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Complete Orbits for Twist Maps with Exponential Generating Functions

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Abstract

In this paper twist symplectic maps with non-periodicity are considered, where the associated generating functions grow exponentially. Under appropriate assumptions the existence of infinitely many bounded complete orbits is established. An application to Toda-like type lattices may help us to explain our results.

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1 Introduction

At the turn of the 1980's, the theory of twist maps received a tremendous boost from the work of Aubry and Mather [3, 5]. Twist maps that are periodic, quasi-periodic or almost periodic in the angle variable has extensively been researched by the highly developed techniques of KAM theory and Aubry-Mather theory.

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However, in many mechanical and physical problems, this hypothesis like periodicity, quasi-periodicity or almost periodicity may be dropped. For example, the forcing terms of the oscillators of second order are only assumed to be bounded. At this time, the previous theory is no longer useful. The pioneering work on twist maps with non-periodic angle is presented by Kunze and Ortega [7]. They connected the maps $(\theta_1, r_1) = f(\theta, r)$ on the plane with the generating functions $h(\theta, \theta_1)$ growing quadratically, i.e.,

$$h(\theta, \theta_1) \sim (\theta_1 - \theta)^2, \tag{1.1}$$

and therefore discuss some special orbits of the associated generating functions. When f is c-exact symplectic and the perturbation is small enough, the existence of infinitely many bounded (in r) complete orbits of the twist map is proven.

The general theory is extended in [8], where the generating function grows like a positive power, that is,

$$h(\theta, \theta_1) \sim (\theta_1 - \theta)^\kappa, \text{ for some } \kappa > 1. \tag{1.2}$$

In case of $\kappa = 3$, this theory is applied to an impact model related to the so-called Fermi-Ulam acceleration.

Recently, twist maps were also considered whose generating functions grow like a negative power [9],

$$h(\theta, \theta_1) \sim (\theta_1 - \theta)^{-\kappa}, \text{ for some } \kappa > 1. \tag{1.3}$$

This hypothesis means that the maps have small twist at infinity. Under some natural assumptions on the first order derivative of h, the existence of infinitely many bounded orbits is established, and it is also proved that the unbounded orbits follow bounded orbits for long times. Furthermore, the authors give an application to the Fermi-Ulam ping-pong model with a non-periodic moving wall.

In this paper, we turn our attention to the exponential case where

$$h(\theta, \theta_1) \sim \mathrm{e}^{\theta_1 - \theta}. \tag{1.4}$$

Our main motivation for this is provided by the problem of the ground states for Toda-like type lattices

$$\theta_n'' + V'(\theta_n) + W'(\theta_{n+1} - \theta_n) - W'(\theta_n - \theta_{n-1}) = 0, \ n \in \mathbb{Z}, \tag{1.5}$$

where the nearest neighbour interaction is assumed to be $W(\theta) = \mathrm{e}^\theta - 1$. It is known that equation (1.5) is the standard Toda lattice with the oscillator potential V vanishing [12]. In the original DNA model, the interaction potential W is taken to be the Morse potential by Peyrard and Bishop[11], i.e., $W(\theta) = (\mathrm{e}^{-a\theta} - 1)^2$. When considering the ground states for the Frenkel-Kontorova model, where W is supposed to be a quadratic function, Aubry establishes a global variational approach relative to twist maps [3]. Together with Mather's result, subsequently it is the famous Aubry-Mather theory.

To begin with, we give some definitions which come from [7, 10].

Definition 1.1 *Let $(\theta_1, r_1) = f(\theta, r)$, $f : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ be a C^1-map.*

(a) *We say f is symplectic, if there exists a C^2-function $\hat{h} : \mathbb{R} \times [a, b] \to \mathbb{R}$ such that*

$$r_1 \mathrm{d}\theta_1 - r\mathrm{d}\theta = \mathrm{d}\hat{h}, \ \text{ where } \ (\theta_1, r_1) = f(\theta, r).$$

(b) *A sequence $(\theta_n, r_n)_{n\in\mathbb{Z}} \subset \mathbb{R} \times [a, b]$ is called a complete orbit for f, if*

$$(\theta_{n+1}, r_{n+1}) = f(\theta_n, r_n) \text{ for } n \in \mathbb{Z}.$$

(c) *We say f has twist, if $\partial_r \theta_1 > 0$ (the reversed inequality $\partial_r \theta_1 < 0$ is also admissible as a twist condition.), where ∂_r denotes the partial derivative with respect to r.*

Especially, consider the generating function with exponential growth

$$h(\theta, \theta_1) = \mathrm{e}^{\theta_1 - \theta},$$

which is defined on the domain $\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\}$, where $\Delta > \delta > 0$, then the associated map on the plane is

$$\tilde{f} : \ (\theta_1, r_1) = (\theta + \ln r, r), \tag{1.6}$$

defined on $D = \{(\theta, r) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \mathrm{e}^{\Delta} \ge r \ge \delta > 1\}$. Obivously, for any positive constants $b > a > 0$, $\tilde{f} : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ defined on the strip is a twist symplectic map with the C^2-function $\hat{h}(\theta, r) = r$. The function $\mathcal{H}(\theta, r) = r$ is a first integral, that is $\mathcal{H}(\theta_1, r_1) = \mathcal{H}(\theta, r)$, and each straight line $r = r_*$ is invariant under $\tilde{f}$. The rotation number $\omega = \ln r_*$ associated to each of these straight lines increases with r_*.

Definition 1.2 *Let $f = f(\theta, r) : \mathbb{R} \times [a, +\infty) \to \mathbb{R}^2$ be a C^1-map and suppose that $c : [a, +\infty) \to \mathbb{R}$ is a function. We say that f is c-exact symplectic, if there exists a C^2-function $\hat{h} : \mathbb{R} \times [a, +\infty) \to \mathbb{R}$ such that*

$$r_1 \mathrm{d}\theta_1 - r\mathrm{d}\theta = \mathrm{d}\hat{h}, \ \ \textit{where} \ \ (\theta_1, r_1) = f(\theta, r),$$

and

$$\sup_{\theta_0 \in \mathbb{R}} \sup_{r \in [a, +\infty)} \min_{\theta \in [\theta_0, \theta_0 + T]} |\hat{h}(\theta, r) - c(r)| < +\infty$$

for some $T > 0$.

Definition 1.2 is a proper extension of the notion of exact symplectic maps from the periodic case or quasi-periodic case [7, 13]. In [7], the authors have shown that, for any symplectic C^1-map $f = f(\theta, r) : (\mathbb{R}/T\mathbb{Z}) \times [a, +\infty) \to \mathbb{R}^2$, f is exact symplectic on the cylinder if and only if the condition from Definition 1.2 holds.

This paper, which consists of five sections, is organized as follows. The second section is dedicated to the theory of exponent-type generating function, i.e., the generating function is governed by one prescribed exponential function. The fundamental theorem is proven that there exists one solution of a sequence for the difference equation given by the generating function. In Section 3, we will show that such sequences are just the complete orbits for the associated symplectic map with boundary preserved. Moreover, a bound of the size of the rotation number will be given, which is dominated by the definition domain of the associated generating function. In the subsequent fourth section, we drop the hypothesis on the invariance of the boundaries. However, a stronger and more qualitative assumption on the generating function will be imposed. In the final section, we will give another interesting application for the ground states of Toda-like type lattices arising from physics, biodynamics and other branches of natural science. The method is based on a global variational technique, which was proposed by Aubry. It has an independent significance.

2 Exponential Generating Functions

Now we will devise a theory to construct some special orbits for maps with the generating function growing exponentially

$$h(\theta, \theta_1) \sim \mathrm{e}^{\theta_1 - \theta},$$

where $h = h(\theta, \theta_1)$ is a C^1-function satisfying

$$r = -\partial_\theta h(\theta, \theta_1), \ \ r_1 = \partial_{\theta_1} h(\theta, \theta_1).$$

The associated map

$$f:\ (\theta_1, r_1) = (\theta + \ln r + F(\theta, r), r + G(\theta, r))$$

generated by h is close to the integrable map $\tilde{f}$ as (1.6). Note that h denotes a general class of functions, e.g., $h(\theta, \theta_1) = ae^{\theta_1 - \theta} + 2b/\pi \arctan\theta$, for any constants $a > b > 0$. The main result of this section is in the following.

Theorem 2.1 *Let $\Delta > \delta > 0$. Suppose that the function $h : \Omega = \{(\theta, \theta') \in \mathbb{R}^2 : \delta \le \theta' - \theta \le \Delta\} \to \mathbb{R}$ is C^1 and such that*

$$\underline{\alpha}\, e^{\theta' - \theta} \le h(\theta, \theta') \le \overline{\alpha}\, e^{\theta' - \theta},\ \ (\theta, \theta') \in \Omega, \tag{2.1}$$

for some constants $\overline{\alpha} \ge \underline{\alpha} > 0$. Then there is a constant $\sigma_{**} \ge 1$ (depending only on $\overline{\alpha}/\underline{\alpha}$) with the following property. If

$$\Delta > 4\delta + 3\ln\sigma_{**}, \tag{2.2}$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ such that $|\theta_0^*| \le \Delta, \delta \le \theta_{n+1}^* - \theta_n^* \le \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}^*, \theta_n^*) + \partial_1 h(\theta_n^*, \theta_{n+1}^*) = 0,\ n \in \mathbb{Z}. \tag{2.3}$$

Moreover,

$$\delta \le \liminf_{n\to\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to\infty} \frac{\theta_n^*}{n} \le \Delta,\ \ \delta \le \liminf_{n\to-\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to-\infty} \frac{\theta_n^*}{n} \le \Delta. \tag{2.4}$$

Remark 2.1 We remark that Theorem 2.1 can be extended to the more general exponent-type generating function

$$h(\theta, \theta_1) \sim e^{\kappa(\theta_1 - \theta)},\ \ \kappa > 0.$$

In fact, we only replace $\kappa\theta_1, \kappa\theta, h$ with $\theta_1, \theta, h/\kappa$, respectively.

The proof of Theorem 2.1 is along the lines of the proof to [7, Thm. 2.1], [8, Thm. 3.1] and [9, Thm. 2.3]. Therefore we keep the presentation short, but indicate at which places changes are needed. For fixed $A > 0, N \in \mathbb{N}$, and $\Delta > 4\delta > 0$, define

$$\Sigma^{(N)} = \Big\{\Theta = (\theta_n)_{-N\le n\le N} : \theta_{\pm N} = \pm A, \delta \le \theta_{n+1} - \theta_n \le \Delta, \\ \text{for } n = -N, \cdots, N-1\Big\}.$$

Since later $A = A_N$ will be chosen to depend on N, the dependence of $\Sigma^{(N)}$ on A is suppressed in our notation. The following lemma can be found in [7, Lemma 2.3].

Lemma 2.1 *If $\delta \le \frac{A}{N} \le \Delta$, then $\Sigma^{(N)} \ne \emptyset$, and $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is compact.*

Define the function $S : \Sigma^{(N)} \to \mathbb{R}$ by

$$S(\Theta) = \sum_{n=-N}^{N-1} h(\theta_n, \theta_{n+1}),\ \ \Theta = (\theta_n)_{-N\le n\le N} \in \Sigma^{(N)}. \tag{2.5}$$

Since S is continuous and $\Sigma^{(N)}$ is compact, there exists a minimizer, i.e.,

$$S(\Theta^{(N)}) = \min_{\Theta\in\Sigma^{(N)}} S(\Theta) \tag{2.6}$$

for a suitable $\Theta^{(N)} = (\theta_n^{(N)})_{-N\le n\le N} \in \Sigma^{(N)}$, which henceforth we consider to be fixed. The bounds obtained in the next lemma are of central importance to the proof.

Lemma 2.2 *Suppose that $\overline{\alpha} \geq \underline{\alpha} > 0$. There exists a constant $\sigma_* = \sigma_*(\overline{\alpha}/\underline{\alpha}) \geq 1$ such that for all $N \in \mathbb{N}$,*

$$\sigma_*^{-1} \mathrm{e}^{\theta_n^{(N)} - \theta_{n-1}^{(N)}} \leq \mathrm{e}^{\theta_{n+1}^{(N)} - \theta_n^{(N)}} \leq \sigma_* \mathrm{e}^{\theta_n^{(N)} - \theta_{n-1}^{(N)}}, \quad -N+1 \leq n \leq N-1.$$

Proof. Write $\theta_n^{(N)} - \theta_{n-1}^{(N)} = L_1$ and $\theta_{n+1}^{(N)} - \theta_n^{(N)} = L_2$ for some positive constants $L_1, L_2 > 0$. Let $\tilde{\Theta} = (\tilde{\theta}_k)_{-N \leq k \leq N} = \{\theta_{-N}^{(N)}, \cdots, \theta_{n-1}^{(N)}, s, \theta_{n+1}^{(N)}, \cdots, \theta_N^{(N)}\}$, where $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_{n-1}^{(N)})$. Consequently, we can verify that $s - \theta_{n-1}^{(N)} = \frac{1}{2}(\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)}) = \theta_{n+1}^{(N)} - s \in [\delta, \Delta]$. Therefore $\tilde{\Theta} \in \Sigma^{(N)}$, together with (2.6), which leads to $S(\Theta^{(N)}) \leq S(\tilde{\Theta})$. By the definition of S, it follows that

$$h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \leq h(\theta_{n-1}^{(N)}, s) + h(s, \theta_{n+1}^{(N)}),$$

since all the other terms cancel. Therefore the assumption (2.1) leads to

$$\begin{aligned} \underline{\alpha}(\mathrm{e}^{L_1} + \mathrm{e}^{L_2}) &= \underline{\alpha}\mathrm{e}^{\theta_n^{(N)} - \theta_{n-1}^{(N)}} + \underline{\alpha}\mathrm{e}^{\theta_{n+1}^{(N)} - \theta_n^{(N)}} \\ &\leq \overline{\alpha}\mathrm{e}^{s - \theta_{n-1}^{(N)}} + \overline{\alpha}\mathrm{e}^{\theta_{n+1}^{(N)} - s} = 2\overline{\alpha}\mathrm{e}^{\frac{1}{2}(\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)})} \\ &= 2\overline{\alpha}\mathrm{e}^{\frac{1}{2}[(\theta_{n+1}^{(N)} - \theta_n^{(N)}) + (\theta_n^{(N)} - \theta_{n-1}^{(N)})]} = 2\overline{\alpha}\mathrm{e}^{\frac{1}{2}(L_1 + L_2)}. \end{aligned}$$

Let $q = \overline{\alpha}/\underline{\alpha}$, then we have $\mathrm{e}^{L_1} + \mathrm{e}^{L_2} - 2q\mathrm{e}^{\frac{1}{2}(L_1+L_2)} \leq 0$. Thus we obtain the inequality

$$\left(\mathrm{e}^{\frac{1}{2}(L_2 - L_1)}\right)^2 - 2q\mathrm{e}^{\frac{1}{2}(L_2 - L_1)} + 1 \leq 0.$$

By solving the quadratic equation corresponding to this inequality, it yields

$$(q - \sqrt{q^2 - 1}\,)^2 \mathrm{e}^{L_1} \leq \mathrm{e}^{L_2} \leq (q + \sqrt{q^2 - 1}\,)^2 \mathrm{e}^{L_1}.$$

Consequencely, we have the explicit expression

$$\sigma_* = (q + \sqrt{q^2 - 1}\,)^2, \quad q = \overline{\alpha}/\underline{\alpha}. \tag{2.7}$$

So the desired bound has been proved. □

For fixed $N \in \mathbb{N}$, define

$$\delta^{(N)} = \min_{-N \leq n \leq N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \text{ and } \Delta^{(N)} = \max_{-N \leq n \leq N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}). \tag{2.8}$$

Then $\delta \leq \delta^{(N)} \leq \Delta^{(N)} \leq \Delta$, since $\Theta^{(N)} = (\theta_n^{(N)})_{-N \leq n \leq N} \in \Sigma^{(N)}$.

Lemma 2.3 *Suppose that $\overline{\alpha} \geq \underline{\alpha} > 0$. There exists a constant $\sigma_{**} = \sigma_{**}(\overline{\alpha}/\underline{\alpha}) \geq 1$ such that for all $N \in \mathbb{N}$,*

$$\mathrm{e}^{\Delta^{(N)}} \leq \sigma_{**} \mathrm{e}^{2\delta^{(N)}}.$$

Proof. Take

$$\sigma_{**} = (q + \sqrt{q^2 + q\sigma_*}\,)^2. \tag{2.9}$$

Obviously, $\sigma_{**} \geq \sigma_* \geq 1$. If $\sigma_* \mathrm{e}^{2\delta^{(N)}} \geq \mathrm{e}^{\Delta^{(N)}}$, then it leads to $\mathrm{e}^{2\delta^{(N)}} \geq \sigma_*^{-1} \mathrm{e}^{\Delta^{(N)}} \geq \sigma_{**}^{-1} \mathrm{e}^{\Delta^{(N)}}$. Similarly, if $\Delta^{(N)} \leq 2\delta^{(N)}$, then $\mathrm{e}^{\Delta^{(N)}} \leq \mathrm{e}^{2\delta^{(N)}} \leq \sigma_{**} \mathrm{e}^{2\delta^{(N)}}$.

Now we consider the other cases where

$$\sigma_* \mathrm{e}^{2\delta^{(N)}} \leq \mathrm{e}^{\Delta^{(N)}} \text{ and } \Delta^{(N)} \geq 2\delta^{(N)}. \tag{2.10}$$

Let $-N \le m, n \le N-1$ be such that

$$\delta^{(N)} = \theta^{(N)}_{m+1} - \theta^{(N)}_m, \quad \Delta^{(N)} = \theta^{(N)}_{n+1} - \theta^{(N)}_n.$$

Without loss of generality, we can suppose that $m \le n$, since the argument is similar for $m \ge n$. If $n = m$, then $\Delta^{(N)} = \delta^{(N)}$, the conclusion holds; If $n = m+1$, then by Lemma 2.2, $\mathrm{e}^{\Delta^{(N)}} = \mathrm{e}^{\theta^{(N)}_{n+1} - \theta^{(N)}_n} \le \sigma_* \mathrm{e}^{\theta^{(N)}_n - \theta^{(N)}_{n-1}} = \sigma_* \mathrm{e}^{\delta^{(N)}} \le \sigma_{**} \mathrm{e}^{2\delta^{(N)}}$. In the following, we restrict ourselves to the case $n \ge m+2$. Now we put

$$\bar{\Theta} = (\bar{\theta}_k)_{-N \le k \le N} = \{\theta^{(N)}_{-N}, \cdots, \theta^{(N)}_m, \theta^{(N)}_{m+2} \cdots, \theta^{(N)}_n, s, \theta^{(N)}_{n+1}, \cdots, \theta^{(N)}_N\},$$

where $s = (\theta^{(N)}_{n+1} + \theta^{(N)}_n)/2$. That is, we remove $\theta^{(N)}_{m+1}$ from $\Theta^{(N)}$, then shift the block $(\theta^{(N)}_{n+2}, \cdots, \theta^{(N)}_n)$ one place to the left, and finally insert s for $\theta^{(N)}_n$.

First we check that $\theta^{(N)}_{m+2} - \theta^{(N)}_m \in [\delta, \Delta]$. On the one hand,

$$\theta^{(N)}_{m+2} - \theta^{(N)}_m = (\theta^{(N)}_{m+2} - \theta^{(N)}_{m+1}) + (\theta^{(N)}_{m+1} - \theta^{(N)}_m)$$

leads to $\theta^{(N)}_{m+2} - \theta^{(N)}_m \ge 2\delta^{(N)} \ge \delta$. One the other hand, by Lemma 2.2, we have

$$\mathrm{e}^{\theta^{(N)}_{m+2} - \theta^{(N)}_m} = \mathrm{e}^{\theta^{(N)}_{m+2} - \theta^{(N)}_{m+1}} \cdot \mathrm{e}^{\theta^{(N)}_{m+1} - \theta^{(N)}_m} \le \sigma^* \mathrm{e}^{2(\theta^{(N)}_{m+1} - \theta^{(N)}_m)} = \sigma^* \mathrm{e}^{2\delta^{(N)}} \le \mathrm{e}^{\Delta^{(N)}} \le \mathrm{e}^{\Delta},$$

which yields $\theta^{(N)}_{m+2} - \theta^{(N)}_m \le \Delta$.

Also, we can verify $s - \theta^{(N)}_n = \theta^{(N)}_{n+1} - s = (\theta^{(N)}_{n+1} - \theta^{(N)}_n)/2 \in [\delta, \Delta]$. In fact, $(\theta_{n+1} - \theta_n)/2 = \Delta^{(N)}/2 \le \Delta/2 \le \Delta$, and $\Delta^{(N)}/2 \ge \delta$ by (2.10). Therefore, we have shown that $\bar{\Theta} \in \Sigma^{(N)}$. Using (2.6), we have $S(\Theta^{(N)}) \le S(\bar{\Theta})$. In view of the definition of S, it follows that

$$\begin{aligned} h(\theta^{(N)}_m, \theta^{(N)}_{m+1}) + h(\theta^{(N)}_{m+1}, \theta^{(N)}_{m+2}) + h(\theta^{(N)}_n, \theta^{(N)}_{n+1}) \\ \le h(\theta^{(N)}_m, \theta^{(N)}_{m+2}) + h(\theta^{(N)}_n, s) + h(s, \theta^{(N)}_{n+1}), \end{aligned}$$

since all the other terms cancel. Since $h \ge 0$, by the assumption (2.1) we obtain that

$$\begin{aligned} \underline{\alpha} \mathrm{e}^{\Delta^{(N)}} &= \underline{\alpha} \mathrm{e}^{\theta^{(N)}_{n+1} - \theta^{(N)}_n} \le h(\theta^{(N)}_m, \theta^{(N)}_{m+2}) + h(\theta^{(N)}_n, s) + h(s, \theta^{(N)}_{n+1}) \\ &\le \overline{\alpha} \mathrm{e}^{\theta^{(N)}_{m+2} - \theta^{(N)}_m} + 2\overline{\alpha} \mathrm{e}^{(\theta^{(N)}_{n+1} - \theta^{(N)}_n)/2} \\ &\le \overline{\alpha} \mathrm{e}^{\theta^{(N)}_{m+2} - \theta^{(N)}_{m+1}} \cdot \mathrm{e}^{\theta^{(N)}_{m+1} - \theta^{(N)}_m} + 2\overline{\alpha} \mathrm{e}^{(\theta^{(N)}_{n+1} - \theta^{(N)}_n)/2} \\ &\le \overline{\alpha} \sigma_* \mathrm{e}^{2\delta^{(N)}} + 2\overline{\alpha} \mathrm{e}^{\Delta^{(N)}/2}, \end{aligned}$$

i.e., $\mathrm{e}^{\Delta^{(N)}} - 2q\mathrm{e}^{\Delta^{(N)}/2} - q\sigma_* \mathrm{e}^{2\delta^{(N)}} \le 0$, where $q = \overline{\alpha}/\underline{\alpha}$. By solving the quadratic equation corresponding to this inequality, we have

$$\begin{aligned} \mathrm{e}^{\frac{\Delta^{(N)}}{2}} &\le q + \sqrt{q^2 + q\sigma_* \mathrm{e}^{2\delta^{(N)}}} \\ &\le \left(q + \sqrt{q^2 + q\sigma_*}\right) \mathrm{e}^{\delta^{(N)}} = \sigma_{**}^{\frac{1}{2}} \mathrm{e}^{\delta^{(N)}}. \end{aligned}$$

Thus we end the proof. □

Corollary 2.1 *Suppose that $\overline{\alpha} \ge \underline{\alpha}$. If*

$$2\delta + \ln \sigma_{**} < \frac{A}{N} < \frac{1}{2}\Delta - \frac{1}{2} \ln \sigma_{**}, \tag{2.11}$$

then for all $N \in \mathbb{N}$ and $-N \le n \le N$,

$$\delta < \delta^{(N)} \le \theta^{(N)}_{n+1} - \theta^{(N)}_n \le \Delta^{(N)} < \Delta. \tag{2.12}$$

Proof. If $\Delta^{(N)} = \Delta$, then by Lemma 2.3,

$$e^{2A} = e^{\theta_N^{(N)} - \theta_{-N}^{(N)}} = \prod_{n=-N}^{n=N-1} e^{\theta_{n+1}^{(N)} - \theta_n^{(N)}} \geq e^{2N\delta^{(N)}} \geq \frac{e^{N\Delta^{(N)}}}{\sigma_{**}^N} = \frac{e^{N\Delta}}{\sigma_{**}^N}.$$

Therefore, it follows that $2A + N \ln \sigma_{**} \geq N\Delta$, which is contradiction with (2.11). Similarly, if $\delta = \delta^{(N)}$, then also

$$e^{2A} = e^{\theta_N^{(N)} - \theta_{-N}^{(N)}} = \prod_{n=-N}^{n=N-1} e^{\theta_{n+1}^{(N)} - \theta_n^{(N)}} \leq e^{2N\Delta^{(N)}} \leq \sigma_{**}^{2N} e^{4N\delta^{(N)}} = \sigma_{**}^{2N} e^{4N\delta}$$

yields $2A \leq 2N \ln \sigma_{**} + 4N\delta$, which contradicts (2.11). □

Now we are ready for the

Proof of Theorem 2.1. We define the constant σ_{**} as in Lemma 2.3, and suppose that $2\delta + \ln \sigma_{**} < \frac{A}{N} < \frac{1}{2}\Delta - \frac{1}{2} \ln \sigma_{**}$ is satisfied. For $N \in \mathbb{N}$, we take $A = A_N = (\delta + \frac{1}{4}\Delta + \frac{1}{4} \ln \sigma_{**})N$, which yields that $\delta \leq 2\delta + \ln \sigma_{**} < \frac{A_N}{N} < \frac{1}{2}\Delta - \frac{1}{2} \ln \sigma_{**} \leq \Delta$, since $\Delta > 4\delta + 3 \ln \sigma_{**}$ and $\sigma_{**} \geq 1$. From Corollary 2.1 we get $\delta < \theta_{n+1}^{(N)} - \theta_n^{(N)} < \Delta$, for all $N \in \mathbb{N}$ and $-N \leq n \leq N-1$. This means that the minimizer $S(\Theta^{(N)})$ of $S(\Theta)$ can not be obtained on the boundary of the compact set $\Sigma^{(N)}$. For fixed $N \in \mathbb{N}$, define

$$\Theta(\varepsilon) = (\theta_k(\varepsilon))_{-N \leq n \leq N} = \left(\theta_{-N}^{(N)}, \cdots, \theta_{n-1}^{(N)}, \theta_n^{(N)} + \varepsilon, \theta_{n+1}^{(N)}, \cdots, \theta_N^{(N)}\right).$$

Recalling

$$S(\Theta(\varepsilon)) = S(\Theta^{(N)}) = \min_{\Theta \in \Sigma^{(N)}} S(\Theta),$$

see (2.6), it follows from differentiating $S(\Theta(\varepsilon))$ with respect to ε that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} S(\Theta(\varepsilon))\Big|_{\varepsilon=0} = \partial_2 h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + \partial_1 h(\theta_n^{(N)}, \theta_{n+1}^{(N)}), \quad -N+1 \leq n \leq N-1.$$

The rest of the proof is the passage to the limit $N \to \infty$, which can now be copied verbatim from [7], and also we can refer to [8, 9].

3 Nonperiodic twist maps preserving the boundary

In this section we are going to apply Theorem 2.1 to twist symplectic maps with the associated generating function growing exponentially, which leave invariant the boundary of the strip where they are defined.

The main result of this section is the following.

Theorem 3.1 *Let $b > a > 1$. Suppose that $f : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ is a symplectic C^1-map given by*

$$\theta_1 = \theta + \ln r + F(\theta, r), \quad r_1 = r + G(\theta, r), \tag{3.1}$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, b] \to \mathbb{R}$, we assume that $F, G \in C^1$ preserve the boundary

$$F(\theta, a) = F(\theta, b) = G(\theta, a) = G(\theta, b) = 0, \quad \theta \in \mathbb{R}, \tag{3.2}$$

and satisfy that there exist constants γ_1, γ_2 such that

$$\frac{1}{2} < \gamma_1 \leq \partial_r \left(r \cdot e^{F(\theta, r)}\right) \leq \gamma_2 < +\infty, \quad \theta \in \mathbb{R}, \ r \in [a, b]. \tag{3.3}$$

If the constant

$$\gamma = \max\left\{|1 - \frac{1}{\gamma_1}|, |1 - \frac{1}{\gamma_2}|\right\} \tag{3.4}$$

also satisfies that

$$\ln b > 4\ln a + 6\ln(q + \sqrt{q^2 + q(q + \sqrt{q^2 - 1})^2}), \tag{3.5}$$

where $q = (1+\gamma)/(1-\gamma)$*, then there exists a complete orbit* $(\theta_n, r_n)_{n\in\mathbb{Z}}$ *for* f *such that* $|\theta_0| < \ln b$ *and* $\ln a \le \inf_{n\in\mathbb{Z}}(\theta_{n+1} - \theta_n) \le \sup_{n\in\mathbb{Z}}(\theta_{n+1} - \theta_n) \le \ln b$ *for* $n \in \mathbb{Z}$*. In addition,*

$$\ln a \le \liminf_{n\to\pm\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n^*}{n} \le \ln b. \tag{3.6}$$

Note that the assumption (3.3) not only ensures that the map f has twist on the strip $\mathbb{R}\times[a,b]$, but also guarantees that the perturbation F is small enough compared to the dominant term. The assumption in term of the constant γ_2 in (3.3) is reasonable, since F and the first derivative $\partial_r F$ is continuous on any bounded and closed subinterval $[a,b] \subset \mathbb{R}$. The proof of Theorem 3.1 is an application of Theorem 2.1, and the main difficulty is to estimate the size of the associated generating function. The following technical lemma will be used in the proof of Theorem 3.1.

Lemma 3.1 *Let* $\Delta > \delta > 0$*. Suppose that* $V : \Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\} \to \mathbb{R}$ *is* C^2 *and such that*

$$V(\theta, \theta+\delta) = \partial_{\theta_1} V(\theta, \theta+\delta) = 0, \ \ |\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \le \gamma \mathrm{e}^{\theta_1 - \theta},$$

for $\theta \in \mathbb{R}$ *and* $(\theta, \theta_1) \in \Omega$*, respectively. Then*

$$|V(\theta, \theta_1)| \le \gamma \mathrm{e}^{\theta_1 - \theta}, \ (\theta, \theta_1) \in \Omega. \tag{3.7}$$

Proof. Using the d'Alembert's formula, we have

$$\begin{aligned} -\int_{\theta+\delta}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta-\delta} \mathrm{d}\xi \partial^2_{\theta\theta_1} V(\xi, \eta) &= -\int_{\theta+\delta}^{\theta_1} \mathrm{d}\eta (\partial_{\theta_1} V(\eta - \delta, \eta) - \partial_{\theta_1} V(\theta, \eta)) \\ &= \int_{\theta+\delta}^{\theta_1} \mathrm{d}\eta \partial_{\theta_1} V(\theta, \eta) \\ &= V(\theta, \theta_1) - V(\theta, \theta+\delta) = V(\theta, \theta_1). \end{aligned}$$

Therefore, by a direct computation, we have the estimate

$$\begin{aligned} |V(\theta, \theta_1)| &\le \int_{\theta+\delta}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta-\delta} \mathrm{d}\xi |\partial^2_{\theta\theta_1} V(\xi, \eta)| \\ &\le \int_{\theta+\delta}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta-\delta} \mathrm{d}\xi \gamma \mathrm{e}^{\eta - \xi} \\ &= \gamma \left[\mathrm{e}^{\theta_1 - \theta} - \mathrm{e}^{\delta}(\theta_1 - \theta) + \mathrm{e}^{\delta}(\delta - 1)\right] \le \gamma \mathrm{e}^{\theta_1 - \theta}, \end{aligned}$$

where the last inequality follows from the formula $\mathrm{e}^{\delta}(\theta_1 - \theta) - \mathrm{e}^{\delta}(\delta - 1) > 0$ for $\theta_1 - \theta \in [\delta, \Delta]$. □

Proof of Theorem 3.1. At first, it is standard to construct the generating function h for f, see [1, 6]. The assumption (3.3), together with (3.1), implies that

$$\partial_r \theta_1 = \frac{1}{r} + \partial_r F(\theta, r) > 0.$$

Hence, by the implicit function theorem and (3.2), the function $[\theta + \ln a, \theta + \ln b] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [a, b]$ is well-defined, strictly increasing and onto. Since f is a symplectic C^1-map, there exists a C^2-function $\hat{h} : \mathbb{R} \times [a, b] \to \mathbb{R}$ such that

$$r_1 \mathrm{d}\theta_1 - r\mathrm{d}\theta = \mathrm{d}\hat{h}, \quad \text{where } (\theta_1, r_1) = f(\theta, r).$$

Define the function

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1)) + a - \hat{h}(0, a)$$

and

$$V(\theta, \theta_1) = h(\theta, \theta_1) - \mathrm{e}^{\theta_1 - \theta}, \quad (\theta, \theta_1) \in \Omega_0, \tag{3.8}$$

where $\Omega_0 := \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \theta + \ln a \le \theta_1 \le \theta + \ln b\}$. Then we have

$$\partial_\theta h = -r, \ \partial_{\theta_1} h = r_1, \ \partial_\theta V = -r(\theta, \theta_1) + \mathrm{e}^{\theta_1 - \theta}, \ \partial_{\theta_1} V = r(\theta, \theta_1) + G - \mathrm{e}^{\theta_1 - \theta}. \tag{3.9}$$

Note that $\theta_1(\theta, a) = \theta + \ln a$ and $\theta_1(\theta, b) = \theta + \ln b$ by (3.2). In particular, $r(0, \ln a) = a$ yields $V(0, \ln a) = h(0, \ln a) - a = (\hat{h}(0, a) + a - \hat{h}(0, a)) - a = 0$. Also from (3.9), we can verify that

$$\partial_\theta V(\theta, \theta + \ln a) = \partial_{\theta_1} V(\theta, \theta + \ln a) = 0, \ \theta \in \mathbb{R}.$$

Note that

$$\begin{aligned} V(\theta, \theta_1) &= V(0, \ln a) + \int_\Gamma \mathrm{d}V \\ &= \int_\Gamma \left(-r(\theta, \theta_1) + \mathrm{e}^{\theta_1 - \theta}\right) \mathrm{d}\theta + \left(r(\theta, \theta_1) + G - \mathrm{e}^{\theta_1 - \theta}\right) \mathrm{d}\theta_1 \end{aligned}$$

for any path Γ in the simply connected Ω_0 which connects $(0, \ln a)$ to (θ, θ_1). While taking Γ to be part of the lower boundary $\{\theta_1 = \theta + \ln a : \theta \in \mathbb{R}\}$ of Ω_0, we have $V(\theta, \theta + \ln a) = 0$ for all $\theta \in \mathbb{R}$. On the other hand, we have

$$\partial^2_{\theta\theta_1} V(\theta, \theta_1) = -\frac{1}{\frac{1}{r} + \partial_r F} + r\mathrm{e}^F = r\mathrm{e}^F \left(1 - \frac{1}{r\mathrm{e}^F(\frac{1}{r} + \partial_r F)}\right).$$

The assumption (3.3) implies that

$$\frac{1}{\gamma_2} \le \frac{1}{r\mathrm{e}^F(\frac{1}{r} + \partial_r F)} \le \frac{1}{\gamma_1},$$

which leads to

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \le \max\{|1 - \gamma_1^{-1}|, |1 - \gamma_2^{-1}|\} r\mathrm{e}^F = \gamma \mathrm{e}^{\theta_1 - \theta}.$$

By Lemma 3.1, taking $\delta = \ln a$, we have

$$|V(\theta, \theta_1)| \le \gamma \mathrm{e}^{\theta_1 - \theta}, \ (\theta, \theta_1) \in \Omega.$$

Thus from (3.8) we obtain that

$$(1 - \gamma)\mathrm{e}^{\theta_1 - \theta} \le h(\theta, \theta_1) \le (1 + \gamma)\gamma \mathrm{e}^{\theta_1 - \theta}, \ (\theta, \theta_1) \in \Omega.$$

Let

$$\underline{\alpha} = 1 - \gamma, \ \overline{\alpha} = 1 + \gamma, \ q = \frac{1 + \gamma}{1 - \gamma}, \ \delta = \ln a, \ \Delta = \ln b.$$

Taking σ_{**} as defined in Lemma 2.3, in view of (3.5) and (2.7), we have that

$$\Delta > 4\delta + 3\ln\sigma_{**}.$$

Then $\Delta > 4\delta > 0$ and

$$\Omega = \{(\theta, \theta') \in \mathbb{R}^2 : \delta \le \theta' - \theta \le \Delta\} \subseteq \Omega_0.$$

By Theorem 2.1, there exists $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0| \le \Delta, \delta \le \theta_{n+1} - \theta_n \le \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0,\ n \in \mathbb{Z}. \tag{3.10}$$

Moreover,

$$\delta \le \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} \le \Delta.$$

From (3.10),

$$\begin{aligned} r_{n+1} &= -\partial_\theta h(\theta_{n+1}, \theta_{n+2}) = \partial_{\theta_1} h(\theta_n, \theta_{n+1}) \\ &= \partial_{\theta_1} V(\theta_n, \theta_{n+1}) + \mathrm{e}^{\theta_1 - \theta} = r(\theta_n, \theta_{n+1}) + G(\theta_n, r_n) = r_n + G(\theta_n, r_n). \end{aligned}$$

Also $r_n = -\partial_\theta h(\theta_n, \theta_{n+1}) = r(\theta_n, \theta_{n+1})$ is equivalent to

$$\theta_n + \ln r_n + F(\theta_n, r_n) = \theta_1(\theta_n, r_n) = \theta_{n+1}.$$

Since $\theta_n + \ln a \le \theta_{n+1} \le \theta_n + \ln b$, we have $r_n = r(\theta_n, \theta_{n+1}) \in [a, b]$. Hence $(\theta_n, r_n)_{n\in\mathbb{Z}} \in [a, b] \times \mathbb{R}$ is a complete orbit for the map of f. This completes the proof.

4 Infinitely many complete orbits

In this section we will consider the map defined on the half plane, where the variable r extends to infinity, and infinitely many complete orbits will be found. While the assumption (3.3) on the invariance of the boundaries is lost, a stronger and more qualitative hypothesis on the generating function h is needed, such as, f is assumed to be c-exact symplectic and satisfies (4.2) below.

Theorem 4.1 *Suppose that $f : \mathbb{R} \times [a, +\infty) \to \mathbb{R}^2$ is c-exact symplectic with some continuous function $c(r)$. Let f be given by*

$$\theta_1 = \theta + \ln r + F(\theta, r), \quad r_1 = r + G(\theta, r), \tag{4.1}$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, b] \to \mathbb{R}$, we assume that $F, G \in C^1$ and

$$\|F\|_\infty < +\infty,\ \ \|G\|_\infty < +\infty,\ \ \textit{and}\ \ \frac{1}{2} < \gamma_1 \le \partial_r \left(r \cdot \mathrm{e}^{F(\theta,r)}\right) \le \gamma_2 < +\infty, \tag{4.2}$$

for $\theta \in \mathbb{R}$, some positive constants γ_1, γ_2 and $r \in [a, +\infty)$. Then f has infinitely many complete orbits $(\theta_n^j, r_n^j)_{n\in\mathbb{Z}}$ such that $R_j \le r_n^j \le R_{j+1}$ for $n \in \mathbb{Z}$ and $j \in \mathbb{N}$, where $R_j < R_{j+1} \to +\infty$ as $j \to +\infty$. Moreover,

$$R_j \le \liminf_{n\to\pm\infty} \frac{\theta_n^j}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n^j}{n} \le R_{j+1},\quad j \in \mathbb{N}. \tag{4.3}$$

Proof. Let

$$M = \sup_{\theta_0\in\mathbb{R}} \sup_{r\in[a,+\infty)} \min_{\theta\in[\theta_0,\theta_0+T]} |\hat{h}(\theta, r) - c(r)| \tag{4.4}$$

and

$$m = \max\{\|F\|_\infty, \|G\|_\infty\} < \infty,$$

where $\hat{h}$ is given by Definition 1.2. Fix $0 < \epsilon < 1-\gamma$ and $a_1 > \max\{a, \frac{1}{1-\gamma-\epsilon}\}$, and define the constant

$$\sigma_{**}(q) = \left(q + \sqrt{q^2 + q(q + \sqrt{q^2-1}\,)^2}\right)^2 > 1$$

with $q = (1-\gamma-\epsilon-a_1^{-1})^{-1}(1+\gamma+\epsilon+a_1^{-1}) > 1$ and $0 \le \gamma < 1$ defined in Theorem 3.1. Then choose b_1 large enough such that

$$\ln b_1 > 4\ln a_1 + 3\ln\sigma_{**}(q), \quad b_1 \mathrm{e}^{-m} > a_1 \mathrm{e}^{2m}, \tag{4.5}$$

$$43^2 \cdot \frac{\gamma_2 m(c_1\mathrm{e}^m + m)\mathrm{e}^{3m} + M + c_1(\mathrm{e}^m+1) + m(\ln b_1 c_1 + m)}{(\ln b_1)^2} + 21 \cdot \frac{m}{\ln b_1} < \frac{1}{2}$$

and

$$43^2 \cdot \frac{\mathrm{e}^m+1}{(\ln b_1)^2} + 42 \cdot \frac{\mathrm{e}^m+1}{\ln b_1} \le \epsilon, \tag{4.6}$$

where $c_1 = \mathrm{e}^{2m}a_1$.

Similar to the proof of Theorem 2.1, we consider the strictly increasing maps $[a_1, b_1] \ni r \mapsto \theta_1(\theta, r) = \theta + \ln r + F(\theta, r) \in \mathbb{R}$ and their inverses

$$[\theta_1(\theta, a_1), \theta_1(\theta, b_1)] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [\mathrm{e}^m a_1, \mathrm{e}^{-m} b_1].$$

Moreover, we have

$$\ln a_1 \le \ln(\mathrm{e}^m a_1) + F(\theta, r) \le \theta_1 - \theta \le \mathrm{e}^{-m} b_1 + F(\theta, r) \le \ln b_1.$$

Define the function

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1))$$

and

$$V(\theta, \theta_1) = h(\theta, \theta_1) - \mathrm{e}^{\theta_1-\theta}, \quad (\theta, \theta_1) \in \Omega_0, \tag{4.7}$$

where $\Omega_0 := \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \theta + \ln a_1 \le \theta_1 \le \theta + \ln b_1\}$ and $\hat{h}$ is taken from Definition 1.2. Then we have

$$\partial_\theta h = -r, \ \partial_{\theta_1} h = r_1, \ \partial_\theta V = -r(\theta, \theta_1) + r(\theta, \theta_1)\mathrm{e}^F, \ \partial_{\theta_1} V = r(\theta, \theta_1) + G - r(\theta, \theta_1)\mathrm{e}^F. \tag{4.8}$$

Claim 1: For any $(\theta, \theta_1) \in \Omega_0$, we have the following rough estimate

$$|V(\theta, \theta_1)| \le \gamma_2 m(c_1\mathrm{e}^m + m)\mathrm{e}^{3m} + M + (\mathrm{e}^m+1)(\mathrm{e}^{\theta_1-\theta} + c_1) + m(\ln b_1 c_1 + m).$$

Consider a fixed $(\theta, \theta_1) \in \Omega_0$. As a consequence of (4.4), for $\theta \in \mathbb{R}$ and $\ln c_1 = \ln a_1 + 2m$ ($c_1 = \mathrm{e}^{2m}a_1 \in [\mathrm{e}^m a_1, \mathrm{e}^{-m} b_1]$), there is $\vartheta \in [\theta, \theta + T]$ such that $|\hat{h}(\vartheta, c_1) - \mathrm{e}^{c_1}| \le M$. Also $\vartheta + \ln c_1 =$

$\vartheta + \ln r + F(\vartheta, r)$ for $r = r(\vartheta, \vartheta + \ln c_1)$ leads to $|\ln r(\vartheta, \vartheta + \ln c_1) - \ln c_1| \le \|F\|_\infty < m$, then $c_1 \mathrm{e}^{-m} \le r(\vartheta, \vartheta + \ln c_1) \le c_1 \mathrm{e}^m$. From (4.2), we have that

$$|\partial_r \hat{h}(\vartheta, r)| = |r_1 \partial_r \theta_1| = \left|(r + G)(\frac{1}{r} + \partial_r F)\right| \le \gamma_2 (c_1 \mathrm{e}^m + m)\mathrm{e}^{2m}/c_1,$$

for $|\ln r - \ln c_1| \le m$. Then it follows that

$$\begin{aligned}
|V(\vartheta, \vartheta + \ln c_1)| &= |\hat{h}(\vartheta, r(\vartheta, \vartheta + \ln c_1)) - \mathrm{e}^{c_1}| \\
&\le |\hat{h}(\vartheta, r(\vartheta, \vartheta + \ln c_1)) - \hat{h}(\vartheta, c_1)| + |\hat{h}(\vartheta, c_1) - \mathrm{e}^{c_1}| \\
&\le \left| \int_{c_1}^{r(\vartheta, \vartheta + \ln c_1)} \partial_r \hat{h}(\vartheta, r) \mathrm{d}r \right| + M \\
&= \left| \int_{\ln c_1}^{\ln r(\vartheta, \vartheta + \ln c_1)} \partial_r \hat{h}(\vartheta, r) \mathrm{e}^\rho \mathrm{d}\rho \right| + M \ (\text{let } r = \mathrm{e}^\rho) \\
&\le \gamma_2 m (c_1 \mathrm{e}^m + m)\mathrm{e}^{3m} + M.
\end{aligned}$$

According to (4.7),

$$\begin{aligned}
V(\theta, \theta_1) &= V(\vartheta, \vartheta + \ln c_1) + \int_\Gamma \mathrm{d}V \\
&= V(\vartheta, \vartheta + \ln c_1) + \int_\Gamma \left(-r + r\mathrm{e}^F\right) \mathrm{d}\theta + \left(r + G - r\mathrm{e}^F\right) \mathrm{d}\theta_1
\end{aligned}$$

for the path Γ connecting first (θ, θ_1) along the vertical line $\Gamma_1 : \{\vartheta = \theta\}$ to the point $(\theta, \theta + \ln c_1)$, thereafter connecting $(\theta, \theta + \ln c_1)$ to $(\vartheta, \vartheta + \ln c_1)$ along the straight line $\Gamma_2 : \{\vartheta_1 - \vartheta = \ln c_1\}$. Therefore, we have

$$\begin{aligned}
|V(\theta, \theta_1)| &= |V(\vartheta, \vartheta + \ln c_1)| + \left| \int_\Gamma \mathrm{d}V \right| \\
&\le |V(\vartheta, \vartheta + \ln c_1)| + \left| \int_{\theta + \ln c_1}^{\theta_1} \left(r(\theta, \theta_1) + G - r(\theta, \theta_1)\mathrm{e}^F\right) \mathrm{d}\theta_1 \right| + \left| \int_\vartheta^\theta G \mathrm{d}\theta \right| \\
&= |V(\vartheta, \vartheta + \ln c_1)| + \left| \int_{\theta + \ln c_1}^{\theta_1} \left(\mathrm{e}^{\theta_1 - \theta} \cdot \mathrm{e}^{-F} + G - \mathrm{e}^{\theta_1 - \theta}\right) \mathrm{d}\theta_1 \right| + \left| \int_\vartheta^\theta G \mathrm{d}\theta \right| \\
&\le \gamma_2 m (c_1 \mathrm{e}^m + m)\mathrm{e}^{3m} + M + \int_{\theta + \ln c_1}^{\theta_1} \left[(1 + \mathrm{e}^m)\mathrm{e}^{\theta_1 - \theta} + m\right] \mathrm{d}\theta_1 + Tm \\
&\le \gamma_2 m (c_1 \mathrm{e}^m + m)\mathrm{e}^{3m} + M + (\mathrm{e}^m + 1)(\mathrm{e}^{\theta_1 - \theta} + c_1) + m(\ln b_1 c_1 + m).
\end{aligned}$$

Claim 2: For any $(\theta, \theta_1) \in \Omega_0$, we have the following estimates for the derivatives of V

$$|\partial_\theta V(\theta, \theta_1) - \partial_{\theta_1} V(\theta, \theta_1)| \le 2(1 + \mathrm{e}^m)\mathrm{e}^{\theta_1 - \theta} + m, \tag{4.9}$$

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \le \gamma \mathrm{e}^{\theta_1 - \theta}, \tag{4.10}$$

where the constant γ is defined in Theorem 3.1.

In fact, in view of (4.8), we have

$$\begin{aligned}
|\partial_\theta V(\theta, \theta_1) - \partial_{\theta_1} V(\theta, \theta_1)| &= |2r(\theta, \theta_1) + G - 2r(\theta, \theta_1)\mathrm{e}^F| \\
&\le 2(1 + \mathrm{e}^m)\mathrm{e}^{\theta_1 - \theta} + m,
\end{aligned}$$

which yields (4.9). The inequality (4.10) has been proved in the proof of Theorem 3.1.

Let $\tilde{a}_1 = a_1 b_1^{1/20}, \tilde{b}_1 = b_1^{19/20}$ and $\widetilde{\Omega}_0 := \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \theta + \ln \tilde{a}_1 \le \theta_1 \le \theta + \ln \tilde{b}_1\}$. Obviously, $\widetilde{\Omega}_0 \subset \Omega_0$. Next we modify the function V and fix a cut-off function $\chi \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that $\chi(s) = 1$ for $s \in [\ln \tilde{a}_1, \ln \tilde{b}_1]$, $\chi(s) = 0$ for $s \le \ln a_1$ or $s \ge \ln b_1$, as well as

$$|\chi(s)| \le 1, \ \ |\chi'(s)| \le \frac{21}{\eta}, \ \text{ and } \ |\chi''(s)| \le \frac{43}{\eta^2}, \ s \in \mathbb{R}$$

are satisfied, where $\eta = \ln b_1$. Let $\tilde{V}(\theta, \theta_1) = \chi(\theta_1 - \theta) V(\theta, \theta_1)$ for $(\theta, \theta_1) \in \Omega_0$.

Claim 3: For any $(\theta, \theta_1) \in \Omega_0$, we have the following estimates for the derivatives of $\tilde{V}$

$$|\partial^2_{\theta\theta_1} \tilde{V}(\theta, \theta_1)| \le \frac{1}{2} + (\gamma + \epsilon) \mathrm{e}^{\theta_1 - \theta}. \tag{4.11}$$

Moreover,

$$\tilde{V}(\theta, \theta + \ln a_1) = 0, \ \ \partial_{\theta_1} \tilde{V}(\theta, \theta + \ln a_1) = 0, \ \text{ for } \theta \in \mathbb{R}. \tag{4.12}$$

To prove Claim 3, firstly we compute the second order derivative of $\tilde{V}$

$$\begin{aligned} \partial^2_{\theta\theta_1} \tilde{V}(\theta, \theta_1) = &-\chi''(\theta_1 - \theta) V(\theta, \theta_1) + \chi'(\theta_1 - \theta) \left[\partial_\theta V(\theta, \theta_1) - \partial_{\theta_1} V(\theta, \theta_1)\right] \\ &+ \chi(\theta_1 - \theta) \partial^2_{\theta\theta_1} V(\theta, \theta_1) \end{aligned}$$

By Claim 1 and Claim 2, we have

$$\begin{aligned} |\partial^2_{\theta\theta_1} \tilde{V}(\theta, \theta_1)| &\le \frac{1}{\eta^2} |V(\theta, \theta_1)| + \frac{1}{\eta} |\partial_\theta V(\theta, \theta_1) - \partial_{\theta_1} V(\theta, \theta_1)| + |\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \\ &\le \frac{1}{2} + (\gamma + \epsilon) \mathrm{e}^{\theta_1 - \theta}. \end{aligned}$$

Obivously, $\tilde{V}(\theta, \theta + \ln a_1) = \chi[(\theta + \ln a_1) - \theta] V(\theta, \theta + \ln a_1) = 0$. Since $\chi(s) \in C^\infty$ has a minimum at $s = \ln a_1$, then $\chi'(\ln a_1) = 0$. Together with the equality $\partial_{\theta_1} \tilde{V}(\theta, \theta_1) = \chi'(\theta_1 - \theta) V(\theta_1 - \theta) + \chi(\theta_1 - \theta) \partial_{\theta_1} V(\theta_1 - \theta)$, we have obtained the second inequality of (4.12).

Claim 4: For any $(\theta, \theta_1) \in \Omega_0$, we have the following estimates for $\tilde{V}$

$$|\tilde{V}(\theta, \theta_1)| \le 1 + (\gamma + \epsilon) \mathrm{e}^{\theta_1 - \theta}. \tag{4.13}$$

Similar to the proof of Lemma 3.1, using the d'Alembert's formula with (4.12), we have

$$\begin{aligned} &\int_{\theta + \ln a_1}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta - \ln a_1} \mathrm{d}\xi \left[-\partial^2_{\theta\theta_1} \tilde{V}(\xi, \eta)\right] \\ =& -\int_{\theta + \ln a_1}^{\theta_1} \mathrm{d}\eta (\partial_{\theta_1} \tilde{V}(\eta - \ln a_1, \eta) - \partial_{\theta_1} \tilde{V}(\theta, \eta)) \\ =& \int_{\theta + \ln a_1}^{\theta_1} \mathrm{d}\eta \partial_{\theta_1} \tilde{V}(\theta, \eta) \\ =& V(\theta, \theta_1) - \tilde{V}(\theta, \theta + \ln a_1) = \tilde{V}(\theta, \theta_1). \end{aligned}$$

Therefore we have the estimate

$$\begin{aligned}|\tilde{V}(\theta,\theta_1)| &\le \int_{\theta+\ln a_1}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta-\ln a_1} \mathrm{d}\xi |\partial^2_{\theta\theta_1}\tilde{V}(\xi,\eta)| \\ &\le \int_{\theta+\ln a_1}^{\theta_1} \mathrm{d}\eta \int_{\theta}^{\eta-\ln a_1} \mathrm{d}\xi \left[\frac{1}{2}+(\gamma+\epsilon)\mathrm{e}^{\eta-\xi}\right] \\ &\le \frac{(\theta_1-\theta)^2}{(\ln b_1)^2} + (\gamma+\epsilon)\mathrm{e}^{\theta_1-\theta} = 1+(\gamma+\epsilon)\mathrm{e}^{\theta_1-\theta}.\end{aligned}$$

Now defining the function $\tilde{h}(\theta,\theta_1) = \mathrm{e}^{\theta_1-\theta} + \tilde{V}(\theta,\theta_1)$, it follows that

$$\begin{aligned}(1-\gamma-\epsilon-a_1^{-1})\mathrm{e}^{\theta_1-\theta} &\le (1-\gamma-\epsilon)\mathrm{e}^{\theta_1-\theta} - \frac{(\theta_1-\theta)^2}{(\ln b_1)^2} \le \tilde{h}(\theta,\theta_1) \\ &\le (1+\gamma+\epsilon)\mathrm{e}^{\theta_1-\theta} + \frac{(\theta_1-\theta)^2}{(\ln b_1)^2} \le (1+\gamma+\epsilon+a_1^{-1})\mathrm{e}^{\theta_1-\theta}.\end{aligned}$$

Note that $\tilde{h} = h$ on $\tilde{\Omega}_0$ by the definition of χ. Take $\underline{\alpha} = 1-\gamma-\epsilon-a_1^{-1}, \overline{\alpha} = 1+\gamma+\epsilon+a_1^{-1}$. Let $q = \overline{\alpha}/\underline{\alpha}$ and select $\sigma_{**}(q)$ as in (2.9). Consequently we can use Theorem 2.1 for h on $\tilde{\Omega}_0$ to get a sequence $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0| \le \ln \tilde{b}_1$, $\ln \tilde{a}_1 \le \theta_{n+1}-\theta_n \le \ln \tilde{b}_1$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1},\theta_n) + \partial_1 h(\theta_n,\theta_{n+1}) = 0,\ n \in \mathbb{Z}.$$

Moreover, $\ln \tilde{a}_1 \le \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} \le \ln \tilde{b}_1$. If we put $r_n = -\partial_\theta h(\theta_n,\theta_{n+1})$, then $(\theta_n, r_n)_{n\in\mathbb{Z}}$ is a complete orbit for the map f.

Write $R_1 = \ln \tilde{a}_1, R_2 = \ln \tilde{b}_1$. In the next step we replace $R_1 = \ln \tilde{a}_1$ by $R_1 = \ln \tilde{b}_1$ in the preceding argument and select a new $\tilde{b}_1$. Continuing this way, we get infinite many complete orbits for f which have the desired properties. □

In the following, we will give an application of our results for the twist symplectic map in a special case.

Theorem 4.2 *Let $f : \mathbb{R}\times[\varrho,+\infty) \to \mathbb{R}^2$ be given by*

$$\theta_1 = \theta + \ln r + \ln\left(1+\frac{1}{r}\Phi'(\theta)\right), \quad r_1 = r + \Phi'(\theta), \tag{4.14}$$

where $(\theta_1, r_1) = f(\theta, r)$, $\varrho > 1$ is a positive constant and $\Phi(\theta) \in C^2(\mathbb{R},\mathbb{R})$ satisfies that

$$\|\Phi(\theta)\|_\infty < +\infty, \ \ \|\Phi'(\theta)\|_\infty \le \varrho < +\infty.$$

Then f has infinitely many complete orbits $(\theta_n^j, r_n^j)_{n\in\mathbb{Z}}$ such that $R_j \le r_n^j \le R_{j+1}$ for $n \in \mathbb{Z}$ and $j \in \mathbb{N}$, where $R_j < R_{j+1} \to +\infty$ as $j \to +\infty$. Moreover,

$$R_j \le \liminf_{n\to\pm\infty} \frac{\theta_n^j}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n^j}{n} \le R_{j+1}, \quad j \in \mathbb{N}. \tag{4.15}$$

Proof. First the condition $\|\Phi'(\theta)\|_\infty \le \varrho$ guarantees that the map f is well defined. Let

$$\hat{h}(\theta, r) = r + \Phi(\theta) + \Phi'(\theta).$$

We can verify that $r_1 \mathrm{d}\theta_1 - r\mathrm{d}\theta = \mathrm{d}\hat{h}$, for $(\theta_1, r_1) = f(\theta, r)$. Choose $c(r) = r$ so that $|\hat{h}(\theta, r) - c(r)| < A$, for $r \in [\underline{\varrho}, +\infty)$ and a fixed constant A. Therefore, f is a c-exact symplectic map with $c(r) = r$. By the definition of the map,

$$\partial_r(r \cdot \mathrm{e}^{F(\theta,r)}) = \partial_r(r + \Phi'(\theta)) = 1.$$

Now the conditions of Theorem 4.1 holds, and we complete the proof. □

In Theorem 4.2, we make no hypothesis on periodicity, quasi-periodicity or almost periodicity for $\Phi'(\theta)$. However, this has to be compensated by a boundedness assumption on Φ. This assumption is necessary and indispensable, otherwise the conclusion of the theorem does not hold. For example, when $\Phi'(\theta)$ has constant sign, every orbit of the map f is unbounded.

5 The variational principle: the ground states of infinite lattices

One central class of systems (1.5) is the model of a chain of interacting oscillators with the Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \left(\frac{p_n^2}{2} + V(\theta_n) + W(\theta_n - \theta_{n-1}) \right), \tag{5.1}$$

where p_n and θ_n are pairs of canonically conjugate momenta and displacements of particles at lattice site n, W denotes the nearest neighbour interaction and the oscillator potential V is C^1. In case of W being the quadratic function $W(r) = \frac{1}{2}r^2$, (5.1) is the Hamiltonian of the Frenkel-Kontorowa (FK) model. The global variational techniques of studying the ground states for FK model in condensed matter physics was first presented by Aubry [2, 3], which is surprisingly relevant to twist maps. See also [10, 4].

In this section, W is assumed to be a nonnegative function $W(r) = \mathrm{e}^r - 1$. When the oscillator potential vanishes, (5.1) presents the standard Toda lattice [12]. To find the equilibrium positions of the bi-infinite lattices, we can look for critical points of the total potential energy

$$\Phi((\theta_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} (V(\theta_n) + W(\theta_n - \theta_{n-1})). \tag{5.2}$$

It is straightforward to verify that $\frac{\partial\Phi}{\partial\theta_n} = 0$ leads to that

$$\mathrm{e}^{\theta_{n+1}-\theta_n} - \mathrm{e}^{\theta_n-\theta_{n-1}} = V'(\theta_n). \tag{5.3}$$

If $\theta = \theta_0$ is a critical point of V, we will see that $(\theta_i)_{i\in\mathbb{Z}}$ with $\theta_i = \theta_0$ is a complete solution of (5.3). However, while V' has constant sign, it is not obvious.

Now let us assume that $V \in C^2(\mathbb{R})$ and define the function

$$h(\theta, \theta_1) = \mathrm{e}^{\theta_1-\theta} + V(\theta).$$

Then $h_{\theta\theta_1}(\theta, \theta_1) = -\mathrm{e}^{\theta_1-\theta} < 0$, for $(\theta, \theta_1) \in \mathbb{R}^2$. Thus $h(\theta, \theta_1)$ can be interpreted as the generating function of a symplectic twist map. The associated map is defined by

$$r = -\partial_\theta h(\theta, \theta_1), \quad r_1 = \partial_{\theta_1} h(\theta, \theta_1)$$

or

$$f : \Omega = \{(\theta, r) : \theta \in \mathbb{R}, r > 0\} \to \mathbb{R}^2 : \ \theta_1 = \theta + \ln r, \ \ r_1 = r + V'(\theta).$$

Therefore, we conclude that, given a "critical point" $(\theta_n^*)_{n\in\mathbb{Z}}$ of Φ, the sequence $(\theta_n^*, r_n^*)_{n\in\mathbb{Z}}$ with $r_{n+1}^* = r_n^* + V'(\theta_n^*)$ is an orbit of the map f.

As an application of Theorem 2.1, we will prove the existence of equilibria, namely, the ground states for the chain of infinite lattices.

Theorem 5.1 *Assume that the oscillator potential V is bounded and of class C^1, then the total potential energy $\Phi((\theta_n)_{n\in\mathbb{Z}})$ has infinitely many critical points. That is,*

$$e^{\theta_{n+1}-\theta_n} - e^{\theta_n-\theta_{n-1}} = V'(\theta_n), \quad n \in \mathbb{Z}$$

has infinitely many complete solutions $(\theta_{n,k})_{n\in\mathbb{Z}}$ with $k = 1, 2, \ldots$. Moreover, the upper and lower rotation numbers

$$\underline{\omega}_k \le \liminf_{n\to\pm\infty} \frac{\theta_{n,k}}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_{n,k}}{n} \le \overline{\omega}_k$$

satisfy $\underline{\omega}_1 \le \overline{\omega}_1 \le \underline{\omega}_2 \le \overline{\omega}_2 \le \cdots \le \underline{\omega}_k \le \overline{\omega}_k \le \cdots$, for each $k \in \mathbb{N}$ and $\underline{\omega}_k \to +\infty$ as $k \to +\infty$.

Proof. Let

$$\delta_0 = \max\{\ln(100 \cdot \|V\|_\infty), \ln 100\}.$$

Define the fucntion $h : \Omega := \{(\theta, \theta_1) \in \mathbb{R}^2 : \ \delta_0 \le \theta_1 - \theta < +\infty\} \to \mathbb{R}^2$ by

$$h(\theta, \theta_1) = e^{\theta_1-\theta} + V(\theta). \tag{5.4}$$

If $(\theta, \theta_1) \in \Omega$, then

$$h(\theta, \theta_1) \le e^{\theta_1-\theta} + \frac{\|V\|_\infty}{e^{\delta_0}} e^{\theta_1-\theta} \le \frac{101}{100} e^{\theta_1-\theta}.$$

A similar lower estimate can be obtained by

$$\frac{99}{100} e^{\theta_1-\theta} \le e^{\theta_1-\theta} - \frac{\|V\|_\infty}{e^{\delta_0}} e^{\theta_1-\theta} \le h(\theta, \theta_1).$$

Take $\underline{\alpha} = \frac{99}{100}, \overline{\alpha} = \frac{101}{100}$. Let $q = \overline{\alpha}/\underline{\alpha} = \frac{101}{99}$. From (2.7) and (2.9), we select $\sigma_{**}(q) = 6.8736$. Now fix sequences (δ_j) and (Δ_j) such that $\delta_1 = \delta_0, \Delta_1 = 4\delta_1 + 4\ln\sigma_{**}(q), \cdots, \delta_k = \Delta_{k-1}, \Delta_k = 4\delta_k + 4\ln\sigma_{**}(q)$, for $k \ge 2$ and $k \in \mathbb{N}$. Then Theorem 2.1 can be used for $h : \Omega_k \to \mathbb{R}$ and $k \in \mathbb{N}$, where $\Omega_k := \{(\theta, \theta_1) \in \mathbb{R}^2 : \ \delta_k \le \theta_1 - \theta \le \Delta_k\}$. Therefore by Theorem 2.1, for every $k \in \mathbb{N}$, there is a sequences $(\theta_{n,k})_{n\in\mathbb{Z}}$ such that

$$\delta_k \le \theta_{n+1,k} - \theta_{n,k} \le \Delta_k, \quad k \in \mathbb{N}, \quad n \in \mathbb{Z},$$

and

$$\partial_2 h(\theta_{n-1,k}, \theta_{n,k}) + \partial_1 h(\theta_{n,k}, \theta_{n+1,k}) = 0, \quad k \in \mathbb{N}, \quad n \in \mathbb{Z},$$

hold, where $\partial_1 h = \partial_\theta h, \partial_2 h = \partial_{\theta_1} h$. The assertions on the rotation numbers are easily seen to be verified. □

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