

Spinorial Yamabe Type Equations on S^3 via Conley Index

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Abstract

In this paper, we consider the existence of solutions to spinorial Yamabe type equations on S^3 with a prescribed function H . We give a new condition about H under which at least one non-trivial solution exists. It is proved by a method based on Conley index theory applied to a reduced functional defined on a certain finite dimensional non-compact manifold. Our proof is also applicable to similar equations on S^m for all $m \geq 2$.

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1 Introduction

Let (M, g, ρ) be an m -dimensional compact spin manifold with a metric g and a spin structure $\rho : P_{Spin}(M) \rightarrow P_{SO}(M)$. We denote by $\mathbb{S}(M) \rightarrow M$ the associated spinor bundle. We are concerned in this paper with spinorial Yamabe type equations of the following form

$$D\psi = H(x)|\psi|^{2^*-2}\psi \quad \text{on } M, \tag{1.1}$$

where D is the (Atiyah-Singer-) Dirac operator, $\psi \in C^1(M, \mathbb{S}(M))$ is a spinor on M , $H : M \rightarrow \mathbb{R}$ is a given function on M and $2^* = \frac{2m}{m-1}$ is the critical Sobolev exponent of the Sobolev embedding $H^{1/2}(M, \mathbb{S}(M)) \subset L^p(M, \mathbb{S}(M))$ for $1 \leq p \leq 2^*$. Note that the embedding is compact for $1 \leq p < 2^*$ while it is not for $p = 2^*$. Thus the equation (1.1) is a kind of critical equation. Throughout this paper, we shall use the same notation used in [22]. Please refer to [22, Section 2] about these notations, if necessary. For geometrical and physical motivation to consider equation (1.1) and some

basic definitions and facts about spin structures on manifolds, Dirac operators and $H^{1/2}$ -spinors, please consult [5], [17], [18], [20], [24], [22, sections 1,2].

A fundamental analytical problem concerning the equation (1.1) is to determine conditions about H under which (1.1) admits a non-trivial solution. This and related problems were considered before by some authors ([5], [6], [7], [8], [9],[21], [28]). As was known by the work of Ammann, Humbert and Ahmedou in [9], if there exists a solution ψ to the equation (1.1), then the following holds

$$\int_M \nabla_X H |\psi|^{2^*} d\text{vol}_g = 0 \quad (1.2)$$

for any conformal vector field X on M . Thus, for some class of functions H on M , there are no non-trivial solutions to (1.1).

Recall that the equation (1.1) is the Euler-Lagrange equation for the action functional

$$\mathcal{L}_H(\psi) = \frac{1}{2} \int_M \langle \psi, D\psi \rangle d\text{vol}_g - \frac{1}{2^*} \int_M H(x) |\psi|^{2^*} d\text{vol}_g \quad (1.3)$$

defined on $H^{1/2}(M, \mathbb{S}(M))$, see [22] for details. By the regularity result proved in [21, Appendix], any $H^{1/2}$ -weak solution to (1.1) is in fact a classical solution, so it is sufficient to find a non-trivial critical point of \mathcal{L}_H on $H^{1/2}(M, \mathbb{S}(M))$ in order to find non-trivial solution to (1.1). As explained in [22, section 1], finding non-trivial critical points, however, is not so straightforward due to the strong indefiniteness of the functional \mathcal{L}_H and the criticality of the exponent 2^* in the Lagrangian. Due to these, the associated variational problem is not compact and any critical point has infinite Morse index. As a result, the standard variational methods are not directly applicable to the functional (1.3).

In our previous paper [22], we gave some sufficient conditions about H under which there exists a non-trivial solution to (1.1) on S^m ($m \geq 2$), the round m -sphere with the canonical spin structure. More precisely, we considered the case $H(x) = \frac{m}{2}(1 + \epsilon K(x))$ for some $K \in C^2(S^m)$ and small $\epsilon > 0$. (Of course, the constant $\frac{m}{2}$ here is replaced by any other positive constant by a simple scaling argument). For particular cases $m = 2, 3$, it leads to geometrically interesting conclusions. First for the case $m = 2$, through Weierstrass representation (see [7]), it gives a condition about H under which there exists a conformal immersion $F : S^2 \rightarrow \mathbb{R}^3$ with prescribed mean curvature H . The problem of the existence of immersions $F : S^2 \rightarrow \mathbb{R}^3$ with prescribed mean curvature function was also considered, for example, in [11], [14], where other type of existence results were proven by different methods. An extension of the result just mentioned for $m = 2$ was proved by Ammann et al. [10] for the case $m = 3$. After a conformal change from g to $\tilde{g} = |\psi|^2 g$ (where ψ is a solution to the equation (1.1)), the solution ψ turns into a generalized Killing spinor on $(S^3, \tilde{g}, \rho_{\text{can}})$, where ρ_{can} is the canonical spin structure on S^3 , see [10, Proposition 4.10]. This combined with [10, Corollary 3.1] imply that if the conformal class and H are analytic, there exists a Ricci flat metric on a neighborhood of $\{0\} \times S^3$ in $\mathbb{R} \times S^3$ whose restriction to $\{0\} \times S^3$ has mean curvature H . Note that this interpretation does not continue to hold for $m \geq 4$. For more details, please refer to [10].

In order to state the result proved in [22], we introduce the following notation: For $H \in C^2(S^m)$, we denote by $\text{Crit}(H) = \{p \in S^m : \nabla H(p) = 0\}$ the set of critical points of H . We define, for $0 \leq j \leq m$, $F_j = \#\{p \in \text{Crit}(H) : m(p, H) = j, \Delta_{g_{S^m}} H(p) > 0\}$, where $m(p, H)$ is the Morse index of H at $p \in \text{Crit}(H)$. Notice that $F_m = 0$ since $\Delta_{g_{S^m}} H(p) < 0$ for any $p \in \text{Crit}(H)$ with $m(p, H) = m$. We always assume that H is a Morse function and $\Delta_{g_{S^m}} H(p) \neq 0$ for any $p \in \text{Crit}(H)$. For small $\epsilon > 0$, the existence of a non-trivial solution to (1.1) on S^m has been proven for H which satisfies the following conditions (see [22, Theorem 1.1–1.3] for more details):

1. When $m \geq 2$, $m \neq 3$: H satisfies at least one of the following conditions (i), (ii):

(i) For some $1 \leq k \leq m - 2$, there holds

$$\sum_{j=0}^k (-1)^{k-j} F_j \leq (-1)^k - 1,$$

or

(ii) There holds

$$\sum_{j=0}^{m-1} (-1)^j F_j \neq 1.$$

2. When $m = 3$: H satisfies one of the following conditions (i)–(iii):

$$(i) -F_1 + F_0 \geq 2, \quad \text{or } (ii) F_2 - F_1 + F_0 \leq 0, \quad \text{or } (iii) F_2 - F_1 \geq 1.$$

Note that, by the Poincaré-Hopf index theorem, it is easy to see that the index counting condition 1-(ii)

$$\sum_{j=0}^{m-1} (-1)^j F_j \neq 1 \tag{1.4}$$

is equivalent to a well-known index counting condition

$$\sum_{\{p: \nabla H(p)=0, \Delta_{g_m} H(p)<0\}} (-1)^{m(K,p)} \neq (-1)^m \tag{1.5}$$

which was first introduced in the scalar curvature problem in [12], [16].

Note the difference of the results for $m \neq 3$ and $m = 3$. Observe that the condition (1.4) follows from any one of the conditions 2-(i), 2-(ii), 2-(iii). As was remarked in [22], there is a function on S^3 which satisfies (1.4) but none of 2-(i), (ii), (iii). In this sense, in the list above, $m = 3$ is exceptional: in this dimension the methods in [22] only provide a weaker result than for $m \neq 3$, $m \geq 2$. It is thus natural to ask whether the stronger statement can be expected to $m = 3$ by other methods, and this is what we do in the present work.

Our purpose in this paper is to give a new approach to the existence problem and, as a conclusion, we give a new existence result for the case $m = 3$. We shall prove that with an additional technical assumption, the existence of non-trivial solution under the condition (1.4) holds true. The following is the main result of this paper:

Theorem 1.1 *Assume that $H \in C^2(S^3)$ is a Morse function on S^3 and $\Delta_{g_{S^3}} H(p) \neq 0$ for all $p \in \text{Crit}(H)$. We assume that $\text{Crit}(H)$ satisfies (1.4) (or, equivalently (1.5)) and the following:*

(A) *For any pair of critical points $x_1, x_2 \in \text{Crit}(H)$ with $H(x_1) < H(x_2)$, there holds $\Delta_{g_{S^3}} H(x_1) > 0$ or $\Delta_{g_{S^3}} H(x_2) < 0$.*

Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a non-trivial solution to the equation $D\psi = \frac{3}{2}(1 + \epsilon H(x))|\psi|$ on S^3 .

As a corollary, we have:

Corollary 1.1 *Assume $H \in C^2(S^3)$ is a Morse function on S^3 and $\Delta_{g_{S^3}} H(p) \neq 0$ for all $p \in \text{Crit}(H)$. We assume that $\text{Crit}(H)$ satisfies (1.4) (or, equivalently (1.5)) and one of the following (1), (2):*

(1) *The number of critical values of H is at most three, or*

(2) *There are more than three critical values of H and denoting by $d_1 > d_2 > \dots > d_p$ ($p \geq 4$) all the critical values of H , define $\text{Crit}_j(H) = \{x \in \text{Crit}(H) : H(x) = d_j\}$. There exists $1 \leq j_0 \leq p$ such that the following holds:*

$$\begin{aligned} \Delta_{g_{S^3}} H(x_i) &< 0 \text{ for any } i < j_0 \text{ and any } x_i \in \text{Crit}_i(H), \\ \Delta_{g_{S^3}} H(x_j) &> 0 \text{ for any } j_0 < j \text{ and any } x_j \in \text{Crit}_j(H) \text{ and} \\ &\text{there exists } x_{j_0} \in \text{Crit}_{j_0}(H) \text{ such that } \Delta_{g_{S^3}} H(x_{j_0}) > 0. \end{aligned}$$

Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a non-trivial solution to the equation $D\psi = \frac{3}{2}(1 + \epsilon H(x))|\psi|\psi$ on S^3 .

The proof of Theorem 1.1 consists of two main steps. We first reduce, via the Lyapunov-Schmidt reduction, the problem to a certain finite dimensional variational problem. This step is common to many nonlinear problems and for the present problem, it was carried out in detail in our previous paper [22]. For the reader's convenience, we will quickly review this step. Notice that, due to non-compactness of our problem, our reduced problem is defined on a non-compact manifold. The next step consists of the study of the finite dimensional problem. At first glance, our reduced problem is quite similar to the corresponding problem for Yamabe type equations considered, for example, in [3], [4], [25]. But this is not the case. As was pointed out in [22], contrary to the Yamabe type equations, our finite dimensional problem is degenerate, that is, any critical point of the main term of the reduced functional is always degenerate. New difficulties arise from here: By the degeneracy and the non-compactness of our finite dimensional problem, we can not apply implicit function theorem, or degree theoretic argument in order to come to a conclusion that critical points of the main term of the reduced functional produce true critical points. To overcome this difficulty, in [22], we developed a variant of Morse-Bott theory for such functionals defined on manifolds with boundary. Since this method gives only a weaker result for $m = 3$, we need other approach here. We develop the Conley index theory for the reduced functional. We will observe that the index counting condition (1.4) (or (1.5)) is equivalent to the non-vanishing of the Euler characteristic of a certain index pair of the gradient flow of the reduced functional. For the existence of such an index pair (more precisely, the existence of an isolating neighborhood), we need an additional technical condition (A) as stated in Theorem 1.3. We should point out here that our method is also applicable for all $m \geq 2$ and provides another approach to the existence problem for all $m \geq 2$, but in order to simplify a presentation, we restrict our attention mainly to the case $m = 3$.

This paper is organized as follows: In the next section, we quickly review the process of the reduction of our problem to a finite dimensional variational problem. §3 is the main part of the present paper. In that section, we first collect basic definitions and results from the Conley index theory. We then study topological and dynamical properties of our reduced functional and develop the Conley index theory suitable for our problem.

2 Preliminaries

Let $H \in L^\infty(M)$ and $\epsilon > 0$. We define

$$\mathcal{L}_{\epsilon, H}(\psi) := \mathcal{L}_{1+\epsilon H}(\psi) = \mathcal{L}(\psi) - \frac{m-1}{4} \epsilon \mathcal{G}_H(\psi), \quad (2.1)$$

where

$$\mathcal{L}(\psi) = \frac{1}{2} \int_M \langle \psi, D\psi \rangle d\text{vol}_g - \frac{m-1}{4} \int_M |\psi|^{2^*} d\text{vol}_g \quad (2.2)$$

and

$$\mathcal{G}_H(\psi) = \int_M H(x)|\psi|^{2^*} d\text{vol}_g. \quad (2.3)$$

We call \mathcal{L} the unperturbed functional. Its critical points are solutions to pure spinorial Yamabe equations considered in [5], [6], [7], [8]. For small $\epsilon > 0$, it is natural to expect that critical points of the functional $\mathcal{L}_{\epsilon,H}$ can be obtained as perturbations of critical points of \mathcal{L} . In general, this is the case if there is a suitable non-degenerate compact critical manifold of \mathcal{L} , or there is a critical manifold (not necessary compact) on which the perturbation term \mathcal{G}_H is non-degenerate in a certain sense. If we were in such a case, as was proved in the books by Mawhin-Willem [26, Chapter 10] and Chang [15, II, 6] for the compact case, or by Ambrosetti et al [1], [2], [4] for the non-compact case, a variant of a well-known Lyapunov-Schmidt reduction provides a powerful tool. Even though both of the conditions are not satisfied for our problem (see §3 for details), it still provides a useful tool and, for the reader's convenience, we quickly review its general construction here.

Let $(H, (\cdot, \cdot))$ be a Hilbert space with the corresponding norm $\|u\| = (u, u)^{1/2}$ for $u \in H$. Let us assume that $L_0 \in C^2(H, \mathbb{R})$ and $G \in C^2(H, \mathbb{R})$ are given. For $\epsilon > 0$, consider the functional defined as

$$L_\epsilon(u) := L_0(u) + \epsilon G(u).$$

We assume that L_0 has a non-degenerate critical manifold $\mathcal{M} \subset H$, that is,

(A-1) There exists a d -dimensional C^2 -submanifold $\mathcal{M} \subset H$ such that $L'_0(u) = 0$ for all $u \in \mathcal{M}$, and

(A-2) \mathcal{M} is non-degenerate in the sense that for all $u \in \mathcal{M}$, we have $T_u\mathcal{M} = \ker L''_0(u)$.

Moreover, we assume

(A-3) $L''_0(u) : H \rightarrow H$ is a Fredholm operator with index $L''_0(u) = 0$ for all $u \in \mathcal{M}$, where index $L''_0(u)$ is the Fredholm index of $L''_0(u)$.

If the conditions (A-1)–(A-3) are satisfied, the problem of finding critical points of L_ϵ on H is reduced to the same problem on \mathcal{M} for the reduced functional \mathcal{L}_ϵ defined in the following way. We shall find critical points of L_ϵ which are small perturbations of critical points of L_0 . Thus we set $u_\epsilon = z + w$, where $z \in \mathcal{M}$ and $w \in W := T_z\mathcal{M}^\perp$ ($T_z\mathcal{M}^\perp$ is the orthogonal complement of $T_z\mathcal{M}$ in H) and find the condition about w under which u_ϵ becomes a critical point of L_ϵ . For this, let $P : H \rightarrow W$ be the orthogonal projection. Projecting the Euler-Lagrange equation $L'_\epsilon(u) = 0$ onto W and W^\perp , we see that $u_\epsilon = z + w$ is a critical point of L_ϵ if and only if the following equations are satisfied:

$$PI'_\epsilon(z + w) = 0, \quad (2.4)$$

$$(I - P)I'_\epsilon(z + w) = 0. \quad (2.5)$$

The equation (2.4) is solved by means of the implicit function theorem: Under assuming (A-2) and (A-3), (2.4) has a unique small solution $w = w_\epsilon(z) \in W$ for any $z \in \mathcal{M}$. Moreover, the following uniform estimate holds on any compact subset $\mathcal{M}_c \subset \mathcal{M}$: There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, we have

$$\begin{aligned} \mathcal{M}_c \ni z \mapsto w_\epsilon(z) \in W \text{ is } C^1 \text{ and} \\ \|w_\epsilon(z)\|, \|w'_\epsilon(z)\| = O(\epsilon) \text{ uniformly for } z \in \mathcal{M}_c. \end{aligned} \quad (2.6)$$

For the proof of (2.6) and more detailed exposition about this construction, see [4, Chapter 2].

After introducing $w_\epsilon(z)$, the reduced functional $\mathcal{L}_\epsilon : \mathcal{M} \rightarrow \mathbb{R}$ is defined as $\mathcal{L}_\epsilon(z) = L_\epsilon(z + w_\epsilon(z))$. The following theorem gives the correspondence between critical points of L_ϵ and \mathcal{L}_ϵ :

Theorem 2.1 (Theorem 2.12 in [4]) *Assume that (A-1), (A-2) and (A-3) are satisfied. For a compact subset $\mathcal{M}_c \subset \mathcal{M}$ and $0 < \epsilon < \epsilon_0$ as above, assume that \mathcal{L}_ϵ has a critical point $z_\epsilon \in \mathcal{M}_c$. Then $u_\epsilon = z_\epsilon + w_\epsilon(z_\epsilon)$ is a critical point of L_ϵ on H .*

The above theorem reduces the problem of finding critical points of L_ϵ on H to the same problem of \mathcal{L}_ϵ on \mathcal{M}_c for a suitable compact subset $\mathcal{M}_c \subset \mathcal{M}$. Under suitable circumstances, the latter problem is well approximated by the problem of finding critical points of G on \mathcal{M}_c . In fact, by (2.6), we have

$$\mathcal{L}_\epsilon(z) = L_0(z) + \epsilon G(z) + o(\epsilon), \quad \mathcal{L}'_\epsilon(z) = \epsilon G'(z) + o(\epsilon). \quad (2.7)$$

Note that $L_0(z)$ is constant on any connected component of \mathcal{M} . If \mathcal{M} is compact or, by the second equation of (2.7), if $z \in \mathcal{M}$ is a non-degenerate critical point of G on \mathcal{M} or, more generally, the local degree of G' at z is non-zero, then there exists a critical point u_ϵ of L_ϵ on H which converges to z as $\epsilon \rightarrow 0$. See [26], [15], [4] for more details.

For our problem, the above abstract framework is applied to functionals defined by $L_0 = \mathcal{L}$ and $G = -\frac{m-1}{4}\mathcal{G}_H$ on $H = H^{1/2}(M, \mathbb{S}(M))$, here $H^{1/2}(M, \mathbb{S}(M))$ is a Hilbert space of $H^{1/2}$ -spinors on M , see [22, section 2.2] for the definition. As we have already shown in [22, sections 5, 6], for $M = S^m$ ($m \geq 2$), \mathcal{L} satisfies (A-1)–(A-3) for a critical manifold \mathcal{M} of \mathcal{L} defined as follows: First, we note that the equation (1.1) is conformally invariant so that (1.1) on S^m is transformed into the equation

$$D_{g_{\mathbb{R}^m}}\varphi = K(x)|\varphi|^{2^*-2}\varphi \quad \text{on } \mathbb{R}^m \quad (2.8)$$

via the conformal equivalence $\pi_p : S^m \setminus \{p\} \cong \mathbb{R}^m$, where $p \in S^m$, π_p is the stereographic projection from p and $H(x) = K(\pi_p(x))$ for $x \in S^m$. In fact, by the regularity result proved in [21, Appendix], the equation (1.1) is equivalent to (2.8), i.e., there is a one to one correspondence between solutions of (1.1) and those of (2.8) via π_p . See [22] for more details. Thus, in the following we shall consider equation (1.1) on S^m and \mathbb{R}^m interchangeably. Under this identification, the critical manifold \mathcal{M} of \mathcal{L} is defined as

$$\mathcal{M} := \{\varphi_{\lambda,\xi,a} : \lambda > 0, \xi \in \mathbb{R}^m, a \in \mathbb{S}_m, |a| = 1\}, \quad (2.9)$$

where

$$\varphi_{\lambda,\xi,a}(x) := \frac{2^{\frac{m-1}{2}} \lambda^{\frac{m-1}{2}}}{(\lambda^2 + |x - \xi|^2)^{\frac{m}{2}}} (\lambda - (x - \xi)) \cdot a$$

for $\lambda > 0$, $\xi \in \mathbb{R}^m$, $a \in \mathbb{S}_m$ with $|a| = 1$ (\mathbb{S}_m is the spinor module, see [20], [24]) and \cdot denotes the Clifford multiplication. Note that \mathcal{M} is diffeomorphic to $(0, \infty) \times \mathbb{R}^m \times S^{2^{\lfloor \frac{m}{2} \rfloor + 1} - 1}(\mathbb{S}_m)$ via $(\lambda, \xi, a) \mapsto \varphi_{\lambda,\xi,a}$, where $S^{2^{\lfloor \frac{m}{2} \rfloor + 1} - 1}(\mathbb{S}_m)$ is the $(2^{\lfloor \frac{m}{2} \rfloor + 1} - 1)$ -dimensional unit sphere in \mathbb{S}_m . Thus \mathcal{M} is a non-compact manifold and the dimension of \mathcal{M} is $m + 2^{\lfloor \frac{m}{2} \rfloor + 1}$.

By Theorem 2.1, we can reduce the problem to a finite dimensional variational problem on \mathcal{M} . But, unfortunately, \mathcal{M} is non-compact and, moreover, the reduced functional is very degenerate: Any critical point of \mathcal{G}_H on \mathcal{M} is degenerate. In fact, it degenerates along directions of $S^{2^{\lfloor \frac{m}{2} \rfloor + 1} - 1}(\mathbb{S}_m)$. Thus we can not apply implicit function theorem, or degree theory, or a variant of Morse theory as in [3], [4], [15], [25], [26] in order to prove the existence of a critical point of \mathcal{L}_ϵ on \mathcal{M} . To overcome this difficulty, in the rest of this paper, we shall develop the Conley index theory for the reduced functional. In subsequent sections, we restrict our attention mainly to the case $\dim M = m = 3$.

3 The Conley index theory for the reduced problem

In the previous section, we have reduced our problem to the search for critical points of the reduced functional $L_\epsilon(\psi) := \mathcal{L}_{\epsilon,H}(\psi + w_\epsilon(\psi))$ defined on \mathcal{M} , where $w_\epsilon(\psi)$ is defined by solving the equation

(2.4). By (2.6) and (2.7), we have

$$L_\epsilon(\psi) = \mathcal{L}(\psi) - \frac{m-1}{4} \epsilon \mathcal{G}_K(\psi) + o(\epsilon) \quad (3.1)$$

and

$$L'_\epsilon(\psi) = -\frac{m-1}{4} \epsilon \mathcal{G}'_K(\psi) + o(\epsilon) \quad (3.2)$$

which hold uniformly on any compact subset $\psi \in \mathcal{M}_c \subset \mathcal{M}$. Note that $\mathcal{L}(\psi) = C_0$ is a constant independent of $\psi \in \mathcal{M}$ and $\mathcal{G}_K(\psi) = \int_{\mathbb{R}^m} K(x)|\psi|^2 dx$.

By (3.1) and (3.2), \mathcal{G}_K is considered as the first approximation of L_ϵ on \mathcal{M} . Thus if \mathcal{G}_K on the critical manifold \mathcal{M} has a non-degenerate critical point, then applying implicit function theorem, we will obtain a critical point of L_ϵ on \mathcal{M} and this will complete the proof of the existence of solution. But, as mentioned before, any critical point of \mathcal{G}_K on \mathcal{M} is degenerate. This is easily seen by observing that $\mathcal{G}_K(\varphi_{\lambda, \xi, a})$ is independent of $a \in \mathbb{S}_m$ with $|a| = 1$. Thus it always degenerate along directions of $S^{2^{\frac{m}{2}+1}-1}(\mathbb{S}_m)$. In order to treat such a degenerate case, we need a more subtle stability result for critical points of \mathcal{G}_K under perturbations. In this section, we show that the Conley index theory provides such a result under suitable conditions on K . Thus, we begin this section with recalling some basic notions and results from the Conley index theory. For a detailed exposition of this theory, see [19], [13], [23], [30].

3.1 Conley's index theory

Throughout this subsection, we assume that X is a locally compact metric space. Let $\phi : \mathbb{R} \times X \rightarrow X$ be a flow. This means that it is continuous and for all $x \in X$ and $s, t \in \mathbb{R}$ there holds $\phi(0, x) = x$ and $\phi(s, \phi(t, x)) = \phi(s+t, x)$. For simplicity, we write $\phi^t(x) = \phi(t, x)$.

For $N \subset X$, we define the maximal invariant set contained in N as

$$I(N, \{\phi^t\}) := \{x \in N : \phi^t(x) \in N, \text{ for all } t \in \mathbb{R}\}.$$

It is the set of points of N that remains in N for all positive and negative times under the flow. N is called invariant if $I(N, \{\phi^t\}) = N$. We now introduce the main objects of the Conley index theory. These are isolated invariant sets and isolating neighborhoods associated with them. A compact set $N \subset X$ is called an isolating neighborhood and $I(N, \{\phi^t\}) \subset N$ is called an isolated invariant set if $I(N, \{\phi^t\}) \subset \text{int}(N)$ holds. In other words, $I \subset X$ is an isolated invariant set if it is invariant and there exists an open neighborhood N such that N contains no other invariant sets other than I .

A pair (N, L) of compact sets $L \subset N \subset X$ is called an index pair for the invariant set $I \subset X$ if the following conditions are satisfied:

- (i) $\overline{N \setminus L}$ is an isolating neighborhood of I with $L \cap I = \emptyset$.
- (ii) L is positively invariant relative to N , i.e., if $x \in L$ and $\phi^s(x) \in N$ for $0 \leq s \leq t$, then $\phi^s(x) \in L$ for $0 \leq s \leq t$.
- (iii) If $x \in N$ and there exists $t > 0$ such that $\phi^t(x) \notin N$, then there exists $0 \leq t_0 < t$ with $\phi^s(x) \in N$ for $0 \leq s \leq t_0$ and $\phi^{t_0}(x) \in L$.

(iii) says that L is the exit set from N for the flow $\{\phi^t\}$.

The following basic existence result is proved in [19], [30].

Theorem 3.1 *For any isolated invariant set $I \subset X$, there exists an index pair (N, L) .*

We consider pairs (Y, A) of topological spaces, where $A \subset Y$ is closed. We abbreviate $Y = (Y, \phi)$. For the special case $A = \{a\}$, $(Y, \{a\})$ is called a pointed space and a is called a base point. For such a pair of sets (Y, A) , the quotient Y/A is a pointed space obtained from Y by collapsing A to a single point denoted by $\langle A \rangle$. $\langle A \rangle$ is the base point of Y/A . The important special case is the case $A = \emptyset$ and in this case we have $X/\emptyset = (X \sqcup \{p\}, \{p\})$, where $p \notin X$.

Let $f : (X, A) \rightarrow (Y, B)$ be continuous, i.e., $f : X \rightarrow Y$ is continuous and $f(A) \subset B$. Two such maps $f_0, f_1 : (X, A) \rightarrow (Y, B)$ are called homotopic if there exists a continuous $F : [0, 1] \times (X, A) \rightarrow (Y, B)$ such that $F(t, A) \subset B$ for all $t \in [0, 1]$ and $f_0 = F(0, \cdot)$, $f_1 = F(1, \cdot)$. In this case, we write $f_0 \sim f_1$. A homotopy equivalence between pairs of spaces (Y, A) and (Z, B) is defined by requiring that there exists $f : (Y, A) \rightarrow (Z, B)$ and $g : (Z, B) \rightarrow (Y, A)$ such that $f \circ g \sim \mathbf{1}_Z$ and $g \circ f \sim \mathbf{1}_Y$, where $\mathbf{1}_Z$ and $\mathbf{1}_Y$ denote the identity of Z and Y . In such a case, we write $(Y, A) \sim (Z, B)$. Note that $f : (Y, A) \rightarrow (Z, B)$ induces a map $\tilde{f} : Y/A \rightarrow Z/B$ and if $(Y, A) \sim (Z, B)$, then we have $Y/A \sim Z/B$. The homotopy equivalence class containing (Y, A) is denoted by $[(Y, A)]$ and we simply call it the homotopy type of (Y, A) .

After the basic existence result (Theorem 3.1), the following result is fundamental to the Conley theory (see [19], [23], [30]).

Theorem 3.2 *Let (N_0, L_0) and (N_1, L_1) be index pairs for an isolated invariant set I . Then N_0/L_0 and N_1/L_1 are homotopically equivalent, that is, $[N_0/L_0] = [N_1/L_1]$.*

By Theorem 3.2, we define:

Definition 3.1 Let I be an isolated invariant set of a flow $\{\phi^t\}$ and (N, L) any index pair for I .

- (i) The homotopy Conley index $h(I, \{\phi^t\})$ is defined as $h(I, \{\phi^t\}) := [N/L]$.
- (ii) The cohomological Conley index $CH^*(I, \{\phi^t\})$ is defined as $CH^*(I, \{\phi^t\}) := H^*(N, L)$.

An index pair (N, L) is called regular if the inclusion map $L \subset N$ is a cofibration. By [30], it is known that any index pair can always be modified to create an index pair (N, \tilde{L}) that is regular. Hence, by Theorem 3.2, $H^*(N, L) \cong H^*(N/L, \langle L \rangle)$ is an invariant associated to an isolated invariant set I , where (N, L) is any regular index pair for I . We henceforth assume that all index pairs appearing in the rest of this paper are regular unless otherwise stated.

Let us now consider the invariance property of the Conley index under small perturbations. Let Λ be a compact locally contractible connected metric space with metric d_Λ . Suppose we have a family of flows parametrized by $\lambda \in \Lambda$, $\phi_\lambda : X \times \Lambda \rightarrow X$, which is continuous with respect to $\lambda \in \Lambda$. We have (see [23], [30]):

Lemma 3.1 *Let $\lambda_0 \in \Lambda$. Let N be an isolating neighborhood for $\{\phi_{\lambda_0}^t\}$. Then there exists $\epsilon > 0$ such that for all $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \epsilon$, N is also an isolating neighborhood for $\{\phi_\lambda^t\}$.*

The following continuation property of the Conley index is fundamental importance for the proof of our main result (see [23], [30]):

Theorem 3.3 *Under the assumption of Lemma 3.1, we have*

$$h(I(N, \{\phi_\lambda^t\})) = h(I(N, \{\phi_{\lambda_0}^t\}))$$

for all $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \epsilon$.

For a pair (Y, A) , the Euler characteristic $\chi(Y, A)$ is defined as

$$\chi(Y, A) = \sum_{k=0}^{\infty} (-1)^k \text{rank } H^k(Y, A; \mathbb{F}),$$

where \mathbb{F} is a field. (Notice that $\chi(Y, A)$ is independent of the choice of the field \mathbb{F}). Let $\{\phi_{-F}^t\}$ be the flow generated by $-F$, where $F : \Omega \rightarrow \mathbb{R}^n$ is locally Lipschitz and $\Omega \subset \mathbb{R}^n$ is open and bounded. The Euler characteristic of the homotopy Conley index $h(I(N, \{\phi_{-F}^t\}))$ for an isolating neighborhood $N \subset \Omega$ is intimately related to the mapping degree $\text{deg}(F, \text{int}(N), 0)$ of F with respect to $0 \in \mathbb{R}^n$ by the following theorem (for the proof, see [27], [29], [31]):

Theorem 3.4 *Let $F : \Omega \rightarrow \mathbb{R}^n$ be locally Lipschitz and $N \subset \Omega$ an isolating neighborhood for the flow $\{\phi_{-F}^t\}$ generated by $-F$. Assume that F does not vanish on the boundary ∂N . Then we have*

$$\chi(h(I(N, \{\phi_{-F}^t\}))) = \text{deg}(F, \text{int}(N), 0).$$

3.2 Proof of the main result, I: The index calculation.

We now specialize to the case $m = 3$. (We should remark, however, that all of our arguments apply to the general case $m \geq 3$ without any difficulty. And, with some additional arguments, it also applies to the case $m = 2$). We assume that a function H on S^3 satisfies the following conditions:

(H-1) $H \in C^2(S^3)$ and is a Morse function on S^3 . Moreover, for any $p \in S^3$ with $\nabla H(p) = 0$, there holds $\Delta_{g_{S^3}} H(p) \neq 0$.

(H-2) H satisfies the following index counting condition

$$\sum_{\{p \in S^3 : \nabla H(p) = 0, \Delta_{g_{S^3}} H(p) < 0\}} (-1)^{m(H,p)} \neq -1.$$

Let us assume that H takes its minimum on S^3 at $p_0 \in S^3$. Consider the stereographic projection from p_0 , $\pi_{p_0} : S^3 \setminus \{p_0\} \rightarrow \mathbb{R}^3$. We define a function K on \mathbb{R}^3 by $K(x) = H(\pi_{p_0}^{-1}(x))$. In the following lemma, we collect basic properties of K which will be used in the sequel.

Lemma 3.2 *K has the following properties:*

(K-1) $K \in L^\infty(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ and there are constants $C_1, C_2 > 0$ such that

$$|D_i K(x)| \leq C_1(1 + |x|^2)^{-1}, \quad |D_{ij}^2 K(x)| \leq C_2(1 + |x|^2)^{-\frac{3}{2}} \quad (3.3)$$

for all $x \in \mathbb{R}^3$, where $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$ ($1 \leq i, j \leq 3$).

(K-2) *Critical points of K are non-degenerate, the number of critical points of K on \mathbb{R}^3 is finite and for any critical point $x \in \mathbb{R}^3$ of K , there holds $\Delta_{g_{\mathbb{R}^3}} K(x) \neq 0$. Moreover, there holds*

$$\sum_{\{x \in \mathbb{R}^3 : \nabla K(x) = 0, \Delta_{g_{\mathbb{R}^3}} K(x) < 0\}} (-1)^{m(K,x)} \neq -1. \quad (3.4)$$

(K-3) *There exists $R_1 > 0$ such that $(DK(x), x) < 0$ for any $x \in \mathbb{R}^3$ with $|x| \geq R_1$.*

(K-4) There exists $R_2 > 0$ such that for any $R \geq R_2$, there holds

$$\int_{B_R} (DK(x), x) dx < 0. \quad (3.5)$$

Proof. Similar statements of (K-1) and (K-2) were already proved in [22, Lemma 4.1]. Though the choice of $p_0 \in S^3$ here is different from that of [22, Lemma 4.1], the same argument applies and we omit the proof.

(K-3): Since H is a Morse function on S^3 and takes the minimum at $p_0 \in S^3$, there exists $R_1 > 0$ such that for any $|x| \geq R_1$, $t \mapsto K(tx) = H(\pi_{p_0}^{-1}(tx))$ is strictly decreasing for $t \geq 1/2$. Differentiating this at $t = 1$, the assertion follows immediately.

(K-4): By (K-3), as a function of $r = |x|$, the function $K(x) = K(r, \omega)$ ($x = r\omega$, $\omega \in S^2$) is monotonically decreasing for $r \geq R_1$. Let $R > R_1$. We have

$$\int_{B_R} (DK(x), x) dx = \int_0^R \int_{S^2} \frac{\partial K}{\partial r}(r, \omega) r^3 dr d\text{vol}_{S^2}(\omega). \quad (3.6)$$

Integrating by parts with respect to r , we have

$$\begin{aligned} \int_0^R \frac{\partial K}{\partial r}(r, \omega) r^3 dr &= R^3 K(R, \omega) - 3 \int_0^R r^2 K(r, \omega) dr \\ &= R^3 K(R, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega) dr - 3 \int_{R_1}^R r^2 K(r, \omega) dr \\ &< R^3 K(R, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega) dr - 3K(R, \omega) \int_{R_1}^R r^2 dr \\ &= R^3 K(R, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega) dr - R^3 K(R, \omega) + R_1^3 K(R, \omega) \\ &= R_1^3 K(R, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega) dr \\ &\leq R_1^3 K(R, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega_0) dr, \end{aligned} \quad (3.7)$$

where $\omega_0 \in S^2$ is such that the function $S^2 \ni \omega \mapsto \int_0^{R_1} r^2 K(r, \omega) dr$ takes its minimum at $\omega = \omega_0$.

The last term of (3.7) converges as $R \rightarrow \infty$, uniformly for $\omega \in S^2$, to

$$R_1^3 K(\infty, \omega) - 3 \int_0^{R_1} r^2 K(r, \omega_0) dr < 0,$$

since $K(r, \omega_0) \geq K(\infty, \omega) = \min_{S^3} H$ and $K(r, \omega_0) \neq \min_{S^3} H$ on the set $\{r : 0 \leq r \leq R_1\}$.

Thus, there exists $R_2 \geq R_1$ such that

$$\int_0^R \frac{\partial K}{\partial r}(r, \omega) r^3 dr < 0 \quad (3.8)$$

for any $\omega \in S^3$ and $R \geq R_2$. (3.5) follows from (3.6) and (3.8). \square

We now begin the study of L_ϵ on \mathcal{M} . By (3.1) and (3.2) the first approximation of L_ϵ (up to a constant factor) is \mathcal{G}_K . Thus, we first study \mathcal{G}_K on \mathcal{M} . For this purpose, define for $(\lambda, \xi, a) \in (0, \infty) \times \mathbb{R}^3 \times S^3(\mathbb{S}_3)$

$$\Phi_K(\lambda, \xi, a) = \mathcal{G}_K(\varphi_{\lambda, a, \xi}) = \int_{\mathbb{R}^3} K(x) |\varphi_{\lambda, \xi, a}|^{2^*} dx,$$

where we recall that $\varphi_{\lambda, \xi, a}(x) := \frac{2\lambda}{(\lambda^2 + |x - \xi|^2)^{\frac{3}{2}}} (\lambda - (x - \xi)) \cdot a \in \mathcal{M}$. In the sequel, we identify (λ, ξ, a) with $\varphi_{\lambda, \xi, a}$ and write $(\lambda, \xi, a) \in \mathcal{M}$ instead of writing $\varphi_{\lambda, \xi, a} \in \mathcal{M}$.

We observe that $|\varphi_{\lambda, \xi, a}|$ is independent of $a \in S^3(\mathbb{S}_3)$. Thus, $\Phi_K(\lambda, \xi, a)$ is independent of $a \in S^3(\mathbb{S}_3)$ and we write $\Gamma_K(\lambda, \xi) = \Phi_K(\lambda, \xi, a)$. Note that, by change of variable,

$$\Gamma_K(\lambda, \xi) = 8 \int_{\mathbb{R}^3} \frac{K(\xi + \lambda y)}{(1 + |y|^2)^3} dy. \quad (3.9)$$

The following is easily checked by Lemma 3.2 (K-1): For any $\xi \in \mathbb{R}^3$, we have

$$\lim_{\lambda \downarrow 0} \Gamma_K(\lambda, \xi) = C_0 K(\xi), \quad \text{where } C_0 := 8 \int_{\mathbb{R}^3} \frac{dy}{(1 + |y|^2)^3}, \quad (3.10)$$

$$D_\lambda \Gamma_K(\lambda, \xi) = 8 \int_{\mathbb{R}^3} (DK(\lambda y + \xi), y) \frac{1}{(1 + |y|^2)^3} dy$$

and

$$\lim_{\lambda \downarrow 0} D_\lambda \Gamma_K(\lambda, \xi) = 8 \int_{\mathbb{R}^3} (DK(\xi), y) \frac{1}{(1 + |y|^2)^3} dy = 0. \quad (3.11)$$

In the following (for the proof of Lemma 3.3, Lemma 3.4 and Corollary 3.1), we shall use some of the ideas of Ambrosetti et al [3]. In fact, the function (3.9) is very similar to the one which appeared in the Yamabe problem in the perturbative setting. But please keep in mind that we are in fact interested in the function Φ_K defined on \mathcal{M} . Thus, unlike the previous paper [22, section 7], following [3] we prefer to work on $(-\infty, \infty) \times \mathbb{R}^3$ by extending the function Γ_K as

$$\tilde{\Gamma}_K(0, \xi) = C_0 K(\xi), \quad \tilde{\Gamma}_K(\lambda, \xi) = \Gamma_K(-\lambda, \xi) \quad \text{for } \lambda < 0.$$

By (3.10) and (3.11), $\tilde{\Gamma}_K$ is C^1 on $(-\infty, \infty) \times \mathbb{R}^3$ and

$$D_\lambda \tilde{\Gamma}_K(0, \xi) = 0 \quad \text{for } \xi \in \mathbb{R}^3. \quad (3.12)$$

Furthermore, we have

$$D_\xi \tilde{\Gamma}_K(0, \xi) = C_0 DK(\xi) \quad \text{for } \xi \in \mathbb{R}^3. \quad (3.13)$$

By (3.12) and (3.13), we have

$$\xi \in \text{Crit}(K) \quad \text{if and only if} \quad (0, \xi) \in \text{Crit}(\tilde{\Gamma}_K). \quad (3.14)$$

We calculate the local index of $\tilde{\Gamma}_K$ at critical points of the form $(0, \xi)$. To do this, we first observe that the function $\tilde{\Gamma}_K$ is C^2 on $(-\infty, \infty) \times \mathbb{R}^3$. In fact, by (3.3), we have

$$D_{\lambda \lambda}^2 \tilde{\Gamma}_K(\lambda, \xi) = 8 \sum_{i, j=1}^3 \int_{\mathbb{R}^3} D_{ij}^2 K(\lambda y + \xi) y_i y_j \frac{1}{(1 + |y|^2)^3} dy, \quad (3.15)$$

$$D_{\lambda \xi_i}^2 \tilde{\Gamma}_K(\lambda, \xi) = 8 \sum_{j=1}^3 \int_{\mathbb{R}^3} D_{ij}^2 K(\lambda y + \xi) y_j \frac{1}{(1 + |y|^2)^3} dy \quad (3.16)$$

and

$$D_{\xi_i \xi_j}^2 \tilde{\Gamma}_K(\lambda, \xi) = 8 \int_{\mathbb{R}^3} D_{ij}^2 K(\lambda y + \xi) \frac{1}{(1 + |y|^2)^3} dy. \quad (3.17)$$

The following lemma gives a formula of the local index of $\tilde{\Gamma}_K$ at critical point $(0, \xi)$ which we denote by $\text{ind}(D\tilde{\Gamma}_K, (0, \xi))$.

Lemma 3.3 *Assume K satisfies (K-1) and the first half of (K-2) in Lemma 3.2. Then there are at most a finite number of critical points of $\tilde{\Gamma}_K$ of the form $(0, \xi) \in (-\infty, \infty) \times \mathbb{R}^3$. Moreover, they are isolated as critical points of $\tilde{\Gamma}_K$ in $(-\infty, \infty) \times \mathbb{R}^3$ and the local index of $\tilde{\Gamma}_K$ at critical point $(0, \xi)$ is given by the formula*

$$\text{ind}(D\tilde{\Gamma}_K, (0, \xi)) = \begin{cases} (-1)^{m(K, \xi)} & \text{if } \Delta_{g_{\mathbb{R}^3}} K(\xi) > 0 \\ (-1)^{m(K, \xi)+1} & \text{if } \Delta_{g_{\mathbb{R}^3}} K(\xi) < 0. \end{cases} \quad (3.18)$$

Proof. By (3.15)–(3.17), the Hessian of $\tilde{\Gamma}_K$ at $(0, \xi)$ is given by

$$D^2\tilde{\Gamma}_K(0, \xi) = \begin{pmatrix} C_1 \Delta_{g_{\mathbb{R}^3}} K(\xi) & O \\ O & C_0 D^2 K(\xi) \end{pmatrix},$$

where $C_1 = \frac{8}{3} \int_{\mathbb{R}^3} \frac{|y|^2}{(1+|y|^2)^3} dy > 0$. The assertion of the lemma easily follows from this. \square

In the next lemma, we calculate the total degree of $D\tilde{\Gamma}_K$ on a large ball in $(-\infty, \infty) \times \mathbb{R}^3$. The calculation is essentially the same as [3]. For completeness, we give a proof.

Lemma 3.4 *There exists $R > 0$ such that $\text{Crit}(\tilde{\Gamma}_K) := \{(\lambda, \xi) \in (-\infty, \infty) \times \mathbb{R}^3 : D\tilde{\Gamma}_K(\lambda, \xi) = 0\} \subset B_R^4 := \{(\lambda, \xi) \in (-\infty, \infty) \times \mathbb{R}^3 : \lambda^2 + |\xi|^2 \leq R^2\}$ and*

$$\deg(D\tilde{\Gamma}_K, B_R^4, 0) = 1.$$

Proof. We prove that there exists $R > 0$ such that

$$(D\tilde{\Gamma}_K(\lambda, \xi), (\lambda, \xi)) < 0 \quad (3.19)$$

for $(\lambda, \xi) \in (-\infty, \infty) \times \mathbb{R}^3$ with $\lambda^2 + |\xi|^2 \geq R^2$. For this purpose, we compute

$$\begin{aligned} (D\tilde{\Gamma}_K(\lambda, \xi), (\lambda, \xi)) &= \lambda D_\lambda \tilde{\Gamma}_K(\lambda, \xi) + (D_\xi \tilde{\Gamma}_K(\lambda, \xi), \xi) \\ &= 8 \int_{\mathbb{R}^3} (DK(\lambda y + \xi), \lambda y + \xi) \frac{1}{(1 + |y|^2)^3} dy \\ &= 8\lambda^{-3} \int_{\mathbb{R}^3} (DK(y), y) \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy. \end{aligned} \quad (3.20)$$

To estimate (3.20), we write

$$\begin{aligned} \int_{\mathbb{R}^3} (DK(y), y) \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy &= \int_{|y| \leq \rho} (DK(y), y) \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy \\ &\quad + \int_{|y| \geq \rho} (DK(y), y) \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy \\ &=: I_1(\rho; \lambda, \xi) + I_2(\rho; \lambda, \xi). \end{aligned} \quad (3.21)$$

By (K-3) in Lemma 3.2, for $\rho \geq R_1$, we have

$$I_2(\rho; \lambda, \xi) < 0. \quad (3.22)$$

On the other hand, we have

$$\begin{aligned} I_1(\rho; \lambda, \xi) &= \int_{|y| \leq \rho} (DK(y), y)_+ \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy - \int_{|y| \leq \rho} (DK(y), y)_- \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} dy \\ &\leq \max_{|y| \leq \rho} \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} \int_{|y| \leq \rho} (DK(y), y)_+ dy \\ &\quad - \min_{|y| \leq \rho} \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} \int_{|y| \leq \rho} (DK(y), y)_- dy. \end{aligned} \quad (3.23)$$

where $(DK(y), y)_+$ and $(DK(y), y)_-$ are the positive and the negative parts of $(DK(y), y)$, respectively.

Define

$$M_\rho(\lambda, \xi) := \max_{|y| \leq \rho} \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} = \begin{cases} \frac{\lambda^6}{(\lambda^2 + (\rho - |\xi|)^2)^3} & \text{if } |\xi| > \rho \\ 1 & \text{if } |\xi| \leq \rho \end{cases}$$

and

$$m_\rho(\lambda, \xi) := \min_{|y| \leq \rho} \frac{\lambda^6}{(\lambda^2 + |y - \xi|^2)^3} = \frac{\lambda^6}{(\lambda^2 + (\rho + |\xi|)^2)^3}.$$

We take $\rho = \max\{R_1, R_2\}$, where $R_2 > 0$ is as in Lemma 3.2 (K-4). If $\epsilon > 0$ is small, we have, by (K-4),

$$\epsilon \int_{|y| \leq \rho} (DK(y), y)_+ dy + \int_{|y| \leq \rho} (DK(y), y) dy < 0. \quad (3.24)$$

Since $\lim_{\lambda^2 + |\xi|^2 \rightarrow \infty} \frac{M_\rho(\lambda, \xi)}{m_\rho(\lambda, \xi)} = 1$, there exists $R_3 \geq \rho$ such that

$$1 \leq \frac{M_\rho(\lambda, \xi)}{m_\rho(\lambda, \xi)} < 1 + \epsilon \quad (3.25)$$

for $\lambda^2 + |\xi|^2 \geq R_3^2$.

For $\lambda^2 + |\xi|^2 \geq R_3^2$, we have, by (3.23), (3.24) and (3.25)

$$\begin{aligned} I_1(\rho; \lambda, \xi) &\leq M_\rho(\lambda, \xi) \int_{|y| \leq \rho} (DK(y), y)_+ dy - m_\rho(\lambda, \xi) \int_{|y| \leq \rho} (DK(y), y)_- dy \\ &\leq \left((1 + \epsilon) \int_{|y| \leq \rho} (DK(y), y)_+ dy - \int_{|y| \leq \rho} (DK(y), y)_- dy \right) m_\rho(\lambda, \xi) \\ &\leq \left(\epsilon \int_{|y| \leq \rho} (DK(y), y)_+ dy + \int_{|y| \leq \rho} (DK(y), y) dy \right) m_\rho(\lambda, \xi) < 0. \end{aligned} \quad (3.26)$$

From (3.20), (3.21), (3.22) and (3.26), we have (3.19) for $R = R_3$. By (3.19), we have, in particular, $\text{Crit}(\tilde{\Gamma}_K) \subset B_R^4$.

To prove the degree formula, we observe that $\frac{D\tilde{\Gamma}_K}{\|D\tilde{\Gamma}_K\|} : \partial B_R^4 \rightarrow S^3$ is homotopic to $-\frac{1}{R} \text{id}_{\partial B_R^4} : \partial B_R^4 \rightarrow S^3$. To see this, by (3.19), we have $-t(\lambda, \xi) + (1-t)D\tilde{\Gamma}_K(\lambda, \xi) \neq 0$ for any $(\lambda, \xi) \in \partial B_R^4$ and $0 \leq t \leq 1$. Thus the homotopy $H_t(\lambda, \xi) = \frac{-t(\lambda, \xi) + (1-t)D\tilde{\Gamma}_K(\lambda, \xi)}{\| -t(\lambda, \xi) + (1-t)D\tilde{\Gamma}_K(\lambda, \xi) \|}$ is well-defined on $[0, 1] \times \partial B_R^4$ and the assertion follows. Therefore, we have

$$\deg(D\tilde{\Gamma}_K, B_R^4, 0) = \deg\left(-\frac{1}{R} \text{id}_{B_R^4}, B_R^4, 0\right) = 1.$$

This completes the proof. \square

We decompose $\text{Crit}(\tilde{\Gamma}_K)$ as

$$\text{Crit}(\tilde{\Gamma}_K) = \text{Crit}_+(\tilde{\Gamma}_K) \cup \text{Crit}_0(\tilde{\Gamma}_K) \cup \text{Crit}_-(\tilde{\Gamma}_K), \quad (3.27)$$

where

$$\text{Crit}_\pm(\tilde{\Gamma}_K) = \{(\lambda, \xi) \in \text{Crit}(\tilde{\Gamma}_K) : \pm\lambda > 0\}, \quad \text{Crit}_0(\tilde{\Gamma}_K) = \{(0, \xi) \in \text{Crit}(\tilde{\Gamma}_K)\}.$$

By the symmetry of $\tilde{\Gamma}_K$, we obviously have

$$\text{Crit}_-(\tilde{\Gamma}_K) = \{(\lambda, \xi) : (-\lambda, \xi) \in \text{Crit}_+(\tilde{\Gamma}_K)\}. \quad (3.28)$$

Moreover, by (3.14),

$$\text{Crit}_0(\tilde{\Gamma}_K) = \{(0, \xi) : \xi \in \text{Crit}(K)\}. \quad (3.29)$$

By Lemma 3.3 and Lemma 3.4, there exists a bounded open set $\mathcal{O}_+ \subset (0, \infty) \times \mathbb{R}^3$ such that

$$\text{Crit}_+(\tilde{\Gamma}_K) \subset \mathcal{O}_+. \quad (3.30)$$

As a corollary of Lemma 3.3 and Lemma 3.4, we have

Corollary 3.1 *Assume $H \in C^2(S^3)$ is a Morse function, $\Delta_{g_{S^3}} H(p) \neq 0$ for $p \in \text{Crit}(H)$ and satisfies the index counting condition (1.4) (or, equivalently (1.5)). Then we have $\deg(D\tilde{\Gamma}_K, \mathcal{O}_+, 0) \neq 0$.*

Proof. Assume, to the contrary, that $\deg(D\tilde{\Gamma}_K, \mathcal{O}_+, 0) = 0$. We define

$$\mathcal{O}_- = \{(\lambda, \xi) : (-\lambda, \xi) \in \mathcal{O}_+\}.$$

Notice that $\mathcal{O}_- \subset (-\infty, 0) \times \mathbb{R}^3$, $\text{Crit}_-(\tilde{\Gamma}_K) \subset \mathcal{O}_-$ and $\deg(D\tilde{\Gamma}_K, \mathcal{O}_-, 0) = 0$ under the assumption. By the additive property of the degree, we have, by Lemma 3.3 and Lemma 3.4,

$$\begin{aligned} 1 &= \deg(D\tilde{\Gamma}_K, B_R^4, 0) \\ &= \deg(D\tilde{\Gamma}_K, \mathcal{O}_+, 0) + \deg(D\tilde{\Gamma}_K, \mathcal{O}_-, 0) + \deg(D\tilde{\Gamma}_K, B_R^4 \setminus (\mathcal{O}_+ \cup \mathcal{O}_-), 0) \\ &= 0 + 0 + \sum_{x \in \text{Crit}_0(\tilde{\Gamma}_K)} \text{ind}(D\tilde{\Gamma}_K, x) \\ &= \sum_{x \in \text{Crit}(K), \Delta_{g_{\mathbb{R}^3}} K(x) > 0} \text{ind}(DK, x) - \sum_{x \in \text{Crit}(K), \Delta_{g_{\mathbb{R}^3}} K(x) < 0} \text{ind}(DK, x), \end{aligned} \quad (3.31)$$

where, taking $R > 0$ large if necessary, we may assume $B_R^4 \supset \mathcal{O}_+ \cup \mathcal{O}_-$.

On the other hand, by (K-3) in Lemma 3.2 and arguing as in the proof of Lemma 3.4, we have

$$\deg(DK, B_R^3, 0) = -1. \quad (3.32)$$

Since

$$\deg(DK, B_R^3, 0) = \sum_{x \in \text{Crit}(K), \Delta_{g_{\mathbb{R}^3}} K(x) > 0} \text{ind}(DK, x) + \sum_{x \in \text{Crit}(K), \Delta_{g_{\mathbb{R}^3}} K(x) < 0} \text{ind}(DK, x),$$

this combined with (3.31) and (3.32) imply that

$$\sum_{x \in \text{Crit}(K), \Delta_{g_{\mathbb{R}^3}} K(x) < 0} \text{ind}(DK, x) = -1. \quad (3.33)$$

As noted in Lemma 3.2, there is a one to one correspondence between (non-degenerate) critical points of H on S^3 with $\Delta_{g_{S^3}} H(x) < 0$ and the ones of K on \mathbb{R}^3 with $\Delta_{g_{\mathbb{R}^3}} K(x) < 0$. Therefore, (3.33) implies

$$\sum_{x \in \text{Crit}(H), \Delta_{g_{S^3}} H(x) < 0} (-1)^{m(H,x)} = \sum_{x \in \text{Crit}(H), \Delta_{g_{S^3}} H(x) < 0} \text{ind}(DH, x) = -1.$$

This contradicts our assumption (1.5). This completes the proof. \square

3.3 Proof of the main result, II: Isolating neighborhood $\mathcal{N}_{+,n}$.

In this subsection, we search a suitable isolating neighborhood for the negative gradient flow $\{\phi^t_{-D\Gamma_K}\}$ of Γ_K on $(0, \infty) \times \mathbb{R}^3$. This will provide us an isolating neighborhood for Φ_K on $(0, \infty) \times \mathbb{R}^3 \times S^3(\mathbb{S}_3)$ which we are primary interested in.

For $n \in \mathbb{N}$, we define

$$\mathcal{N}_{+,n} := \left\{ (\lambda, \xi) \in (0, \infty) \times \mathbb{R}^3 : \lambda^2 + |\xi|^2 \leq n^2, \lambda \geq \frac{1}{n} \right\}.$$

For large $n \in \mathbb{N}$ (say $n \geq N$, $N \in \mathbb{N}$), we have $\mathcal{O}_+ \subset \mathcal{N}_{+,n}$. As we have observed (see the proof of Lemma 3.4), there are no critical points of Γ_K on $\mathcal{N}_{+,n} \setminus \mathcal{O}_+$ and we have

$$\deg(D\Gamma_K, \mathcal{N}_{+,n}, 0) = \deg(D\Gamma_K, \mathcal{O}_+, 0) \neq 0 \quad (3.34)$$

by Corollary 3.1.

We shall prove the following:

Lemma 3.5 *Let K be as in Lemma 3.2. Assume further that K satisfies the following condition:*

(A) *For any pair of critical points $\xi_1, \xi_2 \in \text{Crit}(K)$ with $K(\xi_1) < K(\xi_2)$, there holds $\Delta_{g_{\mathbb{R}^3}} K(\xi_1) > 0$ or $\Delta_{g_{\mathbb{R}^3}} K(\xi_2) < 0$.*

Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathcal{N}_{+,n}$ is an isolating neighborhood for the negative gradient flow of Γ_K on $(0, \infty) \times \mathbb{R}^3$. That is, we have

$$S_{+,n} := I(\mathcal{N}_{+,n}, \{\phi^t_{-D\Gamma_K}\}) \subset \text{int}(\mathcal{N}_{+,n}).$$

Proof. By (3.19), we have

$$S_{+,n} \subset B_R^4(0, 0), \quad (3.35)$$

where $R > 0$ is as in Lemma 3.4.

Assume that on the contrary, for any large n , $\mathcal{N}_{+,n}$ is not an isolating neighborhood for $S_{+,n}$. Then, for any large n , there exists $x_n := (\lambda_n, \xi_n) \in S_{+,n}$ such that $x_n \in \partial\mathcal{N}_{+,n}$. Assume, without loss of generality that $n \geq R$. Then, by (3.35), the possibility $|x_n|^2 = \lambda_n^2 + |\xi_n|^2 = n^2$ does not arise and we have $\lambda_n = \frac{1}{n}$. For simplicity, we write $x_n(t) = \phi^t_{-D\Gamma_K}(x_n)$. By definition, $x_n(t)$ is the solution to the Cauchy problem

$$\frac{dx_n}{dt}(t) = -D\Gamma_K(x_n(t)), \quad x_n(0) = x_n. \quad (3.36)$$

By (3.35), we have $|x_n(t)| \leq R$ for any $t \in \mathbb{R}$ and by the regularity estimate of the solution to (3.36), passing to a subsequence if necessary, we may assume that there exists $x_\infty(t) = (\lambda_\infty(t), \xi_\infty(t)) \in C^2(\mathbb{R}, \mathbb{R}^4)$ such that $x_n \rightarrow x_\infty$ in $C^2_{\text{loc}}(\mathbb{R})$, i.e., $\sup_{t \in [a,b]} \sum_{k=0}^2 |x_n^{(k)}(t) - x_\infty^{(k)}(t)| \rightarrow 0$ if $-\infty < a < b < +\infty$.

In the following, convergence in $C_{\text{loc}}^2(\mathbb{R})$ means with this topology. Then x_∞ is the solution to the Cauchy problem

$$\frac{dx_\infty}{dt}(t) = -D\Gamma_K(x_\infty(t)), \quad x_\infty(0) = (0, \xi_\infty) \in \{0\} \times \mathbb{R}^3 \quad (3.37)$$

for some $\xi_\infty \in \mathbb{R}^3$.

Recall that $D_1\tilde{\Gamma}_K(0, \xi) = 0$ for any $\xi \in \mathbb{R}^3$ (see (3.12)). Thus, by the uniqueness of the solution to the Cauchy problem (3.37), we have $\lambda_\infty(t) = 0$ for any $t \in \mathbb{R}$, i.e., $x_\infty(t) = (0, \xi_\infty(t))$ and ξ_∞ satisfies

$$\frac{d\xi_\infty}{dt}(t) = -C_0DK(\xi_\infty(t)), \quad \xi_\infty(0) = \xi_\infty. \quad (3.38)$$

This implies that ξ_∞ is a (time scaled) negative gradient flow of K .

Observe that by Lemma 3.3 and (3.19), $\text{Crit}(\Gamma_K) \cap (0, \infty) \times \mathbb{R}^3$ is a compact subset of $(0, \infty) \times \mathbb{R}^3$ and there exists a compact neighborhood $\mathcal{N} \subset (0, \infty) \times \mathbb{R}^3$ such that $\text{Crit}(\Gamma_K) \cap (0, \infty) \times \mathbb{R}^3 \subset \mathcal{N}$. We take $\eta > 0$ such that $\mathcal{N} \subset (\eta, \infty) \times \mathbb{R}^3$. Observe also that, by (3.34), $\emptyset \neq \text{Crit}(\Gamma_K) \cap (0, \infty) \times \mathbb{R}^3 \subset S_{+,n}$ and rest points of the flow $\{x_n(t)\}$ are critical points of Γ_K contained in \mathcal{N} . Thus, for $n > 1/\eta$ there exist $t_n > 0$ and $s_n < 0$ such that the λ -components of $x_n(t_n)$ and $x_n(s_n)$ are η and $x_n(t_n) = (\eta, \xi_n(t_n))$, $x_n(s_n) = (\eta, \xi_n(s_n))$. Moreover, we may assume that t_n is the smallest one among $t > 0$'s with the property that the λ -component of $x_n(t)$ is η . Similarly, we may assume that s_n is the largest one among $s < 0$'s with the property that the λ -component of $x_n(s)$ is η .

We claim:

Claim 3.1 *We have $t_n \rightarrow \infty$, $s_n \rightarrow -\infty$.*

Proof. Assume that, on the contrary, for some subsequence (which we still denoted by $\{t_n\}$ for simplicity of the notation), $\{t_n\}$ is bounded above. Then taking a subsequence if necessary, we may assume that $\{t_n\}$ converges to some $t_\infty \geq 0$. Consider the translation $x_n^+(t) = x_n(t_n + t)$. This is also a flow of $-D\Gamma_K$ and contained in $S_{+,n} \subset B_R^4(0, 0)$. Thus, as above, taking a further subsequence if necessary, we may assume that there exists a flow x_∞^+ of $-D\Gamma_K$ such that $x_n^+ \rightarrow x_\infty^+$ in $C_{\text{loc}}^2(\mathbb{R})$. We then have $x_n^+ = x_n(t_n + \cdot) \rightarrow x_\infty(t_\infty + \cdot)$ in $C_{\text{loc}}^2(\mathbb{R})$ and

$$x_\infty^+(0) = x_n(t_n) \rightarrow x_\infty(t_\infty) \in \{0\} \times \mathbb{R}^3.$$

This is a contradiction since the λ -components of $x_n(t_n)$ are $\eta > 0$ for all n . $s_n \rightarrow -\infty$ is proved similarly. \square

We consider the translated flows $\{x_n^+ = x_n(t_n + \cdot)\}$ as defined in the proof of the above Claim 3.1. Then, as above, there exists a $-D\Gamma_K$ -flow x_∞^+ such that (passing a subsequence if necessary) $x_n^+ \rightarrow x_\infty^+$ in $C_{\text{loc}}^2(\mathbb{R})$. By our choice of t_n , the λ -component of $x_n^+(t)$, denoted $\lambda_n^+(t)$, satisfies $\lambda_n^+(t) \leq \eta$ for $-t_n \leq t \leq 0$. By Claim 3.1, it follows that x_∞^+ is a flow of $-D\Gamma_K$ which satisfies

$$\lambda_\infty^+(0) = \eta, \quad \lambda_\infty^+(t) \leq \eta \quad \text{for } t \leq 0, \quad (3.39)$$

where λ_∞^+ is the λ -component of x_∞^+ .

Recall that, by Lemma 3.3, critical points of $\tilde{\Gamma}_K$ contained in the set $\{(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^3 : 0 \leq \lambda \leq \eta\} \cap B_R^4(0, 0)$ are of the form $(0, \xi)$, where ξ is a critical point of K . Moreover, they are non-degenerate. Thus, as $t \rightarrow -\infty$, the flow x_∞^+ converges (exponentially fast) to some critical point of $\tilde{\Gamma}_K$ of the form $(0, \xi^+)$, where $\xi^+ \in \text{Crit}(K)$.

Similarly, for the translated flows $x_n^- = x_n(s_n + \cdot)$, taking a subsequence if necessary, there exists a flow x_∞^- of $-D\Gamma_K$ such that $x_n^- \rightarrow x_\infty^-$ in $C_{\text{loc}}^2(\mathbb{R})$ and

$$\lambda_\infty^-(0) = \eta, \quad \lambda_\infty^-(t) \leq \eta \quad \text{for } t \geq 0. \quad (3.40)$$

Similar to the case x_∞^+ , as $t \rightarrow +\infty$, $x_\infty^-(t)$ converges to a critical point of $\tilde{\Gamma}_K$ of the form $(0, \xi^-)$, where $\xi^- \in \text{Crit}(K)$.

Summing up the above argument, we have obtained three flow lines of $-D\tilde{\Gamma}_K$, x_∞ , x_∞^+ and x_∞^- . Recall that x_∞ is of the form $x_\infty(t) = (0, \xi_\infty(t))$ and ξ_∞ is a flow of $-C_0DK$, i.e., it satisfies (3.38). We also recall that, by (3.35), $x_\infty(\mathbb{R})$ is contained in $B_R^4(0, 0)$ and therefore $\xi_\infty(\mathbb{R}) \subset B_R^3(0)$. Since all the critical points of K are non-degenerate, $\xi_\infty(t)$ converges to some critical points of K as $t \rightarrow \pm\infty$, i.e., there exist $\xi_0^\pm \in \text{Crit}(K)$ such that $\xi_\infty(t) \rightarrow \xi_0^\pm$ as $t \rightarrow \pm\infty$.

Let $T > 0$ be an arbitrary given constant. By Claim 3.1, there exists $n_0 = n_0(T) \in \mathbb{N}$ such that $s_n + T < -T < T < t_n - T$ for $n \geq n_0$. Since Γ_K is non-increasing along the negative gradient flows, we have

$$\Gamma_K(x_n(s_n + T)) \geq \Gamma_K(x_n(-T)) \geq \Gamma_K(x_n(T)) \geq \Gamma_K(x_n(t_n - T)) \quad (3.41)$$

for $n \geq n_0$.

Taking the limit $n \rightarrow \infty$ in (3.41), we obtain

$$\Gamma_K(x_\infty^-(T)) \geq \Gamma_K(x_\infty(-T)) \geq \Gamma_K(x_\infty(T)) \geq \Gamma_K(x_\infty^+(-T)). \quad (3.42)$$

Since $T > 0$ is arbitrary, taking the limit $T \rightarrow \infty$ in (3.42), we obtain

$$\tilde{\Gamma}_K(0, \xi^-) \geq \tilde{\Gamma}_K(0, \xi_0^-) \geq \tilde{\Gamma}_K(0, \xi_0^+) \geq \tilde{\Gamma}_K(0, \xi^+).$$

Recall that $\Gamma_K(0, \xi) = C_0K(\xi)$ (for $\xi \in \mathbb{R}^3$) and we therefore have the following relations:

$$K(\xi^-) \geq K(\xi_0^-) \geq K(\xi_0^+) \geq K(\xi^+). \quad (3.43)$$

(3.43) implies that $K(\xi^-) \geq K(\xi^+)$. On the other hand, we have the following:

Claim 3.2 *We have $\Delta_{g_{\mathbb{R}^3}}K(\xi^-) > 0$ and $\Delta_{g_{\mathbb{R}^3}}K(\xi^+) < 0$.*

Proof. By Lemma 3.3 (see (3.15)), we have

$$D_\lambda \tilde{\Gamma}_K(\lambda, \xi) = C_1 \Delta_{g_{\mathbb{R}^3}}K(\xi)\lambda + o(\lambda)$$

and

$$\begin{aligned} \frac{d\lambda_\infty^-}{dt}(t) &= -D_\lambda \tilde{\Gamma}_K(\lambda_\infty^-(t), \xi_\infty^-(t)) \\ &= -C_1 \Delta_{g_{\mathbb{R}^3}}K(\xi_\infty^-(t))\lambda_\infty^-(t) + o(\lambda_\infty^-(t)). \end{aligned} \quad (3.44)$$

Suppose $\Delta_{g_{\mathbb{R}^3}}K(\xi^-) < 0$. Since $\xi_\infty^-(t) \rightarrow \xi^-$ and $\lambda_\infty^-(t) \rightarrow 0$ as $t \rightarrow \infty$, by (3.44), there exists $T > 0$ such that

$$\frac{d\lambda_\infty^-}{dt}(t) > -\frac{C_1}{2} \Delta_{g_{\mathbb{R}^3}}K(\xi^-)\lambda_\infty^-(t) \quad (3.45)$$

for $t \geq T$. Integrating (3.45), we obtain

$$\lambda_\infty^-(t) > \lambda_\infty^-(T)e^{-\frac{C_1}{2} \Delta_{g_{\mathbb{R}^3}}K(\xi^-)(t-T)}$$

for $t \geq T$. But it contradicts the fact that $\lambda_\infty^-(t) \rightarrow 0$ as $t \rightarrow \infty$. Since we have $\Delta_{g_{\mathbb{R}^3}}K(\xi^-) \neq 0$ by our assumption, we therefore have $\Delta_{g_{\mathbb{R}^3}}K(\xi^-) > 0$. Similar argument also shows that $\Delta_{g_{\mathbb{R}^3}}K(\xi^+) < 0$ and completes the proof of the claim. \square

The following claim completes the proof of Lemma 3.5.

Claim 3.3 *We have $K(\xi^-) > K(\xi^+)$.*

Proof. If $x_\infty \not\equiv \text{const.}$, $\tilde{\Gamma}_K(x_\infty(t))$ is strictly decreasing and we have

$$C_0K(\xi_0^-) = \tilde{\Gamma}_K(0, \xi_0^-) = \tilde{\Gamma}_K(x_\infty(-\infty)) > \tilde{\Gamma}_K(x_\infty(+\infty)) = \tilde{\Gamma}_K(0, \xi_0^+) = C_0K(\xi_0^+).$$

By (3.43), this implies $K(\xi^-) > K(\xi^+)$. So suppose $x_\infty \equiv \text{const.}$ In this case, we have $\xi_0^- = \xi_0^+$. We denote $\xi_0 := \xi_0^\pm$. We observe that the case $\xi^- = \xi_0 = \xi^+$ does not occur: Otherwise, Claim 3.2 implies that $\Delta_{g_{\mathbb{R}^3}}K(\xi_0) = \Delta_{g_{\mathbb{R}^3}}K(\xi^+) < 0 < \Delta_{g_{\mathbb{R}^3}}K(\xi^-) = \Delta_{g_{\mathbb{R}^3}}K(\xi_0)$, a contradiction. So we have $\xi_0 \neq \xi^+$ or $\xi_0 \neq \xi^-$. We consider the first case (the other case is treated similarly).

We take $\delta > 0$ such that $\delta < \eta$ and $B_\delta^4(0, \xi_i) \cap B_\delta^4(0, \xi_j) = \emptyset$ for any $\xi_i, \xi_j \in \text{Crit}(K)$ with $\xi_i \neq \xi_j$. From the fact that $x_n^+ = x_n(t_n + \cdot) \rightarrow x_\infty^+$ in $C_{\text{loc}}^2(\mathbb{R})$ and $x_\infty^+(-\infty) = \xi^+$, there exist $N_1 \in \mathbb{N}$ and $T > 0$ such that

$$x_n(t_n - T) \in B_\delta^4(0, \xi^+) \quad \text{for all } n \geq N_1. \quad (3.46)$$

On the other hand, since $x_n \rightarrow x_\infty \equiv \xi_0$ in $C_{\text{loc}}^2(\mathbb{R})$, for any $M > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$x_n(t) \in B_\delta^4(0, \xi_0) \quad \text{for all } n \geq N_2 \text{ and } |t| \leq M. \quad (3.47)$$

Let $t'_n := \min\{t > 0 : x_n(t) \in \partial B_\delta^4(0, \xi_0)\}$. By (3.47), we have $t'_n \rightarrow \infty$ as $n \rightarrow \infty$. In addition, we claim $t_n - t'_n \rightarrow \infty$ as $n \rightarrow \infty$. To prove this, since $x_n(t_n) \notin B_\delta^4(0, \xi_0)$, we have $t_n > t'_n$. Thus if $\{t_n - t'_n\}$ does not diverge, taking a subsequence if necessary, we may assume that there exists $a \geq 0$ such that $t_n - t'_n \rightarrow a$ as $n \rightarrow \infty$. Then we have

$$x_n(t'_n) = x_n^+(t'_n - t_n) \rightarrow x_\infty^+(-a)$$

as $n \rightarrow \infty$. By our choice of t'_n , we have $x_n(t'_n) \in \partial B_\delta^4(0, \xi_0)$ and therefore $x_\infty^+(-a) \in \partial B_\delta^4(0, \xi_0)$. Now, we observe that $x_\infty^+([-\infty, 0]) \subset [0, \infty) \times \mathbb{R}^3$ is compact and $(0, \xi_0) \notin x_\infty^+([-\infty, 0])$, since otherwise we have $\xi_0 = \xi^+$ and this is a contradiction. Thus, if we take $\delta > 0$ small, we have $\overline{B_\delta^4(0, \xi_0)} \cap x_\infty^+([-\infty, 0]) = \emptyset$. This contradicts $x_\infty^+(-a) \in \partial B_\delta^4(0, \xi_0)$ and we have $t_n - t'_n \rightarrow \infty$ ($n \rightarrow \infty$) as claimed. Combining this with $t'_n \rightarrow \infty$ as $n \rightarrow \infty$, for any $t, T \in \mathbb{R}$, there holds

$$t_n - T > t'_n + t > 0 \quad (3.48)$$

for large n .

On the other hand, as before, passing to a subsequence if necessary, we may assume that

$$\tilde{x}_n := x_n(t'_n + \cdot) \rightarrow \tilde{x}_\infty \quad \text{in } C_{\text{loc}}^2(\mathbb{R}) \quad (3.49)$$

for some $\tilde{x}_\infty \in C^2(\mathbb{R})$. \tilde{x}_∞ is a flow of $-D\tilde{\Gamma}_K$ with $\tilde{x}_\infty(0) \in \partial B_\delta^4(0, \xi_0)$. By our choice of $\delta > 0$, we have $\tilde{x}_\infty(0) \notin \text{Crit}(\tilde{\Gamma}_K)$ and $\tilde{x}_\infty \not\equiv \text{const.}$. On the other hand, by (3.48), we have

$$\tilde{\Gamma}_K(x_n(0)) \geq \tilde{\Gamma}_K(x_n(t'_n + t)) \geq \tilde{\Gamma}_K(x_n(t_n - T)) \quad (3.50)$$

for any $t, T \in \mathbb{R}$ provided if $n \geq N(t, T)$ for some large $N(t, T) \in \mathbb{N}$. Passing to the limit $n \rightarrow \infty$ in (3.50), we thus obtain

$$\tilde{\Gamma}_K(0, \xi_0) \geq \tilde{\Gamma}_K(\tilde{x}_\infty(t)) \geq \tilde{\Gamma}_K(x_\infty^+(-T)). \quad (3.51)$$

Since $T \in \mathbb{R}$ is arbitrary, taking the limit $T \rightarrow \infty$ in (3.51), we have

$$\tilde{\Gamma}_K(0, \xi_0) \geq \tilde{\Gamma}_K(x_\infty^+(t)) \geq \tilde{\Gamma}_K(0, \xi^+) \quad (3.52)$$

for any $t \in \mathbb{R}$. Since $\tilde{x}_\infty \neq \text{const.}$, (3.52) implies that

$$\tilde{\Gamma}_K(0, \xi_0) \geq \tilde{\Gamma}_K(\tilde{x}_\infty(-\infty)) > \tilde{\Gamma}_K(\tilde{x}_\infty(+\infty)) \geq \tilde{\Gamma}_K(0, \xi^+).$$

Thus we have $K(\xi_0) > K(\xi^+)$. Combining this with (3.43), we obtain $K(\xi^-) > K(\xi^+)$ and this completes the proof of Claim 3.3. \square

Now we complete the proof of Lemma 3.5. Under the assumption at the beginning of the proof of Lemma 3.5, by Claim 3.2 and Claim 3.3, we have $\xi^+, \xi^- \in \text{Crit}(K)$ with $K(\xi^-) > K(\xi^+)$ and $\Delta_{g_{\mathbb{R}^3}} K(\xi^+) < 0 < \Delta_{g_{\mathbb{R}^3}} K(\xi^-)$. This contradicts our assumption about K . Therefore, our first assumption is false. This completes the proof of Lemma 3.5. \square

Now we consider the case where K satisfies the assumption (A) of Lemma 3.5. Numbering the critical values of K in decreasing order as $c_1 > c_2 > \cdots > c_p$ ($c_1 = \max K$, $c_p \geq \inf K$). Notice that by our choice of p_0 in §3.2, it is not necessary that $c_p = \inf K$. We set $\text{Crit}_j(K) = \{x \in \text{Crit}(K) : K(x) = c_j\}$ for $1 \leq j \leq p$. The assumption (A) of Lemma 3.5 becomes

$$\begin{aligned} &\text{For any } \xi_i \in \text{Crit}_i(K), \xi_j \in \text{Crit}_j(K) \text{ with } i > j, \\ &\text{we have } \Delta_{g_{\mathbb{R}^3}} K(\xi_i) > 0 \text{ or } \Delta_{g_{\mathbb{R}^3}} K(\xi_j) < 0. \end{aligned} \quad (3.53)$$

When $j = 1$, we always have $\Delta_{g_{\mathbb{R}^3}} K(\xi_j) < 0$, because c_1 is the maximum of K and $\Delta_{g_{\mathbb{R}^3}} K(\xi_j) \neq 0$. Thus (3.53) is automatically satisfied for any $i > j$. For the general case, let j_0 be the smallest j such that there exists $\xi_j \in \text{Crit}_j(K)$ with $\Delta_{g_{\mathbb{R}^3}} K(\xi_j) > 0$. We take $j_0 = p + 1$ if such j_0 does not exist. Note that $j_0 \geq 2$. Then by (3.53), we have $\Delta_{g_{\mathbb{R}^3}} K(\xi_i) > 0$ for any $i > j_0$ and any $\xi_i \in \text{Crit}_i(K)$. That is, there holds the following:

$$\begin{aligned} &\Delta_{g_{\mathbb{R}^3}} K(\xi_j) < 0 \text{ (for any } \xi_j \in \text{Crit}_j(K) \text{ with } j < j_0), \\ &\Delta_{g_{\mathbb{R}^3}} K(\xi_i) > 0 \text{ (for any } \xi_i \in \text{Crit}_i(K) \text{ with } i > j_0). \end{aligned} \quad (3.54)$$

Conversely, assume (3.54) is satisfied for some $j_0 \geq 2$. For any $\xi_i \in \text{Crit}_i(K)$, $\xi_j \in \text{Crit}_j(K)$ with $i > j$, we have $K(\xi_i) < K(\xi_j)$. If $j < j_0$, we have $\Delta_{g_{\mathbb{R}^3}} K(\xi_j) < 0$ and (3.53) is satisfied. Otherwise, we have $j \geq j_0$ and $i > j_0$. This implies $\Delta_{g_{\mathbb{R}^3}} K(\xi_i) > 0$ and (3.53) is satisfied. Thus, the condition (3.53) is equivalent to the condition (3.54) for some $2 \leq j_0 \leq p + 1$. Note that if $j_0 = p$ or $j_0 = p + 1$, (3.54) is automatically satisfied because there is no $1 \leq i \leq p$ with $i > j_0$. When $p = 2$, we have $j_0 = 2$ or $j_0 = 3$ and (3.54) is satisfied. Also, for $p = 1$ the conclusion of Lemma 3.5 holds, because otherwise, as we have observed in the proof of Lemma 3.5, we obtain two distinct critical values of K in \mathbb{R}^3 . Thus, the conclusion of Lemma 3.5 holds if the number of critical values of K is at most two. Equivalently, in terms of H , if the critical values of H is at most three. Summing up the argument, we have proved the following:

Corollary 3.2 *Assume that one of the following (i), (ii) is satisfied:*

(i) *The number p of distinct critical values of K is $p = 1$, or $p = 2$,*

(ii) *$p \geq 3$ and there exists $1 \leq j_0 \leq p + 1$ such that the following condition is satisfied for K :*

$$\begin{aligned} &\Delta_{g_{\mathbb{R}^3}} K(\xi_i) < 0 \text{ for any } i < j_0 \text{ and any } \xi_i \in \text{Crit}_i(K), \\ &\Delta_{g_{\mathbb{R}^3}} K(\xi_j) > 0 \text{ for any } j_0 < j \text{ and any } \xi_j \in \text{Crit}_j(K) \text{ and} \\ &\text{there exists } \xi_{j_0} \in \text{Crit}_{j_0}(K) \text{ such that } \Delta_{g_{\mathbb{R}^3}} K(\xi_{j_0}) > 0. \end{aligned}$$

Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathcal{N}_{+,n}$ is an isolating neighborhood for the negative gradient flow of Γ_K on $(0, \infty) \times \mathbb{R}^3$. That is, we have

$$S_{+,n} := I(\mathcal{N}_{+,n}, \{\phi_{-D\Gamma_K}^t\}) \subset \text{int}(\mathcal{N}_{+,n}).$$

3.4 Proof of the main result, III: Completion of the proof.

Recall that we are searching critical points of L_ϵ on \mathcal{M} defined at the beginning of §3. For $(\lambda, \xi, a) \in \mathcal{M}$, define

$$\Phi_\epsilon(\lambda, \xi, a) = -\frac{4}{m-1} \epsilon^{-1} (L_\epsilon(\varphi_{\lambda, \xi, a}) - C_0).$$

By (3.1) and (3.2), we have

$$\Phi_\epsilon(\lambda, \xi, a) = \Phi_K(\lambda, \xi, a) + r_\epsilon(\lambda, \xi, a), \quad (3.55)$$

where

$$r_\epsilon(\lambda, \xi, a) = o(1), \quad Dr_\epsilon(\lambda, \xi, a) = o(1) \quad (3.56)$$

as $\epsilon \rightarrow 0$ uniformly for $(\lambda, \xi, a) \in \mathcal{M}_c \subset \mathcal{M}$ for any compact subset $\mathcal{M}_c \subset \mathcal{M}$.

Because $(\lambda, \xi, a) \in \mathcal{M}$ is a critical point of L_ϵ if and only if it is a critical point of Φ_ϵ , we thus, in the following, search critical points of Φ_ϵ instead of those of L_ϵ .

The main result follows from the following:

Proposition 3.1 *Assume that for large n , say $n \geq N$, $\mathcal{N} := \mathcal{N}_{+,n}$ is an isolating neighborhood for the isolating invariant set $S := S_{+,n} = I(\mathcal{N}_{+,n}, \{\phi_{-D\Gamma_K}^t\}) \subset \text{int}(\mathcal{N}_{+,n})$. Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, Φ_ϵ has a critical point in $\mathcal{N} \times S^3(\mathbb{S}_3)$.*

Proof. Under the assumption, $\mathcal{N} \times S^3(\mathbb{S}_3)$ is an isolating neighborhood for the invariant set $S \times S^3(\mathbb{S}_3) = I(\mathcal{N} \times S^3(\mathbb{S}_3), \{\phi_{-D\Phi_K}^t\})$. Let $h(S)$ be the homotopy Conley index of S with respect to the flow $\{\phi_{-D\Gamma_K}^t\}$. That is, for an (and any) index pair (N, L) of S with respect to the flow $\{\phi_{-D\Gamma_K}^t\}$ (the existence is guaranteed by Theorem 3.1), $h(S) = [(N/L, [L])]$. See §3.1 for details. By Theorem 3.4, we have

$$\chi(h(S)) = \text{deg}(D_{\lambda, \xi} \Gamma_K, \mathcal{N}, 0). \quad (3.57)$$

By (3.57) and Corollary 3.1 (see (3.34)), we have $\chi(h(S)) \neq 0$. This implies, in particular, that $h(S)$ is not trivial, i.e., $h(S) \neq (*, *)$.

Note that, for an index pair (N, L) of S , $(N, L) \times S^3(\mathbb{S}_3)$ is an index pair for the invariant set $S \times S^3(\mathbb{S}_3)$ with respect to the flow $\{\phi_{-D\Phi_K}^t\}$. We claim that $(N, L) \times S^3(\mathbb{S}_3)$ is also non-trivial. In fact, since $\chi(h(S)) \neq 0$, there exists k such that $H^k(N, L; \mathbb{F}) \neq 0$ (\mathbb{F} is an arbitrary field). By the Künneth formula, $H^{k+3}((N, L) \times S^3(\mathbb{S}_3); \mathbb{F})$ contains as a direct summand a non-trivial factor $H^k(N, L; \mathbb{F}) \otimes_{\mathbb{F}} H^3(S^3(\mathbb{S}_3); \mathbb{F}) \cong H^k(N, L; \mathbb{F}) \neq 0$. Since (N, L) is a regular pair (see §3.1), $(N, L) \times S^3(\mathbb{S}_3)$ is also regular and

$$\begin{aligned} H^*(h(S \times S^3(\mathbb{S}_3)); \mathbb{F}) &= H^*(N \times S^3(\mathbb{S}_3)/L \times S^3(\mathbb{S}_3), [L \times S^3(\mathbb{S}_3)]; \mathbb{F}) \\ &= H^*((N, L) \times S^3(\mathbb{S}_3); \mathbb{F}) \neq 0 \end{aligned} \quad (3.58)$$

for $* = k + 3$. (3.58) implies that $h(S \times S^3(\mathbb{S}_3)) = h(I(\mathcal{N} \times S^3(\mathbb{S}_3), \{\phi_{-D\Phi_K}^t\})) \neq (*, *)$. By the continuation property of the Conley index (see Theorem 3.3), (3.55) and (3.56), there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, $\mathcal{N} \times S^3(\mathbb{S}_3)$ is also an isolating neighborhood for $\{\phi_{-D\Phi_\epsilon}^t\}$ (see Lemma 3.1) and we have $h(I(\mathcal{N} \times S^3(\mathbb{S}_3), \{\phi_{-D\Phi_\epsilon}^t\})) \neq (*, *)$. This implies that $I(\mathcal{N} \times S^3(\mathbb{S}_3), \{\phi_{-D\Phi_\epsilon}^t\}) \neq \emptyset$ and

there exists a rest point of the flow $\{\phi_{-D\Phi_\epsilon}^t\}$ in $\mathcal{N} \times S^3(\mathbb{S}_3)$, i.e., there exists a critical point of Φ_ϵ in $\mathcal{N} \times S^3(\mathbb{S}_3)$. This completes the proof. \square

By Proposition 3.1, Lemma 3.5 and the correspondence between critical points of H on S^3 and those of K on \mathbb{R}^3 (see Lemma 3.2), we finally have the following existence result (see Theorem 1.3 in §1):

Theorem 3.5 *Assume that $H \in C^2(S^3)$ is a Morse function on S^3 with $\Delta_{g_{S^3}}H(p) \neq 0$ for any $p \in \text{Crit}(H)$. We assume that $\text{Crit}(H)$ satisfies (1.4) (or, equivalently (1.5)) and the following:*

(A) *For any pair of critical points $x_1, x_2 \in \text{Crit}(H)$ with $H(x_1) < H(x_2)$, there holds $\Delta_{g_{S^3}}H(x_1) > 0$ or $\Delta_{g_{S^3}}H(x_2) < 0$.*

Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a non-trivial solution to the equation $D\psi = \frac{3}{2}(1 + \epsilon H(x))|\psi|\psi$ on S^3 .

Also, from Corollary 3.2, we have (see Corollary 1.1 in §1)

Corollary 3.3 *Assume that $H \in C^2(S^3)$ is a Morse function on S^3 with $\Delta_{g_{S^3}}H(p) \neq 0$ for any $p \in \text{Crit}(H)$. We assume that $\text{Crit}(H)$ satisfies (1.4) (or, equivalently (1.5)) and one of the following:*

(1) *The number p of critical values of H is at most three, or*

(2) *$p \geq 4$ and denoting by $d_1 > d_2 > \dots > d_p$ all the critical values of H , define $\text{Crit}_j(H) = \{x \in \text{Crit}(H) : H(x) = d_j\}$. There exists $1 \leq j_0 \leq p$ such that the following is satisfied:*

*$\Delta_{g_{S^3}}H(x_i) < 0$ for any $i < j_0$ and any $x_i \in \text{Crit}_i(H)$,
 $\Delta_{g_{S^3}}H(x_j) > 0$ for any $j_0 < j$ and any $x_j \in \text{Crit}_j(H)$ and
there exists $x_{j_0} \in \text{Crit}_{j_0}(H)$ such that $\Delta_{g_{S^3}}H(x_{j_0}) > 0$.*

Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a non-trivial solution to the equation $D\psi = \frac{3}{2}(1 + \epsilon H(x))|\psi|\psi$ on S^3 .

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