Advanced Nonlinear Studies **14** (2014), 339–352

Existence for the Hénon Equation Involving the Fractional Laplacian

Xiaoli Chen, Yu Zhu *

Department of Mathematics
Jiangxi Normal University , Nanchang, Jiangxi 330022, P. R. China
e-mail: littleli_chen@163.com, zyu764@126.com

Received 19 March 2013
Communicated by Yiming Long

Abstract

In this paper, we consider the existence of solutions of the following problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda u + |x|^{\beta}|u|^{2^*_\alpha - 2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2\alpha$, is a smooth bounded domain, $\lambda > 0$, $\beta > 0$, $0 < \alpha < 2$, and $2^*_\alpha = \frac{2N}{N-\alpha}$ is the critical exponent. We show that problem (0.1) possesses a ground state solution provided that $N = 2\alpha$ and $\lambda_m < \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$ and $\beta > 0$ small or $N > (1 + \sqrt{2})\alpha$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$ and $\beta > 0$ small, where λ_m is the eigenvalue of $-\Delta$ with Dirichlet boundary condition.

1991 Mathematics Subject Classification. 35J60, 35J65.

Key words. existence, compactness, critical Hardy-Sobolev exponent, nonlinear system.

1 Introduction

In this paper, we consider the existence of solutions of the following problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda u + |x|^{\beta}|u|^{2^*_\alpha - 2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

*The first author is supported by NNSF of China(No:11261023,11326092), the foundation of teacher's division of Jiangxi Province(GJJ12203), Startup Foundation for Doctors of Jiangxi Normal University. The second author acknowledges the support by NNSF of China(No:11271170), GAN PO 555 program of Jiangxi, NSF of Jiangxi Province(20122BAB201008).

where $\Omega \subset \mathbb{R}^N$, $N \geq 2\alpha$, is a smooth bounded domain, $\lambda > 0, \beta > 0$, $0 < \alpha < 2$, and $2^*_\alpha = \frac{2N}{N-\alpha}$ is the critical exponent. By $N = 2\alpha$ we mean that $\alpha = \frac{1}{2}, 1, \frac{3}{2}$, or $N = 1, 2, 3$.

Problem (1.1) is a Brezis-Nirenberg type problem with the fractional Laplacian. In the case that $\alpha = 2$ and $\beta = 0$, such a problem has been extensively studied, see [1], [4],[11], [12] and [13] etc. If $\alpha = 2$ and $\beta \neq 0$, problem (1.1) is reminiscent of the Hénon equation in [14]

$$-\Delta u = |x|^\beta u^{p-1}, \quad u > 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

which stems from the study of rotating stellar structures. Existence results of problem (1.2) were obtained in [15, 19, 20] etc. Asymptotic behavior of the ground state solution was considered in [5, 6, 8, 9, 20] as either $p < 2^*$, $p \to 2^*$ or $\alpha \to \infty$. The Brezis-Nirenberg problem for the critical Hénon equation

$$\begin{cases} -\Delta u = \lambda u + |x|^\beta |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.3}$$

has been studied in [16] and [18]. A particular feature of problem (1.3) is that the weighted function $|x|^\beta$ attains its maximum value at a point on the boundary of the domain, in contrast with problems usually concerned, which generally assume the weighted function having the maximum value at an interior point of the domain. It was proved in [16] that problem (1.3) always has a solution provided that $N \geq 7$ and $\beta > 0$ small. This result was improved in [18] for $N \geq 5$ and $\beta > 0$ small.

In this paper, we study the existence of the ground state solution of critical Hénon problem (1.1) with the fractional Laplacian. Critical problems with the fractional Laplacian were investigated in [23] for $\alpha = 1$, and in [3] for general case. The fractional Laplacian is a nonlocal operator. Using the idea in [7], by Dirichlet to Neumann mapping, one usually transforms such problems to local one, then variational method can be applied. Without loss for generality, we assume that $(1, 0, \cdots, 0) \in \partial\Omega$ and Ω is contained in the unit ball of $\mathbb{R}^n$ centered at the origin. Our main result is the following.

Theorem 1.1 *Let $N = 2\alpha$ and $\lambda_m < \lambda < \lambda_{m+1}$ or $N > (1 + \sqrt{2})\alpha$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Then, for every $\beta > 0$ sufficiently small, there exists a ground state solution to problem* (1.1).

In section 2, we describe a variational framework to study the ground state solution of problem (1.1). We prove Theorem 1.1 in section 3.

2 Preliminaries

Let (φ_j, ρ_j) be the eigenfunctions and eigenvalues of $-\Delta$ in Ω with zero Dirichlet boundary condition. For any $u = \sum_{j=1}^\infty c_j\varphi_j$ in the space

$$H_0^{\frac{\alpha}{2}}(\Omega) = \{u = \sum_{j=1}^\infty c_j\varphi_j \in L^2(\Omega) : \|u\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = (\sum_{j=1}^\infty c_j^2\rho_j^{\frac{\alpha}{2}})^{\frac{1}{2}} < +\infty\},$$

we define

$$(-\Delta)^{\frac{\alpha}{2}}u = \sum_{j=1}^\infty c_j\rho_j^{\frac{\alpha}{2}}\varphi_j,$$

and we have

$$\|u\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = \|(-\Delta)^{\frac{\alpha}{4}}u\|_{L^2(\Omega)}.$$

The nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$ can be realized as [7]. Precisely, for any $u \in H_0^{\frac{\alpha}{2}}(\Omega)$, the solution $w \in H^1_{0,L}(C_\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(y^{1-\alpha}\nabla w) = 0, & C_\Omega = \Omega \times (0,\infty), \\ w = 0, & \partial_L C_\Omega = \partial\Omega \times (0,\infty), \\ w = u, & \Omega \times \{0\} \end{cases} \tag{2.1}$$

is referred to the extension $w = E_\alpha(u)$ of u, where

$$H^1_{0,L}(C_\Omega) = \{w \in L^2(C_\Omega) : w = 0 \text{ on } \partial_L C_\Omega,\ \kappa_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy < \infty\}$$

is a space equipped with the norm $\|w\|^2_{H^1_{0,L}(C_\Omega)} = \kappa_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy$, here κ_α is a normalization constant. It was shown in [7], see also [2], that

$$-\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y} = (-\Delta)^{\frac{\alpha}{2}} u.$$

Therefore, problem (1.1) can be equivalently reduced to the problem

$$\begin{cases} \mathrm{div}(y^{1-\alpha}\nabla w) = 0, & \text{in} \quad C_\Omega, \\ w = 0, & \text{on} \quad \partial_L C_\Omega, \\ -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y} = \lambda w + |x|^\beta |w|^{2^*_\alpha - 2} w, & \text{in} \quad \Omega \times \{y = 0\}. \end{cases} \tag{2.2}$$

The energy functional associated to (2.2) is given by

$$J(w) = \frac{\kappa_\alpha}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \frac{\lambda}{2}\int_\Omega |w|^2\, dx - \frac{1}{2^*_\alpha}\int_\Omega |x|^\beta |w|^{2^*_\alpha}\, dx, \tag{2.3}$$

which is a C^1 functional on $H^1_{0,L}(C_\Omega)$. Critical points w of J correspond to solutions of (2.2), and $u = tr_\Omega w$ is a solution of (1.1).

It was proved in [2] that $(\varphi_j, \rho_j^{\frac{\alpha}{2}})$ are the eigenfunctions and eigenvalues of $(-\Delta)^{\frac{\alpha}{2}}$ with zero Dirichlet boundary condition. Moreover, $E_\alpha(\varphi_j) = \varphi_j(x)\psi(\rho_j^{\frac{1}{2}} y)$, where ψ_j satisfies

$$\begin{cases} \psi'' + \frac{1-\alpha}{s}\psi' = \psi, \quad s > 0, \\ -\lim_{s\to 0^+} s^{1-\alpha}\psi' = \kappa_\alpha, \\ \psi(0) = 1. \end{cases} \tag{2.4}$$

Hence, for any $u = \sum_{j=1}^\infty c_j\varphi_j \in H_0^{\frac{\alpha}{2}}(\Omega)$, we have $E_\alpha(x,y) = \sum_{j=1}^\infty c_j\varphi_j(x)\psi(\rho_j^{\frac{1}{2}} y)$. Denote $\lambda_j = \rho_j^{\frac{\alpha}{2}}$. To treat problem (1.1) in the case $\lambda_m < \lambda < \lambda_{m+1}$, we may decompose $H_0^{\frac{\alpha}{2}}(\Omega)$ as

$$H_0^{\frac{\alpha}{2}}(\Omega) = H^+ \oplus H^-,$$

where $H^- = \mathrm{span}\{\varphi_j, 1 \le j \le m\}$ and $H^+ = \mathrm{span}\{\varphi_j, j \ge m+1\}$. Let $\Phi_j = E_\alpha(\varphi_j), j = 1, \cdots$. Correspondingly, we have

$$E = E(C_\Omega) = H^1_{0,L}(C_\Omega) = E^+ \oplus E^-$$

with $E^- = \text{span}\{\Phi_j, 1 \le j \le m\}$ and $E^+ = \text{span}\{\Phi_j, j \ge m+1\}$.

In order to find ground state solutions of (2.2), we introduce as [17] a sub-manifold of $H^1_{0,L}(C_\Omega)$:

$$\mathcal{N} = \{w \in H^1_{0,L}(C_\Omega) \setminus \{0\} : \langle \nabla J(w), w\rangle = 0, \nabla J(w) \in E^+\}. \tag{2.5}$$

The set $\mathcal{N}$ is the intersection of the standard Nehari manifold $\{w \in H^1_{0,L}(C_\Omega) \setminus \{0\} : \langle \nabla J(w), w\rangle = 0\}$ with the pre-image $(\nabla J)^{-1}(E^+)$.

Proposition 2.1 *The set $\mathcal{N}$ is a C^1 submanifold of E with codimension $m+1$. Moreover, every critical point of the restriction $J|_{\mathcal{N}}$ is a nontrivial critical point of J.*

Proof. The result can be proved as in [22], see also [18]. We sketch the proof here for reader's convenience. Let $F : E \setminus \{0\} \to \mathbb{R} \times E^-$ be a map defined by

$$F(w) = (\langle \nabla J(w), w\rangle, Q\nabla J(w)),$$

where Q is the orthogonal projection of E onto E^-, then $\mathcal{N} = F^{-1}(0)$. On the Cartesian product $\mathbb{R} \times E^- \simeq \mathbb{R}^{m+1}$, we consider the inner product

$$(t_1, z_1) \cdot (t_2, z_2) = t_1 t_2 + \langle z_1, z_2\rangle \quad \text{for } t_1, t_2 \in \mathbb{R},\ z_1, z_2 \in E^-.$$

We claim that for every $(t, z) \in \mathbb{R} \times E^-$, $(t, z) \neq (0, 0)$,

$$(DF(w)(tw + z)) \cdot (t, z) < 0. \tag{2.6}$$

This implies the first part of the proposition. Indeed, for $(t, z) \neq (0, 0)$, since

$$\langle \nabla J(w), w\rangle = \langle \nabla J(w), z\rangle = 0,$$

we may deduce that

$$\begin{aligned}
&(DF(w)(tw + z)) \cdot (t, z)\\
&= \left[\kappa_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla z|^2 \, dxdy - \lambda \int_\Omega |z|^2 \, dx\right]\\
&\quad - \int_\Omega \Big((2^*_\alpha - 2)t^2|w|^2 + 2(2^*_\alpha - 2)tzw + (2^*_\alpha - 1)|z|^2\Big)|x|^\beta |w|^{2^*_\alpha - 2} dx.
\end{aligned} \tag{2.7}$$

For $\lambda_m \le \lambda < \lambda_{m+1}$, it is ready to verify that (2.6) holds.

Next, we may verify as [18] that $w \in E$ is a critical point of J if and only if $w \in \mathcal{N}$ and $DJ(w)|_{T_w\mathcal{N}} = 0$. The proof is complete.

We recall that a ground state solution w to (2.2) is any element of $\mathcal{N}$ such that $DJ(w)$ vanishes on $T_w\mathcal{N}$ and $J(w) = c$, where

$$c = \inf_{\mathcal{N}} J. \tag{2.8}$$

By the argument in [21], for every $v \in E^+ \setminus \{0\}$, there is a unique continuous map pair $(f(v), g(v)) \in (0, \infty) \times E^-$ such that $F(f(v)v + g(v)) = 0$, and

$$J(f(v)v + g(v)) = \max_{t>0, z\in E^-} J(tv + z).$$

Then,

$$c = \inf_{\mathcal{N}} J = \inf_{v\neq 0, v\in E^+} J(f(v)v + g(v)) = \inf_{\{v\neq 0, v\in E^+\}} \max_{\{t>0, z\in E^-\}} J(tv + z). \tag{2.9}$$

3 Existence results

In this section, we will show that the variational problem (2.8) is achieved, and the minimizer of problem (2.8) is actually a solution of (2.2). Let

$$S_{\beta,p}(C_\Omega) = \inf_{w\in E, w\neq 0}\left\{\frac{\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dxdy}{(\int_\Omega |x|^\beta |w(x,0)|^p\,dx)^{2/p}}\right\}. \tag{3.1}$$

We known from [2] that $S_\alpha := S_{0,2^*_\alpha}(C_\Omega)$ is achieved if and only if $C_\Omega = \mathbb{R}^{N+1}_+$ and it is attained by w if $u = w(x,0)$ takes the form

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-\alpha}{4}}}{(\varepsilon + |x|^2)^{\frac{N-\alpha}{2}}}$$

with $\varepsilon > 0$ arbitrary. Denote by W the extension of the function

$$U(x) = \frac{1}{(1+|x|^2)^{\frac{N-\alpha}{2}}}, \quad x \in \mathbb{R}^N.$$

Then

$$W(x,y) = E_\alpha(U) = P^\alpha_y * U(x) = cy^\alpha \int_{\mathbb{R}^N} \frac{U(t)}{(|x-t|^2+y^2)^{\frac{N+\alpha}{2}}}\,dt.$$

We know from [3] that

$$|\nabla W(x,y)| \le \frac{C}{y} W(x,y) \tag{3.2}$$

for $(x,y) \in \mathbb{R}^{N+1}_+$. The extension of the function

$$U_\varepsilon(x) = \frac{\varepsilon^{\frac{N-\alpha}{4}}}{(\varepsilon + |x-x_0|^2)^{\frac{N-\alpha}{2}}} \tag{3.3}$$

has the form

$$W_\varepsilon(x,y) = \varepsilon^{\frac{\alpha-N}{4}} W\left(\frac{x-x_0}{\sqrt{\varepsilon}}, \frac{y}{\sqrt{\varepsilon}}\right)$$

for $\varepsilon > 0$.

Lemma 3.1 *Suppose that*

$$c < \frac{\alpha}{2N}(\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}}. \tag{3.4}$$

Then there exists a $v \in E^+ \setminus \{0\}$ such that

$$\max_{t>0, z\in E^-} J(tv+z) = J(f(v)v + g(v)) = c.$$

Proof. The proof is similar to that of Proposition 3.1 in [18], we omit it.

From Lemma 3.1, we know that there exists a minimizer of problem (2.8) provided that (3.4) holds. By Proposition 2.1, such a minimizer is actually a solution of problem (1.1). Therefore, to prove Theorem 1.1, it is sufficient to verify the condition (3.4).

Let

$$x_0 = (y_0, 0) = (1-\rho, 0, \cdots, 0) \in \mathbb{R}^N,\ z_0 = (x_0, 0) \in \mathbb{R}^{N+1},$$

where $\rho > 0$ is small and to be determined. Denote

$$\mathcal{A}_\rho = \{(x,y) : |(x,y) - z_0| < \rho, y \geq 0\}, \quad \mathcal{B}_\rho = \{(x,y) : |(x,y)| < \rho, y \geq 0\},$$

$$B_\rho(x_0) = \{x : |x - x_0| < \rho\}.$$

Let $\varphi \in C_0^\infty(C_\Omega)$ be a cut-off function satisfying

$$\varphi(x,y) = \begin{cases} 1, & (x,y) \in \mathcal{A}_{\frac{1}{2}\rho} \\ 0, & (x,y) \notin \mathcal{A}_\rho, \end{cases}$$

$0 \leq \varphi(x,y) \leq 1$ and $|\nabla\varphi(x,y)| \leq \frac{C}{\rho}$ for $(x,y) \in C_\Omega$. Let $w_\varepsilon = \varphi W_\varepsilon$. Then, $w_\varepsilon \in E$ for $\varepsilon > 0$ small. We have the following estimates for w_ε.

Lemma 3.2 *Supposing $N \geq 2\alpha$ and $\frac{\rho}{\sqrt{\varepsilon}} > 0$ large. Then we have*

$$\|w_\varepsilon\|_E^2 \leq (\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}} + O(\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}), \tag{3.5}$$

$$\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*} dx \geq (1-2\rho)^\beta \left((\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}} + O(\varepsilon^{\frac{N}{2}}\rho^{-N})\right), \tag{3.6}$$

$$\int_\Omega |w_\varepsilon(x,0)|^2 dx \geq \begin{cases} C\varepsilon^{\frac{\alpha}{2}} + O(\varepsilon^{\frac{N-\alpha}{2}}\rho^{2\alpha-N}), & N > 2\alpha, \\ C\varepsilon^{\frac{\alpha}{2}}\left(\ln\frac{\rho}{2} + \frac{1}{2}|\ln\varepsilon|\right) + O(\varepsilon^{\frac{\alpha}{2}}), & N = 2\alpha, \end{cases} \tag{3.7}$$

$$\int_\Omega |w_\varepsilon(x,0)| dx \leq C\varepsilon^{\frac{N-\alpha}{4}}\rho^\alpha, \tag{3.8}$$

and

$$\int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*-1} dx \leq C\varepsilon^{\frac{N-\alpha}{4}} + C\varepsilon^{\frac{N+\alpha}{4}}\rho^{-\alpha}, \tag{3.9}$$

where $N = 2\alpha$ implies that $\alpha = \frac{1}{2}, 1, \frac{3}{2}$, or correspondingly $N = 1, 2, 3$.

Proof. There holds

$$\begin{aligned} &\kappa_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla w_\varepsilon|^2 \, dxdy \\ &= \kappa_\alpha \int_{C_\Omega} y^{1-\alpha} (|\varphi\nabla W_\varepsilon|^2 + 2\langle W_\varepsilon \nabla\varphi, \varphi\nabla W_\varepsilon\rangle + |W_\varepsilon\nabla\varphi|^2) \, dxdy. \end{aligned} \tag{3.10}$$

Since $0 \leq U_\varepsilon(x) \leq \varepsilon^{\frac{N-\alpha}{4}}|x - x_0|^{\alpha-N}$, and the extension of $|x-x_0|^{\alpha-N}$ is $(|x-x_0|^2 + y^2)^{\frac{\alpha-N}{2}}$, we have

$$W(x,y) \leq \frac{1}{(|x|^2 + y^2)^{\frac{N-\alpha}{2}}} \tag{3.11}$$

and

$$W_\varepsilon(x,y) \leq \varepsilon^{\frac{\alpha-N}{4}} \frac{1}{(\frac{|x-x_0|^2}{\varepsilon} + \frac{y^2}{\varepsilon})^{\frac{N-\alpha}{2}}} = \frac{\varepsilon^{\frac{N-\alpha}{4}}}{(|x-x_0|^2 + y^2)^{\frac{N-\alpha}{2}}}.$$

For $(x,y)\in\mathcal{A}_\rho\setminus\mathcal{A}_{\frac{1}{2}\rho}$, we have

$$\begin{aligned}
&\int_{C_\Omega} y^{1-\alpha}|W_\varepsilon\nabla\varphi|^2\,dxdy\\
\le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{-2}\int_{\mathcal{A}_\rho\setminus\mathcal{A}_{\frac{1}{2}\rho}}\frac{y^{1-\alpha}}{\left(|x-x_0|^2+y^2\right)^{N-\alpha}}\,dxdy\\
=& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{\mathcal{B}_1\setminus\mathcal{B}_{\frac{1}{2}}}\frac{y^{1-\alpha}}{\left(|x|^2+y^2\right)^{N-\alpha}}\,dxdy\\
=& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}.
\end{aligned}\tag{3.12}$$

On the other hand, by (3.2),

$$\begin{aligned}
&2\int_{\mathcal{A}_\rho\setminus\mathcal{A}_{\frac{1}{2}\rho}} y^{1-\alpha}\langle W_\varepsilon\nabla\varphi,\varphi\nabla W_\varepsilon\rangle\,dxdy\\
\le& C\rho^{-1}\int_{\mathcal{A}_\rho\setminus\mathcal{A}_{\frac{1}{2}\rho}} y^{1-\alpha}|W_\varepsilon||\nabla W_\varepsilon|\,dxdy\\
=& C\varepsilon^{\frac{1}{2}}\rho^{-1}\int_{\mathcal{B}_{\frac{\rho}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} y^{1-\alpha}|W(x,y)||\nabla W(x,y)|\,dxdy\\
\le& C\varepsilon^{\frac{1}{2}}\rho^{-1}\int_{\mathcal{B}_{\frac{\rho}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} y^{-\alpha}|W(x,y)|^2\,dxdy.
\end{aligned}\tag{3.13}$$

For $(x,y)\in\mathcal{B}_{\frac{\rho}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}$, we have

$$\begin{aligned}
W(x,y) =& Cy^\alpha\int_{\mathbb{R}^N}\frac{U(z)}{\left(|x-z|^2+y^2\right)^{\frac{N+\alpha}{2}}}\,dz\\
=& Cy^\alpha\Big(\int_{\{|z|\le\mathcal{B}_{\frac{\rho}{4\sqrt{\varepsilon}}}\}}+\int_{\{|z|\ge\mathcal{B}_{\frac{\rho}{4\sqrt{\varepsilon}}}\}}\Big)\frac{1}{\left(|x-z|^2+y^2\right)^{\frac{N+\alpha}{2}}}\frac{1}{\left(1+|z|^2\right)^{\frac{N-\alpha}{2}}}\,dz\\
=& I_1+I_2.
\end{aligned}\tag{3.14}$$

If $|z|\le\frac{\rho}{4\sqrt{\varepsilon}}$, we have $(|x-z|^2+y^2)^{\frac{1}{2}}\ge\frac{\rho}{4\sqrt{\varepsilon}}$. Hence,

$$I_1\le C\varepsilon^{\frac{N+\alpha}{2}}\rho^{-(N+\alpha)}y^\alpha\int_{\{|z|\le\frac{\rho}{4\sqrt{\varepsilon}}\}}\frac{1}{|z|^{N-\alpha}}\,dz\le C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}.\tag{3.15}$$

Furthermore,

$$\begin{aligned}
I_2 \le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{\{|z|\ge\frac{\rho}{4\sqrt{\varepsilon}}\}}\frac{y^\alpha}{\left(|x-z|^2+y^2\right)^{\frac{N+\alpha}{2}}}\,dz\\
\le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{\mathbb{R}^N}\frac{y^\alpha}{\left(|z|^2+y^2\right)^{\frac{N+\alpha}{2}}}\,dz\\
=& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{\mathbb{R}^N}\frac{1}{\left(|z|^2+1\right)^{\frac{N+\alpha}{2}}}\,dz\\
=& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}.
\end{aligned}\tag{3.16}$$

Hence,

$$W(x,y) \le C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N} \tag{3.17}$$

for $(x,y) \in \mathcal{B}_{\frac{\rho}{\sqrt{\varepsilon}}} \setminus \mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}$. By (3.13) and (3.17),

$$\begin{aligned} &2\int_{C_\Omega} y^{1-\alpha}\langle W_\varepsilon \nabla\varphi, \varphi\nabla W_\varepsilon\rangle\, dxdy \\ \le& C\varepsilon^{\frac{2(N-\alpha)+1}{2}}\rho^{-(2(N-\alpha)+1)}\int_{\mathcal{B}_{\frac{\rho}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{1}{2}\frac{\rho}{\sqrt{\varepsilon}}}} y^{-\alpha}\, dxdy \\ =& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{\mathcal{B}_1\setminus\mathcal{B}_{\frac{1}{2}}} y^{-\alpha}\, dxdy = C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}. \end{aligned} \tag{3.18}$$

We also have

$$\begin{aligned} &\int_{C_\Omega} y^{1-\alpha}|\varphi\nabla W_\varepsilon|^2\, dxdy \\ =& \int_{C_\Omega} y^{1-\alpha}|\nabla W_\varepsilon|^2\, dxdy + \int_{C_\Omega} y^{1-\alpha}(\varphi^2-1)|\nabla W_\varepsilon|^2\, dxdy \\ \le& \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla W_\varepsilon|^2\, dxdy + \int_{C_\Omega\setminus\mathcal{A}_{\frac{1}{2}\rho}} y^{1-\alpha}|\nabla W_\varepsilon|^2\, dxdy \\ =& \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla W|^2\, dxdy + \int_{C_{\frac{\Omega-x_0}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} y^{1-\alpha}|\nabla W|^2\, dxdy. \end{aligned} \tag{3.19}$$

For $(x,y) \in C_{\frac{\Omega-x_0}{\sqrt{\varepsilon}}} \setminus \mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}$, we have $y > \frac{\rho}{2\sqrt{\varepsilon}}C_0$ if $\frac{\rho}{\sqrt{\varepsilon}}$ large. Hence, we deduce by (3.2) and (3.11) that,

$$\begin{aligned} &\int_{C_{\frac{\Omega-x_0}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} y^{1-\alpha}|\nabla W|^2\, dxdy \le C\int_{C_{\frac{\Omega-x_0}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} \frac{1}{y^{1+\alpha}}W^2\, dxdy \\ \le& C\int_{C_{\frac{\Omega-x_0}{\sqrt{\varepsilon}}}\setminus\mathcal{B}_{\frac{\rho}{2\sqrt{\varepsilon}}}} \frac{1}{y^{1+\alpha}}\frac{1}{(|x|^2+y^2)^{N-\alpha}}\, dxdy \\ \le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{(\mathbb{R}^{N+1}_+\setminus\mathcal{B}_1)\cap\{y>C_0\}} \frac{1}{y^{1+\alpha}}\frac{1}{(|x|^2+y^2)^{N-\alpha}}\, dxdy \\ \le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}\int_{(\mathbb{R}^{N+1}_+\setminus\mathcal{B}_1)\cap\{y>C_0\}} \frac{1}{y^{2N-\alpha+1}}\, dxdy \\ \le& C\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}. \end{aligned} \tag{3.20}$$

Thus, we obtain from (3.19) and (3.20) that

$$\begin{aligned} \kappa_\alpha\int_{C_\Omega} y^{1-\alpha}|\varphi\nabla W_\varepsilon|^2\, dxdy &\le \|W\|_{E(\mathbb{R}^{N+1}_+)} + O(\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}) \\ &= (\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}} + O(\varepsilon^{\frac{N-\alpha}{2}}\rho^{\alpha-N}). \end{aligned} \tag{3.21}$$

Equation (3.5) follows from (3.10), (3.12), (3.18) and (3.21). On the other hand,

$$\begin{aligned}
&\Big|\int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*}\,dx - \int_{\mathbb{R}^N} |U_\varepsilon|^{2_\alpha^*}\,dx\Big| \\
&\le \int_{\mathbb{R}^N\setminus B_{\frac{1}{2}\rho}(x_0)} |U_\varepsilon(x)|^{2_\alpha^*}\,dx \\
&= \int_{\mathbb{R}^N\setminus B_{\frac{1}{2}\rho}(x_0)} \frac{\varepsilon^{\frac{N}{2}}}{(\varepsilon+|x-x_0|^2)^N}\,dx \\
&\le C\varepsilon^{\frac{N}{2}} \int_{\frac{1}{2}\rho}^{+\infty} r^{-N-1}\,dr = C\varepsilon^{\frac{N}{2}}\rho - N = O\Big(\varepsilon^{\frac{N}{2}}\rho^{-N}\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^N} |U_\varepsilon|^{2_\alpha^*} dx &= \int_{\mathbb{R}^N} \frac{\varepsilon^{\frac{N}{2}}}{(\varepsilon+|x-x_0|^2)^N}\,dx = \int_{\mathbb{R}^N} \frac{\varepsilon^{\frac{N}{2}}}{\varepsilon^N(1+|\frac{x-x_0}{\sqrt{\varepsilon}}|^2)^N}\,dx \\
&= \|U\|^{2_\alpha^*}_{L^{2_\alpha^*}(\mathbb{R}^N)} = (\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}},
\end{aligned}$$

we obtain

$$\int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*}\,dx = (\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}} + O\Big(\varepsilon^{\frac{N}{2}}\rho^{-N}\Big).$$

Observe that Ω is contained in the unit ball centered at the origin, so $|x|^\beta \ge (1-2\rho)^\beta$ and

$$\begin{aligned}
\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*} dx &\ge (1-2\rho)^\beta \int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*} dx \\
&= (1-2\rho)^\beta \Big((\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}} + O\Big(\varepsilon^{\frac{N}{2}}\rho^{-N}\Big)\Big),
\end{aligned}$$

namely, (3.6) holds. Now, we estimate

$$\begin{aligned}
\int_\Omega |w_\varepsilon(x,0)|^2 dx &= \int_{B_{\frac{1}{2}\rho}(x_0)} \frac{\varepsilon^{\frac{N-\alpha}{2}}}{(\varepsilon+|x-x_0|^2)^{N-\alpha}} dx \\
&\quad + \int_{B_\rho(x_0)\setminus B_{\frac{1}{2}\rho}(x_0)} \frac{\varepsilon^{\frac{N-\alpha}{2}}\varphi(x,0)^2}{(\varepsilon+|x-x_0|^2)^{N-\alpha}} dx \\
&= J_1 + J_2.
\end{aligned} \tag{3.22}$$

In the case $N > 2\alpha$,

$$\begin{aligned}
J_1 &\ge \int_{B_{\varepsilon^{\frac{1}{2}}}(x_0)} \frac{\varepsilon^{\frac{N-\alpha}{2}}}{(2\varepsilon)^{N-\alpha}} dx + \int_{B_{\frac{1}{2}\rho}(x_0)\setminus B_{\varepsilon^{\frac{1}{2}}}(x_0)} \frac{\varepsilon^{\frac{N-\alpha}{2}}}{(2|x-x_0|^2)^{N-\alpha}} dx \\
&= C\varepsilon^{\frac{\alpha-N}{2}} \int_0^{\varepsilon^{\frac{1}{2}}} r^{N-1} dr + C\varepsilon^{\frac{N-\alpha}{2}} \int_{\varepsilon^{\frac{1}{2}}}^{\rho} r^{2\alpha-N-1} dr \\
&= C\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{\frac{N-\alpha}{2}}\rho^{2\alpha-N} = C\varepsilon^{\frac{\alpha}{2}} + O\Big(\varepsilon^{\frac{N-\alpha}{2}}\rho^{2\alpha-N}\Big)
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned} J_2 &\le \varepsilon^{\frac{N-\alpha}{2}} \int_{B_\rho(x_0)\setminus B_{\frac{1}{2}\rho}(x_0)} \frac{1}{|x-x_0|^{2(N-\alpha)}} dx \\ &= C\varepsilon^{\frac{N-\alpha}{2}} \int_{\frac{1}{2}\rho}^{\rho} r^{2\alpha-N-1} dr = C\varepsilon^{\frac{N-\alpha}{2}} \rho^{2\alpha-N}. \end{aligned} \tag{3.24}$$

In the case $N = 2\alpha$, that is, $\alpha = \frac{1}{2}, 1, \frac{3}{2}$, or correspondingly $N = 1, 2, 3$, we have

$$\begin{aligned} J_1 &\ge \varepsilon^{\frac{\alpha}{2}} \int_{B_{\varepsilon^{\frac{1}{2}}}(x_0)} \frac{1}{(2\varepsilon)^{N-\alpha}} dx + \varepsilon^{\frac{\alpha}{2}} \int_{B_{\frac{1}{2}\rho}(x_0)\setminus B_{\varepsilon^{\frac{1}{2}}}(x_0)} \frac{1}{(2|x-x_0|^2)^{N-\alpha}} dx \\ &= C\varepsilon^{-\frac{\alpha}{2}} \int_0^{\varepsilon^{\frac{1}{2}}} r^{2\alpha-1} dr + C\varepsilon^{\frac{\alpha}{2}} \int_{\varepsilon^{\frac{1}{2}}}^{\frac{1}{2}\rho} r^{-1} dr \\ &= C\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{\frac{\alpha}{2}} \left(\ln \frac{\rho}{2} + \frac{1}{2} |\ln \varepsilon| \right) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} J_2 &\le \varepsilon^{\frac{\alpha}{2}} \int_{B_\rho(x_0)\setminus B_{\frac{1}{2}\rho}(x_0)} \frac{1}{|x-x_0|^{2(N-\alpha)}} dx \\ &= C\varepsilon^{\frac{\alpha}{2}} \int_{\frac{1}{2}\rho}^{\rho} r^{-1} dr = C\varepsilon^{\frac{\alpha}{2}}. \end{aligned} \tag{3.26}$$

Thus, (3.7) follows from (3.22)-(3.26). In the same way, we obtain

$$\begin{aligned} \int_\Omega |w_\varepsilon(x,0)| dx &= \int_{B_\rho(x_0)} \frac{\varphi \varepsilon^{\frac{N-\alpha}{4}}}{(\varepsilon + |x-x_0|^2)^{\frac{N-\alpha}{2}}} dx \\ &\le C\varepsilon^{\frac{N-\alpha}{4}} \int_0^\rho \frac{1}{r^{N-\alpha}} r^{N-1} dr = C\varepsilon^{\frac{N-\alpha}{4}} \rho^\alpha. \end{aligned}$$

which is (3.8). Finally, there holds

$$\begin{aligned} &\int_\Omega |w_\varepsilon(x,0)|^{2^*_\alpha - 1} dx \\ &\le \left(\int_{B_{\varepsilon^{\frac{1}{2}}}(x_0)} + \int_{B_\rho(x_0)\setminus B_{\varepsilon^{\frac{1}{2}}}(x_0)} \right) \frac{\varepsilon^{\frac{N+\alpha}{4}}}{(\varepsilon + |x-x_0|^2)^{\frac{N+\alpha}{2}}} dx \\ &\le C\varepsilon^{\frac{N+\alpha}{4}} \int_0^{\varepsilon^{\frac{1}{2}}} \varepsilon^{-\frac{N+\alpha}{2}} r^{N-1} dr + C\varepsilon^{\frac{N+\alpha}{4}} \int_{\varepsilon^{\frac{1}{2}}}^\rho \frac{1}{r^{N+\alpha}} r^{N-1} dr \\ &= C\varepsilon^{\frac{N-\alpha}{4}} + C\varepsilon^{\frac{N+\alpha}{4}} \rho^{-\alpha}, \end{aligned}$$

which implies (3.9).

Proposition 3.1 *There results*

$$c < \frac{\alpha}{2N} (\kappa_\alpha S_\alpha)^{N/\alpha}.$$

Proof. We will check that

$$\max_{t>0,v\in E^-} J(tw_\varepsilon + v) < \frac{\alpha}{2N}(\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}}. \tag{3.27}$$

Let $\omega = \Omega \setminus \sup\varphi(\cdot,0)$. By Lemma 3.3 in [22], $h \mapsto \|h\|_{L^{2^*_\alpha}(\omega)}$ defines a norm on H^- if $h \in H^-$. Since $\dim H^- = m < +\infty$, all the norms are equivalent on H^-. Using the fact that$\|v\|_E = \|v(\cdot,0)\|_{H_0^{\frac{\alpha}{2}}}$ for $v \in E$, we have $\int_\omega |v(x,0)|^{2^*_\alpha}|x|^\beta\,dx \le C\|v\|_E$ and $\|v(\cdot,0)\|_{L^\infty} \le C\|v\|_E$.

Moreover, $0 \notin \omega$, so $|x|^\beta$ is bounded away from zero for $x \in \omega$. For every $t > 0$ and every $v \in E^-$, by convexity we deduce

$$\begin{aligned}
&\int_\Omega |x|^\beta |tw_\varepsilon(x,0) + v(x,0)|^{2^*_\alpha}\,dx \\
&= \int_{\Omega\setminus\omega} |x|^\beta |tw_\varepsilon(x,0) + v(x,0)|^{2^*_\alpha}\,dx + \int_\omega |x|^\beta |v(x,0)|^{2^*_\alpha}\,dx \\
&\ge t^{2^*_\alpha}\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2^*_\alpha}\,dx + 2^*_\alpha t^{2^*_\alpha-1}\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2^*_\alpha-1}v(x,0)\,dx \\
&\quad + 2^*_\alpha C\|v\|_E^{2^*_\alpha}.
\end{aligned} \tag{3.28}$$

It follows that

$$\begin{aligned}
J(tw_\varepsilon + v) \le J(tw_\varepsilon) &+ \kappa_\alpha \int_{C_\Omega} ty^{1-\alpha}\nabla w_\varepsilon \nabla v + \frac{\kappa_\alpha}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla v|^2\,dxdy \\
&- \lambda t\int_\Omega w_\varepsilon(x,0)v(x,0)\,dx - \frac{\lambda}{2}\int_\Omega |v(x,0)|^2\,dx \\
&- t^{2^*_\alpha-1}\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2^*_\alpha-1}v(x,0)\,dx - C\|v\|_E^{2^*_\alpha}.
\end{aligned} \tag{3.29}$$

By the assumption $\lambda_m \le \lambda < \lambda_{m+1}$,

$$\kappa_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla v|^2\,dxdy - \lambda\int_\Omega |v(x,0)|^2\,dx \le (\lambda_m - \lambda)\|v\|_E^2 \le 0. \tag{3.30}$$

In particular, we can write

$$J(tw_\varepsilon + v) \le A(t^2 + t\|v\|_E + t^{2^*_\alpha-1}\|v\|_E) - B(t^{2^*_\alpha} + \|v\|_E^{2^*_\alpha})$$

for suitable constants $A > 0$ and $B > 0$. Hence there exists $R > 0$ such that, for ε small, $t > R$ and $v \in E^-$ there holds $J(tw_\varepsilon + v) \le 0$.

We claim that

$$J(tw_\varepsilon + v) \le J(tw_\varepsilon) + O\left(\varepsilon^{\frac{N-\alpha}{4}}\right)\|v\|_E - C\|v\|_E^{2^*_\alpha} \le J(tw_\varepsilon) + O\left(\varepsilon^{\frac{N(N-\alpha)}{2(N+\alpha)}}\right). \tag{3.31}$$

whenever $t \le R$. Indeed, integrating by part and using the definition of E^-, we obtain

$$\begin{aligned}
&\kappa_\alpha\int_{C_\Omega} y^{1-\alpha}\nabla w_\varepsilon\nabla v\,dxdy - \lambda\int_\Omega w_\varepsilon(x,0)v(x,0)\,dx \\
&= -\kappa_\alpha\int_{C_\Omega}\operatorname{div}(y^{1-\alpha}\nabla v)w_\varepsilon\,dxdy + \kappa_\alpha\int_{\partial C_\Omega} y^{1-\alpha}\frac{\partial v}{\partial \nu}w_\varepsilon\,dS - \lambda\int_\Omega w_\varepsilon(x,0)v(x,0)dx \\
&= \int_\Omega w_\varepsilon(x,0)(-\Delta)^{\frac{\alpha}{2}}v(x,0)\,dx - \lambda\int_\Omega w_\varepsilon(x,0)v(x,0)\,dx \le |\lambda_m - \lambda|\int_\Omega |w_\varepsilon(x,0)v(x,0)|\,dx \\
&\le |\lambda_m - \lambda|\|v(\cdot,0)\|_{L^\infty}\int_\Omega |w_\varepsilon(x,0)|\,dx \le C|\lambda_m - \lambda|\|v\|_E\int_\Omega |w_\varepsilon(x,0)|\,dx
\end{aligned}$$

and

$$|\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*-1} v\,dx| \le \|v(\cdot,0)\|_{L^\infty} \int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*-1}\,dx \le C\|v\|_E \int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*-1}\,dx.$$

By (3.8), (3.9) and choosing $\rho = \varepsilon^\gamma$, where $0 < \gamma < \frac{N-2\alpha}{2(N-\alpha)}$, since $\frac{N(N-\alpha)}{N+\alpha} > \alpha$ whenever $N > (1+\sqrt{2})\alpha$ and $\frac{N-2\alpha}{2(N-\alpha)} < \frac{1}{2}$, we find

$$\int_\Omega |w_\varepsilon(x,0)|\,dx \le C\varepsilon^{\frac{N-\alpha}{4}}, \quad \int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*-1}\,dx \le C\varepsilon^{\frac{N-\alpha}{4}}.$$

By the Young inequality,

$$O\left(\varepsilon^{\frac{N-\alpha}{4}}\right)\|v\|_E \le O\left(\varepsilon^{\frac{N(N-\alpha)}{2(N+\alpha)}}\right) + C\|v\|_E^{2_\alpha^*}.$$

Therefore, together with (3.30), we see that (3.31) holds.

By Lemma 3.2, for $\beta > 0$ and $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\max_{t>0, v\in E^-} J(tw_\varepsilon + v)\\
&\le \max_{t>0} J(tw_\varepsilon) + O\left(\varepsilon^{\frac{N(N-\alpha)}{2(N+\alpha)}}\right)\\
&= \frac{\alpha}{2N}\left(\frac{\|w_\varepsilon\|_E^2 - \lambda\|w_\varepsilon(x,0)\|_{L^2(\Omega)}^2}{(\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*}\,dx)^{2/2_\alpha^*}}\right)^{\frac{N}{\alpha}} + O\left(\varepsilon^{\frac{N(N-\alpha)}{2(N+\alpha)}}\right)\\
&\le \frac{\alpha}{2N}\left(\frac{1}{(1-2\varepsilon^\gamma)^{2\beta/2_\alpha^*}}\left(\kappa_\alpha S_\alpha - C\lambda\varepsilon^{\frac{\alpha}{2}} + O\left(\varepsilon^{\frac{N-\alpha}{2}}\varepsilon^{\gamma(\alpha-N)}\right)\right)\right)^{\frac{N}{\alpha}} + O\left(\varepsilon^{\frac{N(N-\alpha)}{2(N+\alpha)}}\right)\\
&< \frac{\alpha}{2N}(\kappa_\alpha S_\alpha)^{\frac{N}{\alpha}}.
\end{aligned}$$

Assume now that $N = 2\alpha$, and $\lambda_m < \lambda < \lambda_{m+1}$. By (3.8), (3.9) and choose $\rho = r^\gamma$, where $1 > r > \varepsilon$ is a constant , $0 < \gamma < \frac{1}{2}$, we get

$$\int_\Omega |w_\varepsilon(x,0)|\,dx \le C\varepsilon^{\frac{\alpha}{4}}, \quad \int_\Omega |w_\varepsilon(x,0)|^{2_\alpha^*-1}\,dx \le C\varepsilon^{\frac{\alpha}{4}}.$$

By the assumption $\lambda_m < \lambda < \lambda_{m+1}$,

$$\kappa_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla v|^2\,dxdy - \lambda\int_\Omega |v(x,0)|^2\,dx \le (\lambda_m - \lambda)\|v\|_E^2 = -C_1\|v\|_E^2. \tag{3.32}$$

Inequalities (3.29) and (3.32) imply that, for $t \le R$,

$$J(tw_\varepsilon + v) \le J(tw_\varepsilon) + O(\varepsilon^{\frac{\alpha}{4}})\|v\|_E - C_1\|v\|_E^2 < J(tw_\varepsilon) + O(\varepsilon^{\frac{\alpha}{2}})$$

for $\varepsilon > 0$ small. We obtain, from Lemma 3.2, for $\beta > 0$ and $\varepsilon > 0$ small enough,

$$\begin{aligned}
\max_{t>0, v\in E^-} J(tw_\varepsilon + v) &\le \frac{1}{4}\left(\frac{\|w_\varepsilon\|_E^2 - \lambda\|w_\varepsilon(x,0)\|_{L^2(\Omega)}^2}{(\int_\Omega |x|^\beta |w_\varepsilon(x,0)|^{2_\alpha^*}\,dx)^{2/2_\alpha^*}}\right)^2 + O(\varepsilon^{\frac{\alpha}{2}})\\
&\le \frac{1}{4}\left(\frac{1}{(1-2\rho)^{2\beta/2_\alpha^*}}\left(\kappa_\alpha S_\alpha - C\lambda\varepsilon^{\frac{\alpha}{2}}|\ln\varepsilon| + O\left(\varepsilon^{\frac{\alpha}{2}}\right)\right)\right)^2 + O(\varepsilon^{\frac{\alpha}{2}})\\
&< \frac{1}{4}(\kappa_\alpha S_\alpha)^2.
\end{aligned}$$

□

Proof of Theorem 1.1. By Lemma 3.1 and Proposition 3.1, there exists $w \in \mathcal{N}$ such that $J(w) = c$ and $DJ(w)|_{T_w\mathcal{N}} = 0$. It follows from Proposition 2.1 that $DJ(w) = 0$ on E. □

References

[1] A. Ambrosetti and M. Struwe, *A note on the problem* $-\Delta u = \lambda u + u|u|^{2^*-2}$, Manuscripta Math., **54** (1986), 373–379.

[2] C. Brändle, E. Colorado, A. De Pablo and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinb., **143** (2013), 39–71.

[3] B. Barrios, E. Colorado, A. De Pablo and U. Sánchez, *On some critical problems for the fractional Laplacian oprator*, J. Differential Equations, **252** (2012), 6133–6162.

[4] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), 437–477.

[5] J. Byeon and Z. Wang, *On the Hénon equation: asymptotic profile of ground states I*, Ann. Inst. H. Poincaré. Annales: Analyse Non Linéaire, **23** (2006), 803–828.

[6] J. Byeon and Z. Wang, *On the Hénon equation: asymptotic profile of ground states II*, J. Differential Equations, **216** (2005), 78–108.

[7] L. Caffarelli and L. Silvestre, *An extention problem related to the fractional Laplacian*, Comm. in Part. Diff. Equa., **32** (2007), 1245–1260.

[8] D. Cao and S. Peng, *The asymptotic behavior of the ground state solutions for Hénon equation*, J. Math. Anal. Appl., **278** (2003), 1–17.

[9] D. Cao, S. Peng and S. Yan, *Asymptotic behaviour of ground state solutions for the Hénon equation*, IMA Journal of Applied Mathematics, **74** (2009), 468–480.

[10] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, *Regularity of radial extremal solutions for some non local semilinear equations*, Comm. in Part. Diff. Equa., **36** (2011), 1353–1384.

[11] A. Capozzi, D. Fortunato, and G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincarie Anal. Non Linéaire, **2** (1985), 463–470.

[12] M. Clapp and T. Weth, *Multiple solutions for the Brezis-Nirenberg problem*, Adv. Differential Equations, **10** (2005), 463–480.

[13] F. Gazzola and B. Ruf, *Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations*, Adv. Differential Equations, **2** (1997), 555–572.

[14] M. Hénon, *Numerical experiments on the stability of spherical stellar systems*, Astronom. Astrophys, **24** (1973), 229–238.

[15] W. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J., **31** (1982), 801–807.

[16] W. Long and J. Yang, *Existence for critical Hénon type equations*, Diff. and Int. Equas., **25** (2012), 567–578.

[17] A. Pankov,*Periodic Nonlinear Schrödinger equation with application to photonic crystals*, Milan J. Math., **73** (2005), 259–287.

[18] S. Secchi,*The Brezis-Nirenberg problem for the Hénon equation: ground state solutions*, Adv. Nonli. Stud., **12** (2012), 383–394.

[19] E. Serra, *Non radial positive solutions for the Hénon equation with critical growth*, Calc. Var. Partial Differential Equations, **23** (2005), 301–326.

[20] D. Smets, J. B. Su and M. Willem, *Non-radial ground states for the Hénon equation*, Commun. Contemp. Math., **4** (2002), 467–480.

[21] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal., **257** (2009), 3802–3822.

[22] A. Szulkin, T. Weth and M. Willem, *Ground state solutions for a semilinear problem with critical exponent*, Differential Integral Equations, **22** (2009), 913–926.

[23] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, **42** (2011), 21–41.