\tag{3.11}$$

By (A_1) and (A_2), using the similar arguments as in the proof of Proposition 2.4 we have

$$\lim_{n\to\infty} I_{s_n}(u_n, v_n) \geq I(u, v).$$

(5) We prove that if $(u, v) = (0, 0)$, then $d \notin (0, S_{\alpha,\beta})$.

Suppose that $(u, v) = (0, 0)$, then $u_n \to 0, v_n \to 0$ in $L^p(\Omega)$ for $1 \leq p < \frac{4N}{N-2}$. Since (u_n, v_n) is the critical point of I_{s_n}, we have

$$\begin{aligned}
&s_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \varphi dx + \int_\Omega \sum_{ij=1}^N a_{ij}(x, u_n) D_i u_n D_j \varphi dx \\
&\quad + \frac{1}{2} \int_\Omega \sum_{ij=1}^N D_s a_{ij}(x, u_n) D_i u_n D_j u_n \varphi dx \\
&= \alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta \varphi dx + \lambda \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu \varphi dx, \forall \varphi \in W_0^{1,4}(\Omega).
\end{aligned} \tag{3.12}$$

For $T > 0$, take $\varphi = u_n^T$ as the test function in (3.12) we have

$$\begin{aligned}
&s_n \int_\Omega |\nabla u_n^T|^4 dx + \int_\Omega \sum_{ij=1}^N a_{ij}(x, u_n^T) D_i u_n^T D_j u_n^T dx \\
&+ \frac{1}{2} \int_\Omega \sum_{ij=1}^N D_s a_{ij}(x, u_n^T) D_i u_n^T D_j u_n^T u_n^T dx + \frac{1}{2} \int_{|u_n| \geq T} \sum_{ij=1}^N u_n^T D_s a_{ij}(x, u_n) D_i u_n D_j u_n dx \\
= &\alpha \int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta u_n^T dx + \lambda \int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu u_n^T dx.
\end{aligned} \tag{3.13}$$

We have

$$|\int_\Omega |u_n|^{\alpha-2} u_n |v_n|^\beta u_n^T dx| \leq T \int_\Omega |u_n|^{\alpha-1} |v_n|^\beta dx \leq T |u_n|_{\alpha+\beta-1}^{\alpha-1} |v_n|_{\alpha+\beta-1}^\beta \to 0,$$

and similarly

$$|\int_\Omega |u_n|^{\lambda-2} u_n |v_n|^\mu u_n^T dx| \to 0.$$

On the other hand, by (A_3), we have

$$\int_{|u_n| \geq T} \sum_{ij=1}^N u_n^T D_s a_{ij}(x, u_n) D_i u_n D_j u_n dx \geq 0,$$

for T large enough. Since the right hand of (3.13) converges to zero, thus we obtain by $(A_1), (A_2)$ that

$$\int_\Omega (1 + |u_n^T|^2) |\nabla u_n^T|^2 dx = o(1). \tag{3.14}$$

If we take $\varphi = u_n$ as the test function in (3.12), then we get

$$\begin{aligned} & s_n \int_\Omega |\nabla u_n|^4 dx + \int_\Omega \sum_{ij=1}^N a_{ij}(x,u_n) D_i u_n D_j u_n dx \\ & + \frac{1}{2} \int_\Omega \sum_{ij=1}^N D_s a_{ij}(x,u_n) D_i u_n D_j u_n u_n dx \\ = {} & \alpha \int_\Omega |u_n|^\alpha |v_n|^\beta dx + \lambda \int_\Omega |u_n|^\lambda |v_n|^\mu dx \\ = {} & \alpha \int_\Omega |u_n|^\alpha |v_n|^\beta dx + o(1). \end{aligned} \tag{3.15}$$

But for $T > 0$, we have

$$\begin{aligned} & \int_\Omega \sum_{ij=1}^N (a_{ij}(x,u_n) + \frac{1}{2} u_n D_s a_{ij}(x,u_n)) D_i u_n D_j u_n dx \\ = {} & o_n(1) + \int_{|u_n| \geq T} \sum_{ij=1}^N (a_{ij}(x,u_n) + \frac{1}{2} u_n D_s a_{ij}(x,u_n)) D_i u_n D_j u_n dx \\ = {} & o_n(1) + \int_{|u_n| \geq T} (4u_n^2 + O(|u_n|^\gamma)) |Du_n|^2 dx \\ = {} & \int_\Omega 4u_n^2 |\nabla u_n|^2 dx + o_n(1) + \frac{c}{T^{2-\gamma}} \\ = {} & \int_\Omega 4u_n^2 |\nabla u_n|^2 dx + o_n(1). \end{aligned} \tag{3.16}$$

Thus we have

$$s_n \int_\Omega |\nabla u_n|^4 dx + 4 \int_\Omega u_n^2 |\nabla u_n|^2 dx = \alpha \int_\Omega |u_n|^\alpha |v_n|^\beta dx + o(1). \tag{3.17}$$

Similarly,

$$s_n \int_\Omega |\nabla v_n|^4 dx + 4 \int_\Omega v_n^2 |\nabla v_n|^2 dx = \beta \int_\Omega |u_n|^\alpha |v_n|^\beta dx + o(1). \tag{3.18}$$

Hence

$$\begin{aligned} & s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4) dx + 4 \int_\Omega u_n^2 |\nabla u_n|^2 dx + 4 \int_\Omega v_n^2 |\nabla v_n|^2 dx \\ = {} & (\alpha + \beta) \int_\Omega |u_n|^\alpha |v_n|^\beta dx + o(1). \end{aligned} \tag{3.19}$$

And

$$\begin{aligned} & I_{s_n}(u_n, v_n) \\ = {} & \frac{1}{4} s_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4) dx + \frac{1}{2} \int_\Omega \sum_{ij}^N a_{ij}(x,u_n) D_i u_n D_j u_n dx \\ & + \frac{1}{2} \int_\Omega \sum_{ij}^N a_{ij}(x,v_n) D_i v_n D_j v_n dx - \int_\Omega |u_n|^\alpha |v_n|^\beta dx - \int_\Omega |u_n|^\lambda |v_n|^\mu dx. \end{aligned} \tag{3.20}$$

Similar as in the proof of (3.16), for $T > 0$ large enough, we have

$$\begin{aligned}\int_\Omega \sum_{ij}^N a_{ij}(x,u_n)D_iu_nD_ju_ndx &= \int_{|u_n|\geq T} \textstyle\sum_{ij}^N a_{ij}(x,u_n)D_iu_nD_ju_ndx + o_n(1)\\ &= \displaystyle\int_{|u_n|\geq T} (2u_n^2 + O(|u_n|^\gamma))|\nabla u_n|^2dx + o_n(1)\\ &= \int_\Omega 2|u_n|^2|\nabla u_n|^2dx + o_n(1),\end{aligned} \tag{3.21}$$

and

$$\int_\Omega \sum_{ij}^N a_{ij}(x,v_n)D_iv_nD_jv_ndx = \int_\Omega 2|v_n|^2|\nabla v_n|^2dx + o_n(1). \tag{3.22}$$

It follows that

$$\begin{aligned}I_{s_n}(u_n,v_n) &= \frac{1}{4}s_n\int_\Omega(|\nabla u_n|^4 + |\nabla v_n|^4)dx + \int_\Omega u_n^2|\nabla u_n|^2dx + \int_\Omega v_n^2|\nabla v_n|^2dx\\ &\quad - \int_\Omega |u_n|^\alpha|v_n|^\beta dx + o(1).\end{aligned} \tag{3.23}$$

The rest of the proof is the same as that of Proposition 2.1.

□

Proof of Theorem 4.1 The proof is similar to the proof of Theorem 1.1. We only give the sketch. Use the similar notations as before.

(1) Define the mountain pass value c_s for the perturbation functional I_s.

(2) Note that, in this case we have

$$\int_\Omega a_{ij}(x,Z_\epsilon)D_iZ_\epsilon D_jZ_\epsilon dx = 2\int_\Omega Z_\epsilon^2|\nabla Z_\epsilon|^2dx + O(\int_\Omega(1+|Z_\epsilon|^\gamma)|\nabla Z_\epsilon|^2dx).$$

On the other hand, a direct computation shows that

$$\int_\Omega |Z_\epsilon|^\gamma|\nabla Z_\epsilon|^2dx = O(\epsilon^{\frac{(N-2)(2-\alpha)}{8}}),\ \text{for } \alpha\in(0,2).$$

By using the similar arguments as in Lemma 3.1, we have

$$d = \lim_{s\to 0} c_s \in (0, S_{\alpha,\beta}).$$

Then by using the Mountain pass theorem, we obtain a positive critical point (u_s, v_s) of I_s.

(3) Apply Proposition 4.1 to the sequence $\{(u_s, v_s)\}$, we obtain a positive critical point of I_0.

□

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