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Existence of a Least Energy Nodal Solution for a Class of $p\&q$-Quasilinear Elliptic Equations

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Abstract

In this paper we prove an existence result for a least energy nodal (or sign-changing) solution for the class of $p\&q$ problems given by

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) & = & f(u) & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $2 \leq p < N$. The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ grows like $t^{\frac{q-p}{p}}$ as $t \to +\infty$ for some $p \leq q < N$, the case $q = p$ meaning that a is bounded away from zero and infinity. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ grows like $|t|^{m-1}$ at infinity with $q < m < q^* = \frac{Nq}{N-q}$. Moreover, we find that u has exactly two nodal domains or changes sign exactly once in Ω. The functions f and a satisfy suitable additional growth and monotonicity conditions which allow this result to extend previous ones to a larger class of $p\&q$ type problems. The proof is based on a minimization argument and a variant of quantitative deformation lemma.

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1 Introduction

This article deals mainly with the existence of a least energy solution to the following class of quasilinear problems

$$(P) \qquad \begin{cases} -\mathrm{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) & = & f(u) & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a smooth bounded domain and $2 \leq p < N$. Actually, we are interested in the search of a least energy solution of (P) which is nodal or sign-changing in Ω namely with $u^+ \neq 0$ and $u^- \neq 0$ in Ω where

$$u^+(x) := \max\{u(x), 0\} \ \text{ and } \ u^-(x) := \min\{u(x), 0\},$$

for every $x \in \Omega$. Notice that, in this case, $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

The hypotheses on function $a : \mathbb{R}^+ \to \mathbb{R}^+$ of C^1 class are the following:

(a_1) There exist constants $0 < k_0, k_1, k_2$, $0 \leq k_3$, $2 \leq p \leq q < N$ such that

$$k_0 + k_2 t^{(q-p)/p} \leq a(t) \leq k_1 + k_3 t^{(q-p)/p}, \quad \text{for all } t \geq 0,$$

with k_3 which is eventually allowed to be zero only if $q = p$.

(a_2) There exist positive real constants α and θ such that

$$\frac{1}{\alpha} a(t)t \leq A(t) = \int_0^t a(s)ds, \quad \text{for all } t \geq 0,$$

with

$$\frac{q}{p} \leq \alpha < \frac{\theta}{p},$$

where θ is also defined in (f_3) and satisfies $q < \theta < q^*$ with $q^* = \frac{Nq}{N-q}$.

(a_3) The map

$$t \mapsto \frac{a(t)}{t^{(q-p)/p}} \text{ is decreasing for all } t > 0$$

or, equivalently, the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \text{ for all } t > 0.$$

We assume that the function $f : \mathbb{R} \to \mathbb{R}$ is of C^1 class and satisfies

(f_1)

$$\lim_{|t| \to 0^+} \frac{f(t)}{|t|^{p-1}} = 0.$$

(f_2) There exists $m \in (q, q^*)$ such that

$$\lim_{|t| \to +\infty} \frac{f(t)}{|t|^{m-1}} = 0.$$

(f_3) There exists $\theta \in (q, q^*)$ so that

$$0 < \theta F(t) \leq f(t)t, \text{ for all } |t| > 0,$$

where θ is the same in (a_2) and $F(t) = \displaystyle\int_0^t f(s)ds$ denotes the primitive of f.

(f_4) The map

$$t \mapsto \frac{f(t)}{|t|^{q-1}} \text{ is strictly increasing for all } |t| > 0$$

or, equivalently,

$$f'(t) > (q-1)\frac{f(t)}{t}, \text{ for all } t \neq 0.$$

The main result of this paper is the following.

Theorem 1.1 *Suppose that* $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies* $(a_1)-(a_3)$ *and* $f \in C^1(\mathbb{R}, \mathbb{R})$ *satisfies* $(f_1)-(f_4)$. *Then problem* (P) *possesses a least energy nodal solution, which has precisely two nodal domains.*

Remark 1.1 Introduced the function $H : \mathbb{R}^+ \to \{0, 1\}$ such that $H(\xi) = 1$ if $\xi > 0$ and $H(\xi) = 0$ if $\xi = 0$, hypothesis (a_1) can be written equivalently as

(a_1') There exist constants $0 < k_0, k_1, k_2$, $0 \leq k_3$, $2 \leq p < q < N$ such that

$$k_0 + H(k_3)k_2 t^{(q-p)/p} \leq a(t) \leq k_1 + k_3 t^{(q-p)/p}, \quad \text{for all } t \geq 0.$$

Furthermore, defined $\gamma = (1 - H(k_3))p + H(k_3)q$ and $\gamma^* = (1 - H(k_3))p^* + H(k_3)q^* = \frac{\gamma N}{N-\gamma}$ being p^*, q^* the critical exponents of p and q, (a_2) and (a_3) are respectively equivalent to

(a_2') there exist positive real constants α and θ such that

$$\frac{1}{\alpha}a(t)t \leq A(t) = \int_0^t a(s)ds, \quad \text{for all } t \geq 0,$$

with

$$\frac{\gamma}{p} \leq \alpha < \frac{\theta}{p},$$

where θ is also defined in (f_3') and satisfies $\gamma < \theta < \gamma^*$;

(a_3') the map

$$t \mapsto \frac{a(t)}{t^{H(k_3)(\gamma-p)/p}} \text{ is decreasing for all } t > 0$$

or, equivalently, the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(\gamma - p)}{p}a(t) \text{ for all } t > 0.$$

Substituting q and q^* by γ and γ^* in assumptions (f_2), (f_3) and (f_4) we obtain the equivalent (f_2'), (f_3') and (f_4'). Clearly (f_1) does not change.

The study on the class of problems of type $p\&q$ Laplacian has increased considerably in the last ten years, as it can be seen in [1], [8], [10], [11], [12], [14], [15], [16], [17], [18], [19], [21], [22] [23], [25], [26] and references therein. These studies are greatly justified in view of two basic aspects of mathematical research, namely:

a) Applications

The case

$$\left\{\begin{array}{rcll} -\Delta_p u - \Delta_q u & = & f(u) & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array}\right. \tag{pq}$$

comes, for example, from a general reaction-diffusion system:

$$u_t = div[D(u)\nabla u] + c(x,u), \tag{1.1}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration and the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ is a polynomial of u with variable coefficients (see [15], [18], [26]).

b) Mathematical techniques

The second aspect of the relevance of (pq) problems is related to the mathematical techniques used to approach them. For instance, using the Leray-Lions theorem for monotone operators, the authors in [8] prove the existence of at least one weak nontrivial solution to the resonance problem of the type (pq). Resonance problems also are studied in [25].

In [10], [17] and [23], equation (pq) is studied by applying Morse theory to the associated Euler functional; more precisely, the authors exhibit a saddle geometry, which implies that a suitable Poincaré polynomial is nontrivial, and show that a certain critical group at zero is trivial. A more general case is studied in [14].

In [11], [18], [22] and [26] the proof relies on variational arguments and on Mountain Pass Theorem of Ambrosetti and Rabinowitz. Multiplicity results are obtained in [1] and [16] via category theory and in [12] and [19] via genus theory.

In [15] the authors establish Hölder regularity and exponential decay estimates for the solutions of the problem (pq).

The study on nodal solutions is treated in [14] and [21]. In [14] the authors study a nonlinear elliptic problem with a nonsmooth potential (hemivariational inequality) of the type (pq). Since the energy functional of the problem is coercive, they showed the existence of three solutions. Two solutions of which have constant sign (one positive and the other negative) and another solution which is nodal.

In [21] the authors show the existence of a greatest negative, a smallest positive and a nodal weak solution to problem (pq) where the reaction term makes coercive the corresponding energy functional.

In this paper, we aim to complete the study started in [14] and [21] by showing an existence result of a least energy nodal solution which has precisely two nodal domains. Furthermore, in the more recent paper [3], we extend this study to existence of positive, negative and nodal solutions to problem (P) on unbounded domains also in presence of potentials vanishing at infinity. As our nonlinearity is superlinear, the functional associated to our problem is unbounded below. Moreover, our result is true for a general class of problems that include but are not restricted to the type of (pq). Just to illustrate the degree of generality of problem (P), let us consider some special cases, depending on the function a, that are covered in this article, since a satisfies assumptions $(a_1) - (a_3)$.

Example 1.1 If $a \equiv 1$, our operator is the p-Laplacian and so problem (P) becomes

$$\begin{cases} -\Delta_p u &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

with $q = p$ and $k_0 + k_2 = 1 = k_1 + k_3$.

Example 1.2 If $a(t) = 1 + t^{\frac{q-p}{p}}$ we obtain

$$\begin{cases} -\Delta_p u - \Delta_q u &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

with $k_0 = k_1 = k_2 = k_3 = 1$.

Example 1.3 Taking $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ we get

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

with $q = p$, $k_0 + k_2 = 1$ and $k_1 + k_3 = 2$.

Example 1.4 If we consider $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, we obtain

$$\begin{cases} -\Delta_p u - \Delta_q u - div(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $k_0 = k_2 = k_3 = 1$ and $k_1 = 2$.

In recent years, the study of nodal solutions for problems involving Laplacian operator has received a special attention, as can be seen in [2], [4], [5], [7] and [9]. In the proof of Theorem 1.1 we base on arguments that can be found in [4]. Clearly, due to the presence of the more general function a, some estimates more refined are needed, such as in Lemma 2.3 for instance.

2 Variational framework and preliminary results

In order to prove that problem (P) has a variational structure, let us consider the space $X := W_0^{1,q}(\Omega)$ endowed with the norm

$$\|u\|_{1,q} = \left(\int_\Omega |\nabla u|^q dx\right)^{1/q}.$$

Let us remark that the norm on $W^{1,q}(\Omega)$ given by

$$\left(\int_\Omega |\nabla u|^q dx + \int_\Omega |u|^q dx\right)^{1/q}$$

is equivalent to $\|\cdot\|_{1,q}$ on X. Indeed, since Ω is open bounded, by Poincarѐ inequality a positive constant $C > 0$ exists such that

$$\int_\Omega |\nabla u|^q dx + \int_\Omega |u|^q dx \le C \int_\Omega |\nabla u|^q dx$$

for every $u \in X$. Furthermore, X is continuosly embedded in $L^r(\Omega)$ for all $r \in [1, q^*]$ and compactly embedded for all $r \in [1, q^*)$.

Here we search a nodal or sign-changing weak solution of problem (P), i.e., a function $u \in X$ such that $u^+ \neq 0$, $u^- \neq 0$ in Ω and

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx - \int_\Omega f(u)\phi \, dx = 0, \text{ for all } \phi \in X. \tag{2.2}$$

Observe that, if $u \in X$, also $u^+, u^- \in X$. In particular, we look for $u \in X$ which has exactly two nodal domains or equivalently changes sign exactly once, precisely

$$u = u^+ + u^-, \text{ with } u^+ \geq 0, u^- \leq 0 \text{ and } supp(u^+) \bigcap supp(u^-) = \emptyset.$$

At this point, in order to use variational methods, we observe that by assumption (a_1) we have

$$a(t^p)t^{p-1} \leq k_1 t^{p-1} + k_3 t^{q-1}, \text{ for all } t \geq 0, \tag{2.3}$$

and, by $(f_1) - (f_2)$ for any $\epsilon > 0$, a positive constant $C_\epsilon > 0$ exists such that

$$|f(t)| \leq \epsilon|t|^{p-1} + C_\epsilon|t|^{m-1}, \text{ for all } t \in \mathbb{R}. \tag{2.4}$$

By these subcritical growth conditions the energy functional $J : X \to \mathbb{R}$ given by

$$J(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p)dx - \int_\Omega F(u)dx$$

and the expression (2.2) are well defined. Moreover, by standard arguments, $J \in C^1(X)$ with the following Fréchet derivative

$$\langle J'(u), v\rangle = \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \int_\Omega f(u)v dx$$

for every $u, v \in X$ and the weak solutions of (P) are precisely the critical points of J and conversely. So, we define the Nehari manifold associated to the functional J

$$\mathcal{N} := \left\{ u \in X\backslash\{0\} : \langle J'(u), u\rangle = 0 \right\}$$

and we can look for critical points of J constrained on $\mathcal{N}$ still denoting, for simplicity of notations, $J_{|\mathcal{N}}$ by J.

Recall that a non zero critical point w of J is a least energy weak solution of (P) if

$$J(w) = \min_{v\in\mathcal{N}} J(v)$$

and, since our purpose is to prove the existence of a least energy sign-changing weak solution of (P), in particular, we look for $w \in \mathcal{M}$ such that

$$J(w) = \min_{v\in\mathcal{M}} J(v),$$

where $\mathcal{M}$ is the subset of $\mathcal{N}$ containing all sign-changing weak solutions of (P), i.e.,

$$\begin{aligned} \mathcal{M} &:= \left\{ w \in \mathcal{N} : w^+ \neq 0, w^- \neq 0, \langle J'(w), w^+\rangle = 0 = \langle J'(w), w^-\rangle \right\} \\ &= \left\{ w \in \mathcal{N} : w^+ \neq 0, w^- \neq 0, \langle J'(w^+), w^+\rangle = 0 = \langle J'(w^-), w^-\rangle \right\}. \end{aligned}$$

For the sake of simplicity, in the following we often denote

$$\langle J'(w^\pm), w^\pm\rangle = \int_\Omega a(|\nabla w^\pm|^p)|\nabla w^\pm|^p dx - \int_\Omega f(w^\pm)w^\pm dx. \tag{2.5}$$

Let us observe that minimizers or maximizers of J on $\mathcal{M}$ are not automatically (weak) solutions of problem (P). Indeed, in [6] it has been proved that $\mathcal{M}$ is not a submanifold of X with $X = W_0^{1,2}(\Omega)$ since the map $u \to u^\pm$ loses differentiability and $\mathcal{M}$ is a codimension 2 submanifold of $W^{2,2}(\Omega)$. So one cannot talk about vector fields on $\mathcal{M}$ and deformations cannot be easily constructed on $\mathcal{M}$. A similar situation occurs in our case. However as in [4] we are able to prove that every minimizer on $\mathcal{M}$ of $J|_{\mathcal{M}}$ is a critical point of J.

So, let us begin by establishing some preliminary results which will be exploited in the last section for a minimization argument. In particular, in this first lemma, we prove that J is strictly positive on $\mathcal{N}$ then on $\mathcal{M}$, $\|\cdot\|_{1,q}$ is uniformly bounded from below by a strictly positive radius on $\mathcal{N}$ thus on $\mathcal{M}$ and the same applies to the positive and negative part $w^\pm$ of every $w \in \mathcal{M}$. It follows that J is coercive on $\mathcal{N}$ and in particular on $\mathcal{M}$ since $J(u) \to +\infty$ as $\|u\|_{1,q} \to +\infty$, for every $u \in \mathcal{N}$.

Lemma 2.1 **(i)** *For all $u \in \mathcal{N}$ we have*

$$J(u) \geq \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)\overline{k}\,\|u\|_{1,q}^q,$$

where $\overline{k} = \min\{k_0, k_2\}$.

(ii) *There exists $\rho > 0$ such that*

$$\|u\|_{1,q} \geq \rho,$$

for all $u \in \mathcal{N}$ and

$$\|w^\pm\|_{1,q} \geq \rho,$$

for all $w \in \mathcal{M}$.

Proof. Since $u \in \mathcal{N}$ and (f_3) holds, we have that

$$J(u) = J(u) - \frac{1}{\theta}\langle J'(u), u\rangle \geq \frac{1}{p}\int_\Omega A(|\nabla u|^p)dx - \frac{1}{\theta}\int_\Omega a(|\nabla u|^p)|\nabla u|^p dx$$

and, by using (a_2),

$$J(u) \geq \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)\int_\Omega a(|\nabla u|^p)|\nabla u|^p dx.$$

By the left-hand side inequality of (a_1) we get

$$J(u) \geq \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)\left(k_0\|u\|_{1,p}^p + k_2\|u\|_{1,q}^q\right) \geq \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)\overline{k}\|u\|_{1,q}^q,$$

for all $u \in \mathcal{N}$, which proves (i) denoting $\overline{k} = \min\{k_0, k_2\}$.

In order to prove (ii), let us observe that, by (2.4), for any $\epsilon > 0$, a positive constant $C_\epsilon > 0$ exists such that

$$|f(t)t| \leq \epsilon|t|^p + C_\epsilon|t|^m, \text{ for all } t \in \mathbb{R}. \tag{2.6}$$

Since for every $u \in \mathcal{N}$ we have

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^p dx = \int_\Omega f(u)u dx,$$

from (a_1) , (2.6) and Sobolev's embeddings, there exists $C > 0$ such that

$$(k_0 - \varepsilon C)\|u\|_{1,p}^p + k_2\|u\|_{1,q}^q \le C_\varepsilon C\|u\|_{1,q}^m.$$

Choosing $\varepsilon = \frac{k_0}{2C}$, it follows that

$$0 < \min\left\{\frac{k_0}{2}, k_2\right\}\|u\|_{1,q}^q \le C\|u\|_{1,q}^m,$$

thus

$$\rho \le \|u\|_{1,q},$$

where $\rho = \left(\frac{\overline{k}'}{C}\right)^{1/(m-q)}$ and $\overline{k}' = \min\{\frac{k_0}{2}, k_2\}$.

Now, if $w \in \mathcal{M}$, we have that $\langle J'(w^\pm), w^\pm\rangle = 0$ namely $w^\pm \in \mathcal{N}$, hence by the previous estimate we obtain

$$0 < \rho \le \|w^\pm\|_{1,q}.$$

□

From the previous lemma we deduce a result valid for every sequence in $\mathcal{M}$ that we apply in the last section to every bounded minimizing sequence of J on $\mathcal{M}$ so that the candidate minimizer is different from zero.

Remark 2.1 If (w_n) is a sequence in $\mathcal{M}$ and $m \in (q, q^*)$, we have that

$$\liminf_{n\to\infty} \int_\Omega |w_n^\pm|^m dx > 0.$$

In the following, by means of Miranda Theorem, we are able to prove that every sign-changing function in X corresponds to a suitable sign-changing solution of (P).

Lemma 2.2 *If $v \in X$ with $v^\pm \neq 0$, then there exist $t, s > 0$ such that*

$$\langle J'(tv^+ + sv^-), v^+\rangle = 0 \textit{ and } \langle J'(tv^+ + sv^-), v^-\rangle = 0.$$

Consequently, $tv^+ + sv^- \in \mathcal{M}$.

Proof. Let $V : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^2$ be a continuous vector field given by

$$V(t, s) = \Big(\langle J'(tv^+ + sv^-), (tv^+)\rangle, \langle J'(tv^+ + sv^-), (sv^-)\rangle\Big)$$

for every $(t, s) \in (0, +\infty) \times (0, +\infty)$. Note that, (a_1), (2.6) and Sobolev's embeddings imply

$$\begin{aligned}
\langle J'(tv^+ + sv^-), (tv^+)\rangle &= \langle J'(tv^+), (tv^+)\rangle \\
&= t^p \int_\Omega a(|t\nabla v^+|^p)|\nabla v^+|^p dx - \int_\Omega f(tv^+)tv^+ dx \\
&\ge t^p k_0\|v^+\|_{1,p}^p + t^q k_2\|v^+\|_{1,q}^q - t^p \epsilon C\|v^+\|_{1,p}^p - t^m C_\epsilon C\|v^+\|_{1,q}^m \\
&\ge t^q(\overline{k} - \epsilon C)\|v^+\|_{1,q}^q - t^m C_\epsilon C\|v^+\|_{1,q}^m. \qquad (2.7)
\end{aligned}$$

So, for $r > 0$ sufficiently small we have

$$\langle J'(rv^+ + sv^-), (rv^+)\rangle > 0, \text{ for all } s > 0.$$

Similarly, starting by

$$\langle J'(tv^+ + sv^-), (sv^-)\rangle = \langle J'(sv^-), (sv^-)\rangle,$$

we get

$$\langle J'(tv^+ + rv^-), (rv^-)\rangle > 0, \text{ for all } t > 0.$$

On the other hand, by (f_3), there exist $C_1, C_2 > 0$ such that

$$F(t) \geq C_1|t|^\theta - C_2, \text{ for every } t \in \mathbb{R}. \tag{2.8}$$

Using the right-hand side inequality in (a_1), (2.8) and (f_3), we have

$$\begin{aligned} \langle J'(tv^+ + sv^-), (tv^+)\rangle &= \langle J'(tv^+), (tv^+)\rangle \leq k_1 t^p \|v^+\|_{1,p}^p + k_3 t^q \|v^+\|_{1,q}^q \\ &- t^\theta \theta C_1 \int_\Omega |v^+|^\theta dx + C_2\theta|\Omega| \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus, since $q < \theta < q^*$, for $R > 0$ sufficiently large, we get

$$\langle J'(Rv^+ + sv^-), (Rv^+)\rangle < 0, \text{ for all } s > 0.$$

Similarly starting from

$$\langle J'(tv^+ + sv^-), (sv^-)\rangle = \langle J'(sv^-), (sv^-)\rangle,$$

we get

$$\langle J'(tv^+ + Rv^-), (Rv^-)\rangle < 0, \text{ for all } t > 0.$$

Consequently we have proved the existence of suitable $0 < r < R$ such that, for all $t, s \in [r, R]$, we have

$$\langle J'(rv^+ + sv^-), (rv^+)\rangle > 0 \text{ and } \langle J'(tv^+ + rv^-), (rv^-)\rangle > 0,$$

$$\langle J'(Rv^+ + sv^-), (Rv^+)\rangle < 0 \text{ and } \langle J'(tv^+ + Rv^-), (Rv^-)\rangle < 0,$$

and the lemma follows applying Miranda's theorem [20]. □

At this point, some useful remarks follow. First, let us observe that assumptions (a_3) and (f_4) imply the following monotonicity conditions:

$$t \longmapsto \frac{1}{p}A(t) - \frac{1}{q}a(t)t \text{ is increasing for all } t > 0 \tag{2.9}$$

and

$$t \longmapsto \frac{1}{q}f(t)t - F(t) \text{ is strictly increasing (resp. decreasing) for } t > 0 \text{ (resp. } t < 0). \tag{2.10}$$

Furthermore, let us point out that, when $w^{\pm} \neq 0$ and the supports of w^+ and w^- are disjoint in Ω it follows that

$$\begin{aligned}
|\nabla w|^p &= |\nabla(w^+ + w^-)|^p = |\nabla(w^+ + w^-)|^{p-2}|\nabla(w^+ + w^-)|^2 \\
&= |\nabla(w^+ + w^-)|^{p-2}\nabla(w^+ + w^-) \cdot \nabla(w^+ + w^-) \\
&= |\nabla(w^+ + w^-)|^{p-2}\nabla(w^+ + w^-) \cdot \nabla(w^+) \\
&+ |\nabla(w^+ + w^-)|^{p-2}\nabla(w^+ + w^-) \cdot \nabla(w^-) \\
&= |\nabla(w^+)|^{p-2}\nabla(w^+) \cdot \nabla(w^+) + |\nabla(w^-)|^{p-2}\nabla(w^-) \cdot \nabla(w^-) \\
&= |\nabla w^+|^p + |\nabla w^-|^p
\end{aligned}$$

where, for simplicity of notations, we omit the dependence of $x \in \Omega$. Consequently,

$$\int_\Omega A(|\nabla(w^+ + w^-)|^p \, dx = \int_\Omega (A(|\nabla(w^+)|^p + |\nabla(w^-)|^p)) \, dx. \tag{2.11}$$

Now, we can define a suitable function and its gradient vector field which are related to functional J and will be involved in particular in the application of the deformation lemma. Indeed, for each $v \in X$ with $v^{\pm} \neq 0$, let us consider $h^v : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ given by

$$h^v(t,s) = J(tv^+ + sv^-), \text{ for every } (t,s) \in [0,+\infty) \times [0,+\infty),$$

and its gradient $\Phi^v : [0,+\infty) \times [0,+\infty) \to \mathbb{R}^2$ defined by

$$\begin{aligned}
\Phi^v(t,s) &= \big(\Phi_1^v(t,s), \Phi_2^v(t,s)\big) = \Big(\frac{\partial h^v}{\partial t}(t,s), \frac{\partial h^v}{\partial s}(t,s)\Big) \\
&= \big(\langle J'(tv^+ + sv^-), v^+\rangle, \langle J'(tv^+ + sv^-), v^-\rangle\big)
\end{aligned}$$

for every $(t,s) \in [0,+\infty) \times [0,+\infty)$. Furthermore, we consider the Hessian matrix of h^v or the Jacobian matrix of Φ^v, i.e.

$$(\Phi^v)'(t,s) = \begin{pmatrix} \frac{\partial \Phi_1^v}{\partial t}(t,s) & \frac{\partial \Phi_1^v}{\partial s}(t,s) \\ \frac{\partial \Phi_2^v}{\partial t}(t,s) & \frac{\partial \Phi_2^v}{\partial s}(t,s) \end{pmatrix}$$

for every $(t,s) \in [0,+\infty)\times[0,+\infty)$. Indeed, in the following we aim to prove that, if $w \in \mathcal{M}$, function h^w has a critical point and in particular a global maximum in $(t,s) = (1,1)$.

Lemma 2.3 *If* $w \in \mathcal{M}$, *then*

(a) $h^w(t,s) < h^w(1,1) = J(w)$, *for all* $t, s \geq 0$ *such that* $(t,s) \neq (1,1)$*;*

(b) $\frac{\partial \Phi_1^v}{\partial t}(t,s) < 0$ *and* $det(\Phi^w)'(1,1) > 0$.

Proof. Let us observe that, since $w \in \mathcal{M}$, we have

$$0 = \langle J'(w), w^{\pm}\rangle = \langle J'(w^+ + w^-), w^{\pm}\rangle .$$

Thus

$$\Phi^w(1,1) = \Big(\frac{\partial h^w}{\partial t}(1,1), \frac{\partial h^w}{\partial s}(1,1)\Big) = (0,0)$$

implies that $(1,1)$ is a critical point of h^w.

Now, let $t, s \geq 0$. By the right-hand side inequality of (a_1) and (2.8) we get

$$\begin{aligned} h^w(t,s) = J(tw^+ + sw^-) &\leq \frac{1}{p}k_1\|tw^+ + sw^-\|_{1,p}^p + \frac{1}{q}k_3\|tw^+ + sw^-\|_{1,q}^q \\ &- C_1\int_\Omega |tw^+ + sw^-|^\theta dx + C_2|\Omega|, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Since $q < \theta$, then

$$\lim_{|(t,s)|\to+\infty} h^w(t,s) = -\infty,$$

and h^w has a global maximum point in some $(\bar{t}, \bar{s}) \in [0,+\infty) \times [0,+\infty)$.

First we prove that $\bar{t}, \bar{s} > 0$. Suppose, by contradiction that $\bar{s} = 0$. Thus, $\left\langle J'(\bar{t}w^+), \bar{t}w^+ \right\rangle = 0$ and it can be written equivalently as

$$\int_\Omega \frac{a(\overline{t^p}|\nabla w^+|^p)}{|\bar{t}\nabla w^+|^{q-p}}|\nabla w^+|^q dx = \int_\Omega \frac{f(\bar{t}w^+)}{(\bar{t}w^+)^{q-1}}(w^+)^q dx. \tag{2.12}$$

Moreover, for $\langle J'(w), w^+\rangle = \langle J'(w^+), w^+\rangle = 0$ we similarly get

$$\int_\Omega \frac{a(|\nabla w^+|^p)}{|\nabla w^+|^{q-p}}|\nabla w^+|^q dx = \int_\Omega \frac{f(w^+)}{(w^+)^{q-1}}(w^+)^q dx. \tag{2.13}$$

By subtracting (2.13) from (2.12), we have

$$\begin{aligned} &\int_\Omega \left[\frac{a(\overline{t^p}|\nabla w^+|^p)}{|\bar{t}\nabla w^+|^{q-p}} - \frac{a(|\nabla w^+|^p)}{|\nabla w^+|^{q-p}}\right]|\nabla w^+|^q dx \\ = &\int_\Omega \left[\frac{f(\bar{t}w^+)}{(\bar{t}w^+)^{q-1}} - \frac{f(w^+)}{(w^+)^{q-1}}\right](w^+)^q dx. \end{aligned}$$

and, from (a_3) and (f_4), it follows that $\bar{t} \leq 1$. Consequently, also by (2.9) and (2.10), we obtain

$$\begin{aligned} h^w(\bar{t},0) &= J(\bar{t}w^+) = J(\bar{t}w^+) - \frac{1}{q}\left\langle J'(\bar{t}w^+), (\bar{t}w^+)\right\rangle \\ &= \int_\Omega \left[\frac{1}{p}A(|\nabla(\bar{t}w^+)|^p) - \frac{1}{q}a(|\nabla(\bar{t}w^+)|^p)|\nabla(\bar{t}w^+)|^p\right]dx \\ &+ \int_\Omega \left[\frac{1}{q}f(\bar{t}w^+)\bar{t}w^+ - F(\bar{t}w^+)\right]dx \\ &\leq \int_\Omega \left[\frac{1}{p}A(|\nabla(w^+)|^p) - \frac{1}{q}a(|\nabla(w^+)|^p)|\nabla(w^+)|^p\right]dx \\ &+ \int_\Omega \left[\frac{1}{q}f(w^+)w^+ - F(w^+)\right]dx \\ &= J(w^+) - \frac{1}{q}\langle J'(w^+)w^+\rangle = J(w^+) = h^w(1,0). \end{aligned} \tag{2.14}$$

At this point, in order to have a contradiction with the fact that $(\bar{t},0)$ is a global maximum point, it is sufficient to prove that

$$J(w^+) = h^w(1,0) < J(w) = h^w(1,1).$$

Indeed, since $w \in \mathcal{M}$ and then $w^- \in \mathcal{N}$, by Lemma 2.1, $J > 0$ on $\mathcal{N}$ and

$$\begin{aligned} J(w^+) < J(w^+) + J(w^-) &= \frac{1}{p}\int_\Omega \Big[A(|\nabla(w^+)|^p) + A(|\nabla(w^-)|^p)\Big]dx \\ &- \int_\Omega \Big[F(w^+) + F(w^-)\Big]dx. \end{aligned} \tag{2.15}$$

By the linearity of F, it follows easily that

$$\int_\Omega (F(w^+) + F(w^-))\, dx = \int_\Omega F(w)dx, \tag{2.16}$$

and, by using the definition of the operator A and (2.11), it results

$$\begin{aligned} \int_\Omega (A(|\nabla(w^+)|^p) + A(|\nabla(w^-)|^p))\, dx &= \int_\Omega A(|\nabla(w^+)|^p + |\nabla(w^-)|^p)dx \\ &= \int_\Omega A(|\nabla(w^+ + w^-)|^p dx = \int_\Omega A(|\nabla w|^p dx. \end{aligned} \tag{2.17}$$

Then we can conclude that

$$h^w(\bar{t}, 0) \leq h^w(1, 0) = J(w^+) < J(w^+) + J(w^-) = J(w) = h^w(1, 1),$$

which is an absurd because $(\bar{t}, 0)$ is a global maximum point. In a similarly way, we prove that $\bar{t} > 0$. Now, since $(1, 1)$ and $(\bar{t}, \bar{s})$ are both critical points of h^w, it results

$$\left\langle J'(\bar{t}w^+), \bar{t}w^+\right\rangle = 0, \text{ and } \langle J'(\bar{s}w^-), \bar{s}w^-\rangle = 0,$$

$$\langle J'(w^+), w^+\rangle = 0 \text{ and} \langle J'(w^-), w^-\rangle = 0$$

and by previous arguments again we can conclude that $0 < \bar{t}, \bar{s} \leq 1$.
Next we prove that h^w does not assume a global maximum in $[0, 1] \times [0, 1]\backslash\{(1, 1)\}$, namely

$$h^w(\bar{t}, \bar{s}) < h^w(1, 1), \text{ for every } (\bar{t}, \bar{s}) \in [0, 1] \times [0, 1]\backslash\{(1, 1)\}.$$

Indeed, by adapting to the function $\bar{t}w^+ + \bar{s}w^-$ formulas (2.16) and (2.17) and applying monotonicity conditions (2.9) and (2.10) we find that

$$
\begin{aligned}
h^w(\bar{t},\bar{s}) = J(\bar{t}w^+ + \bar{s}w^-) &= J(\bar{t}w^+ + \bar{s}w^-) - \frac{1}{q}\left\langle J'(\bar{t}w^+ + \bar{s}w^-), (\bar{t}w^+ + \bar{s}w^-)\right\rangle \\
&= \frac{1}{p}\int_\Omega A(|\nabla \bar{t}w^+ + \bar{s}w^-|^p)dx - \int_\Omega F(\bar{t}w^+ + \bar{s}w^-)dx \\
&\quad - \frac{1}{q}\Big[\int_\Omega a(|\nabla \bar{t}w^+ + \bar{s}w^-|^p)|\nabla \bar{t}w^+ + \bar{s}w^-|^p dx - \int_\Omega f(\bar{t}w^+ + \bar{s}w^-)(\bar{t}w^+ + \bar{s}w^-)dx\Big] \\
&= \int_\Omega\Big[\frac{1}{p}A(|\nabla(\bar{t}w^+)|^p) - \frac{1}{q}a(|\nabla(\bar{t}w^+)|^p)|\nabla(\bar{t}w^+)|^p\Big]dx + \int_\Omega\Big[\frac{1}{q}f(\bar{t}w^+)\bar{t}w^+ - F(\bar{t}w^+)\Big]dx \\
&\quad + \int_\Omega\Big[\frac{1}{p}A(|\nabla(\bar{s}w^-)|^p) - \frac{1}{q}a(|\nabla(\bar{s}w^-)|^p)|\nabla(\bar{s}w^-)|^p\Big]dx \\
&\quad + \int_\Omega\Big[\frac{1}{q}f(\bar{s}w^-)\bar{s}w^- - F(\bar{s}w^-)\Big]dx \\
&< \int_\Omega\Big[\frac{1}{p}A(|\nabla(w^+)|^p) - \frac{1}{q}a(|\nabla(w^+)|^p)|\nabla(w^+)|^p\Big]dx \\
&\quad + \int_\Omega\Big[\frac{1}{q}f(w^+)w^+ - F(w^+)dx\Big]dx \\
&\quad + \int_\Omega\Big[\frac{1}{p}A(|\nabla(w^-)|^p) - \frac{1}{q}a(|\nabla(w^-)|^p)|\nabla(w^-)|^p\Big]dx \\
&\quad + \int_\Omega\Big[\frac{1}{q}f(w^-)w^- - F(w^-)\Big]dx \\
&= \Big(J(w^+) - \frac{1}{q}\langle J'(w^+)w^+\rangle\Big) + \Big(J(w^-) - \frac{1}{q}\langle J'(w^-)w^-\rangle\Big) \\
&= J(w^+) + J(w^-)
\end{aligned}
$$

Clearly now we are in position to take into account the right-hand side equality in (2.15) and, by exploiting (2.16) and (2.17) again we conclude that

$$h^w(\bar{t},\bar{s}) < J(w^+ + w^-) = J(w) = h^w(1,1)$$

and item (a) is proved.

At this point, let us prove item (b). Taking into account of the definition of $(\Phi^w)'$ given before and denoting by

$$
\begin{aligned}
g_1(t) &= \Phi_1^w(t,s) = \langle J'(tw^+ + sw^-), w^+\rangle = \langle J'(tw^+), w^+\rangle \\
&= t^{p-1}\int_\Omega a(|t\nabla w^+|^p)|\nabla w^+|^p dx - \int_\Omega f(tw^+)w^+dx
\end{aligned}
$$

and

$$
\begin{aligned}
g_2(s) &= \Phi_2^w(t,s) = \langle J'(tw^+ + sw^-), w^-\rangle = \langle J'(sw^-), w^-\rangle \\
&= s^{p-1}\int_\Omega a(|s\nabla w^-|^p)|\nabla w^-|^p dx - \int_\Omega f(sw^-)w^-dx,
\end{aligned}
$$

we have

$$(\Phi^w)'(t,s) = \begin{pmatrix} g_1'(t) & 0 \\ 0 & g_2'(s) \end{pmatrix},$$

being $\frac{\partial \Phi_1^w}{\partial s}(t,s) = \frac{\partial \Phi_2^w}{\partial t}(t,s) = 0$,

$$\begin{aligned} g_1'(t) &= \frac{\partial \Phi_1^w}{\partial t}(t,s) = (p-1)t^{p-2}\int_\Omega a(|t\nabla w^+|^p)|\nabla w^+|^p dx \\ &+ pt^{p-1}\int_\Omega a'(|t\nabla w^+|^p)t^{p-1}|\nabla w^+|^{2p}dx - \int_\Omega f'(tw^+)(w^+)^2 dx \end{aligned}$$

and

$$\begin{aligned} g_2'(s) &= \frac{\partial \Phi_2^w}{\partial s}(t,s) = (p-1)s^{p-2}\int_\Omega a(|s\nabla w^-|^p)|\nabla w^-|^p dx \\ &+ ps^{p-1}\int_\Omega a'(|s\nabla w^-|^p)s^{p-1}|\nabla w^-|^{2p}dx - \int_\Omega f'(sw^-)(w^-)^2 dx. \end{aligned}$$

Since in particular by (a_3), the fact that $(1,1)$ is a critical point of h^w and (f_4) it follows that

$$\begin{aligned} g_1'(1) = (p-1)&\int_\Omega a(|\nabla w^+|^p)|\nabla w^+|^p dx + p\int_\Omega a'(|\nabla w^+|^p)|\nabla w^+|^{2p}dx \\ &- \int_\Omega f'(w^+)(w^+)^2 dx \le \int_\Omega \Big[(q-1)f(w^+)w^+ - f'(w^+)(w^+)^2\Big]dx < 0 \end{aligned}$$

and

$$\begin{aligned} g_2'(1) = (p-1)&\int_\Omega a(|\nabla w^-|^p)|\nabla w^-|^p dx + p\int_\Omega a'(|\nabla w^-|^p)|\nabla w^-|^{2p}dx \\ &- \int_\Omega f'(w^-)(w^-)^2 dx \le \int_\Omega \Big[(q-1)f(w^-)w^- - f'(w^-)(w^-)^2\Big]dx < 0, \end{aligned}$$

we conclude that

$$det(\Phi^w)'(1,1) = g_1'(1)g_2'(1) > 0$$

and item (b) is proved. □

3 Proof of Theorem 1.1

At this point, we can finally prove the existence of $w \in \mathcal{M}$ in which the infimum of J is attained on $\mathcal{M}$. After, following some arguments used in [4] by Bartsch, Weth and Willem (see also [2]) and, in particular, applying a deformation lemma, we find that w is a critical point of J and then a least energy nodal solution of (P). In order to complete the proof of Theorem 1.1, we conclude by showing that w has exactly two nodal domains.

First, let us start with the existence of a minimizer $w \in \mathcal{M}$ of J. By Lemma 2.1, there exists $c_0 \in \mathbb{R}$ such that

$$0 < c_0 = \inf_{v \in \mathcal{M}} J(v).$$

Thus, there exists a bounded minimizing sequence (w_n) in $\mathcal{M}$ and J is coercive on $\mathcal{M}$ from Lemma 2.1. Hence, by Sobolev Imbedding Theorem, without loss of generality, we can assume up to a subsequence that there exist $w, w_1, w_2 \in X$ such that

$$\begin{aligned} &w_n \rightharpoonup w, \quad w_n^+ \rightharpoonup w_1, \quad w_n^- \rightharpoonup w_2 \quad \text{in } X, \\ &w_n \to w, \quad w_n^+ \to w_1, \quad w_n^- \to w_2 \quad \text{in } L^m(\Omega), m \in (q, q^*). \end{aligned}$$

Since the transformations $w \to w^+$ and $w \to w^-$ are continuous from $L^m(\Omega)$ in $L^m(\Omega)$ (see Lemma 2.3 in [9] with suitable adaptations), we have that $w^+ = w_1 \geq 0$ and $w^- = w_2 \leq 0$. At this point, we can prove that $w \in \mathcal{M}$. Indeed, by $w_n^+ \to w^+$ and $w_n^- \to w^-$ in $L^m(\Omega)$ it is, as $n \to +\infty$

$$\int_\Omega |(w_n)^\pm|^m dx \to \int_\Omega |w^\pm|^m dx.$$

Then, by Remark 2.1, we conclude that $w^\pm \neq 0$ and consequently $w = w^+ + w^-$ is sign-changing. By Lemma 2.2, there exist $t, s > 0$ such that

$$\begin{aligned} \langle J'(tw^+ + sw^-), w^+\rangle &= \langle J'(tw^+), w^+\rangle = \langle J'(tw^+), tw^+\rangle = 0, \\ \langle J'(tw^+ + sw^-), w^-\rangle &= \langle J'(sw^-), w^-\rangle = \langle J'(sw^-), sw^-\rangle = 0, \end{aligned} \tag{3.18}$$

then $tw^+ + sw^- \in \mathcal{M}$. Now, let us prove that $t, s \leq 1$. First let us observe that, by (2.3) and (a_3), the functionals

$$\int_\Omega A(|\nabla u|^p)dx \text{ and } \int_\Omega a(|\nabla u|^p)|\nabla u|^p dx$$

are weakly lower semicontinuous on X. Then since $w_n^\pm \rightharpoonup w^\pm$ as $n \to +\infty$,

$$\int_\Omega A(|\nabla w^\pm|^p)dx \leq \liminf_{n\to+\infty} \int_\Omega A(|\nabla w_n^\pm|^p)dx$$

and

$$\int_\Omega a(|\nabla w^\pm|^p)|\nabla w^\pm|^p\, dx \leq \liminf_{n\to+\infty} \int_\Omega a(|\nabla w_n^\pm|^p)|\nabla w_n^\pm|^p dx.$$

Moreover, since (2.4) implies that f has a subcritical growth, both

$$\int_\Omega F(u)dx \quad \text{and} \quad \int_\Omega f(u)udx$$

are weakly continuous on X. Then, as $n \to +\infty$

$$\int_\Omega f((w_n)^\pm)(w_n)^\pm dx \to \int_\Omega f(w^\pm)w^\pm dx$$

and

$$\int_\Omega F((w_n)^\pm)dx \to \int_\Omega F(w^\pm)dx.$$

Thus, since $\langle J'(w_n), w_n^\pm\rangle = \langle J'(w_n^\pm), w_n^\pm\rangle = 0$ we get

$$\langle J'(w^+), w^+\rangle \leq 0 \ \text{ and } \ \langle J'(w^-), w^-\rangle \leq 0. \tag{3.19}$$

Consequently, combining (3.18) and (3.19) and arguing as in the proof of Lemma 2.3 item (a), we obtain $0 < t, s \leq 1$.

In the next step we show that $J(tw^+ + sw^-) = c_0$ and $t = s = 1$ or better $J(w) = c_0$. Indeed, since $t, s \leq 1$ and $w_n \rightharpoonup w$ as $n \to +\infty$, exploiting the arguments used in the proof of Lemma 2.3 item (a) and the weak lower semicontinuity of both J and K with $K(u) = \langle J'(u), u\rangle$ on X described above we get

$$
\begin{aligned}
c_0 \le J(tw^+ + sw^-) &= J(tw^+ + sw^-) - \frac{1}{q}\left\langle J'(tw^+ + sw^-), (tw^+ + sw^-)\right\rangle \\
&\le J(w^+ + w^-) - \frac{1}{q}\left\langle J'(w^+ + w^-), (w^+ + w^-)\right\rangle \\
&\le \liminf_{n\to+\infty}\Big[J(w_n^+ + w_n^-) - \frac{1}{q}\left\langle J'(w_n^+ + w_n^-), (w_n^+ + w_n^-)\right\rangle\Big] \\
&= \lim_{n\to+\infty} J(w_n) = c_0.
\end{aligned}
$$

Then we have found that

$$J(tw^+ + sw^-) = c_0$$

or, equivalently, that there exist $0 < t, s \le 1$ such that $tw^+ + sw^- \in \mathcal{M}$ and $J(tw^+ + sw^-) = c_0$. Let us observe that, if $t \neq 1$ or $s \neq 1$ by above calculations we would obtain a contradiction. Thus, $t = s = 1$, $w^+ + w^- \in \mathcal{M}$ and $J(w) = c_0$.

At this point, by using a quantitative deformation lemma and adapting the arguments used in [4] with slight technical changes, we point out that w is a critical point of J, i.e. $J'(w) = 0$. If we reason by contradiction, we find that there exist a positive constant $\alpha > 0$ and $v_0 \in X$, $\|v_0\|_{1,q} = 1$ such that

$$\langle J'(w), v_0\rangle = 2\alpha > 0.$$

By the continuity of J', we can choose a radius $r > 0$ so that

$$\langle J'(v), v_0\rangle = \alpha > 0, \text{ for every } v \in B_r(w) \subset X \text{ with } v^\pm \neq 0.$$

Let us fix $D = (\xi, \chi) \times (\xi, \chi) \subset \mathbb{R}^2$ with $0 < \xi < 1 < \chi$ such that

(i) $(1, 1) \in D$ and $\Phi^w(t, s) = (0, 0)$ in $\overline{D}$ if and only if $(t, s) = (1, 1)$;

(ii) $c_0 \notin h^w(\partial D)$;

(iii) $\left\{tw^+ + sw^- : (t, s) \in \overline{D}\right\} \subset B_r(w)$,

where h^w and Φ^w are defined as in Section 2 and satisfy Lemma 2.3. At this point, we can choose a smaller radius $r' > 0$ such that

$$\mathcal{B} = \overline{B_{r'}(w)} \subset B_r(w) \text{ and } \mathcal{B} \cap \{tw^+ + sw^- : (t, s) \in \partial D\} = \emptyset. \tag{3.20}$$

Now define a continuous mapping $\rho : X \to [0, +\infty)$ such that

$$\rho(u) := \operatorname{dist}(u, \mathcal{B}^c), \text{ for all } u \in X,$$

then a bounded Lipschitz vector field $V : X \to X$ given by

$$V(u) = -\rho(u)v_0$$

and, for every $u \in X$, denoting by $\eta(\tau) = \eta(\tau, u)$ we consider the following Cauchy problem

$$\begin{cases} \eta'(\tau) = V(\eta(\tau)), \text{ for all } \tau > 0, \\ \eta(0) = u. \end{cases}$$

Now, we observe that there exist a continuous deformation $\eta(\tau, u)$ and $\tau_0 > 0$ such that for all $\tau \in [0, \tau_0]$ the following properties hold:

(a) $\eta(\tau, u) = u$ for all $u \notin \mathcal{B}$;

(b) $\tau \to J(\eta(\tau, u))$ is decreasing for all $\eta(\tau, u) \in \mathcal{B}$;

(c) $J(\eta(\tau, w)) \leq J(w) - \frac{r'\alpha}{2}\tau$.

Item (a) follows immediately by the definition of ρ. Indeed, $u \notin \mathcal{B}$ implies $\rho(u) = 0$ and the unique solution satisfying the above Cauchy problem is constant with constant value u.

Concerning item (b), let us first observe that, since $\eta(\tau) \in \mathcal{B} \subset B_r(w)$, it follows that $\langle J'(\eta(\tau)), v_0\rangle = \alpha > 0$ and, by definition of ρ, it follows that $\rho(\eta(\tau)) > 0$. Now, differentiating J with respect to τ, for all $\eta(\tau) \in \mathcal{B}$, we have that

$$\begin{aligned}\frac{d}{d\tau}(J(\eta(\tau)) &= \langle J'(\eta(\tau)), \eta'(\tau)\rangle = \langle J'(\eta(\tau)), -\rho(\eta(\tau))v_0\rangle \\ &= -\rho(\eta(\tau))\langle J'(\eta(\tau)), v_0\rangle = -\rho(\eta(\tau))\alpha < 0\end{aligned}$$

thus concluding that $J(\eta(\tau, u))$ is decreasing with respect to τ.

In order to prove item (c), being $\tau_0 > 0$ such that $\eta(\tau, u) \in \mathcal{B}$ for every $0 \leq \tau \leq \tau_0$, we can assume without loss of generality that

$$\|\eta(\tau, w) - w\| \leq \frac{r'}{2} \Longleftrightarrow \eta(\tau, w) \in \overline{B_{\frac{r'}{2}}(w)}, \text{ for every } 0 \leq \tau \leq \tau_0.$$

Thus, since $\rho(\eta(\tau, w)) = \text{dist}(\eta(\tau, w), \mathcal{B}^c) \geq \frac{r'}{2}$ it follows that

$$\frac{d}{d\tau}(J(\eta(\tau, w)) = -\rho(\eta(\tau, w))\alpha \leq \frac{r'\alpha}{2}$$

and, integrating in $[0, \tau_0]$ we finally get

$$J(\eta(\tau, w)) - J(w) \leq -\frac{r'\alpha}{2}\tau.$$

At this point, let us consider a suitable deformed path $\overline{\eta}_0 : \overline{D} \to X$ defined by

$$\overline{\eta}_{\tau_0}(t, s) := \eta(\tau_0, tw^+ + sw^-), \quad \text{for all } (t, s) \in \overline{D}$$

so that

$$\max_{(t,s)\in\overline{D}} J(\overline{\eta}_{\tau_0}(t, s)) < c_0.$$

Indeed, by (b) and the fact that η satisfies the initial condition $\eta(0, u) = u$, for all $(t, s) \in \overline{D} - \{(1, 1)\}$ we have

$$\begin{aligned}J(\overline{\eta}_{\tau_0}(t, s)) &= J(\eta(\tau_0, tw^+ + sw^-)) \leq J(\eta(0, tw^+ + sw^-)) \\ &= J(tw^+ + sw^-) = h^w(t, s) < c_0,\end{aligned}$$

and, for $(t, s) = (1, 1)$, by (c) we get

$$\begin{aligned}J(\overline{\eta}_{\tau_0}(1, 1)) &= J(\eta(\tau_0, w^+ + w^-)) = J(\eta(\tau_0, w)) \\ &\leq J(w) - \frac{r'\alpha}{2}\tau_0 < J(w) < c_0.\end{aligned}$$

Then, $\overline{\eta}_{\tau_0}(\overline{D}) \cap \mathcal{M} = \emptyset$, i.e.

$$\overline{\eta}_{\tau_0}(t,s) \notin \mathcal{M}, \text{ for all } (t,s) \in \overline{D}. \tag{3.21}$$

On the other side, defined $\Psi_{\tau_0} : \overline{D} \to \mathbb{R}^2$ such that

$$\Psi_{\tau_0} := \Big(\frac{\big\langle J'(\overline{\eta}_{\tau_0}(t,s)), (\overline{\eta}_{\tau_0}(t,s))^+\big\rangle}{t}, \frac{\big\langle J'(\overline{\eta}_{\tau_0}(t,s)), (\overline{\eta}_{\tau_0}(t,s))^-\big\rangle}{s}\Big),$$

we observe that, for all $(t,s) \in \partial D$, by (3.20) and (a) for $\tau = \tau_0$, one has

$$\Psi_{\tau_0}(t,s) = \Big(\langle J'(tw^+ + sw^-), w^+\rangle, \langle J'(tw^+ + sw^-), w^-\rangle\Big) = \Phi^w(t,s).$$

Then, since by Brouwer's topological degree

$$\deg(\Psi_{\tau_0}, D, (0,0)) = \deg(\Phi^w, D, (0,0)) = \mathrm{sgn}(\det(\Phi^w)'(1,1)) = 1,$$

we get that Ψ_{τ_0} has a zero $(\bar{t}, \bar{s}) \in D$ namely

$$\Psi_{\tau_0}(\bar{t},\bar{s}) = (0,0) \Longleftrightarrow \big\langle J'(\overline{\eta}_{\tau_0}(\bar{t},\bar{s})), (\overline{\eta}_{\tau_0}(\bar{t},\bar{s}))^{\pm}\big\rangle = 0.$$

Consequently there exists $(\bar{t},\bar{s}) \in D$ such that $\overline{\eta}_{\tau_0}(\bar{t},\bar{s}) \in \mathcal{M}$ and we have a contradiction with (3.21). We conclude that w is a critical point of J.

Finally, we prove that w has exactly two nodal domains or equivalently it changes sign exactly once. Let us observe that assumptions (a_1), (f_1) and (f_2) ensure that w is continuous and then $\widetilde{\Omega} = \{x \in \Omega : w(x) \neq 0\}$ is open. Suppose by contradiction that $\widetilde{\Omega}$ has more than two components or w has more than two nodal domains and, since w changes sign, without loss of generality, we can assume that

$$w = w_1 + w_2 + w_3, \text{ where } w_1 \geq 0, w_2 \leq 0, w_3 \neq 0,$$

and

$$\mathrm{supp}(w_i) \cap \mathrm{supp}(w_j) = \emptyset, \text{ for } i \neq j, i, j = 1,2,3.$$

Clearly it is understood that $w_i = 0$ on $\Omega \setminus \mathrm{supp}(w_i)$ for $i = 1,2,3$. So the disjointness of the supports combined with $J'(w) = 0$ implies

$$\langle J'(w_1 + w_2), w_1\rangle = 0 = \langle J'(w_1 + w_2), w_2\rangle.$$

Since $0 \neq w_1 = (w_1 + w_2)^+$ and $0 \neq w_2 = (w_1 + w_2)^-$, by previous arguments, there exist $t, s \in (0,1]$ such that $t(w_1 + w_2)^+ + s(w_1 + w_2)^+ \in \mathcal{M}$ namely $tw_1 + sw_2 \in \mathcal{M}$ and then $J(tw_1 + sw_2) \geq c_0$.

On the other side, $0 \neq w_3 \in \mathcal{N}$, Lemma 2.1 (i) and the arguments used in the proof of Lemma 2.3 imply that

$$J(tw_1 + sw_2) \leq J(w_1 + w_2) < J(w_1 + w_2) + J(w_3) = J(w) = c_0$$

then there is a contradiction and we conclude that $w_3 = 0$. Thus, the proof of Theorem 1.1 is complete.

□

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