Advanced Nonlinear Studies **15** (2015), 447–496

A Pseudo-gradient Flow
Arising In Contact Form Geometry

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Received 10 August 2014
Communicated by Abbas Bahri

Abstract

In this work we construct a pseudo-gradient flow on a subspace C_β of the dual Legendrian curves on a closed contact 3-manifold. These results improve and refine earlier results proved in this framework by A. Bahri [3]. The variational problem involved is not compact, thus asymptotes and critical points at infinity might occur. The flow in this work deforms curves in C_β into a special set $\bigcup_{k\in\mathbf{N}}\Gamma_{2k}$ of curves at infinity and thus allows us a better understanding of this non-compact problem.

1991 Mathematics Subject Classification. 53D35,35A15,58E05.
Key words. contact form, pseudo-gradient flow, Legendrian loops, variational problem

1 Introduction

Let M be a compact three dimensional manifold without boundary and α be a contact form on M, that is

$$\alpha \wedge d\alpha \neq 0.$$

There always exists a vector field ξ satisfing $\alpha(\xi) = 1, d\alpha(\xi, \cdot) = 0$ which is called the Reeb vector field associated to α.

The well-known Weinstein conjecture states that the associated Reeb vector field always admits periodic orbits. In dimension 3, P. H. Rabinowitz [14] proved the conjecture for S^3 with its standard contact structure. Then H. Hofer [11] using techniques from

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[*] The author acknowledges the support of CNSF grant #11101271.

pseudo-holomorphic curves methods established the conjecture for all overtwisted contact structures and some other cases on three dimensional closed manifolds. C. H. Taubes [16], using the Seiberg-Witten equations, has announced the proof of the full conjecture in dimension 3.

There is a variational problem associated to the Weinstein conjecture. This variational problem is defined on a subspace C_β of dual Legendrian curves on (M, α). It was developed in [2, 3, 4]. This work starts from [3], clarifies and improves several estimates. In this paper we will work under the assumption (A) that there exists a nontrivial vector field $v \in \ker\alpha$ such that the dual form $\beta = d\alpha(v, \cdot)$ is also a contact form with the same orientation than α. We can always renormalize v by multiplying a function so that $\alpha \wedge d\alpha = \beta \wedge d\beta$.

Whereas the conclusions are essentially the same, in a forthcoming paper [17], we will explore more the properties of C_β. The assumption that β is a contact form will be removed in two ways: we will prove that when this assumption is removed, C_β remains a manifold under a generic assumption on v, see also [3] and we will prove also that, with this assumption removed, if $\ker\alpha$ "turns well" [3] along v, the injection of C_β into the loop space of M is a weak S^1-equivariant homotopy equivalence. This improves considerably the result of A. Maalaoui and V. Martino [12]. After this forthcoming paper, C_β will become a general tool that is available in Contact Form Geometry.

This semi-flow allows us a better understanding of the lack of compactness of this problem. Namely, the flow might blow-up and end up in a special class of curves $\bigcup \Gamma_{2k}$ which will be defined in the sequel. In a second step, we will build a flow at infinity, that is a flow on $\bigcup \Gamma_{2k}$, under some assumption. It will decrease the functional at infinity $\int \alpha_x(\dot{x})dt$, satisfy the Fredholm condition and thus lead to the discovery of the critical points at infinity as critical points and their Morse indices etc.

It is well-known that every slice set $\{B \leq \int |\dot{x}|^2 \leq A\}$ of $H^1(S^1, M)$ can be approximated by pieces of geodesics. As we move to contact framework (M, α), the space of all Legendrian curves $\mathcal{L}_\beta = \{x \in H^1(S^1, M) | \beta(\dot{x}) = 0\}$ is a natural space to consider. We define C_β by

$$C_\beta := \{x \in H^1(S^1, M) | \beta(\dot{x}) = 0 \text{ and } \alpha(\dot{x}) = c\},$$

where c is a constant which might vary with the curve x.

C_β has been a very convenient tool in the study of contact form geometry, see [4, 5, 6]. Maalaoui and Martino proved that $C_\beta \hookrightarrow H^1(S^1, M)$ is a weak homotopy equivalence under the assumption (A) and the assumption that α turns well along v in [12]. This is an extension of the result of S. Smale [15] which states that $\mathcal{L}_\beta \hookrightarrow H^(S^1, M)$ is a homotopy equivalence.

In this framework, we can think of small pieces of Reeb orbits as geodesics in a generalized sense because now the metric is Finsler and in addition there could be several pieces of geodesics connecting two different points. Furthermore, there is an additional dimension that is provided by the vector field v and the approximation of Birkhoff and Milnor by small pieces of geodesics has to be replaced by the space $\bigcup_{k\in\mathbf{N}} \Gamma_{2k}$ where small pieces of "geodesics" (that is Reeb orbit) are connected by pieces of v-orbits.

Denote the Reeb vector fields of α and β by ξ and w respectively. The deformation defined here provides a control on $\int_0^1 |b|$ and a control on the number of zeros of b where b is the v-component of the tangent vector $\dot{x} = a\xi + bv + cw$. The result is a deformation of

C_β onto the unstable manifolds of periodic Reeb orbits and $\bigcup \Gamma_{2k}$.

An interesting point is to compare the flow to the curve shortening flow [7, 8, 9, 10]. The flow having $\eta = b$ in the framework of the unit sphere cotangent bundle of a two dimensional surface Σ with the standard contact form on $ST^*\Sigma$ projects onto the curve shortening flow on Σ. However, there is an essential difference as tiny loops along an immersed curve in Σ lifts in C_β into large pieces of curves almost tangent to v. As cusps develop on the curves deforming on Σ, continuation by such solution is possible ”downstairs” [1], while such a continuation is not possible ”upstairs” in contact form geometry.

Therefore, the curve shortening flow, under the current ”geodesic” version, can be continued on Σ at least curve by curve. The issue of defining the semi-flow on a compact set of curves is different in contact form geometry. The flow with $\eta = b$ is not a good flow for deformation in this framework.

In section 2 we cover the preliminaries needed to construct our pseudo-gradient flow. Then in section 3 we start with the simplest case that b has only one oscillation. We construct the pseudo-gradient flow and derive monotonicity formula for a single oscillation in this section. In section 4, we show that the same construction can be carried out for each suitable oscillation and therefore the same construction works for b with multiple oscillations. Then we show that the monotonicity formulas we derive in section 3 combine well under convex combination and thus we can globalize our construction. In section 5, we deal with the existence of the flow. We indicate how the flow deforms into curves in $\bigcup \Gamma_{2k}$ when blow-up occurs in the last two sections.

2 Preliminaries

In this section, we set up the variational problem and study the structural equations of the tangent space to C_β. These equations will be important in the construction of our pseudo-gradient flow. The critical points of this variational problem are periodic orbits of the Reeb vector field of α.

Let (M, α) be a three-dimensional compact manifold without boundary and ξ be the Reeb vector field associated to the contact form α. We assume that there exists a vector filed $v \in \ker \alpha$ such that $\alpha \wedge d\alpha = \beta \wedge d\beta$, where $\beta = d\alpha(v, \cdot)$ is the dual form of α. Denote the Reeb vector field of β by w. Then the following proposition holds:

Proposition 2.1 ([3] Proposition A2) *Let $\bar\mu = \alpha(w)$. Under the assumption that $\alpha \wedge d\alpha = \beta \wedge d\beta$, we have:*

$$w = -[\xi, v] + \bar\mu\xi, \quad [\xi, [\xi, v]] = -\tau v, \quad d\alpha(v, [\xi, v]) = -1,$$
$$\bar\mu = d\alpha(v, [v, [\xi, v]]), \quad \gamma = -d\alpha(w, \cdot) = d\beta(\xi, \cdot),$$

where τ is a real-valued C^∞ function on M.

$\bar\mu$ and τ are structural parameters determined by the contact form α and our choice of the vector field v, similar to the notion of curvatures in the Riemannian setting. They play an important role in the computation of Morse indices etc in this framework as shown in [4].

As $\{\xi, v, w\}$ provides with a basis to the tangent space of M, $\dot{x}$ can be decomposed on this basis, thus giving

$$\dot{x} = a\xi + bv + cw; \quad a, b, c \in H^1(S^1).$$

$x \in \mathcal{L}_\beta$ if and only if $\beta_x(\dot{x}) = d\alpha_x(v, \dot{x}) = c \equiv 0$; And $x \in C_\beta$ if and only if $c \equiv 0$ and a is a positive constant.

Let z be a tangent vector to C_β at x. We decompose z into $z = \lambda\xi + \mu v + \eta w$, where $\lambda, \mu, \eta \in H^1(S^1; \mathbf{R})$. We derive the formulas for the first variations of a, b and c along z:

Proposition 2.2 *The first variations of a, b and c along z are given by*

$$\left\{ \begin{array}{rcl} \dfrac{\partial a}{\partial s} &=& \partial_z\alpha(\dot{x}) = \overset{\cdot}{\overbrace{\lambda + \bar{\mu}\eta}} - (b\eta - c\mu) \\ \dfrac{\partial b}{\partial s} &=& \partial_z\gamma(\dot{x}) = \dot{\mu} + \tau(a\eta - c\lambda) - \bar{\mu}_\xi(b\eta - c\mu) \\ \dfrac{\partial c}{\partial s} &=& \partial_z\beta(\dot{x}) = \dot{\eta} - (a\mu - b\lambda) \end{array} \right. \tag{2.1}$$

We defer the proof of this proposition to the appendix.

z is tangent to $\mathcal{L}_\beta$ at x if and only if $\dfrac{d}{dt}\beta_x(z) = d\beta_x(\dot{x}, z)$. This implies $\dot{\eta} = \mu a - \lambda b$. If in addition we require that $a \equiv constant$, we need $\delta_z a = \dfrac{d}{dt}\alpha(z) - d\alpha(\dot{x}, z) \equiv constant$. This implies that $\overset{\cdot}{\overbrace{\lambda + \bar{\mu}\eta}} = b\eta + C$. Thus we have:

Proposition 2.3 *A vector field* $z = \lambda\xi + \mu v + \eta w$, *where* $\lambda, \mu, \eta \in H^1(S^1; \mathbf{R})$, *is tangent to* C_β *along a closed loop* x *with* $\dot{x} = a\xi + bv + cw$ *if and only if*

$$\left\{ \begin{array}{rcl} \dot{\eta} &=& \mu a - b\lambda \\ \overset{\cdot}{\overbrace{\lambda + \bar{\mu}\eta}} &=& b\eta - \int_0^1 b\eta \end{array} \right. . \tag{2.2}$$

We consider the functional

$$J(x) = \alpha_x(\dot{x}) = \int_0^1 \alpha_x(\dot{x})dt.$$

on C_β. The following proposition reveals the connection between J and Reeb periodic orbits of α.

Proposition 2.4 *J is a* C^2 *functional on* C_β *whose critical points are periodic orbits to* ξ. *If z is a tangent vector to* C_β *along the curve* $x(\cdot)$, *then*

$$\partial_z J(x) = -\int_0^1 d\alpha_x(\dot{x}, z)dt = -\int_0^1 b\eta dt,$$

and the variation of b along z is given by

$$\partial_z b = \frac{\partial b}{\partial s} = \dot{\mu} + a\eta\tau - b\eta\bar{\mu}_\xi.$$

The proof of Proposition 2.4 is immediate after Proposition 2.2. This implies that the critical points of J are Reeb periodic orbits of α.

3 Construction for a single oscillation, monotonicity formulas

We take $b \in H^1(S^1)$. By Sobolev embedding theorem b is continuous. We assume that b has an oscillation, i.e. there is an interval $[t_1, t_2]$ such that

$$\tilde{\tilde{\mu}} := \max_{t_1 \leq t \leq t_2} |b(t)| > \sup(|b(t_1)|, |b(t_2)|) =: \underline{\tilde{\mu}}$$

and we assume that b does not take the value zero on $[t_1, t_2]$.

Let $|b|_\infty$ be the L_∞-norm of b and $\epsilon_0 > 0$ be a small fixed constant less than 1 which will be determined later. We assume that

$$C(|I| + \int_I b^2)|I| < \epsilon_0 \tag{3.3}$$

where $I = [t_1, t_2]$ and C is a universal constant larger than 1 which will be specified later. The choice of the constant C is related to the data of this problem. Observe that (3.3) implies

$$\int_{|I|} |b| \leq (\int_I b^2)^{\frac{1}{2}} |I|^{\frac{1}{2}} \leq \sqrt{\epsilon_0}.$$

In this section we construct a flow locally on $[t_1, t_2]$ which is a pseudo-gradient of the functional $J(x)$. In the next section we will globalize our construction to the case of multiple oscillations and then to the whole space C_β.

3.1 A technical lemma

We take two values μ_1 and μ_2 such that

$$\underline{\tilde{\mu}} < \mu_1, \mu_2 < \frac{\tilde{\tilde{\mu}} + \underline{\tilde{\mu}}}{2}. \tag{3.4}$$

Let $\bar{t}$ be the point in $[t_1, t_2]$ such that $|b(\bar{t})| = \tilde{\tilde{\mu}}$ and

$$t_1^+ := \inf\{t, t_1 \leq t \leq \bar{t}, |b(s)| \geq \mu_1 \text{ for all } s \in [t, \bar{t}]\},$$

$$t_2^+ := \sup\{t, \bar{t} \leq t \leq t_2, |b(s)| \geq \mu_2 \text{ for all } s \in [\bar{t}, t]\}.$$

The following lemma holds:

Lemma 3.1 *μ_1 and μ_2 can be chosen satisfying* (3.4) *and so that: there exists a constant $\gamma \in (0, \infty)$ such that, for any $\tilde{\tilde{\delta}} > 0$, there exists $t_1^- \in [t_1^+ - \tilde{\tilde{\delta}}, t_1^+]$ and $t_2^- \in [t_2, t_2^+ + \tilde{\tilde{\delta}}]$ such that :*

1. $\mu_1^- = |b(t_1^-)| < \mu_1$ *and* $\mu_2^- = |b(t_2^-)| < \mu_2$,
2. $|\frac{b(t_i^+) - b(t_i^-)}{t_i^+ - t_i^-}| \geq \gamma.$

It is well-known in Lebesgue theory that there are strictly increasing functions whose derivative is zero almost everywhere. Lemma 1 states that nevertheless the set of points where the derivative is nonzero is dense.

Proof. Without loss of generality we assume that b is positive on $[t_1, t_2]$. By symmetry, the other case follows. For $t < \bar{t}$, we define

$$\tilde{b}(t) = \inf_{\tau \in [t, \bar{t}]} b(\tau).$$

A similar construction can be completed for $t > \bar{t}$ as well. Clearly, $\tilde{b}(t)$ is continuous and $\tilde{b}(t) \leq b(t)$.

Taking μ_1 a value in $[\underline{\tilde{\mu}}, \frac{\tilde{\tilde{\mu}} + \underline{\tilde{\mu}}}{2}]$, there exists t_0 such that $\tilde{b}(t_0) = b(t_0) = \mu_1$. There are two possibilities:
1. $t_1^+(\mu_1) < t_0$. This implies $\tilde{b}(t) = \mu_1$ on $[t_1^+(\mu_1), t_0]$, or
2. $t_1^+(\mu_1) = t_0$. In this case arguing by contradiction, assume that lemma does not hold at $t_1^+(\mu_1) = t_0$. Then for any $\gamma_k = \frac{1}{k}$, there exists $\delta_k > 0$ such that for any $t \in [t_0 - \delta_k, t_0]$,

$$\frac{b(t) - b(t_0)}{t - t_0} < \frac{1}{k}.$$

Thus the following holds

$$b(t) \geq b(t_0) + \frac{1}{k}(t - t_0) \text{ for } t \in [t_0 - \delta_k, t_0].$$

This yields

$$\tilde{b}(t) \geq \tilde{b}(t_0) + \frac{1}{k}(t - t_0) \text{ for } t \in [t_0 - \delta_k, t_0].$$

In case 1, this inequality also holds true. Thus we have, for all $t \in [t_0 - \delta_k, t_0]$,

$$\tilde{b}(t) \geq \tilde{b}(t_0) + \frac{1}{k}(t - t_0) \tag{3.5}$$

always holds.

Let $[t_0', t_0]$ be the maximal interval that (3.5) holds true. We can always repeat this argument for $\mu_1' = \mu_1 + \frac{1}{k}(t_0' - t_0)$which is less than μ_1 as long as $\mu_1' \geq \underline{\tilde{\mu}}$. Since the interval that (3.5) holds is both open and closed we can always extend the interval until we hit $\underline{\tilde{\mu}}$. This process can be carried for every $k \in \mathbf{N}$. Letting $k \to \infty$, we have

$$\tilde{b}(t) \geq \tilde{b}(t_0) \text{ for all } t \leq t_0.$$

which contradicts with the fact that b has an oscillation on $[t_1, t_2]$, i.e. $\underline{\tilde{\mu}} < \tilde{\tilde{\mu}}$.

Remark 3.1 From the proof, we can see that the value of $\underline{\tilde{\mu}}$ and $\frac{\underline{\tilde{\mu}} + \tilde{\tilde{\mu}}}{2}$ does not play any role in the argument above besides that $\underline{\tilde{\mu}}$ is smaller than μ_1 and μ_2 and $\tilde{\tilde{\mu}}$ is larger than them. As long as there is an oscillation lemma 1 always holds true, and the construction also holds for every curve $\tilde{b}$ which is L^∞-close to b.

3.2 Construction of η_0

All arguments in this section will be applied to the interval $[t_1^-, t_2^-]$ that we derived using Lemma 3.1 in section 3.1. We introduce on $[t_1^-, t_2^-]$ the function η_0 which is the solution of

$$\begin{cases} \dfrac{d^2}{dx^2}(e^{-\int_{t_1^-}^t \bar{\mu} b}\eta_0) + C(1+b^2)\eta_0 e^{-\int_{t_1^-}^t \bar{\mu} b} = \dfrac{-2}{|x_1^- - x_2^-|^2}, \\ \eta_0(x_1^-) = \eta_0(x_2^-) = 0 \end{cases} \tag{3.6}$$

where $x = x(t) = \int_{t_1^-}^t e^{-\int_{t_1^-}^z \bar{\mu} b d\tau} dz$, $x_1^- = 0$ and $x_2^- = \int_{t_1^-}^{t_2^-} e^{-\int_{t_1^-}^t \bar{\mu} b d\tau} dz$. This change of variable will facilitate our computation greatly.

Lemma 3.2 *Under the assumption* (3.3), *the equation* (3.6) *has a unique positive solution. Furthermore, given a bound on* $|b|_\infty$, *we have the following bounds on* $\eta_0 e^{-\int_{t_1^-}^t \bar{\mu} b d\tau}$*:*

$$|\eta_0 e^{-\int_{t_1^-}^t \bar{\mu} b d\tau} dz|_\infty + |x_1^- - x_2^-|\,|\frac{d}{dx}(\eta_0 e^{-\int_{t_1^-}^t \bar{\mu} b d\tau})|_\infty + |x_1^- - x_2^-|^2 |\frac{d^2}{dx^2}(\eta_0 e^{-\int_{t_1^-}^t \bar{\mu} b d\tau})|_\infty \le C_0,$$

and thus the following bounds on η_0*:*

$$\begin{aligned} |\eta_0|_\infty &\le C_0 \\ |x_1^- - x_2^-|\,|\frac{d}{dx}\eta_0|_\infty &\le C_0(\tilde{\bar{\mu}} + |b|_\infty) \\ |x_1^- - x_2^-|^2 |\frac{d^2}{dx^2}\eta_0|_\infty &\le C(|b|_\infty) \end{aligned}$$

where C_0 *is a constant which depends only on the contact form* α, $\bar{\mu}$ *and* ϵ_0, *but not on* b.

Proof. Taking t_1^-, t_2^- very close to t_1 and t_2, by (3.3) we can also assume that

$$C(|I| + \int_{t_1^-}^{t_2^-} b^2)|t_1^- - t_2^-| < \epsilon_0.$$

Since C is larger than 1, we derive that $\int_{t_1^-}^{t_2^-} |b| < 1$. Thus $x = x(t)$ is a change of variable with bounded derivative and inverse derivative, i.e. $\frac{1}{C_1} \le \left|\frac{dx}{dt}\right| \le C_1$, where C_1 depends only on the contact form α and $\bar{\mu}$, but not on $|b|_\infty$. Hence,

$$C\int_{x_1^-}^{x_2^-} (1+b^2)|x_1^- - x_2^-| < C_1\epsilon_0.$$

For a function $f \in H_0^1[x_1^-, x_2^-]$, we have

$$\begin{aligned} \int_{x_1^-}^{x_2^-} \dot{f}^2 dx &\ge \frac{1}{|x_1^- - x_2^-|}|f|_\infty^2 \\ &\ge \frac{C}{C_1\epsilon_0}\int_{x_1^-}^{x_2^-} (1+b^2)|f|_\infty^2 \\ &\ge \frac{C}{C_1\epsilon_0}\int_{x_1^-}^{x_2^-} (1+b^2)f^2. \end{aligned}$$

Denote $f = e^{-\int_{t_1^-}^{t}\bar{\mu}b}\eta_0$. We prove by contradiction and assume that (3.6) has two solutions f_1 and f_2.
Since

$$\frac{d^2}{dx^2}(f_1 - f_2) + C(1 + b^2)(f_1 - f_2) = 0,$$

we have

$$\begin{aligned}\int_{x_1^-}^{x_2^-} |\frac{d}{dx}(f_1 - f_2)|^2 &= \int_{x_1^-}^{x_2^-} C(1 + b^2)(f_1 - f_2)^2 \\ &\geq \frac{1}{C_1\epsilon_0}\int_{x_1^-}^{x_2^-} C(1 + b^2)(f_1 - f_2)^2.\end{aligned}$$

If $C_1\epsilon_0 < 1$, then $\int_{x_1^-}^{x_2^-} C(1 + b^2)(f_1 - f_2)^2 = 0$. This implies (3.6) has a unique solution.

Now we show this solution is positive. Suppose that f is negative on $[\tau_1, \tau_2]$, $f(\tau_1) = f(\tau_2) = 0$, and it achieves its minimum on $[\tau_1, \tau_2]$ at $\bar{\tau}$, then

$$\frac{d}{dx}f(\tau) - \frac{d}{dx}f(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} C(1 + b^2)f = -\frac{2(\tau - \bar{\tau})}{|x_1^- - x_2^-|^2}.$$

Integrating in x, by (3.3) we obtain

$$-\frac{2(\tau - \bar{\tau})}{|x_1^- - x_2^-|^2} - \frac{\epsilon_0}{C_1|I|}f(\bar{\tau}) \geq \frac{d}{dx}f(\tau) \geq -\frac{2(\tau - \bar{\tau})}{|x_1^- - x_2^-|^2}, \tag{3.7}$$

for $\tau > \bar{\tau}$. Integrating in x we get

$$-\frac{(\tau_2 - \bar{\tau})^2}{|x_1^- - x_2^-|^2} - \frac{\epsilon_0}{C_1^2|I|}f(\bar{\tau}) \geq f(\tau_2) - f(\bar{\tau}).$$

Thus

$$(1 - \frac{\epsilon_0}{C_1^2|I|}(\tau_2 - \bar{\tau}))f(\bar{\tau}) \geq \frac{(\tau_2 - \bar{\tau})^2}{|x_1^- - x_2^-|^2},$$

which contradicts with the assumption that f is negative on $[\tau_1, \tau_2]$. Hence η_0 is nonnegative.

We assume that η_0 attains its maximum at $\bar{\tau}$. Then we have

$$-\frac{2(\tau - \bar{\tau})}{|x_1^- - x_2^-|^2} + \frac{C_1\epsilon_0}{|I|}f(\bar{\tau}) \geq \frac{d}{dx}f(\tau) \geq -\frac{2(\tau - \bar{\tau})}{|x_1^- - x_2^-|^2},$$

for $\tau < \bar{\tau}$. Integrating in x on $[x_1^-, \bar{\tau}]$, we get

$$\frac{(x_1^- - \bar{\tau})^2}{|x_1^- - x_2^-|^2} + \frac{C_1^2\epsilon_0}{|I|}(\bar{\tau} - x_1^-)f(\bar{\tau}) \geq f(\bar{\tau}).$$

Thus

$$(1 - \frac{C_1^2\epsilon_0}{|I|}(\bar{\tau} - x_1^-))f(\bar{\tau}) \leq \frac{(\bar{\tau} - (x_1^-)^2}{|x_1^- - x_2^-|^2},$$

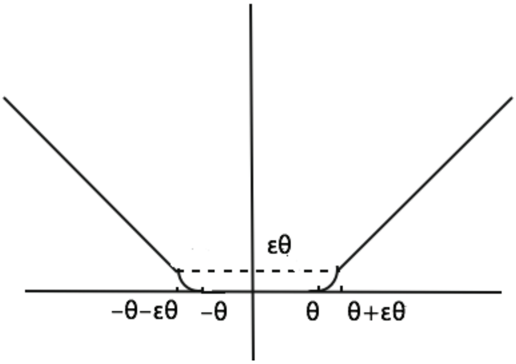

Figure 1: $\psi_{\theta,\epsilon}$

and

$$|\eta_0|_\infty \leq C(\epsilon_0, \alpha, \bar{\mu}).$$

Thanks to (3.7), we get

$$|x_2^- - x_1^-||\frac{d}{dx}f(\tau)| \leq C(\epsilon_0, \alpha, \bar{\mu}).$$

for $\tau < \bar{\tau}$. For $\tau > \bar{\tau}$, a similar estimate holds. Thanks to (3.6), we have

$$|x_2^- - x_1^-|^2|\frac{d^2}{dx^2}f(x)| \leq C(|b|_\infty).$$

This finishes the proof of lemma 3.2.

3.3 Construction of the cut-off function $\psi_{\theta(t),\epsilon(t)}(b)$ and $\psi_{\tilde{\theta}(t),\tilde{\epsilon}(t)}(\tilde{b})$

Given $\theta > 0$, we define $\psi_{\theta,\epsilon}(b)$ to be a C^∞ ϵ- regularization of the function $(|b| - \theta)^+$, which behaves as follows (see Figure 1): $\psi_{\theta,\epsilon}$ is even, C^∞ and $\psi_{\theta,\epsilon}(b) = (|b| - \theta)^+$ when $[-\infty, -\theta - \theta\epsilon] \cup [\theta + \theta\epsilon, \infty]$. It also satisfies $\frac{\partial \psi_{\theta,\epsilon}}{\partial b} \cdot b \geq 0$.

First we construct $\psi_{1,\epsilon}$ so that it satisfies all the conditions listed above. In addition it satisfies $|\psi'_{1,\epsilon}| \leq 2$ and $|\psi''_{1,\epsilon}| \leq \frac{2}{\epsilon}$. Then we define

$$\psi_{\theta,\epsilon}(x) = \theta\psi_{1,\epsilon}(\frac{x}{\theta}).$$

Direct computation shows that:

Lemma 3.3 $\psi_{\theta,\epsilon}$ *satisfies the following estimates:*

$$\begin{cases} \dfrac{\partial}{\partial\theta}\psi_{\theta,\epsilon}(x) = -1 \textit{ for } x \geq \theta + \epsilon\theta, \dfrac{\partial}{\partial\theta}\psi_{\theta,\epsilon}(x) = 0 \textit{ for } x \leq \theta \\ |\dfrac{\partial}{\partial\theta}\psi_{\theta,\epsilon}(x)| \leq 2 \textit{ otherwise} \\ |\dfrac{\partial^2\psi_{\theta,\epsilon}}{\partial\theta^2}|, |\dfrac{\partial^2\psi_{\theta,\epsilon}}{\partial\theta\partial x}|, |\dfrac{\partial^2\psi_{\theta,\epsilon}}{\partial x^2}| \textit{ and } |\dfrac{\partial^2\psi_{\theta,\epsilon}}{\partial x\partial\epsilon}| \leq \dfrac{8}{\theta\epsilon} \\ \dfrac{\partial^2\psi_{\theta,\epsilon}}{\partial x\partial\epsilon} = O(\dfrac{1}{\epsilon}) \end{cases}$$

Now we proceed with the construction of $\theta(t)$ and $\epsilon(t)$. We choose μ_1 and μ_2 extremely close and such that

$$\underline{\tilde{\mu}} < \mu_1 < \mu_2 \leq \frac{\tilde{\bar{\mu}} + \underline{\tilde{\mu}}}{2}.$$

We also choose $\tilde{\tilde{\delta}}$ in Lemma 3.1 small so that $\mu_2 + \tilde{\tilde{\delta}} < \frac{\mu_2 + \tilde{\bar{\mu}}}{2}$. There exists a point $\tilde{t}_2$ in $(t_1, \bar{t})$ such that $|b(t)| \geq \mu_2 + \tilde{\tilde{\delta}}$ on $[\tilde{t}_2, \bar{t}]$, $|b(\tilde{t}_2)| = \mu_2 + \tilde{\tilde{\delta}}$. Thanks to the freedom in Lemma 3.1, we can also require

$$0 < \epsilon_2 := \frac{\mu_2 - \mu_2^-}{2\mu_2^-} < \epsilon_1 := \frac{\mu_1 - \mu_1^-}{2\mu_1^-}. \tag{3.8}$$

In the proof of Lemma 3.1, we can take $|\mu_2 - \mu_1|$ as small as we want and still have a lower bound on $t_2^+ - t_1^+$, since our choice of μ_1 and μ_2 implies

$$t_2^+(\mu_2) - t_1^+(\mu_1) \geq t_2^+(\frac{\tilde{\bar{\mu}} + \underline{\tilde{\mu}}}{2}) - t_1^+(\frac{\tilde{\bar{\mu}} + \underline{\tilde{\mu}}}{2}) > 0.$$

We set

$$\begin{aligned} \epsilon(t) &\equiv \frac{\mu_1 - \mu_1^-}{2\mu_1^-} \text{ on } [t_1^-, \tilde{t}_2^+],\ \epsilon(t) \equiv \frac{\mu_2 - \mu_2^-}{2\mu_2^-} \text{ on } [\bar{t}, t_2^-] \\ \theta(t) &\equiv \mu_1^- \text{ on } [t_1^-, \tilde{t}_2^+],\ \theta(t) \equiv \mu_2^- \text{ on } [\bar{t}, t_2^-]. \end{aligned} \tag{3.9}$$

Notice that

$$\psi_{\theta(t),\epsilon(t)}(b) = |b| - \mu_1^- \text{ on } [t_1^+, \tilde{t}_2^+], \tag{3.10}$$

since $|b(t)| > \mu_1$ on $[t_1^+, \tilde{t}_2^+]$ and $\mu_1 > \mu_1^-(1 + \epsilon_1)$.
Similarly we also have

$$\psi_{\theta(t),\epsilon(t)}(b) = |b| - \mu_2^- \text{ on } [\bar{t}, t_2^+]. \tag{3.11}$$

Now let $\theta(t)$ evolve from μ_1^- to μ_2^- in $[\tilde{t}_2^+, \bar{t}]$ so that $|\theta - \mu_1^-|_{C^3} \to 0$ as $\mu_2^- \to \mu_1^-$, and $\epsilon(t)$ evolve from ϵ_1 to ϵ_2 monotonically. We also require that

$$\theta(t)(1 + \epsilon(t)) < \sup\{\frac{\mu_1 + \mu_1^-}{2}, \frac{\mu_2 + \mu_2^-}{2}\}. \tag{3.12}$$

Thus

$$\psi_{\theta(t),\epsilon(t)}(b) = |b| - \theta(t) \text{ on } [\tilde{t}_2^+, \bar{t}]. \tag{3.13}$$

If $\tilde{b}$ is close to b in H^1, thus in L^∞, we can build using $t_1^-, t_1^+, t_2^-, t_2^+$ of b a related $\psi_{\tilde{\theta}(t),\tilde{\epsilon}(t)}(\tilde{b})$ which satisfies (3.8)-(3.12). We can also require $\psi_{\tilde{\theta}(t),\tilde{\epsilon}(t)}(\tilde{b})$ to satisfy (3.10), (3.11) and (3.13). The previous constructions for $\theta(t)$ and $\epsilon(t)$ can be repeated step by step for $\tilde{\theta}(t)$ and $\tilde{\epsilon}(t)$. Observe that $\tilde{\theta}$ and $\tilde{\epsilon}$ depend only on $\|\tilde{b} - b\|_{L^\infty}$ and can be taken to be Lipschitz as functions of $\tilde{b}$ in H^1.

3.4 Estimates along the flow

In this note, we always use $\dot{f}$ to denote the derivative in t, $\frac{df}{dt}$, where t is the parameter along the curve. And s is the time along a flow-line. Now we derive the estimates with the presence of just one oscillation.

We define

$$\eta_{12} = (\psi_{\theta(t),\epsilon(t)}(b)\eta_0)\ sgnb + cb, \tag{3.14}$$

where we require:

$$0 < c \leq \inf\{-\frac{\tilde{C}a}{1 + |b|_\infty^3}\frac{\partial a}{\partial s}, 1\}. \tag{3.15}$$

$\tilde{C}$ is a fixed universal constant which depends on the contact form α and the choice of the vector field v.
Actually the coefficient c in (3.14) is a function of the time $c(s)$ along our flow. We will analyze the choice of $\tilde{C}$ and the meaning of (3.15) more later.

Let Z_{12} be the H^1-vector field created by η_{12} using (2.2) on C_β. It is obvious that $\partial_z J = -\int b\eta_{12}$ is always negative and thus Z_{12} is a pseudo-gradient flow for J. We derive some monotonicity properties first.

Theorem 3.1 *Along a flow-line generated by Z_{12} which is the H^1-vector field generated by η_{12} using* (2.2) *on C_β, we have the following estimates:*

1. *There exists a constant C_1 depending only on $\tilde{C}$ and on the contact form α, but not on $|b|_\infty$ such that*
$$\frac{\partial}{\partial s}\left(\frac{|b|_\infty}{a}\right) \leq -\frac{C_1}{a}\frac{\partial a}{\partial s}, \tag{3.16}$$
2. *There exists a universal constant C_2, which depends only on v and γ_0, not on the choice of $\tilde{C}$ such that*
$$\frac{\partial}{\partial s}\left(\int(|b| - v)^+ + C_2(1 + a)\right) \leq 0 \text{ if } a \geq \gamma_0 \tag{3.17}$$
3. *We also have*
$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 \quad \leq \quad & -\frac{9c}{10}\int_0^1 \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(1 + \frac{1}{\Gamma^2} + \frac{1}{\Gamma}(-\frac{\tilde{\bar{\mu}}}{\partial s} - C\frac{\partial a}{\partial s})\right) \times \\ & \left(\int_{t_1^-}^{t_2^-} \dot{b}^2 + t_2^- - t_1^-\right) - C_4(|b|_\infty)(1 + \int_0^1 \dot{b}^2)\frac{\partial a}{\partial s}\end{aligned} \tag{3.18}$$

And moreover, all the estimates hold for $\tilde{b}$ *as well if* $|b-\tilde{b}|_{H^1}$ *is small enough.*

Notation: We will omit the subscript of η_{12} in (3.14) and simply write η to represent η_{12} in the proof of this theorem. We use C with a subsript to denote constants which depend only on the data given by the contact form α and our choice of the vector field $v \in \ker\alpha$ such as $\bar{\mu}$ and τ, but not on b.

Proof. We prove estimate 1 first. Thanks to Proposition 2.3, we have

$$\begin{aligned}\frac{\partial b}{\partial s} &= \dot{\mu} + a\eta\tau - b\eta\bar{\mu}_\xi \\ &= \frac{\ddot{\eta} + b^2\eta - b\int_0^1 b\eta - \overbrace{\bar{\mu}\eta}^{\cdot}\, b}{a} + a\eta\tau - b\eta\bar{\mu}_\xi. \end{aligned} \tag{3.19}$$

First we prove estimate 1 for $b \in C^1$. Then by density the result extends to the flow defined by Z_{12}. We thus only need in the first step to prove the following estimate:

$$\frac{\partial}{\partial s}\left(\frac{|b(t)|}{a}\right) \leq -\frac{C_1}{a}\frac{\partial a}{\partial s} \text{ at any } t \text{ such that } |b|(t) = |b|_\infty.$$

At such point t, $(sgnb)\ddot{b}(t)$ is always negative and $\dot{b}(t) = 0$. All the computation below are performed at such point.

If $\eta_{12} = cb$, we have

$$\begin{aligned}\frac{\partial}{\partial s}|b|(t) &= \frac{c}{a} sgnb\Big(\ddot{b} + b^3 - \dot{\bar{\mu}}b^2 - b\int_0^1 b^2\Big) + cb(a\tau - b\bar{\mu}_\xi) \\ &\leq \frac{c}{a}(|b|_\infty^3 + 1)C_1 - \frac{c}{a}|b|(t)\int_0^1 b^2. \end{aligned}$$

Thus

$$\frac{\partial}{\partial s}\left(\frac{|b|_\infty}{a}\right) \leq \frac{c}{a^2}\left(|b|_\infty^3 + 1\right)C_1, \tag{3.20}$$

where C_1 is a constant which depends on $\bar{\mu}, \tau, a$, but not on b. By (3.15), we know estimate 1 holds when $a \geq \gamma_0$.

We now consider the case $\eta_{12} = \psi_{\theta(t),\epsilon(t)}(b)\eta_0 sgnb$. The case where $\tilde{\bar{\mu}} < |b|_\infty$ is immediate since our flow Z_{12} has no effect there. Thus we may assume that $\tilde{\bar{\mu}} > \frac{|b|_\infty}{2}$. We compute far from t_1^-, t_2^-, at a point in the interval $[t_1^-, t_2^-]$ with $\dot{b}(t) = 0$ and $\ddot{b}(t)sgnb \leq 0$. By (3.19), at such point we have

$$\frac{\partial}{\partial s}|b| = sgnb\left(\frac{\overbrace{\dot{\eta} - \bar{\mu}b\eta}^{\cdot} + b^2\eta - b\int_0^1 b\eta}{a} + a\eta\tau - b\eta\bar{\mu}_\xi\right). \tag{3.21}$$

First we compute the first term in (3.21), the value of

$$\begin{aligned}
\overbrace{\dot{\eta} - \bar{\mu} b \eta} &= e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau} \frac{d^2}{dx^2}\left(\eta e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}\right) = e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau} \frac{d^2}{dx^2}\left(\psi_{\theta,\epsilon}\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}\right) \\
&= e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}\Bigg(\psi_{\theta,\epsilon}\frac{d^2}{dx^2}(\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}) + 2\frac{d}{dx}\psi_{\theta,\epsilon}\frac{d}{dx}(\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau})+ \\
&\quad +\frac{d^2}{dx^2}\psi_{\theta,\epsilon}(\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau})\Bigg) \\
&= e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}\left(\left(\frac{-2}{|x_1^- - x_2^-|^2} - C(1+b^2)\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}\right)\psi_{\theta,\epsilon} - \right. \\
&\quad \left. - 2\dot{\theta}\frac{d}{dx}(\eta_0 e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau})\right) + (-\ddot{\theta} + \frac{\partial \psi_{\theta,\epsilon}}{\partial b}\ddot{b})(\eta_0 e^{-2\int_{t_1^-}^{t} \bar{\mu} b d\tau}) \\
&\leq \frac{-2}{|x_1^- - x_2^-|^2}\psi_{\theta,\epsilon} e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau} + C_0(\frac{O(|\dot{\theta}|)}{|x_1^- - x_2^-|} + O(|\ddot{\theta}|)) - \\
&\quad -\frac{C}{4}(1+\tilde{\bar{\mu}}^2)\psi_{\theta,\epsilon} e^{-2\int_{t_1^-}^{t} \bar{\mu} b d\tau}. \qquad (3.22)
\end{aligned}$$

In the last inequality, we used the fact that $\frac{\partial \psi_{\theta,\epsilon}}{\partial b}(t)\ddot{b}(t) \leq 0$ at such points t and lemma 3.2 .

Now we estimate the remaining term in (3.21):

$$\left|(a\tau - b\bar{\mu}_\xi + \frac{b^2}{a})\psi_{\theta,\epsilon}\eta_0\right| \leq (a|\tau| + \tilde{\bar{\mu}}_\xi^2 + \frac{1}{a})(1+\bar{\mu}^2)\psi_{\theta,\epsilon}. \qquad (3.23)$$

Thus, at such $b(t)$, we can absorb $(a\tau - b\bar{\mu}_\xi + \frac{b^2}{a})\psi_{\theta,\epsilon}\eta_0$ using $-\frac{C}{2}(1+\tilde{\bar{\mu}}^2)\psi_{\theta,\epsilon} e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}$ by taking $C \geq 4((a|\tau| + \bar{\mu}_\xi^2 + \frac{1}{a})$ in (3.6).

Thanks to (2.1), (3.21) and (3.22) , by taking μ_2 close enough to μ_1, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{|b|(t)}{a}\right) &= \frac{1}{a}\frac{\partial}{\partial s}|b|(t) - \frac{|b|(t)}{a^2}\frac{\partial a}{\partial s} \\
&\leq \frac{-2}{a^2}\left(\frac{1}{|x_1^- - x_2^-|^2}\psi_{\theta,\epsilon} e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau} + \frac{O(|\dot{\theta}|)}{|x_1^- - x_2^-|} + O(|\ddot{\theta}|)\right) \\
&\leq \frac{1}{a^2}\frac{-1}{2|x_1^- - x_2^-|^2}\psi_{\theta,\epsilon} e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau}. \qquad (3.24)
\end{aligned}$$

Thus estimate 1 holds in this case as well. Combining (3.20) and (3.24), we conclude that for η_{12} defined in (3.14), the estimate

$$\frac{\partial}{\partial s}\left(\frac{|b|(t)}{a}\right) \leq \frac{1}{a^2}\frac{-1}{2|x_1^- - x_2^-|^2}\psi_{\theta,\epsilon} e^{-\int_{t_1^-}^{t} \bar{\mu} b d\tau} - C_1\frac{1}{a}\frac{\partial a}{\partial s} \qquad (3.25)$$

always holds. We remark that at any point t where $\dot{b}(t) = 0$ and $sgnb(t)\ddot{b}(t) \leq 0$ the estimate (3.25) holds as well.

Let Γ be the altitude of the oscillation $\tilde{\bar{\mu}} - \mu_2$. Notice that at such t we have $\psi_{\theta,\epsilon} \le \tilde{\bar{\mu}} - \mu_2$. Thus when $a > \gamma_0$ we have

$$\frac{\partial}{\partial s}\left(\frac{\tilde{\bar{\mu}}}{a}\right) \le C_0 \frac{-\Gamma}{a^2|t_1^- - t_2^-|^2} - C_1 \frac{\partial a}{\partial s}. \tag{3.26}$$

Now we give another proof of estimate 1 using integral estimates. Let $\rho = |b|_{L^\infty}(0)$. We estimate for $s > 0$ small, $\frac{\partial}{\partial s}(|b| - \rho)^{+2}$. We have

$$\frac{1}{2}\frac{\partial}{\partial s}(|b| - \rho)^{+2} = \int_0^1 (|b| - \rho)^+ \left(\frac{\overbrace{\dot{\eta} + b\lambda}^{\cdot}}{a} + \eta(a\tau - b\bar{\mu}_\xi)\right)$$
$$= -\frac{c}{a}\int_{|b|>\rho} \dot{b}^2 - \frac{1}{a}\int_{|b|>\rho} \psi_{\theta,\epsilon}^{'}\dot{b}^2\eta_0 + \frac{1}{a}\int_{|b|>\rho} \dot{b}\dot{\theta}\eta_0 - \frac{1}{a}\int_{|b|>\rho} \dot{b}\psi_{\theta,\epsilon}\dot{\eta}_0 - \frac{1}{a}\int_{|b|>\rho} \dot{b}b\eta$$
$$+ \int (|b| - \rho)^+ \left(\psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi) + c(ab\tau - b^2\bar{\mu}_\xi)\right).$$

For s small, we may assume that $\dot{\theta} = 0$ for $|b| > \rho$. On the other hand, by (2.2) we have

$$\int_{|b|>\rho} \dot{b}b\eta = \frac{1}{2}\int_{|b|>\rho} (b^2 - \rho^2)(b\eta - \int_0^1 b\eta - \overbrace{\bar{\mu}\eta}^{\cdot}\ )$$
$$= \frac{1}{2}\int_{|b|>\rho} (b^2 - \rho^2)(b\eta - \int_0^1 b\eta) + \int_{|b|>\rho} b\dot{b}(\psi_{\theta,\epsilon}\eta_0 + cb)$$
$$= \frac{1}{2}\int_{|b|>\rho} (b^2 - \rho^2)(b\eta - \int_0^1 b\eta) - \int_{|b|>\rho} \hat{\psi}_{\theta,\epsilon}\, \overbrace{b\bar{\mu}\eta_0}^{\cdot} - c\int_{|b|>\rho} (a\bar{\mu}_\xi + b\bar{\mu}_\nu)\frac{b^3 - \rho^3}{3},$$

where

$$\hat{\psi}_{\theta,\epsilon} = \int_\rho^x \psi_{\theta,\epsilon}.$$

We also have

$$-\int_{|b|>\rho} \dot{b}\psi_{\theta,\epsilon}\dot{\eta}_0 = \int_{|b|>\rho} \hat{\psi}_{\theta,\epsilon}\ddot{\eta}_0.$$

By (3.6), we obtain that

$$\frac{1}{a}\int_{|b|>\rho} \hat{\psi}_{\theta,\epsilon}\ddot{\eta}_0 - \frac{1}{a}\int_{|b|>\rho} \hat{\psi}_{\theta,\epsilon}\, \overbrace{b\bar{\mu}\eta_0}^{\cdot} + \frac{1}{a}\int_{|b|>\rho} \psi_{\theta,\epsilon}(|b| - \rho)^+ \left((b + \rho)b + a\tau - b\bar{\mu}_\xi\right)\eta_0 \le 0.$$

Now we estimate the remaining terms:

$$c\int (|b| - \rho)^+ \left(ab\tau - b^2\bar{\mu}_\xi + \frac{b + \rho}{2}(b^2 - \int_0^1 b^2) - \frac{1}{3}(b^2 + b\rho + \rho^2)\right) -$$
$$-\left(\int_0^1 (|b| - \rho)^+(|b| + \rho)\right)\int_0^1 b\eta.$$

By (3.15), we can require that $c \leq \frac{-\delta}{1+|b|^3} \int_0^1 b\eta$, thus the above expression is also negative. Hence for s small, we have

$$\frac{\partial}{\partial s}(|b| - \rho)^{+2} \leq 0.$$

This implies that

$$\frac{\partial}{\partial s}|b|_\infty \leq 0, \text{ for } s \text{ small.}$$

We now proceed to prove estimate 2. We will prove that it holds when $|b| - \nu$ has a finite number of zeros. Then we will indicate how to extend this result to any $b \in H^1$.

Assume that on $[t_j, \tilde{t}_{j+1}]$,

$$|b| - \nu > 0, \text{ with } |b| = \nu \text{ at } t_j, \tilde{t}_{j+1}.$$

Observe that, there are two cases since we take μ_1 and μ_2 extremely close. If $\mu_1, \mu_2 > \nu$, we can take $\tilde{\tilde{\delta}}$ small enough so that $\mu_1^-, \mu_2^- > \nu$. Thus η_0 is identically 0 near t_j and $\tilde{t}_{j+1}$.

If $\mu_1, \mu_2 < \nu$, we then have

$$\nu < \tilde{\bar{\mu}} \text{ and } t_1^- < t_j < \tilde{t}_{j+1} < t_2^-.$$

In both cases, we may assume

$$\dot{b}(t_j) sgnb(t_j) \geq 0, \quad \dot{b}(\tilde{t}_{j+1}) sgnb(\tilde{t}_{j+1}) \leq 0. \tag{3.27}$$

For the sake of simplicity, we compute for $b \geq 0$. The other case is the same. By proposition 2.3, we have:

$$\begin{aligned}
\frac{\partial}{\partial s}\int_0^1 (|b| - \nu)^+ &= \sum_j \Big[\frac{\dot{\eta} + \lambda b}{a}\Big]_{t_j}^{\tilde{t}_{j+1}} + \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi) \\
&= \sum_j \Big[\frac{\dot{\eta} - \bar{\mu} b\eta}{a} + \frac{b(\int_0^t b\eta - t\int_0^1 b\eta)}{a}\Big]_{t_j}^{\tilde{t}_{j+1}} + \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi) \\
&= \sum_j \frac{1}{a}\Big[c(\dot{b}(t) - \bar{\mu}\nu^2) + (\dot{\psi}_{\theta,\epsilon}\eta_0 + \psi_{\theta,\epsilon}\dot{\eta}_0 - \bar{\mu}\nu\psi_{\theta,\epsilon}\eta_0) \\
&\quad + \nu\Big(\int_0^t b\eta - t\int_0^1 b\eta\Big)\Big]_{t_j}^{\tilde{t}_{j+1}} + \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi)
\end{aligned}$$

Thanks to (3.27), we have $\sum_j c\dot{b}\Big|_{t_j}^{\tilde{t}_{j+1}} \leq 0$ and $\sum_j \frac{\partial \psi_{\theta,\epsilon}}{\partial b}\dot{b}\eta_0\Big|_{t_j}^{\tilde{t}_{j+1}} \leq 0$. Hence we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\int_0^1 (|b|-\nu)^+ \\
\leq\ & \frac{1}{a}\Big[-c\bar{\mu}\nu^2 - \dot{\theta}\eta_0 + \psi_{\theta,\epsilon}\frac{d}{dx}(e^{-\int_{t_1^-}^t \bar{\mu}b}\eta_0)\Big]_{t_j}^{\tilde{t}_{j+1}} + \frac{\nu}{a}(\int_0^t b\eta - t\int_0^1 b\eta)\Big|_{t_j}^{\tilde{t}_{j+1}} \\
&+ \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi) \\
\leq\ & \frac{1}{a}\Big[-c\bar{\mu}\nu^2 - \dot{\theta}\eta_0\Big]_{t_j}^{\tilde{t}_{j+1}} + C_1(\frac{1}{a|x_1^- - x_2^-|}\int_{t_j}^{\tilde{t}_{j+1}} |\dot{\theta}|) + \frac{\nu}{a}(\int_0^t b\eta - t\int_0^1 b\eta)\Big|_{t_j}^{\tilde{t}_{j+1}} \\
&+ \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi),
\end{aligned}$$

since $e^{-\int_{t_1^-}^t \bar{\mu}b}\eta_0$ is concave by (3.6).

Now we proceed to estimate each term. Since $b \geq \nu$ on $[\tilde{t}_{j+1}, t_j]$, we have

$$\left|\sum_j \int_{t_j}^{\tilde{t}_{j+1}} (a\eta\tau - b\eta\bar{\mu}_\xi)\right| \leq \sum_j \int_{t_j}^{\tilde{t}_{j+1}} (\frac{a}{\nu}|\tau| - |\bar{\mu}_\xi|)b\eta \leq C_2(a,\nu)\sum_j \int_{t_j}^{\tilde{t}_{j+1}} b\eta \leq -C_2(a,\nu)\frac{\partial a}{\partial s},$$

and thanks to the choice of c,

$$\sum_j -c\bar{\mu}\nu^2\Big|_{t_j}^{\tilde{t}_{j+1}} \leq c\nu^2\int_{t_j}^{\tilde{t}_{j+1}} |a\bar{\mu}_\xi + b\bar{\mu}_\nu| \leq c\int_{t_j}^{\tilde{t}_{j+1}} b^3(\frac{a}{\nu}|\bar{\mu}_\xi| + |\bar{\mu}_\nu|) \leq -C_2(a,\nu,\tilde{C})\frac{\partial a}{\partial s}.$$

It is clear that

$$\left|\sum_j \frac{\nu}{a}\left(\int_0^t b\eta - t\int_0^1 b\eta\right)\Big|_{t_j}^{\tilde{t}_{j+1}}\right| \leq 2\frac{\nu}{a}\int_0^1 b\eta \leq -2\frac{\nu}{a}\frac{\partial a}{\partial s}.$$

We can take μ_2 so close to μ_1 so that :

$$\sum_j C_1\left(\frac{1}{|x_1^- - x_2^-|}\int_{t_j}^{\tilde{t}_{j+1}} |\dot{\theta}|\right) + |\dot{\theta}|\eta_0\Big|_{t_j}^{\tilde{t}_{j+1}}| \leq \sum_j \int_{t_j}^{\tilde{t}_{j+1}} b\eta \leq -\frac{\partial a}{\partial s}.$$

Adding all the terms together, we have

$$\frac{\partial}{\partial s}\int_0^1 (|b|-\nu)^+ \leq -C_2(a,\nu,\tilde{C})\frac{\partial a}{\partial s}.$$

The result can be extended to the general case by density argument, see Proposition 1 in Appendix 1 [3]. Namely, given any real ν, any flow-line $b(s,t)$, $s \in [s_1, s_2]$, there exists

$\delta > 0$ such that, for each slice of $[s_1, s_2]$ with size δ, $[\tilde{s}_1, \tilde{s}_1 + \delta]$, there exists a sequence (ν_l) tending to ν and b_0^l tending to $b_0 = b(\tilde{s}_1, t)$ such that for every l, the set $\{t$ such that $b^l(s,t) = \nu_l\}$ is finite almost everywhere in $s \in [\tilde{s}_1, \tilde{s}_1 + \delta]$.

Now we prove estimate 3. We compute in the formal sense. By (2.1) and (2.2), we derive:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 &= -\int_0^1 \ddot{b}\frac{\partial b}{\partial s} = -\int_0^1 \ddot{b}(\dot{\mu} + a\eta\tau - b\eta\bar{\mu}_\xi)\\ &= -\int_0^1 \ddot{b}\Big(\frac{\overbrace{\eta + (\lambda + \bar{\mu}\eta)b}^{\cdot}}{a} + a\eta\tau - b\eta\bar{\mu}_\xi - \frac{\overbrace{b\eta\bar{\mu}}^{\cdot}}{a}\Big),\end{aligned}$$

where $\eta = cb + sgnb\psi_{\theta(t),\epsilon(t)}(b)\eta_0$.

First, we estimate the term $-\int_0^1 \ddot{b}\ddot{\eta}$ in the formal sense. We will make this computation rigorous later. The derivatives of $\psi_{\theta,\epsilon}(b)$ are given by

$$\begin{aligned}\dot{\psi}_{\theta,\epsilon} &= -\dot{\theta} + \frac{\partial\psi_{\theta,\epsilon}}{\partial b}\dot{b}\\ \ddot{\psi}_{\theta,\epsilon} &= -\ddot{\theta} + \frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2 + \frac{\partial\psi_{\theta,\epsilon}}{\partial b}\ddot{b}. \end{aligned}\tag{3.28}$$

Thus we have

$$\begin{aligned}-\int_0^1 \ddot{b}\ddot{\eta} &= -c\int_0^1 \ddot{b}^2 - \int_0^1 \ddot{b}sgnb\overbrace{\psi_{\theta,\epsilon}\eta_0}^{\cdot\cdot}\\ &= \int_0^1 \ddot{b}sgnb\Big[(-\ddot{\theta} + \frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2 + \frac{\partial\psi_{\theta,\epsilon}}{\partial b}\ddot{b})\eta_0 + +2\dot{\eta}_0(-\dot{\theta} + \frac{\partial\psi_{\theta,\epsilon}}{\partial b}\dot{b}) + \psi_{\theta,\epsilon}\ddot{\eta}_0\Big]\\ &\quad -c\int_0^1 \ddot{b}^2\\ &= \int_0^1 sgnb\Big[-\ddot{b}^2\frac{\partial\psi_{\theta,\epsilon}}{\partial b}\eta_0 + \ddot{b}(\ddot{\theta}\eta_0 + 2\dot{\theta}\dot{\eta}_0) - 2\ddot{b}\dot{b}\frac{\partial\psi_{\theta,\epsilon}}{\partial b}\dot{\eta}_0 - \ddot{b}\psi_{\theta,\epsilon}\ddot{\eta}_0 - \ddot{b}\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2\Big]\\ &\quad -c\int_0^1 \ddot{b}^2 \end{aligned}\tag{3.29}$$

Now we compute each term. By the construction in section 3.3, the support of $\ddot{\theta}$ is $[\tilde{t}_2^+, \bar{t}]$. Therefore,

$$|\int_0^1 \ddot{b}\ddot{\theta}\eta_0| \le \int_{\tilde{t}_2^+}^{\bar{t}} \ddot{b}^2\eta_0 + \int_{\tilde{t}_2^+}^{\bar{t}} \ddot{\theta}^2\eta_0.$$

Since $0 \le \frac{\partial\psi_{\theta,\epsilon}}{\partial b} \le 2$ for all t and $\frac{\partial\psi_{\theta,\epsilon}}{\partial b} = 1$ when $t_1^+ \le t \le t_2^+$, and by taking μ_2 so close to μ_1

so that $|\theta|_{C^2}$ is small enough, by Lemma 3.2 we have

$$\begin{aligned}
&\int_0^1 sgnb(-\ddot{b}^2\frac{\partial\psi_{\theta,\epsilon}}{\partial b}\eta_0 + \ddot{b}(\ddot{\theta}\eta_0 + 2\dot{\theta}\dot{\eta}_0) - 2\ddot{b}\dot{b}\frac{\partial\psi_{\theta,\epsilon}}{\partial b}\dot{\eta}_0) \\
\leq\ & -\int_{t_1^+}^{t_2^+} \ddot{b}^2\eta_0(1-\chi_{[\tilde{t}_2^+,\bar{t}]}) + \int_{\tilde{t}_2^+}^{\bar{t}} \ddot{\theta}^2\eta_0 + \int_{\tilde{t}_2^+}^{\bar{t}} 2|\ddot{b}\dot{\theta}\dot{\eta}_0| + \int_{t_1^-}^{t_2^-} 4|\ddot{b}\dot{b}\dot{\eta}_0| \\
\leq\ & \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2 + \frac{C_3}{c}\int_{t_1^-}^{t_2^-}(\dot{\theta}^2 + \dot{b}^2)\dot{\eta}_0^2 + \int_{\tilde{t}_2^+}^{\bar{t}}\ddot{\theta}^2\eta_0 \\
\leq\ & \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2 + \frac{C_3}{c}\int_{t_1^-}^{t_2^-}(1+\dot{b}^2)\left(\tilde{\bar{\mu}}^2 + \frac{1}{|t_2^- - t_1^-|^2}\right) + |t_2^- - t_1^-| \\
\leq\ & \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(1 + \frac{1}{|t_2^- - t_1^-|^2}\right)\int_{t_1^-}^{t_2^-}(1+\dot{b}^2). \qquad (3.30)
\end{aligned}$$

Rewriting $\ddot{\eta}_0$ as $\overbrace{\dot{\eta}_0 - \bar{\mu}b\eta_0}^{\cdot} + \overbrace{\bar{\mu}b\eta_0}^{\cdot}$, by (3.6) we get

$$\begin{aligned}
-\int_0^1 \ddot{b}sgnb\psi_{\theta,\epsilon}\ddot{\eta}_0 &= -\int_{t_1^-}^{t_2^-}\ddot{b}sgnb\psi_{\theta,\epsilon}\overbrace{\dot{\eta}_0 - \bar{\mu}b\eta_0}^{\cdot} - \int_{t_1^-}^{t_2^-}\ddot{b}sgnb\psi_{\theta,\epsilon}\overbrace{\bar{\mu}b\eta_0}^{\cdot} \\
&\leq \int_{t_1^-}^{t_2^-}\ddot{b}sgnb\psi_{\theta,\epsilon}\left(\frac{2}{|x_1^- - x_2^-|^2} + C(1+b^2)\eta_0 e^{-\int_{t_1^-}^{t}\bar{\mu}bd\tau}\right)e^{-\int_{t_1^-}^{t}\bar{\mu}bd\tau} \\
&\quad + \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2 + \frac{C_3}{c}\int_{t_1^-}^{t_2^-}\psi_{\theta,\epsilon}^2(b^2 + \dot{b}^2 + b^2\dot{\eta}_0^2). \qquad (3.31)
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
\frac{2}{|x_1^- - x_2^-|^2}\int_{t_1^-}^{t_2^-}\ddot{b}\psi_{\theta,\epsilon}e^{-\int_{t_1^-}^{t}\bar{\mu}bd\tau} &= \frac{-2}{|x_1^- - x_2^-|^2}\int_{t_1^-}^{t_2^-}\dot{b}\frac{d}{dt}\left(\psi_{\theta,\epsilon}e^{-\int_{t_1^-}^{t}\bar{\mu}bd\tau}\right) \\
&\leq \frac{2}{|x_1^- - x_2^-|^2}\left(\int_{t_1^-}^{t_2^-}\dot{b}^2 + C_3\int_{t_1^-}^{t_2^-}(\dot{\theta}^2 + \dot{b}^2 + b^2\psi_{\theta,\epsilon}^2)\right) \\
&\leq \frac{C_3}{|x_1^- - x_2^-|^2}\int_{t_1^-}^{t_2^-}(1 + \dot{b}^2 + b^2\Gamma^2). \qquad (3.32)
\end{aligned}$$

Observe that

$$\Gamma \leq \int_{t_1^-}^{t_2^-}|\dot{b}| \leq \sqrt{|t_2^- - t_1^-|}\left(\int_{t_1^-}^{t_2^-}\dot{b}^2\right)^{\frac{1}{2}}.$$

Thus

$$\frac{1}{|t_2^- - t_1^-|} \leq \frac{\int_{t_1^-}^{t_2^-}\dot{b}^2}{\Gamma^2}. \qquad (3.33)$$

By (3.31), (3.32), (3.33) and Lemma 3.2, we get

$$\begin{aligned}
-\int_0^1 \ddot{b}\,sgn b\psi_{\theta,\epsilon}\ddot{\eta}_0 &\le \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2+\frac{C_3(|b|_\infty)}{|x_1^- - x_2^-|^2}\int_{t_1^-}^{t_2^-}(1+\dot{b}^2)+\\
&\quad +\frac{C_3(|b|_\infty)}{c}\int_{t_1^-}^{t_2^-}(1+\dot{b}^2+\frac{1}{|t_1^- - t_2^-|})\\
&\le \frac{c}{100}\int_{t_1^-}^{t_2^-}\ddot{b}^2+\frac{C_3(|b|_\infty)}{c}\left(1+\frac{1}{|t_2^- - t_1^-|^2}\right)\int_{t_1^-}^{t_2^-}(1+\dot{b}^2).
\end{aligned}\tag{3.34}$$

We are left with $-\int_0^1 \ddot{b}\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2\eta_0$. By construction, $\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}$ is zero on $[t_1^+, t_2^+]$ and otherwise bounded by $\frac{8}{\epsilon\theta}$. Since $\dot{b}(\bar{t})=0$, we have:

$$|\dot{b}(t)|^2 \le \Big(\int_{\bar{t}}^t |\ddot{b}(\tau)|d\tau\Big)^2 \le \int_{t_1^-}^{t_2^-}|\ddot{b}|^2(t-\bar{t}).$$

We proceed with the computation now:

$$\begin{aligned}
\Big|-\int_0^1 \ddot{b}\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2\eta_0\Big| &\le \frac{8}{\mu_1-\mu_1^-}\int_{t_1^-}^{t_1^+}|\ddot{b}||\dot{b}|^2\eta_0+\frac{8}{\mu_2-\mu_2^-}\int_{t_2^+}^{t_2^-}|\ddot{b}||\dot{b}|^2\eta_0\\
&\le 8\Big(\int_{t_1^-}^{t_2^-}\ddot{b}^2\dot{b}^2\Big)^{\frac12}\Big[\Big(\int_{t_1^+}^{t_1^-}\eta_0^2\dot{b}^2\Big)^{\frac12}\frac{1}{\mu_1-\mu_1^-}+\Big(\int_{t_2^-}^{t_2^+}\eta_0^2\dot{b}^2\Big)^{\frac12}\frac{1}{\mu_2-\mu_2^-}\Big]\\
&\le 8\int_{t_1^-}^{t_2^-}\ddot{b}^2\Big[\Big(\int_{t_1^+}^{t_1^-}\frac{\eta_0^2\dot{b}^2}{(\mu_1-\mu_1^-)^2}\Big)^{\frac12}+\Big(\int_{t_2^-}^{t_2^+}\frac{\eta_0^2\dot{b}^2}{(\mu_2-\mu_2^-)^2}\Big)^{\frac12}\Big].
\end{aligned}$$

Observe that on $[t_i^-, t_i^+]$, thanks to Lemma 3.2, we have:

$$|\eta_0(t)| \le \frac{C_3(|b|_\infty)}{|t_2^- - t_1^-|}|t_i^- - t_i^+|.$$

By lemma 3.1, for any $t \in [t_i^-, t_i^+]$,

$$\frac{\eta_0(t)}{\mu_i-\mu_i^-} \le \frac{C_3(|b|_\infty)}{|t_2^- - t_1^-|}\frac{|t_i^- - t_i^+|}{\mu_i-\mu_i^-} \le \frac{C_3(|b|_\infty)}{|t_2^- - t_1^-|}\frac{1}{\gamma}.$$

Thus

$$\Big|-\int_0^1 \ddot{b}^2\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2\eta_0\Big| \le \frac{8C_3(|b|_\infty)}{|t_2^- - t_1^-|}\frac{1}{\gamma}\int_{t_1^-}^{t_2^-}\ddot{b}^2\Big(\Big(\int_{t_1^-}^{t_1^+}\dot{b}^2\Big)^{\frac12}+\Big(\int_{t_2^+}^{t_2^-}\dot{b}^2\Big)^{\frac12}\Big)$$

Since $\dot{b}$ is L^2, we can choose $\tilde{\delta}$ small in Lemma 1 so that

$$\Big|-\int_0^1 \ddot{b}^2\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\dot{b}^2\eta_0\Big| \le \frac{c}{1000}\int_{t_1^-}^{t_2^-}\ddot{b}^2.\tag{3.35}$$

By (3.29), (3.30), (3.34) , (3.35) and (3.33), we obtain

$$-\int_0^1 \ddot{b}\ddot{\eta} \leq -\frac{9c}{10}\int_{t_1^-}^{t_2^-} \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(1 + \frac{1}{|t_2^- - t_1^-|^2}\right)\int_{t_1^-}^{t_2^-} (1 + \dot{b}^2). \tag{3.36}$$

Now we estimate $-\int_0^1 \frac{\overbrace{b(\lambda + \bar{\mu}\eta)}^{\cdot}}{a}\ddot{b}$. We use $\widetilde{\lambda + \eta\bar{\mu}}$ to denote the part of $\lambda + \eta\bar{\mu}$ related to $sgnb\psi_{\theta,\epsilon}\eta_0$ and $\underline{\lambda + \eta\bar{\mu}}$ to denote the part of $\lambda + \eta\bar{\mu}$ related to cb. Clearly $\lambda + \eta\bar{\mu} = \widetilde{\lambda + \eta\bar{\mu}} + \underline{\lambda + \eta\bar{\mu}}$. We compute

$$\begin{aligned}
\int_0^1 \overbrace{b(\widetilde{\lambda + b\eta\bar{\mu}})}^{\cdot}\,\ddot{b} &= sgnb\left(\int_0^1 (\ddot{b}b - \frac{1}{2}\dot{b}^2)(b\psi_{\theta,\epsilon}\eta_0 - \int_0^1 b\psi_{\theta,\epsilon}\eta_0)\right)\\
&= sgnb\left(\frac{1}{2}\int_0^1 \dot{b}^2 \int_0^1 b\psi_{\theta,\epsilon}\eta_0 + \int_0^1 \ddot{b}b^2\psi_{\theta,\epsilon}\eta_0 - \frac{1}{2}\dot{b}^2 b\psi_{\theta,\epsilon}\eta_0)\right)\\
&\leq \frac{1}{2}\int_0^1 \dot{b}^2 \int_0^1 |b|\psi_{\theta,\epsilon}\eta_0 + \frac{c}{200}\int_{t_1^-}^{t_2^-} \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}|t_2^- - t_1^-| +\\
&\quad +C_3(|b|_\infty)\int_{t_1^-}^{t_2^-} \dot{b}^2,
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
\int_0^1 \overbrace{b(\underline{\lambda + b\eta\bar{\mu}})}^{\cdot}\,\ddot{b} &= c\int_0^1 (-\frac{1}{2}\dot{b}^2 + \ddot{b}\dot{b})(b^2 - \int_0^1 b^2)\\
&\leq \frac{c}{200}\int_{t_1^-}^{t_2^-} \ddot{b}^2 + c\int_0^1 \dot{b}^2\left(\int_0^1 b^2 + |b|_\infty^4\right).
\end{aligned}$$

By the choice of c in (3.15) and (3.37), (3.38), we obtain

$$-\int_0^1 \frac{\overbrace{b(\lambda + \bar{\mu}\eta)}^{\cdot}}{a}\ddot{b} \leq C_3(|b|_\infty)\int_0^1 \dot{b}^2\left(-\frac{\partial a}{\partial s}\right) + \frac{c}{100}\int_{t_1^-}^{t_2^-} \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\int_{t_1^-}^{t_2^-} (1 + \dot{b}^2). \tag{3.38}$$

There are two terms left. We compute first:

$$\begin{aligned}
\int_0^1 \ddot{b}(a\tau - b\bar{\mu}_\xi)\eta &= \int_0^1 \ddot{b}(a\tau - b\bar{\mu}_\xi)(cb + sgnb\psi_{\theta,\epsilon}\eta_0)\\
&\leq \frac{c}{100}\int_{t_1^-}^{t_2^-} \ddot{b}^2 + C_3(|b|_\infty)(c + \frac{|t_1^- - t_2^-|}{c}).
\end{aligned} \tag{3.39}$$

The last term is

$$\begin{aligned}
&\int_0^1 \frac{\ddot{b}}{a}\overbrace{b\eta\bar{\mu}}^{\cdot} = \int_0^1 \frac{\ddot{b}}{a}\overbrace{b((cb + sgnb\psi_{\theta,\epsilon}\eta_0)\bar{\mu}}^{\cdot} \\
\leq\ & C_3(|b|_\infty)\left(\int_{t_1^-}^{t_2^-} |\ddot{b}|(1 + |\dot{b}| + \frac{\psi_{\theta,\epsilon}}{|t_1^- - t_2^-|}\right) + c\frac{1}{a}\int_0^1 |\ddot{b}|(2b\dot{b}\bar{\mu} + b^3\bar{\mu}_\xi + ab^2\bar{\mu}_v) \\
\leq\ & \frac{c}{100}\int_0^1 \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(|t_1^- - t_2^-| + \frac{\Gamma^2}{|t_1^- - t_2^-|^2} + \int_{t_1^-}^{t_2^-} \dot{b}^2\right) + \\
& +cC_3(|b|_\infty)\int_0^1 |\ddot{b}|\left(1 + |b|_\infty^3 + |b\dot{b}|\right) \\
\leq\ & \frac{c}{100}\int_0^1 \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(|t_1^- - t_2^-| + 2\int_{t_1^-}^{t_2^-} \dot{b}^2\right) + C_3(|b|_\infty)\left(1 + \int_0^1 \dot{b}^2\right)\left(-\frac{\partial a}{\partial s}\right).
\end{aligned} \tag{3.40}$$

By (3.36), (3.38), (3.39) and (3.40), we get

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 \quad \leq\ & -\frac{9c}{10}\int_0^1 \ddot{b}^2 + \frac{C_3(|b|_\infty)}{c}\left(1 + \frac{1}{|t_1^- - t_2^-|^2}\right)\int_{t_1^-}^{t_2^-}(1 + \dot{b}^2) + \\
& +C_3(|b|_\infty)(1 + \int_0^1 \dot{b}^2)\left(-\frac{\partial a}{\partial s}\right).
\end{aligned}$$

By (3.26), we replace $\frac{1}{|t_1^- - t_2^-|^2}$ with $-\frac{1}{\Gamma}\left(-\frac{\partial}{\partial s}(\frac{\bar{\mu}}{a}) - C_{12}\frac{\partial a}{\partial s}\right)$ and thus get estimate 3.

When we try to upper-bound $-\int_0^1 \ddot{b}sgnb\psi_{\theta,\epsilon}\ddot{\eta}_0$, $\ddot{\eta}_0$ presents Diracs at t_1^-, t_2^-. In our formal computation, we acted like it is smooth. Now we want to show what meaning it should be given and justify our formal computation.

We multiply η_0 by a cutoff function g_ω with compact support on $[t_1^-, t_2^-]$. We also assume $g_\omega = 1$ on $[t_1^+, t_2^+]$. We show that as g_ω tends to 1 on (t_1^-, t_2^-), $-\int_0^1 \ddot{b}sgnb\psi_{\theta,\epsilon}\overset{\cdot\cdot}{\overline{g_\omega\eta_0}}$ converges to $-\int_0^1 \ddot{b}sgnb\psi_{\theta,\epsilon}\ddot{\eta}_0$. And all the other terms also converge, with $g_\omega\eta_0$ instead of η_0, thus all the related computation can be understood in the distributional sense.

Since $\psi_{\theta,\epsilon}(t_i^-) = 0$ and $|\dot{\psi}_{\theta,\epsilon}| \leq |\dot{\theta}| + |\dot{b}| \leq 2|\dot{b}$, we have:

$$\psi_{\theta,\epsilon}(t) \leq 2\left|\int_{t_i^-}^{t} |\dot{b}|\right| \leq 2|t - t_i^-|\left(\int_{t_1^-}^{t_2^-} \ddot{b}\right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}
&\int_{t_1^-}^{t_2^-} \ddot{b}sgnb\psi_{\theta,\epsilon}(\overset{\cdot\cdot}{\overline{g_\omega\eta_0}} - \ddot{\eta}_0) \\
\leq\ & C\left(\int_{t_1^-}^{t_2^-} \ddot{b}\right)^{\frac{1}{2}}\sum\int_{t_1^-}^{t_2^-} |\ddot{b}|(|\ddot{g}_\omega|(t - t_i^-)^2 + |g_\omega - 1| + |\dot{g}_\omega||t - t_i^-|) \\
\leq\ & 10C\left(\int_{t_1^-}^{t_2^-} \ddot{b}\right)^{\frac{1}{2}}\int_{Supp|g_\omega - 1|\cup[t_1^-, t_2^-]} |\ddot{b}| \leq 10C\int_{t_1^-}^{t_2^-} \ddot{b}\left|Supp|g_\omega - 1| \cup [t_1^-, t_2^-]\right|.
\end{aligned}$$

Thus as ω tends to 0, such a term converges to 0. This finishes the proof of theorem 1.

Remark 3.2 Let $\tilde{x}$ be a curve in the $H^2(S^1)$ neighborhood of x and $\tilde{b}$ be its v-component. Denote

$$\eta(\tilde{b}) = \psi_{\tilde{\theta},\tilde{\epsilon}}\tilde{\eta}_0 sgn\tilde{b} + c\tilde{b} \tag{3.41}$$

where $\psi_{\tilde{\theta},\tilde{\epsilon}}$ is constructed for $\tilde{b}$ using the same $t_1^-, t_1^+, t_2^-, t_2^+$ as in the construction of $\psi_{\theta,\epsilon}(b)$ and

$$\tilde{\eta}_0 = \eta_0 e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)}. \tag{3.42}$$

If $\tilde{b}$ is close enough to b in H^1, we have

$$\overbrace{\dot{\tilde{\eta}}_0 - \tilde{\bar{\mu}}\tilde{b}\tilde{\eta}_0}^{\cdot} + C(1+\tilde{b}^2)\tilde{\eta}_0 e^{-2\int_{t_1^-}^{t}\tilde{\bar{\mu}}\tilde{b}} \leq -\frac{3}{2(x_1^- - x_2^-)^2}e^{\int_{t_1^-}^{t}\tilde{\bar{\mu}}\tilde{b}}. \tag{3.43}$$

(3.43) will replace (3.6) in all our arguments involving $\tilde{b}$.

Now, we verify (3.43). By (3.6) and Lemma 3.2, we have:

$$\begin{aligned}
\overbrace{\dot{\tilde{\eta}}_0 - \tilde{\bar{\mu}}\tilde{b}\tilde{\eta}_0}^{\cdot} &= \overbrace{e^{\int_{t_1^-}^{t}\tilde{\bar{\mu}}\tilde{b}}\frac{d}{dt}\left(e^{-\int_{t_1^-}^{t}\bar{\mu}\tilde{b}}\tilde{\eta}_0\right)}^{\cdot} = \overbrace{e^{\int_{t_1^-}^{t}\tilde{\bar{\mu}}\tilde{b}}\frac{d}{dt}\left(e^{-\int_{t_1^-}^{t}\bar{\mu}b}\eta_0\right)}^{\cdot} \\
&= \overbrace{e^{\int_{t_1^-}^{t}\bar{\mu}b}\frac{d}{dt}\left(e^{-\int_{t_1^-}^{t}\bar{\mu}b}\eta_0\right)}^{\cdot} + \overbrace{\left(e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)} - 1\right)e^{\int_{t_1^-}^{t}\bar{\mu}b}\frac{d}{dt}\left(e^{-\int_{t_1^-}^{t}\bar{\mu}b}\eta_0\right)}^{\cdot} \\
&\leq -\frac{3}{2(x_1^- - x_2^-)^2}e^{\int_{t_1^-}^{t}\tilde{\bar{\mu}}\tilde{b}}.
\end{aligned}$$

Remark 3.3 If $|\tilde{b} - b|_{H^1}$ is small enough, all the computations in Theorem 3.1 hold for $\tilde{b}$.

Theorem 3.2 *Let* $\eta = \eta_{12} + cb$ *as in* (3.14) . *Assume that* b_0 *and* $\dot{b}_0$ *have only simple zeros. Then*

$$\begin{aligned}
&C_0\frac{\partial a}{\partial s} + \frac{\partial}{\partial s}\int \dot{b}^+\Big|_{s=0} \\
\leq\ & -\frac{\tilde{C}_4}{|x_1^- - x_2^-|^2}(\tilde{\bar{\mu}} - \mu_2) + C_4(|b|_\infty)\frac{1}{\tilde{\bar{\mu}} - \mu_2} + C_4(|b|_\infty)(1 + \int_{t_1^-}^{t_2^-}\dot{b}^+) - C_4\frac{\partial a}{\partial s}\int_0^1 \dot{b}^+.
\end{aligned}$$

where C_0, C_4 *and* $\tilde{C}_4$ *do not depend on* b_0.

Proof. Assume that $z_j^+, z_j^-, 1 \leq j \leq m$ are all the critical points of b. $z_j^+, 1 \leq j \leq m$ are local maximums and $z_j^-, 1 \leq j \leq m$ are the preceding local minimums, thus $b'(z_j^+) = b'(z_j^-) = 0$, $b''(z_j^+) \leq 0$ and $b''(z_j^-) \geq 0$.
If b is positive on $[t_1^+, t_2^+]$, by Lemma 3.1, t_1^+ belongs to some interval $[z_{j_0}^-, z_{j_0}^+]$ and t_2^+ belongs to some interval $[z_{j_1}^+, z_{j_1+1}^-]$; Otherwise, t_1^+ belongs to some interval $[z_{j_0}^+, z_{j_0+1}^-]$ and t_2^+

belongs to some interval $[z_{j_1}^-, z_{j_1}^+]$. By Proposition 2.2 and 2.3, we have

$$\frac{\partial}{\partial s}\int_0^1 \dot{b}^+ = \sum_j \frac{\overbrace{\eta + \lambda b}^{\cdot}}{a} + a\eta\tau - \bar{\mu}_\xi b\eta\Big|_{z_j^-}^{z_j^+},$$

where $\eta = cb + sgnb\psi_{\theta,\epsilon}(b)\eta_0$.

We denote

$$\Delta_j = \frac{\overbrace{\eta + \lambda b}^{\cdot}}{a} + a\eta\tau - \bar{\mu}_\xi b\eta\Big|_{z_j^-}^{z_j^+}.$$

Now we proceed to estimate Δ_j. By Proposition 2.3, we have

$$\begin{aligned}
\Delta_j &= \frac{\overbrace{sgnb\psi_{\theta,\epsilon}\eta_0 + cb}^{\cdot\cdot} + \dot{\lambda} b}{a} + (sgnb\psi_{\theta,\epsilon}\eta_0 + cb)(a\tau - b\bar{\mu}_\xi)\Bigg|_{z_j^-}^{z_j^+} \\
&= sgnb\left[\frac{1}{a}\overbrace{\psi_{\theta,\epsilon}\eta_0}^{\cdot\cdot} + \frac{1}{a}b^2\psi_{\theta,\epsilon}\eta_0 - \frac{1}{a}b^{-}\overbrace{\psi_{\theta,\epsilon}\eta_0}^{\cdot} + \psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi)\right]_{z_j^-}^{z_j^+} \\
&\quad - \frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+} + c\left[\frac{b^3}{a} + b(a\tau - b\bar{\mu}_\xi) + \frac{1}{a}b^2\dot{\bar{\mu}}\right]_{z_j^-}^{z_j^+} \\
&\le sgnb\left[\frac{1}{a}(-\ddot{\theta}\eta_0 - 2\dot{\theta}\dot{\eta}_0) + \frac{1}{a}b^2\psi_{\theta,\epsilon}\eta_0 + \psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi) + \frac{1}{a}\dot{\psi}_{\theta,\epsilon}b\bar{\mu}\eta_0 + \right. \\
&\quad \left. + \frac{1}{a}\psi_{\theta,\epsilon}\overbrace{\dot{\eta}_0 - \bar{\mu}b\eta_0}^{\cdot}\right]_{z_j^-}^{z_j^+} + c\left[\frac{b^3}{a} + b(a\tau - b\bar{\mu}_\xi) + \frac{1}{a}b^2\dot{\bar{\mu}}\right]_{z_j^-}^{z_j^+} - \frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+} \\
&= sgnb\left[\frac{1}{a}(-\ddot{\theta}\eta_0 - 2\dot{\theta}\dot{\eta}_0 - \dot{\theta}b\bar{\mu}\eta_0) + \psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi) + \frac{1}{a}b^2\psi_{\theta,\epsilon}\eta_0\right]_{z_j^-}^{z_j^+} \\
&\quad - \frac{C}{a}sgnb\psi_{\theta,\epsilon}(1 + b^2)\eta_0 e^{-2\int_{t_1^-}^t \bar{\mu}b}\Bigg|_{z_j^-}^{z_j^+} - \frac{2}{|x_1^- - x_2^-|^2}sgnb\psi_{\theta,\epsilon}e^{-2\int_{t_1^-}^t \bar{\mu}b}\Bigg|_{z_j^-}^{z_j^+} \\
&\quad + c\left[\frac{b^3}{a} + b(a\tau - b\bar{\mu}_\xi) + \frac{1}{a}b^2\dot{\bar{\mu}})\right]_{z_j^-}^{z_j^+} - \frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+}
\end{aligned} \tag{3.44}$$

At z_j^- and z_j^+, by (3.28), we have

$$\dot{\psi}_{\theta,\epsilon}(b) = -\dot{\theta}, \quad \ddot{\psi}_{\theta,\epsilon}(b) = -\ddot{\theta} + \frac{\partial\psi_{\theta,\epsilon}}{\partial b}\ddot{b}$$

and

$$\frac{\partial\psi_{\theta,\epsilon}}{\partial b}\ddot{b}\Big|_{z_j^-}^{z_j^+} \le 0.$$

An important observation is that for terms involving $\psi_{\theta,\epsilon}$, they are nonzero only in $[t_1^-, t_2^-]$. For example, we consider the case that b is positive on$[t_1^-, t_2^-]$. Then $\dot{b}(t_1^+) > 0$ and $\dot{b}(t_2^+) < 0$, and t_1^+must be contained in an interval of the form $[z^-_{j_0}, z^+_{j_0}]$ and t_2^+ in $[z^+_{j_1}, z^-_{j_1+1}]$. $\psi_{\theta,\epsilon}(b(z_i^\pm))$ is nonzero only when $j_0 \le j \le j_1$. Thus for $j_0 < j < j_1$,

$$\begin{aligned}\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-)) &= b(z_j^+) - b(z_j^-) - (\theta(z_j^+) - \theta(z_j^-)) \\ &\ge -(\theta(z_j^+) - \theta(z_j^-)) \ge -\int_{z_j^-}^{z_j^+} |\dot{\theta}|. \qquad (3.45)\end{aligned}$$

At the two ends of the interval $[t_1^-, t_2^-]$, since $\left|\psi_{\theta,\epsilon} - (b-\theta)^+\right| \le C_0\epsilon_i\theta_i = C_0|\mu_i^+ - \mu_i^-|$ where C_0 is a constant embed in our construction in, we derive that for $j = j_0$ or $j_1 + 1$,

$$\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-)) \ge -\int_{z_j^-}^{z_j^+} |\dot{\theta}| - C_0(|\mu_1^+ - \mu_1^-| + |\mu_2^+ - \mu_2^-|). \qquad (3.46)$$

We also have

$$\begin{aligned}\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-)) &\le \int_{z_j^-}^{z_j^+} \dot{b} + \int_{z_j^-}^{z_j^+} |\dot{\theta}|, \text{ for } j_0 \le j \le j_1 \\ \psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-)) &\le \int_{z_j^-}^{z_j^+} \dot{b} + \int_{z_j^-}^{z_j^+} |\dot{\theta}| + C_0(|\mu_1^+ - \mu_1^-| + |\mu_2^+ - \mu_2^-|) \\ &\quad \text{if } j = j_0, j_1\end{aligned}$$

When b is negative on $[t_1^-, t_2^-]$, similar argument also applies.

First we estimate the last line of (3.44) which are terms involving c and the integral. For example, by (3.15) we have

$$\frac{1}{a}c(b^3(z_j^+) - b^3(z_j^-)) \le \frac{3}{a}c(b(z_j^+) - b(z_j^-))|b|_\infty^2 \le C_4 \int_{z_j^-}^{z_j^+} \dot{b}\left(-\frac{\partial a}{\partial s}\right),$$

and

$$\begin{aligned}|cab\tau|\Big|_{z_j^-}^{z_j^+} &= ca\left|(\tau(z_j^+) - \tau(z_j^-))b(z_j^+) + \tau(z_j^-)(b(z_j^+) - b(z_j^-))\right| \\ &\le C_4 \int_{z_j^-}^{z_j^+} (1 + \dot{b})\left(-\frac{\partial a}{\partial s}\right).\end{aligned}$$

where as before C_4 depends only on $\tilde{C}$ in (3.15) and τ which is determined by our contact form α, but independent of b. We can treat other terms with c similarly. And we also have

$$-\frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+} = -\frac{1}{a}\int_{z_j^-}^{z_j^+} \dot{b}\left(-\frac{\partial a}{\partial s}\right)$$

Thus we obtain:

$$c\left[\frac{b^3}{a} + b(a\tau - b\bar{\mu}_\xi) - \frac{1}{a}b^2\dot{\bar{\mu}}\right]_{z_j^-}^{z_j^+} - \frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+} \le C_4\int_{z_j^-}^{z_j^+} (1 + \dot{b})\left(-\frac{\partial a}{\partial s}\right). \qquad (3.47)$$

Alternatively, we have

$$c\left[\frac{b^3}{a} + b(a\tau - b\bar{\mu}_\xi) - \frac{1}{a}b^2\dot{\bar{\mu}}\right]_{z_j^-}^{z_j^+} - \frac{b}{a}\int_0^1 b\eta\Big|_{z_j^-}^{z_j^+} \leq C_4(|b|_\infty)(1+c)\int_{z_j^-}^{z_j^+}(1+\dot{b}) + \\ +C_4\left(\int_{z_j^-}^{z_j^+}\dot{b}\right)\left(-\frac{\partial a}{\partial s}\right). \tag{3.48}$$

We are finished with the last line of (3.44). Now we deal with the first line. We begin with the difference $sgnb\psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi)\Big|_{z_j^-}^{z_j^+}$. By Lemma 3.2, we have:

$$\begin{aligned}
& sgnb\psi_{\theta,\epsilon}\eta_0 a\tau\Big|_{z_j^-}^{z_j^+} \\
= \quad & asgnb\left(\psi_{\theta,\epsilon}(b(z_j^-))\eta_0(z_j^-)(\tau(z_j^+) - \tau(z_j^-)) + \psi_{\theta,\epsilon}(b(z_j^-))(\eta_0(z_j^+) - \eta_0(z_j^-))\tau(z_j^+)\right) \\
& +asgnb(\psi_{\theta,\epsilon}(b(z_j^+) - \psi_{\theta,\epsilon}(b(z_j^-))\eta_0(z_j^+))\tau(z_j^+) \\
\leq \quad & C_4(|b|_\infty)\frac{z_j^+ - z_j^-}{|t_2^- - t_1^-|} + C_4\left|\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-))\right|.
\end{aligned}$$

Similarly we have

$$-sgnb\psi_{\theta,\epsilon}\eta_0 b\bar{\mu}_\xi\Big|_{z_j^-}^{z_j^+} \leq C_4(|b|_\infty)\frac{z_j^+ - z_j^-}{|t_2^- - t_1^-|} + C_4\Gamma\int_{z_j^-}^{z_j^+}\dot{b} + C_4|b|_\infty\left|\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-))\right|.$$

Thus we derive

$$\begin{aligned}
& sgnb\psi_{\theta,\epsilon}\eta_0(a\tau - b\bar{\mu}_\xi)\Big|_{z_j^-}^{z_j^+} \\
\leq \quad & C_4(|b|_\infty)\frac{z_j^+ - z_j^-}{|t_2^- - t_1^-|} + C_4\Gamma\int_{z_j^-}^{z_j^+}\dot{b} + C_4(|b|_\infty + 1)\left|\psi_{\theta,\epsilon}(b(z_j^+)) - \psi_{\theta,\epsilon}(b(z_j^-))\right|.
\end{aligned} \tag{3.49}$$

We estimate $\frac{1}{a}sgnb(-\ddot{\theta}\eta_0 - 2\dot{\theta}\dot{\eta}_0 - \dot{\theta}b\bar{\mu}\eta_0)\Big|_{z_j^-}^{z_j^+}$ now.

$$\begin{aligned}
& \frac{1}{a}sgnb(-\ddot{\theta}\eta_0 - 2\dot{\theta}\dot{\eta}_0 - \dot{\theta}b\bar{\mu}\eta_0)\Big|_{z_j^-}^{z_j^+} \\
\leq \quad & \frac{C_4}{a}\left(\int_{z_j^-}^{z_j^+}(\theta^{(3)}| + |\ddot{\theta}|)(\eta(z_j^+) + |\dot{\eta}_0|(z_j^+)) + \int_{z_j^-}^{z_j^+}(|\ddot{\eta}_0| + |\dot{\eta}_0|)(|\dot{\theta}|(z_j^-) + |\ddot{\theta}_0|(z_j^-))\right)\times \\
& \times(1 + |b(z_j^-)|) + C_{22}\left(|\dot{\theta}|_{C^0}\int_{z_j^-}^{z_j^+}|\dot{b}| + |b|_\infty\int_{z_j^-}^{z_j^+}|\dot{\eta}_0|\right)
\end{aligned}$$

Thanks to the observation we made earlier, these types of terms are nonzero only when $j_0 \leq j \leq j_1 + 1$. Thus we get

$$\begin{aligned}
&\left|\sum_j \frac{1}{a} sgnb(-\ddot{\theta}\eta_0 - 2\dot{\theta}\dot{\eta}_0 - \dot{\theta}b\bar{\mu}\eta_0)\Big|_{z_j^-}^{z_j^+}\right| \\
\leq\ & C_4\left(|\eta|_{C^1}\int_{t_1^-}^{t_2^-}(|\theta^{(3)}| + |\ddot{\theta}|) + |\theta|_{C^2}\int_{t_1^-}^{t_2^-}(|\ddot{\eta}_0| + |\dot{\eta}_0|)\right)(1 + |b|_\infty) \\
&+C_4\left(|\dot{\theta}|_{C^0}\int_{t_1^-}^{t_2^-}|\dot{b}| + |b|_\infty\int_{t_1^-}^{t_2^-}|\dot{\eta}_0|\right) \\
\leq\ & C_4(|b|_\infty)\frac{1}{|t_2^- - t_1^-|} + C_4|\dot{\theta}|_{C^0}\int_{t_1^-}^{t_2^-}\dot{b}^+.
\end{aligned} \tag{3.50}$$

Now let us rewrite the difference $-\frac{1}{a}C sgnb\psi_{\theta,\epsilon}(1 + b^2)\eta_0 e^{-2\int_{t_1^-}^{t}\bar{\mu}b}\Big|_{z_j^-}^{z_j^+}$.

$$\begin{aligned}
&-\frac{1}{a}C sgnb\psi_{\theta,\epsilon}(1 + b^2)\eta_0 e^{-2\int_{t_1^-}^{t}\bar{\mu}b}\Big|_{z_j^-}^{z_j^+} \\
=\ & -\frac{1}{a}C\,\underbrace{sgnb(b(z_j^+)^2 - b(z_j^-)^2)\psi_{\theta,\epsilon}(b(z_j^+))\eta_0(z_j^+)e^{-2\int_{t_1^-}^{z_j^+}\bar{\mu}b}}_{(I)} \\
&-\frac{1}{a}C\,\underbrace{sgnb(1 + b(z_j^-)^2)(\psi_{\theta,\epsilon}(b(z_j^+) - \psi_{\theta,\epsilon}(b(z_j^-))\eta_0(z_j^+)e^{-2\int_{t_1^-}^{z_j^+}\bar{\mu}b}}_{(II)} \\
&-\frac{1}{a}C\,\underbrace{sgnb(1 + b(z_j^-)^2)\psi_{\theta,\epsilon}(b(z_j^-)(\eta_0(z_j^+) - \eta(z_j^-))e^{-2\int_{t_1^-}^{z_j^+}\bar{\mu}b}}_{(III)} \\
&-\frac{1}{a}C\,\underbrace{sgnb(1 + b(z_j^-)^2)\psi_{\theta,\epsilon}(b(z_j^-))\eta(z_j^-)e^{-2\int_{t_1^-}^{z_j^-}\bar{\mu}b}(e^{-2\int_{z_j^-}^{z_j^+}\bar{\mu}b} - 1)}_{(IV)}
\end{aligned} \tag{3.51}$$

We notice that (I) in (3.51) is always negative. We can use it to cancel out other terms by taking C very large in (3.6) later. By Lemma 3.2 we also notice that:

$$(III) \leq C_4(|b|_\infty)\frac{z_j^+ - z_j^-}{|t_2^- - t_1^-|}, \tag{3.52}$$

and

$$(IV) \leq C_4(|b|_\infty)(z_j^+ - z_j^-). \tag{3.53}$$

Now we estimate $\frac{1}{a}b^2\psi_{\theta,\epsilon}sgnb\eta_0\Big|_{z_j^-}^{z_j^+}$. We rewrite it into:

$$\begin{aligned}\frac{1}{a}b^2\psi_{\theta,\epsilon}sgnb\eta_0\Big|_{z_j^-}^{z_j^+} &= \frac{1}{a}sgnb\left(b(z_j^+)^2-b(z_j^-)^2\right)\psi_{\theta,\epsilon}(b(z_j^+))\eta_0(z_j^+)\\ &+\frac{1}{a}sgnbb(z_j^-)^2\left(\psi_{\theta,\epsilon}(b(z_j^+)-\psi_{\theta,\epsilon}(b(z_j^-))\right)\eta_0(z_j^+)\\ &+\frac{1}{a}sgnbb(z_j^-)^2\psi_{\theta,\epsilon}(b(z_j^+))\left(\eta_0(z_j^+)-\eta(z_j^-)\right). \end{aligned}\tag{3.54}$$

Notice that $sgnb(b(z_j^+)^2-b(z_j^-)^2)$, η_0 and $\psi_{\theta,\epsilon}$ are always non-negative and $e^{-2\int_{t_1^-}^{t}\bar{\mu}b}$ is bounded both from above and below. Comparing the first term in (3.54) with (I) in (3.51), taking C large enough, it can be absorbed. By Lemma 3.2, the last term in (3.54) is upper bounded by $C_4(|b|_\infty)\frac{z_j^+-z_j^-}{|t_2^--t_1^-|}$.

The last term left is $-\frac{2}{|x_1^--x_2^-|^2}sgnb\psi_{\theta,\epsilon}e^{-2\int_{t_1^-}^{t}\bar{\mu}b}\Big|_{z_j^-}^{z_j^+}$. We can rewrite it into:

$$\begin{aligned}&-\frac{2}{|x_1^--x_2^-|^2}sgnb\psi_{\theta,\epsilon}e^{-2\int_{t_1^-}^{t}\bar{\mu}b}\Bigg|_{z_j^-}^{z_j^+} = -\frac{2}{|x_1^--x_2^-|^2}sgnb\Bigg(\psi_{\theta,\epsilon}(b(z_j^+))e^{-2\int_{t_1^-}^{z_j^-}\bar{\mu}b}\Bigg(e^{-2\int_{z_1^-}^{z_j^+}\bar{\mu}b}-1\Bigg)\\ &+\left(\psi_{\theta,\epsilon}(b(z_j^+)-\psi_{\theta,\epsilon}(b(z_j^-))\right)e^{-2\int_{t_1^-}^{z_j^-}\bar{\mu}b}\Bigg)\end{aligned}$$

Thanks to Lemma 3.2, the first term in the expression above satisfies:

$$-\frac{2}{|x_1^--x_2^-|^2}sgnb\psi_{\theta,\epsilon}(b(z_j^+))e^{-2\int_{t_1^-}^{z_j^-}\bar{\mu}b}\Bigg(e^{-2\int_{z_1^-}^{z_j^+}\bar{\mu}b}-1\Bigg)\leq C_4(|b|_\infty)\frac{z_j^+-z_j^-}{|t_2^--t_1^-|^2}.\tag{3.55}$$

Thus by taking μ_1 close enough to μ_2 so that $|\theta_{C^3}$ is small, the first term in (3.55) satisfies:

$$\begin{aligned}&\sum_j-\frac{2}{|x_1^--x_2^-|^2}sgnb\left(\psi_{\theta,\epsilon}(b(z_j^+)-\psi_{\theta,\epsilon}(b(z_j^-))\right)e^{-2\int_{t_1^-}^{z_j^-}\bar{\mu}b}\\ \leq\ &\frac{2C_4}{|x_1^--x_2^-|^2}\left(\int_{t_1^-}^{t_2^-}|\dot{\theta}|-\sum_j\int_{z_1^-}^{z_j^+}\dot{b}+C_4(\mu_1-\mu_2^-+\mu_1-\mu_2^-)\right)\\ \leq\ &-\frac{C_4}{|x_1^--x_2^-|^2}(\tilde{\bar{\mu}}-\mu_1).\end{aligned}\tag{3.56}$$

By (3.33) and (3.44)- (3.56), we get

$$\begin{aligned}&C\frac{\partial a}{\partial s}+\frac{\partial}{\partial s}\int\dot{b}^+|_{s=0}\\ \leq\ &-\frac{C_4}{|x_1^--x_2^-|^2}(\tilde{\bar{\mu}}-\mu_2)+C_4(|b|_\infty)\frac{1}{|t_1^--t_2^-|}+C_4(|b|_\infty)(1+\int_{t_1^-}^{t_2^-}\dot{b}^+)-C_4\frac{\partial a}{\partial s}\int_0^1\dot{b}^+\\ \leq\ &-\frac{\tilde{C}_4}{|x_1^--x_2^-|^2}(\tilde{\bar{\mu}}-\mu_2)+C_4(|b|_\infty)\frac{1}{\tilde{\bar{\mu}}-\mu_2}+C_4(|b|_\infty)(1+\int_{t_1^-}^{t_2^-}\dot{b}^+)-C_4\frac{\partial a}{\partial s}\int_0^1\dot{b}^+.\end{aligned}$$

Using (3.15) and (3.48), alternatively we get

$$C\frac{\partial a}{\partial s} + \frac{\partial}{\partial s}\int \dot{b}^{+}|_{s=0} \leq -\frac{\tilde{C}_4}{|x_1^- - x_2^-|^2}(\tilde{\bar{\mu}} - \mu_2) + C_4(|b|_\infty)\frac{1}{\bar{\tilde{\mu}} - \mu_2} + C_4(|b|_\infty)(1+c)\int_0^1 \dot{b}^{+}.$$

This is the end of the proof.

Remark 3.4 It is easy to see from the proof that the result extends to $\psi_{\tilde{\theta},\tilde{\epsilon}}\tilde{\eta}_0 sgn\tilde{b}$ for $\tilde{b}$ close to b in H^1 as in Remark 3.2 and 3.3.

4 The Case of Multiple oscillations and Convex Combination

In the proof of theorem 3.1 and 3.2, we did not really use the fact that we were considering one single oscillation. Namely, we can take several oscillations with disjoint supports and build an $\eta_{0,i}$ as in (3.6) for each oscillation. Now we consider

$$\eta = \sum_{i=1}^{m} \eta_{12,i} + cb, \tag{4.57}$$

where $\eta_{12,i} = sgnb\psi_{\theta_i,\epsilon_i}\eta_{0,i}$ satisfies
1) The supports of $\eta_{0,i}$ are disjoint,
2) On the support of each $\eta_{0,i}$, b does not change sign. Now, we require that $c(s)$ satisfies

$$0 < c \leq \inf\{\frac{\tilde{C}a}{1+|b|_\infty^3}\sum_{i=1}^{m}\left(\int_{t_{1,i}^-}^{t_{2,i}^-} b\eta_{12,i}\right), 1\}. \tag{4.58}$$

Since each term in Theorem 3.2 behaves well under combination, we derive the following two theorems using the same techniques:

Theorem 4.1 *Let $\eta = \sum_{i=1}^{m}\eta_{12,i} + cb$. Assume that b_0 is C^∞ and $\dot{b}_0$ has only simple zeros. Then*

$$\begin{aligned} & C\frac{\partial a}{\partial s} + \frac{\partial}{\partial s}\int \dot{b}^{+}|_{s=0} \\ \leq \quad & -C_4\sum_{i=1}^{m}\frac{\bar{\tilde{\mu}}_i - \mu_{2,i}}{|x_1^- - x_2^-|^2} + C_4(|b|_\infty)\sum_{i=1}^{m}\frac{1}{\bar{\tilde{\mu}}_i - \mu_{2,i}} + C_4(|b|_\infty)(1+c)\int_0^1 \dot{b}^{+}, \end{aligned}$$

where C and C_4 depend on the contact form α, the choice of the vector v and the lower bound of a.

and

Theorem 4.2 *Assume that, in addition we require that for every i,* $|\tilde{\bar{\mu}}_i - \mu_{2,i}| \geq \zeta > 0$. *Then*

$$\begin{aligned} & C\frac{\partial a}{\partial s} + \frac{\partial}{\partial s}\int \dot{b}^+|_{s=0} \\ \leq\ & -C_4(|b|_\infty)\zeta\sum_{i=1}^{m}\frac{1}{|x_1^- - x_2^-|^2} + C_4(\zeta, |b|_\infty) + C_5\int_0^1 \dot{b}^+\left(-\frac{\partial a}{\partial s}\right) + C_4(|b|_\infty)\int_0^1 \dot{b}^+, \end{aligned}$$

Moreover, Theorem 3.1 can be extended as follows:

Theorem 4.3 *Let* $\eta = \sum_{i=1}^m \alpha_i\eta_{12,i} + cb$ *be a convex combination of several* η_i *which is made of several oscillations* $\sum_j \psi_{\theta_{ij},\epsilon_{ij}}$ *with disjoint support, and with amplitute lower bounded by a uniform* $\Gamma > 0$.
We then have:

1. $\frac{\partial}{\partial s}\left(\frac{|b|_\infty}{a}\right) \leq -\frac{C_1}{a}\frac{\partial a}{\partial s}$, *where* C_1 *is a universal constant depending only on* $\tilde{C}$ *and on the contact form* α.
2. *There exists a universal constant* $\bar{C}$, *which depends only on* γ *and* γ_0, *not on the choice of* $\tilde{C}$ *such that*

$$\frac{\partial}{\partial s}\left(\int(|b| - \gamma)^+ + \bar{C}(1 + a)\right) \leq 0 \text{ if } a \geq \gamma_0$$

3.

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 + \frac{9c}{10}\int_0^1 \ddot{b}^2 \\ \leq\ & \frac{C_3(|b|_\infty)}{c}\left(-\frac{1}{\Gamma}\frac{\partial}{\partial s}\left(Ca + \int_0^1 \dot{b}^+\right) + C_4(|b|_\infty, \Gamma) + \frac{C_4(|b|_\infty)}{\Gamma}\int_0^1 \dot{b}^+ + \right. \\ & \left. + \frac{C_5}{\Gamma}\int_0^1 \dot{b}^+\left(-\frac{\partial a}{\partial s}\right)\right)\left(\int_0^1(1 + \dot{b}^2)\right) - C_3(|b|_\infty)(1 + \int_0^1 \dot{b}^2)\frac{\partial a}{\partial s}. \end{aligned} \tag{4.59}$$

Now we proceed with the proof of Theorem 4.3.

Proof. Estimates 1 and 2 are obvious since estimates 1 and 2 of Theorem 3.1 behave well under convex combination. (3.18) yields

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 + \frac{9c}{10}\int_0^1 \ddot{b}^2 \\ \leq\ & \frac{C_3(|b|_\infty)}{c}\left(1 + \sum_{i=1}^{m}\frac{1}{|x_{1,i} - x_{2,i}|^2}\right)\left(\int_0^1 \dot{b}^2 + 1\right) - C_3(|b|_\infty)\left(\int_0^1 \dot{b}^2 + 1\right)\frac{\partial a}{\partial s} \end{aligned}$$

Thanks to Theorem 4.2, we can replace $\sum_{i=1}^m \frac{1}{|x_{1,i} - x_{2,i}|^2}$ by

$$\frac{\tilde{C}_4(|b|_\infty)}{\Gamma}\left(-\frac{\partial}{\partial s}\left(Ca + \int \dot{b}^+\right) + C_4(\Gamma, |b|_\infty) + C_5\int_0^1 \dot{b}^+\left(-\frac{\partial a}{\partial s}\right) + C_4(|b|_\infty)\int_0^1 \dot{b}^+\right).$$

This yields (4.59). This is the end of the proof.

Corollary 4.1 *When s tends to the explosion time or to $+\infty$, $c\int_0^1 b^2$ has to tend to* 0 *on a subsequence. If there is blow-up in finite time, c has to tend to* 0 *on a subsequence.*

Proof. When s tends to $+\infty$, thanks to Proposition 2.2 , we have

$$\int_0^\infty c\int_0^1 b^2 < \int_0^\infty \int_0^1 b\eta < a(0) < +\infty.$$

Thus, $c\int_0^1 b^2$ has to tend to 0, as claimed.
If there is blow-up in finite time T, we then use (4.59) of Theorem 4.3. Arguing by contradiction, we assume that $c(s)$ can be lower bounded by $\theta > 0$, then we have

$$\begin{aligned}&\frac{\theta\left(\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}^2 + \frac{9c}{10}\int_0^1 \ddot{b}^2\right)}{1+\int_0^1 \dot{b}^2}\\ \leq\ & C_3(|b|_\infty)\left(-\frac{1}{\Gamma}\frac{\partial}{\partial s}\left(Ca + \int_0^1 \dot{b}^+\right) + C_4(|b|_\infty, \Gamma) + \frac{C_4(|b|_\infty)}{\Gamma}\int_0^1 \dot{b}^+ + \frac{C_5}{\Gamma}\int_0^1 \dot{b}^+\left(-\frac{\partial a}{\partial s}\right)\right) - \\ & -C_3(|b|_\infty)\frac{\partial a}{\partial s}. \end{aligned}\tag{4.60}$$

Thanks to estimate 1 of Theorem 4.3, we derive that in finite time $|b|_\infty$ is bounded along the flow. Then by Theorem 4.1, we obtain that $\int_0^1 \dot{b}^+$ satisfy the following:

$$\frac{\partial}{\partial s}\int_0^1 \dot{b}^+ \leq \tilde{C}_4(|b|_\infty) + C_4(|b|_\infty)\int_0^1 \dot{b}^+.$$

This yields that $\int_0^1 \dot{b}^+$ is also bounded in finite time along the flow. Plugging everything into (4.60), we derive that $\int_0^1 \dot{b}^2$ is bounded. There cannot be explosion. Thus $c(s)$ has to tend to 0 on a subsequence. This finishes the proof of corollary 4.1.

5 Existence of the flow

In this section, we show the existence of the globalized flow Z_{12} which is defined in (4.57). We will build a partition of unity $(\alpha_i)_{i\in I}$ of H^1 (on b) so that Theorem 3.1 and 4.3 hold in each neighborhood we use. Namely for every $b \in H^1$, we single out oscillations satisfying (3.3) and build an $\eta_{12,i}$ on these intervals. There is a small neighborhood $U(b)$ in H^1 of b such that for every $\tilde{b} \in U(b)$ we can use the same set of $t^-_{1,i}, t^+_{1,i}, t^-_{2,i}, t^+_{2,i}$ to build an $\tilde{\eta}_{12,i}$ for each oscillation. We also require that the associated $\tilde{\theta}(t), \tilde{\epsilon}(t)$ satisfy (3.8)-(3.13) and their C^2-norm depend in b in a Lipschitz way as we constructed in section 3.3.

We then construct a locally finite partition of unity $(\alpha_i)_{i\in I}$ subject to $U(b)$. On a given ball in H^1, there are only finite number of functions $\eta^i_0\psi^i_{\theta,\epsilon}$ involved for this ball. But the $\tilde{\eta}^i_0\psi^i_{\tilde{\theta},\tilde{\epsilon}}$ depends on $\tilde{b}$, thus are infinitely many of them. Since they are originated from a finite number of $\eta^i_0\psi^i_{\theta,\epsilon}$, Theorem 3.1- 4.3 still hold for them.

In order to show the existence of the globalized flow, first we show the existence of a regularized flow Z_ϵ. Unfortunately they do not stay in C_β, instead they evolve in the whole loop space $H^2(S^1, M)$. Thus we will show that Z_ϵ does converge to Z_{12} as ϵ approaches 0 in the second step. We will also show the continuity of the map b_0 to $b(s, \cdot)$ as a map from $H^1(S^1)$ into L^∞.

5.1 The regularization flow Z_ϵ

In this section we construct a regularization flow Z_ϵ whose limit is Z_{12}. Our construction is completed on each bounded ball $B(0, R) \subset H^2(S^1, M)$. Since these balls are weakly paracompact, we can extract from the related weak covering finite ones. By partition of unity, we then have finitely many of functions $\alpha_i \psi^i_{\theta,\epsilon} \eta^i_0$ on the ball $B(0, R)$. Every α_i is locally finite and Lipschitz with respect to b. Now let us start our construction.

We consider the regularizing operator ϕ_ϵ: $L^2(S^1) \to H^2(S^1)$ such that for $f \in L^2(S^1)$, ϕ_ϵ satisfies the following equation:

$$-\epsilon\ddot{\phi}_\epsilon(f) + \phi_\epsilon(f) = f. \tag{5.61}$$

It is easy to see that for $f \in H^1(S^1)$ we have

1. $||\phi_\epsilon(f)||_{H^1} \leq ||f||_{H^1}$;
2. $||\phi_\epsilon(f) - f||^2_{L^2} \leq \epsilon ||f||^2_{H^1}$.

Let x_0 be a curve in C_β such that $b_0 \in H^1$ and ϵ be a small parameter. For ϵ small enough, $0 < \epsilon < \epsilon_0$, we denote $\phi_\epsilon(b)$ an ϵ-regularization of b, which is obtained by solving (5.61).

To simplify our notation we also denote the regularization parameter ϵ, but please keep in mind that they are different from the ϵ in $\psi_{\theta,\epsilon}$. $\phi_\epsilon(b)$ can be assumed to be in a given bounded weak neighborhood V in $H^1(S^1)$. By the construction above, there are only a finite number of $\psi_{\theta,\epsilon}\eta_0$ involved in V. We can therefore rewrite the η component of $\sum \alpha_i Z_i$ as $\beta_i sgn\phi_\epsilon \psi_{\theta_i,\epsilon_i} \eta_i$. To simplify our notation, we come back to use the notation $\sum \alpha_i \eta_i$.

We consider the map:

$$\begin{array}{ccl} L^2 & \to & H^2 \\ b & \to & \sum \alpha_i(\phi_\epsilon(b))\eta_i(\phi_\epsilon(b)) := N_\epsilon(b). \end{array}$$

With some work it can be shown to be Lipschitz in b. We will construct the flow Z_ϵ using $N_\epsilon(b)$ as the w-component of the tangent vector.

For any curve x close to x_0 in the full space $H^1(S^1, M)$, we define

$$Z_\epsilon(x) = \lambda_\epsilon \xi(x) + \frac{\dot{N}_\epsilon(b) + \lambda_\epsilon \phi_\epsilon(b)}{\int_0^1 a} v(x) + N_\epsilon(b) w(x)$$

to be the H^1 tangent vector field to $H^1(S^1, M)$ at x, where $\dot{x} = a\xi + bv + cw$ and

$$\lambda_\epsilon = \left(\int_0^t \left(bN_\epsilon(b) - c\frac{\dot{N}_\epsilon(b)}{\int_0^1 a} \right) - t \int_0^1 \left(bN_\epsilon(b) - c\frac{\dot{N}_\epsilon(b)}{\int_0^1 a} \right) \right) - \bar{\mu} N_\epsilon(b). \tag{5.62}$$

Because we have smoothened b into $\phi_\epsilon(b)$, Z_ϵ is locally Lipschitz. Hence the differential equation :

$$\begin{cases} \frac{\partial x_\epsilon}{\partial s} & = Z_\epsilon(x_\epsilon) \\ x_\epsilon(0) & = x(0) \end{cases} \tag{5.63}$$

has locally a unique solution.

Let us write $\dot{x}_\epsilon(s) = a_\epsilon(s)\xi + b_\epsilon(s)v + c_\epsilon(s)w$. By Proposition 1, we have:

$$\begin{aligned} \frac{\partial a_\epsilon}{\partial s} &= \overbrace{\lambda_\epsilon + \bar{\mu}N_\epsilon}^{\cdot} - b_\epsilon N_\epsilon + c_\epsilon \frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon} = \frac{c_\epsilon}{\int_0^1 a_\epsilon}\lambda_\epsilon\phi_\epsilon - \int_0^1 \left(b_\epsilon N_\epsilon - \frac{c_\epsilon \dot{N}_\epsilon}{\int_0^1 a_\epsilon} \right) \\ \frac{\partial b_\epsilon}{\partial s} &= \overbrace{\frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon}}^{\cdot} + N_\epsilon(a_\epsilon\tau - b_\epsilon\bar{\mu}_\xi) - c_\epsilon \left(-\tau\lambda_\epsilon + \bar{\mu}_\xi \frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon} \right) \\ \frac{\partial c_\epsilon}{\partial s} &= \dot{N}_\epsilon - \frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon} + \lambda_\epsilon b_\epsilon = \dot{N}_\epsilon \left(1 - \frac{a_\epsilon}{\int_0^1 a_\epsilon} \right) + \lambda_\epsilon \left(b_\epsilon - \frac{\phi_\epsilon a_\epsilon}{\int_0^1 a_\epsilon} \right). \end{aligned} \tag{5.64}$$

Since the initial condition $x(0)$ belongs to C_β, we can assume that $c(0) \equiv 0$, $a(0) \equiv const$ and $b(0) \in B(b_0, \rho) \subset V$. Then for $\epsilon > 0$ small enough, the equation (5.63) has a continuous family of solutions x_ϵ for $0 \leq s \leq s_0(\epsilon, b(0))$. When s is small enough, $\phi_\epsilon(b_\epsilon(s))$ will belong to V because $N_\epsilon, \phi_\epsilon$ are bounded if b_ϵ is bounded in L^2. Let $s_1(\epsilon, b(0))$ be the maximal time that $\phi_\epsilon(b_\epsilon(s))$ stays in V.

Now we show that $s_1(\epsilon, b(0))$ is lower bounded independent of ϵ and $b(0) \in B(b_0, \rho)$.

Lemma 5.1 *There exists $\gamma > 0$, $\bar{\epsilon} > 0$ and $\bar{\rho} > 0$ such that if $\rho < \bar{\rho}$, $\epsilon < \bar{\epsilon}$, then $s_1(\epsilon, b(0)) \geq \gamma$ for any $b(0) \in B(b_0, \rho)$.*

Proof. Assume $\phi_\epsilon(b_\epsilon) \in V$ and $\int_0^1 b_\epsilon^2 \leq C$. Observe that $\phi_\epsilon(b_\epsilon)$ is then bounded in H^1, hence the α_i's are all zero, but a finite number of them. Therefore N_ϵ is bounded in H^1, with a uniform bound C as well. First, by (5.62) we notice that

$$|\lambda_\epsilon|_{L^\infty} \leq C(V) \left(1 + \frac{\left(\int_0^1 c_\epsilon^2 \right)^{\frac{1}{2}}}{\int_0^1 a_\epsilon} \right), \tag{5.65}$$

and

$$|\dot{N}_\epsilon|_\infty \leq C(V) \left(1 + |\overbrace{\phi_\epsilon(b_\epsilon)}^{\cdot}|_\infty \right) \leq C(V) \left(1 + \left(\int_0^1 \overbrace{\phi_\epsilon(b_\epsilon)}^{\cdot\cdot}{}^2 \right)^{\frac{1}{2}} \right). \tag{5.66}$$

The second inequality holds since the $\eta_{0,i}$'s are globally bounded and there are only finitely many of them. Thus by (5.64) and Proposition 2.2 we have,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 a_\epsilon^2 &= \int_0^1 \frac{a_\epsilon c_\epsilon \lambda_\epsilon \phi_\epsilon}{\int_0^1 a_\epsilon} - \left(\int_0^1 a_\epsilon\right)\int_0^1\left(b_\epsilon N_\epsilon - \frac{c_\epsilon \dot{N}_\epsilon}{\int_0^1 a_\epsilon}\right)\\ &\leq C(V)\left(1+\frac{\int_0^1 a_\epsilon^2}{(\int_0^1 a_\epsilon)^2}+\frac{(\int_0^1 c_\epsilon^2)^2}{(\int_0^1 a_\epsilon)^2}+(\int_0^1 a_\epsilon)^2+\int_0^1 c_\epsilon^2\right) \qquad (5.67)\end{aligned}$$

and

$$\frac{\partial}{\partial s}(\int_0^1 a_\epsilon)^2 \leq C(V)\left(1+\int_0^1 c_\epsilon^2+(\int_0^1 a_\epsilon)^2\right). \qquad (5.68)$$

Since $c_\epsilon(0)\equiv 0$, by (5.64)

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 c_\epsilon^2\\ =\ &\frac{1}{2}\int_0^1\left(\int_0^s \dot{N}_\epsilon\left(1-\frac{a_\epsilon}{\int_0^1 a_\epsilon}\right)+\lambda_\epsilon\left(b_\epsilon-\frac{\phi_\epsilon a_\epsilon}{\int_0^1 a_\epsilon}\right)\right)^2\\ \leq\ &C(V)\frac{s}{2}\int_0^s\int_0^1\left(\dot{N}_\epsilon\left(1-\frac{a_\epsilon}{\int_0^1 a_\epsilon}\right)\right)^2+\left(\lambda_\epsilon\left(b_\epsilon-\frac{\phi_\epsilon a_\epsilon}{\int_0^1 a_\epsilon}\right)\right)^2\\ \leq\ &C(V)\frac{s}{2}\int_0^s\left(\left(\int_0^1 \ddot{\phi}_\epsilon^2\right)\left(1+\frac{\int_0^1 a_\epsilon^2}{(\int_0^1 a_\epsilon)^2}\right)+\left(1+\frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2}\right)\left(1+\frac{\int_0^1 a_\epsilon^2}{(\int_0^1 a_\epsilon)^2}\right)\right).\end{aligned}$$
$$(5.69)$$

Now we compute $\frac{\partial}{\partial s}\int_0^1 \dot{b}_\epsilon\overset{\cdot}{\overbrace{\phi_\epsilon(b_\epsilon)}}$:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\int_0^1 \dot{b}_\epsilon\overset{\cdot}{\overbrace{\phi_\epsilon(b_\epsilon)}} &= \int_0^1 \frac{\partial \dot{b}_\epsilon}{\partial s}\overset{\cdot}{\overbrace{\phi_\epsilon(b_\epsilon)}} = -\int_0^1 \frac{\partial b_\epsilon}{\partial s}\overset{\cdot\cdot}{\overbrace{\phi_\epsilon(b_\epsilon)}}\\ &= -\frac{1}{\int_0^1 a_\epsilon}\int_0^1 \overset{\cdot}{\overbrace{\dot{N}_\epsilon+\lambda_\epsilon\phi_\epsilon}}\,\ddot{\phi}_\epsilon - \int_0^1 (a_\epsilon N_\epsilon \tau - b_\epsilon N_\epsilon \bar{\mu}_\xi)\ddot{\phi}_\epsilon +\\ &\quad+\int_0^1 c_\epsilon\left(-\tau\lambda_\epsilon+\bar{\mu}_\xi\frac{\dot{N}_\epsilon(b_\epsilon)+\lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon}\right)\ddot{\phi}_\epsilon\end{aligned}$$

The expression $\int_0^1 \ddot{N}_\epsilon\ddot{\phi}_\epsilon$ can be estimated as (3.18) in theorem 3.1 for $\int \overset{\cdot\cdot}{\overbrace{\psi_{\theta,\epsilon}\eta_0}} sgnb\ddot{b}$. The only difference is that we have now a finite number of such $\psi_{\theta,\epsilon}\eta_0$ combined along the partition of unity. Similar to (3.36), by (5.66) we have

$$-\int_0^1 \ddot{N}_\epsilon\ddot{\phi}_\epsilon \leq -\frac{9c}{10}\int_0^1 \overset{\cdot\cdot}{\overbrace{\phi_\epsilon(b)}} + \frac{C_3}{c}\left(1+\frac{1}{\inf\Gamma_i^2}+\frac{C_4}{\inf\Gamma_i}\right).$$

For any $\delta > 0$, we have

$$\left|\int_0^1 (a_\epsilon N_\epsilon \tau - b_\epsilon N_\epsilon \bar{\mu}_\xi)\ddot{\phi}_\epsilon\right| \leq \delta \int \ddot{\phi}_\epsilon^2 + C(\delta),$$

and

$$\begin{aligned} &\left|\int_0^1 c_\epsilon \left(-\tau\lambda_\epsilon + \bar{\mu}_\xi \frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon \phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon}\right)\ddot{\phi}_\epsilon\right| \\ \leq\;& \delta \int \ddot{\phi}_\epsilon^2 + C(\delta)(1 + \frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2}) \int_0^1 c_\epsilon^2 (1 + \frac{1}{(\int_0^1 a_\epsilon)^2}) + C(\delta)\frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2} \int_0^1 \ddot{\phi}_\epsilon(b_\epsilon)^2. \end{aligned}$$

Finally, we have

$$\begin{aligned} \frac{1}{\int_0^1 a_\epsilon} \int_0^1 \overset{\cdot}{\overbrace{\lambda_\epsilon \phi_\epsilon}}\, \ddot{\phi}_\epsilon &\leq \delta \int \ddot{\phi}_\epsilon^2 + \frac{C(\delta)}{(\int_0^1 a_\epsilon)^2} \left(1 + \frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2} + \frac{C_4}{(\int_0^1 a_\epsilon)^2} \int_0^1 c_\epsilon^2 \dot{N}_\epsilon(b)^2\right) \\ &\leq \delta \int \ddot{\phi}_\epsilon^2 + \frac{C(\delta)}{(\int_0^1 a_\epsilon)^2} \left(1 + \frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2} + \frac{C_4}{(\int_0^1 a_\epsilon)^2} \int_0^1 c_\epsilon^2 \int_0^1 \ddot{\phi}_\epsilon(b_\epsilon))^2\right). \end{aligned}$$

Assuming that $c > c_0 > 0$ on the trajectory, we thus have:

$$\begin{aligned} \frac{\partial}{\partial s} \int_0^1 \dot{b}_\epsilon \overset{\cdot}{\overbrace{\phi_\epsilon(b_\epsilon)}} &\leq -\frac{9c}{10} \int_0^1 \overset{\cdot\cdot}{\overbrace{\phi_\epsilon(b)}} + C(\delta)(1 + \frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2}) \int_0^1 c_\epsilon^2 (1 + \frac{1}{(\int_0^1 a_\epsilon)^2}) \\ &\quad + 2\delta \int_0^1 \ddot{\phi}_\epsilon^2 + C(\delta)\left(1 + \frac{1}{\int_0^1 a_\epsilon^2}\right) \frac{\int_0^1 c_\epsilon^2}{(\int_0^1 a_\epsilon)^2} \int_0^1 \ddot{\phi}_\epsilon(b_\epsilon)^2. \end{aligned} \tag{5.70}$$

Observe that

$$\int_0^1 \dot{b}_\epsilon \dot{\phi}_\epsilon(b_\epsilon) = \int_0^1 \dot{\phi}_\epsilon(b_\epsilon)^2 + \epsilon \int_0^1 \ddot{\phi}_\epsilon(b_\epsilon)^2 \geq \int_0^1 \dot{\phi}_\epsilon(b_\epsilon)^2.$$

We take δ small so that $\delta < \frac{c_0}{10}$. Let $\tilde{s}$ be the maximum positive time for the solution of (5.63) with x_0 as initial data satisfying:

$$\begin{cases} \int_0^1 c_\epsilon^2(s) & \leq \tilde{\delta} < 1 \quad \int_0^1 a_\epsilon^2(s) \leq 1 + a_0^2 \\ \left(\int_0^1 a_\epsilon(s)\right)^2 & \geq \frac{1}{2} a_0^2 \quad \phi_\epsilon(b_\epsilon) \in V \quad \int_0^1 b_\epsilon^2 \leq C' \quad \forall s \in [0, \tilde{s}] \end{cases} \tag{5.71}$$

where $\tilde{\delta}$ is a small constant which will be chosen later.

By (5.67) – (5.71) we have:

$$\begin{aligned}
\left|(\int_0^1 a_\epsilon)^2(s) - (\int_0^1 a_\epsilon)^2(0)\right| &\leq Cs,\\
\int_0^1 a_\epsilon^2(s) &\leq C_1 s + \int_0^1 a_\epsilon^2(0),\\
\int_0^1 c_\epsilon^2(s) &\leq C_2 s + C_3 s\int_0^s\int_0^1 \ddot{\phi}_\epsilon^2,\\
\left(\frac{7c_0}{10} - C_4(\delta)\tilde{\delta}\right)\int_0^s\int_0^1 \ddot{\phi}_\epsilon(b_\epsilon)^2 + \int_0^1 \dot{\phi}_\epsilon^2(b_\epsilon) &\leq \int_0^1 \dot{b}_\epsilon\dot{\phi}_\epsilon(b_\epsilon)(0) + C_5(\delta) \leq C_6(\delta),
\end{aligned}$$

where C, C_1, C_2, C_3 and $C_4(\delta), C_5(\delta), C_4(\delta)$ do not depend on $\tilde{\delta}$ as long as $\tilde{\delta} < 1$. Take $\tilde{\delta}$ small so that $\tilde{\delta} \leq \frac{7c_0}{20C_4(\delta)}$. Since $a_\epsilon(0)$ is close to a_0(ϵ_2 measures how close), we have for $s \in [0, \tilde{s}]$:

$$\begin{aligned}
\left|(\int_0^1 a_\epsilon)^2(s) - a_0^2\right| &\leq Cs + \epsilon_2,\\
\int_0^1 a_\epsilon^2(s) &\leq C_1 s + a_0^2 + a\epsilon_2 a_0, \qquad (5.72)\\
\int_0^1 c_\epsilon^2(s) &\leq C_7(\delta)s,
\end{aligned}$$

where $C_7(\delta)$ does not depend on $\tilde{\delta}$. In addition we have:

$$\frac{2}{5}c_0\int_0^s\int_0^1 \ddot{\phi}_\epsilon(b_\epsilon)^2 + \int_0^1 \dot{\phi}_\epsilon^2(b_\epsilon)(s) \leq \int_0^1 \dot{b}_\epsilon\dot{\phi}_\epsilon(b_\epsilon)(0) + C_6(\delta)s. \qquad (5.73)$$

Since $\int_0^1 \dot{b}_\epsilon\dot{\phi}_\epsilon(b_\epsilon)$ is lower-bounded by $\int_0^1 \dot{\phi}_\epsilon^2(b_\epsilon)$, and $\int_0^1 \dot{\phi}_\epsilon^4(b_\epsilon)$ is upper-bounded by $C\int_0^1 \ddot{\phi}_\epsilon^2(b_\epsilon)$, we derive that

$$\int_0^s \left|\frac{\partial b_\epsilon}{\partial s}\right|_{L^2}^2 + \left|\frac{\partial a_\epsilon}{\partial s}\right|_{L^2}^2 + \left|\frac{\partial c_\epsilon}{\partial s}\right|_{L^2}^2$$

is finite for s finite. Thus, there is no explosion in finite time as long as we keep our bounds. We have also, as long as $\int_0^1 b_\epsilon^2 \leq C'$ and $\phi_\epsilon(b_\epsilon) \in V$:

$$\begin{aligned}
&\frac{\partial}{\partial s}\int_0^1 b_\epsilon^2\\
= &\int_0^1 b_\epsilon\left(\frac{\overbrace{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}^{\cdot}}{\int_0^1 a_\epsilon} + N_\epsilon(a_\epsilon\tau - b_\epsilon\bar{\mu}_\xi) - c_\epsilon\left(-\tau\lambda_\epsilon + \bar{\mu}_\xi\frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon\phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon}\right)\right)\\
\leq &C'''\left(\int_0^1 b_\epsilon^2 + \int_0^1 \ddot{\phi}_\epsilon^2 + \int_0^1 \dot{\phi}_\epsilon^4\right) + C_4\\
\leq &C_1'\left(\int_0^1 b_\epsilon^2 + \int_0^1 \ddot{\phi}_\epsilon^2\right) + \tilde{C}_4.
\end{aligned}$$

By (5.72) and (5.73), we derive that if $\int_0^1 b_\epsilon^2(0) \le \frac{C'}{2}$, taking V to be a ball $B(0,R)$ and assuming that $b_\epsilon(0)$ and ϵ satisfy:

$$\int_0^1 \dot{b}\dot{\phi}_\epsilon(\dot{b}) \le \frac{R}{2}.$$

$\phi_\epsilon(b)$ will still be in $B(0,R)$ by (5.73). And $\int_0^1 b_\epsilon^2$ will be less than C' for some sizable time Δs which depends only on C_1', R and c, but independent of ϵ, provided ϵ is small enough. This is the end of the proof.

5.2 Convergence of Z_ϵ

Now we show that Z_ϵ converges to Z_{12}.

Lemma 5.2 *As $\epsilon \to 0$, c_ϵ tends to* 0 *in $L^1(S^1)$ everywhere in s, $a_\epsilon - \int_0^1 a_\epsilon(s)$ tends to zero in $L^1(S^1)$ everywhere in s.*

Proof. By (5.64), we have,

$$\begin{aligned}\frac{\partial}{\partial s}\int_0^1 |c_\epsilon| &\le \int_0^1 |\dot{N}_\epsilon|\left|1 - \frac{a_\epsilon}{\int_0^1 a_\epsilon}\right| + C\left(\int |b_\epsilon - \phi_\epsilon| + \int_0^1\left|1 - \frac{a_\epsilon}{\int_0^1 a_\epsilon}\right|\right)\\ &\le \left(\left(\int_0^1 |\ddot{\phi}_\epsilon|^2\right) + C'\right)\int_0^1\left|a_\epsilon - \int_0^1 a_\epsilon\right|,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial s}\int_0^1\left|a_\epsilon - \int_0^1 a_\epsilon\right| &\le \int_0^1\left|\frac{c_\epsilon\lambda_\epsilon\phi_\epsilon}{\int_0^1 a_\epsilon} - \int_0^1 \frac{c_\epsilon\lambda_\epsilon\phi_\epsilon}{\int_0^1 a_\epsilon}\right|\\ &\le C\int_0^1 |c_\epsilon|.\end{aligned}$$

Denoting $y_\epsilon(s) = \int_0^s\int_0^1 |c_\epsilon|$, since $a_\epsilon(0) = \int_0^1 a_\epsilon(0)$, we have

$$y_\epsilon''(s) \le C\left(C' + \left(\int_0^1 |\ddot{\phi}_\epsilon|^2\right)^{\frac{1}{2}}\right)y_\epsilon(s) + C\int |b_\epsilon - \phi_\epsilon|.$$

Since $y_\epsilon(0) = \int_0^1 |c_\epsilon(0)| = 0$, we have :

$$\begin{aligned}y_\epsilon'(s) &\le C_{30}\max_{\tau\in[0,s]} y_\epsilon(s)\left(1 + \left(\int_0^1 |\ddot{\phi}_\epsilon|^2\right)^{\frac{1}{2}}\right) + C\int_0^s\int_0^1 |b_\epsilon - \phi_\epsilon|\\ &\le C_{31}\max_{\tau\in[0,s]} y_\epsilon(s) + C\epsilon\int_0^s\int_0^1 |b_\epsilon - \phi_\epsilon|\\ &\le C_{31}\max_{\tau\in[0,s]} y_\epsilon(s) + C_{31}\epsilon.\end{aligned}$$

Thus

$$y_\epsilon(s) \leq C_{31} \max_{\tau_1 \in [0,s]} y_\epsilon(\tau_1)\tau + C_{31}\epsilon\tau$$

and taking $C_{31}s < \frac{1}{2}$, we have:

$$\max_{\tau \in [0,s]} y_\epsilon(\tau) \leq \frac{1}{2} C\epsilon s.$$

Thus as ϵ tends to zero, $\max_{\tau \in [0,s]} y_\epsilon(\tau)$ as well as $y'_\epsilon(s)$ tends to zero. Also $\left(a_\epsilon - \int_0^1 a_\epsilon\right)$ tends to zero in $L^1(S^1)$. Integrating (5.64), we also have:

$$\begin{aligned} \int_0^s \int_0^1 \left| \frac{\partial}{\partial s} + \int_0^1 b_\epsilon N_\epsilon \right| &\leq C \int_0^s \int_0^1 |c_\epsilon| \left(1 + |\dot{N}_\epsilon|_\infty\right) \\ &\leq C \int_0^s \int_0^1 |c_\epsilon| \left(1 + \left(\int_0^1 |\ddot{\phi}_\epsilon|^2\right)^{\frac{1}{2}}\right) \\ &\leq C_1 \sup_{(0,s)} \int_0^1 |c_\epsilon| \end{aligned}$$

also tends to zero. The same argument implies that

$$\int_0^s \int_0^1 |c_\epsilon| \left| -\tau\lambda_\epsilon + \bar{\mu}_\xi \frac{\dot{N}_\epsilon(b_\epsilon) + \lambda_\epsilon \phi_\epsilon(b_\epsilon)}{\int_0^1 a_\epsilon} \right|$$

and

$$\int_0^s \left| \lambda_\epsilon + \bar{\mu} N_\epsilon(b_\epsilon) - \left(\int_0^t b N_\epsilon(b_\epsilon) - t \int_0^t b_\epsilon N_\epsilon(b_\epsilon) \right) \right|_{L^\infty(S^1)}$$

also tend to zero.

Actually, we have the following stronger result:

Lemma 5.3 *1. There exists a constant $C > 0$ and $\bar{\epsilon} > 0$ such that for any $b(0) \in V$ and $s \leq s_1$, we have:*

$$\int_0^s \left| \int_0^1 \left| \frac{\partial b_\epsilon}{\partial s} \right|^2 + \int_0^1 \left(|\ddot{N}_\epsilon|^2 + |\ddot{\phi}_\epsilon|^2 \right) + \int_0^1 \left(\dot{\phi}_\epsilon(b_\epsilon)^2 + \phi_\epsilon(b_\epsilon)^2 \right) \right| \leq C$$

$$\int_0^s \left(\int_0^1 \left| \frac{\partial x_\epsilon}{\partial s} \right|^2 + |\dot{x}_\epsilon|^2 \right) \leq C \text{ for any } s \leq \inf\{s_1, 1\}. \qquad (5.74)$$

2. b_ϵ converges weakly in $L^2([0, s_1] \times S^1)$ to a function $b(s,t)$ and $\phi_\epsilon(b_\epsilon)$ converges strongly to $b(s,t)$ in $H^1(S^1)$ for almost every s. And the limit function $b(s,t)$ satisfies:

- $\int_0^1 |\dot{b}(s,t)|^2 dt \leq C$ *for any* $s \in [0, s_1]$.
- $\int_0^{s_1} \int_0^1 |\ddot{b}(s,t)|^2 ds dt \leq C$.

3. *$x_\epsilon(s,t)$ converges weakly in $H^1([0,s_1]\times S^1)$ to $x(s,t)$ and $x_\epsilon(s,t)$ converges strongly to $x(s,t)$ in $L^\infty(S^1)$ for any s, where $\dot{x}(s,t) = a\xi + b(s,t)v$ where $a(s) = \lim_{\epsilon\to 0}\int_0^1 a_\epsilon(s)$.*

Proof. By (5.73), we know that $\int_0^1 \dot{\phi}_\epsilon(b_\epsilon)^2 + \phi_\epsilon(b_\epsilon)^2$ is bounded for $0 \le s \le s_1$ for $b_\epsilon \in V$ and that $\int_0^s \int_0^1 |\ddot{\phi}_\epsilon(b_\epsilon)^2$ is bounded. Observe that

$$\ddot{\overline{|\psi_{\theta,\epsilon}(\phi_\epsilon)\eta_0|}} \le \frac{C}{|t_1^- - t_2^-|^2} + |\dot{\phi}_\epsilon|^2\left(1 + \left|\frac{\partial^2\psi_{\theta,\epsilon}}{\partial b^2}\right|\right) + |\ddot{\phi}_{\theta,\epsilon}| + |\ddot{\theta}|.$$

When we are in V, only finitely many of the α_i are non-zero. Thus we have

$$\int_0^s\int_0^1 |\ddot{N}_\epsilon|^2 \le C(V)\int_0^s\int_0^1 |\ddot{\phi}_\epsilon|^2$$

Observe that

$$\int_0^s\int_0^1 c_\epsilon^2\dot{N}_\epsilon^2 \le |\dot{N}_\epsilon|_\infty^2 \int_0^1 c_\epsilon^2 \le \int_0^1 |\ddot{N}_\epsilon|^2 \int_0^1 c_\epsilon^2 \le C.$$

By (5.64), we derive that

$$\int_0^s\int_0^1 \left(\frac{\partial b_\epsilon}{\partial s}\right)^2$$

is bounded. By (5.1), we know that

$$\left|\frac{\partial x_\epsilon}{\partial s}\right| \le C\left(\frac{|\dot{N}_\epsilon| + |\lambda_\epsilon||\phi_\epsilon|}{\int_0^1 a_\epsilon} + |N_\epsilon| + |\lambda_\epsilon|\right).$$

The estimates on $\int_0^1 \left|\frac{\partial x_\epsilon}{\partial s}\right|^2$ and $\int_0^1 |\dot{x}_\epsilon|^2$ follow.

The second statement follows from the fact that $\phi_\epsilon(b_\epsilon)$ is bounded in $H^1([0,s_1]\times S^1)$ and that $\phi_\epsilon(b_\epsilon) - b_\epsilon$ tends to zero weakly.

Now we show that $x_\epsilon(s,t)$ converges to $x(s,t)$ strongly in $L^\infty(S^1)$ for any s. Define the curve $\hat{x}_\epsilon$ by the equation:

$$\begin{cases} \dot{\hat{x}}_\epsilon = & \left(\int_0^1 a_\epsilon\right)(s)\xi(\hat{x}_\epsilon) + b_\epsilon v(\hat{x}_\epsilon) \\ \hat{x}_\epsilon(0) = & x_\epsilon(s,0) \end{cases}$$

$\hat{x}_\epsilon$ is bounded in H^1 since b_ϵ is bounded in L^2. And it is alsmost closed because $\hat{x}_\epsilon$ and x_ϵ start at the same point and the L^1 distance(defined locally) of $\dot{\hat{x}}_\epsilon$ and $\dot{x}_\epsilon$ tends to zero by the previous lemma. When ϵ tends to zero, for any s, $\hat{x}_\epsilon$ onverges strongly in $H^1(S^1)$, and hence in L^∞. Hence x_ϵ also converges strongly for any s in L^∞.

We proceed to show the uniqueness of the sets of limits for each s. Notice that

$$\left|\frac{\partial x_\epsilon}{\partial s}\right|_\infty \le C(1 + |\dot{N}|_\infty) \le C'\left(1 + \left(\int_0^1 |\ddot{N}_\epsilon|^2\right)^{\frac{1}{2}}\right).$$

Therefore we have

$$\int_s^{s'} \left|\frac{\partial x_\epsilon}{\partial s}\right|_\infty \leq C'(|s'-s| + (\int_s^{s'}\int_0^1 |\ddot{N}_\epsilon|^2)^{\frac{1}{2}}) \leq C''(|s'-s| + |s'-s|^{\frac{1}{2}}).$$

Thus we conclude that $x_\epsilon(s')$ converges to $x_\epsilon(s)$ in $L^\infty(S^1)$ as s' tends to s. On the other hand, $x_\epsilon(s)$ converges to x in $L^\infty(S^1)$ a.e. in s. Using diagonal argument, we derive the third statement. The last statement follows.

This lemma settles the existence problem. The next lemma proves the uniqueness and continuity of the flow:

Lemma 5.4 *1. The solution defined in the previous lemma is locally unique and continuous from $H^1(S^1)$ into $L^\infty(S^1)$ as a function in b.*

2. The map

$$b(0,t') \to b(s,t)$$

is continuous from $H^1(S^1)$ into $H^1(S^1)$ a.e. in s.

Proof. We consider two distinct solutions b, $\tilde{b} \in V \subset H^1(S^1)$ of the evolution equation. They are both bounded in $H^1(S^1)$ for every s and $\int_0^s\int_0^1 |\ddot{b}|^2 \leq C$. First we estimates the a component:

$$|\frac{\partial}{\partial s}(a-\tilde{a})| \leq \int_0^1 |bN_0(b) - \tilde{b}N_0(\tilde{b})| \leq C\int_0^1 |b-\tilde{b}|.$$

Thus we have

$$|a-\tilde{a}|(s) \leq |a-\tilde{a}|(0) + C\int_0^s \left(\int_0^1 |b-\tilde{b}|^2\right)^{\frac{1}{2}}$$

Now we estimate $\frac{\partial}{\partial s}\int_0^1 (b-\tilde{b})^2$. By Proposition 2.2 we have

$$\frac{\partial}{\partial s}\int_0^1 (b-\tilde{b})^2 = \int_0^1 \left(\overbrace{\frac{1}{a}N_0(b) - \frac{1}{\tilde{a}}N_0(\tilde{b})}^{\cdot\cdot}\right)(b-\tilde{b}) + \int_0^1 \overbrace{\left(\frac{\lambda_0 b}{a} - \frac{\tilde{\lambda}_0\tilde{b}}{\tilde{a}}\right)}^{\cdot}(b-\tilde{b})$$
$$+ \int_0^1 \left(aN_0(b)\tau - bN_0(b)\bar{\mu}_\xi - \tilde{a}N_0(\tilde{b})\tau + \tilde{b}N_0(\tilde{b})\bar{\mu}_\xi\right)(b-\tilde{b}). \qquad (5.75)$$

We are going to analyze each term in (5.75) to determine whether it can be controlled by $|b-\tilde{b}|_{H^1}(0) + |a-\tilde{a}|(0) + d(x(0),\tilde{x}(0))$. It is easy to see that the terms of type $(a-\tilde{a})(b-\tilde{b})$, are good terms and so are the terms involving $\int(N_0(b)-N_0(\tilde{b}))(b-\tilde{b})$.

By (3.42) and Lemma 3.2, $\eta_0 - \tilde{\eta}_0$ and its derivatives can be computed as follows:

$$\begin{aligned}
\tilde{\eta}_0 - \eta_0 &= \eta_0\left(e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)} - 1\right)\\
\overbrace{\tilde{\eta}_0 - \eta_0}^{\cdot} &= \dot{\eta}_0\left(e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)} - 1\right) + \eta_0(\tilde{\bar{\mu}}\tilde{b} - \bar{\mu}b)e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)}\\
\overbrace{\tilde{\eta}_0 - \eta_0}^{\cdot\cdot} &= \ddot{\eta}_0\left(e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)} - 1\right) + 2\dot{\eta}_0(\tilde{\bar{\mu}}\tilde{b} - \bar{\mu}b)e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)}\\
&\quad + \eta_0\left((\tilde{\bar{\mu}}\tilde{b} - \bar{\mu}b)^2 e^{\int_{t_1^-}^{t}(\tilde{\bar{\mu}}\tilde{b}-\bar{\mu}b)} + \overbrace{(\tilde{\bar{\mu}}\tilde{b} - \bar{\mu}b)}^{\cdot}\right)
\end{aligned}$$

We are encountering terms of the type $(\bar{\mu}(x) - \bar{\mu}(\tilde{x}))(b - \tilde{b})$ where $\bar{\mu}$ is evaluated on two different curves, the same phenomena happens for τ, $\bar{\mu}_\xi$ etc, for example terms of type $(\tau - \tilde{\tau})(b - \tilde{b})$ and $(\bar{\mu}_\xi(x) - \bar{\mu}_\xi(\tilde{x}))(b - \tilde{b})$. We analyze this type of terms as follows: Since $\frac{\partial x}{\partial s} = \lambda\xi + \frac{\dot{\eta}+\eta b}{a}v + \eta w$ with $\eta = \sum \alpha_i\eta_{12,i} + cb$, using the face that $|\theta - \tilde{\theta}| + |\epsilon - \tilde{\epsilon}| \le C|b - \tilde{b}|_{H^1}$, we derive that

$$\begin{aligned}
&\int_0^1 |x(s,z) - \tilde{x}(s,z)|dz\\
\le\ & C\left(\int_0^s\int_0^1 |b - \tilde{b}| + \int_0^s\int_0^1 |a - \tilde{a} + \int_0^s\int_0^1 |\dot{b} - \dot{\tilde{b}}|\right) + \int_0^1 |x(0,z) - \tilde{x}(0,z)|.
\end{aligned}$$

Thus for any function $\omega : M \to \mathbf{R}$, we have:

$$\begin{aligned}
&|\omega(x) - \omega(\tilde{x})|_\infty\\
\le\ & C_\omega\left(\int_0^s\int_0^1 |b - \tilde{b}| + \int_0^s\int_0^1 |a - \tilde{a}| + \int_0^s\int_0^1 |\dot{b} - \dot{\tilde{b}}| + \int_0^1 |a - \tilde{a}|\right.\\
&\left. + \int_0^1 |b - \tilde{b}| + \int_0^1 |x(0,z) - \tilde{x}(0,z)|\right)
\end{aligned}$$

Now let us take care of $\psi_{\theta,\epsilon}(b) - \psi_{\tilde{\theta},\tilde{\epsilon}}(\tilde{b})$. Since $\tilde{\theta}$, $\tilde{\epsilon}$ can be chosen in C^2 to depend in a Lipschitz way on $b \in L^\infty$, we have

$$|\psi_{\theta,\epsilon}(b) - \psi_{\tilde{\theta},\tilde{\epsilon}}(\tilde{b})| \le C|b - \tilde{b}|_{L^\infty}.$$

Its derivatives are more complicated to deal with.

We are thus left with $\int_0^1\left(\overbrace{\frac{1}{a}N_0(b) - \frac{1}{\tilde{a}}N_0(\tilde{b})}^{\cdot\cdot}\right)(b - \tilde{b}) + \int_0^1\overbrace{\left(\frac{\lambda_0 b}{a} - \frac{\tilde{\lambda}_0\tilde{b}}{\tilde{a}}\right)}^{\cdot}(b - \tilde{b})$ except than the terms involving $\eta_0 - \tilde{\eta}_0$ and its derivatives.

In $\int_0^1\left(\overbrace{\frac{1}{a}N_0(b) - \frac{1}{\tilde{a}}N_0(\tilde{b})}^{\cdot\cdot}\right)(b - \tilde{b})$, cb term contributes $-c\int_0^1(\dot{b} - \dot{\tilde{b}})^2$. We will use it to absorb similar terms. For these terms involving the difference in the partition of unity coefficients, since

$$|\psi_{\theta_i,\epsilon_i}(b)\eta_{0,i}|_{C^2} + |\psi_{\tilde{\theta}_i,\tilde{\epsilon}_i}(\tilde{b})\tilde{\eta}_{0,i}|_{C^2} \leq C_i(1 + |\ddot{b}| + |\ddot{\tilde{b}}| + |\dot{b}^2| + |\dot{\tilde{b}}|^2),$$

we derive that they are upper bounded by

$$\frac{c}{100}|b - \tilde{b}|^2_{H^2} + C\frac{100}{c}|b - \tilde{b}|^2_{L^2}(1 + \int_0^1 (\ddot{b}^2 + \ddot{\tilde{b}}^2)).$$

We can absorb $\frac{c}{100}|b-\tilde{b}|^2_{H^2}$ in $-c\int_0^1(\dot{b}-\dot{\tilde{b}})^2$, and the second term is a good term. Thus terms of this type are tackled.
By the construction of $\psi_{\theta,\epsilon}$ in section 3.3, we can show that the remaining terms are bounded by $\frac{c}{100}\int_0^1(\dot{b}-\dot{\tilde{b}})^2 + (1 + |\ddot{b}|^2_{L^2} + |\ddot{\tilde{b}}|^2_{L^2})\int_0^1 |b-\tilde{b}|^2$. Summing up, we get

$$\begin{aligned}
& \frac{c}{2}\int_0^1 (\dot{b}-\dot{\tilde{b}})^2 + \frac{\partial}{\partial s}\int_0^1 (b-\tilde{b})^2 \\
\leq\ & C\Bigg(\int_0^s\int_0^1 |b-\tilde{b}|^2 + |a-\tilde{a}|^2 + \int_0^s\int_0^1 |\dot{b}-\dot{\tilde{b}}| + (1 + |\ddot{b}|^2_{L^2} + |\ddot{\tilde{b}}|^2_{L^2})\int_0^1 |b-\tilde{b}|^2 \\
& + \int_0^1 |x(0,z) - \tilde{x}(0,z)|^2\Bigg) \\
\leq\ & C\left(|a-\tilde{a}|^2 + \int_0^1 |x(0,z)-\tilde{x}(0,z)|^2\right) + C'\left(\frac{c}{2}\int_0^s\int_0^1(\dot{b}-\dot{\tilde{b}})^2 + \int_0^1 (b-\tilde{b})^2\right) \\
& + C''\int_0^s\int_0^1 |b-\tilde{b}|^2.
\end{aligned}$$

Setting $w(s) = \frac{c}{2}\int_0^s\int_0^1(\dot{b}-\dot{\tilde{b}})^2 + \int_0^1(b-\tilde{b})^2$, we get

$$w(s) \leq \tilde{C}\left(w(0) + |a-\tilde{a}|^2 + \int_0^1 |x(0,z)-\tilde{x}(0,z)|^2\right) + \tilde{C}'\int_0^s\int_0^\tau\int_0^1 |b-\tilde{b}|^2.$$

For any $s \in [0, s_1]$, we then get:

$$w(s) \leq \tilde{C}\left(w(0) + |a-\tilde{a}|^2 + \int_0^1 |x(0,z)-\tilde{x}(0,z)|^2\right) + \tilde{C}'s^2 \sup_{z\in[0,s_1]} w(z).$$

If $\tilde{C}'s_1^2$ is less than $\frac{1}{2}$ (otherwise, we can take s_1 smaller so that it satisfies this condition), we conclude that :
$\int_0^1(b-\tilde{b})^2 + \int_0^s\int_0^1(\dot{b}-\dot{\tilde{b}})^2$ *tends to zero as* $|a(0)-\tilde{a}(0)| + |b(0,t)-\tilde{b}(0,t)|^2_{L^2} + d(x(0),\tilde{x}(0))$ *tends to zero*. Thus for any $h > 0$, we have

$$\begin{aligned}
b(s,t) - \tilde{b}(s,t) &= \frac{1}{2h}\int_{t-h}^{t+h}(b(s,t') - \tilde{b}(s,t'))dt' + O\left(\sqrt{h}\left(\int_0^1 \dot{b}^2 + \dot{\tilde{b}}^2\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{\sqrt{h}}O\left(|b-\tilde{b}|_{L^2}(s)\right) + O(\sqrt{h}).
\end{aligned}$$

Since $\int_0^1 |b - \tilde{b}|^2$ tends to 0, we get

$$\limsup_{\tilde{b}\to b} \sup_{s\in[0,s_1]} |b(s,t) - \tilde{b}(s,t)|_{L^\infty(S^1)} \leq C\sqrt{h} \text{ for any } h.$$

Therefore the map is continuous from $H^1(S^1)$ into $L^\infty(S^1)$ in b. It is also continuous from $H^1(S^1)$ to $H^1(S^1)$ a.e. in s.

It is obvious that the same statement holds for the curves $x(s,t)$, as a map from $H^2(S^1)$ into $W^{1,\infty}(S^1)$.

6 Arriving to the v-stretched curves

In this section, we combine the η_{12} flow in section 3 and 4 and the ω flow defined by (6.76) below to deform curves in C_β into the v-stretched curves, namely, curves with sizable almost $|b|_\infty$ pieces alternating with pieces with $||b| - v| \leq \delta_1$ where v and δ_1 are prefixed small numbers connected by very sharp descending or ascending pieces in between. We can view v-stretched curves as curves made of v-pieces alternating with $\xi + O(v)v$ pieces in graph. We will make this definition more precise later. In the sequel, we call η_{12} the cancellation flow since the control over $|b|_\infty$ and $\int(|b| - v)^+$ along the flow is kept as shown in Theorem 3.1 and 4.3.

Fix a weak H^2 neighborhood V in C_β. Let $x(t)$ be a curve in V. We introduce a function ω on an interval I with the following properties:

$$\begin{aligned} \omega &\in C_0^\infty(I),\ 0 \leq w \leq 1, \\ |\dot{\omega}|_{C^1} &\leq c_0|b|_\infty^3,\ \int_I \omega \geq c_1|I|, \end{aligned} \tag{6.76}$$

where c_0, c_1 are fixed small positive numbers. Combining the cancellation flow η_{12} and the ω flow $\eta = \omega sgnb$, we define a global flow for such curves to deform them into v-stretched curves.

We prove a technical lemma first. It is needed to determine the shape of our curves.

Let v be a fixed small number, and θ_1 be fixed. We also assume that

$$\begin{aligned} &\bar{\tilde{\mu}} \geq v + 2\theta_1, \\ &\underline{\tilde{\mu}} \leq v + \frac{1}{2}\theta_1. \end{aligned}$$

In addition, we choose μ_1 such that $\mu_1 = v + \frac{5}{6}\theta_1 + o(1)$. Then we have

Lemma 6.1 *For any $\epsilon_1 > 0$, there exists $c(\epsilon_1) > 0$ such that*

$$\int_I (|b| - v)^+ \leq c_0\left(2\bar{\tilde{\mu}}\epsilon_1|I| + \frac{4\bar{\tilde{\mu}}}{\theta_1^2 v c(\epsilon_1)}\int_I b\eta_{12} + \frac{3\theta_1}{2}|I|\right), \tag{6.77}$$

where c_0 is a universal constant which depends only on the contact form α and the choice of the vector field v.

This lemma indicates that $\int_I (|b|-\nu)^+$ is controlled by $\int_I b\eta_{12}$ on pieces where the curve shifts between near $|b|_\infty$ level and ν level. But $\int_I b\eta_{12}$ is in queue controlled by $c(s)$ by our choice of $c(s)$ in (3.14). Thus as $c(s) \to 0$, our curve consists of large pieces which is of the order $|b|_\infty$ and small pieces with $\int_I (|b|-\nu)^+$ small connected by sharp drops which might contain lots oscillation in between. We can think that $c(s)$ determines the shape of our curves in some sense. Now we proceed to prove lemma 6.1.

Proof. First we consider the case that

$$|\{t \in I \big| |b(t)| > \nu + \frac{3\theta_1}{2}\}| \leq \epsilon_1 |I|.$$

Then

$$\begin{aligned} \int_I (|b|-\nu)^+ &\leq \tilde{\tilde{\mu}}\epsilon_1 |I| + \int_{t\in I \big| |b(t)|\leq \nu+\frac{3\theta_1}{2}\}} (|b|-\nu)^+ \\ &\leq (\tilde{\tilde{\mu}}\epsilon_1 + \frac{3\theta_1}{2})|I|. \end{aligned} \tag{6.78}$$

Now we consider the case that

$$|\{t \in I \big| |b(t)| > \nu + \frac{3\theta_1}{2}\}| > \epsilon_1 |I|. \tag{6.79}$$

Since it is an open set, we can assume that

$$\{t \in I \big| |b(t)| > \nu + \frac{3\theta_1}{4}\} = \cup_{j=1}^{\infty} F_j,$$

where F_j's are open intervals. There exists j_1 such that

$$|\bigcup_{j=j_1}^{\infty} F_j| \leq \frac{1}{10}\epsilon_1 |I|. \tag{6.80}$$

We consider the remaining intervals $F_1, F_2, \cdots, F_{j_1}$. We may assume that

$$|\{t \in F_j \big| |b(t)| > \nu + \frac{3\theta_1}{2}\}| \leq \frac{1}{10}\epsilon_1 |F_j|, \tag{6.81}$$

for $j_0 < j \leq j_1$, and

$$|\{t \in F_j \big| |b(t)| > \nu + \frac{3\theta_1}{2}\}| > \frac{1}{10}\epsilon_1 |F_j|, \tag{6.82}$$

for $j \leq j_0$. For $F_{j_0+1}, \cdots, F_{j_1}$, we can proceed as in (6.79).

We are left with $F_1, \cdots, F_{j_0}$. We notice that

$$|\{t \in \bigcup_{j=1}^{j_0} F_j \text{ s.t. } |b(t)| > \nu + \frac{3}{2}\theta_1\}| > \epsilon_1 |I| - |\bigcup_{j>j_1} F_j| - \frac{1}{10}\epsilon_0 |\bigcup_{j_0<j\leq j_1} F_j| > \frac{1}{2}\epsilon_1 |I|. \tag{6.83}$$

Let $E_j = \{t \in F_j \text{ s.t. } |b(t)| > \nu + \frac{2}{3}\theta_1\}$. We may assume that $E_j = \bigcup_{k=1}^{\infty} E_j^k$ where E_j^k are open intervals. Repeating the same process (6.80)-(6.83) with $\frac{1}{20}\epsilon_1$, we keep a finite number of them.

Starting from each of these intervals left, we can build an oscillation where η_{12} can be set thanks to the fact that $|b(t)| \geq \tilde{\mu} + \frac{3\theta}{2}$. $|b|$ take the value $|b(t)| \geq \tilde{\mu} + \frac{2\theta}{3}$ at its boundaries, by construction.

We thus have now a finite number of disjoint intervals $L_j^k = [t_{1,j,k^-}, t_{2,j,k^-}]$ where we have constructed an $\eta_{12,j,k}$. Furthermore,

$$\left|\{t \in \bigcup L_{j,k}, \text{ s.t. } |b(t)| > \nu + \frac{3}{2}\theta_1\}\right| > \frac{\epsilon_1}{4}|I|. \tag{6.84}$$

As we did in (6.81) and (6.82) we already got rid of the other ones. Thus we may assume that

$$\left|\{t \in L_{j,k} \mid |b(t)| > \nu + \frac{3}{2}\theta_1\}\right| > \frac{\epsilon_1}{4}\left|L_{j,k}\right|. \tag{6.85}$$

Denote $F_{j,k} := \{t \in L_{j,k} \text{ s.t. } |b(t)| > \nu + \frac{3\theta_1}{2}\}$. Observe that, by (3.6) and maximum principle we have

$$\eta_{0,j,k} \geq \frac{c_0(t - t^-_{1,j,k})(t^-_{2,j,k} - t)}{(t^-_{1,j,k} - t^-_{2,j,k})^2}, \tag{6.86}$$

where c_0 is an constant which depends on $\bar{\mu}$ and $|b|_\infty$. Thus the following holds:

$$\begin{aligned}
\int_{L_{j,k}} \eta_{0,j,k} dt &\geq \int_{t^-_{1,j,k}}^{t^-_{2,j,k}} \frac{c_0(t - t^-_{1,j,k})(t^-_{2,j,k} - t)}{(t^-_{1,j,k} - t^-_{2,j,k})^2} dt \\
&= c_0 \int_0^1 s(1-s)ds \left|L_{j,k}\right| = c_1 \left|L_{j,k}\right|
\end{aligned}$$

and

$$\begin{aligned}
\int_{F_{j,k}} \eta_{0,j,k} dt &\geq \int_{F_{j,k}} \frac{c_0(t - t^-_{1,j,k})(t^-_{2,j,k} - t)}{(t^-_{1,j,k} - t^-_{2,j,k})^2} dt \\
&= c_0 \int_{\tilde{F}_{j,k}} s(1-s)ds \left|L_{j,k}\right|,
\end{aligned} \tag{6.87}$$

where $\tilde{F}_{j,k} := \frac{F_{j,k}}{|t^-_{1,j,k} - t^-_{2,j,k}|} > \frac{\epsilon_1}{20}$. Therefore we have

$$\int_{F_{j,k}} \eta_{0,j,k} dt \geq c(\epsilon_1)\left|L_{j,k}\right| \tag{6.88}$$

where c_{ϵ_1} is a function of ϵ_1 which can be taken as ϵ_1^3.

Thanks to (6.84) and (6.87), we get

$$\frac{\epsilon_1}{4}|I| < \sum_{j,k} \left|L_{j,k}\right| \leq \sum_{j,k} \frac{1}{c(\epsilon_1)} \int_{F_{j,k}} \eta_{0,j,k} dt.$$

Hence

$$\begin{aligned}\int_I (|b|-\nu)^+ &\leq \tilde{\bar{\mu}}\,|I| \leq \frac{4\tilde{\bar{\mu}}}{\epsilon_1 c(\epsilon_1)} \sum_{j,k} \int_{F_{j,k}} \eta_{0,j,k} \\ &\leq \frac{144\tilde{\bar{\mu}}}{\epsilon_1 c(\epsilon_1)\theta_1^2} \sum_{j,k} \int_{F_{j,k}} |b|\eta_{12,j,k} \\ &\leq \frac{C\tilde{\bar{\mu}}}{\tilde{c}(\epsilon_1)\theta_1^2} \int_I b\eta_{12}, \qquad (6.89)\end{aligned}$$

where we use the fact that $|b(t)| - \mu_1^-, \psi_{\theta_1,\epsilon_1}(|b(t)|) \geq \frac{\theta_1}{6}$.
By (6.78) and (6.89), we obtain (6.77). Thus lemma 6.1 is proved.

First, let us look at the almost $|b|_\infty$-pieces, namely pieces with $\frac{|b|_\infty}{2} \leq |b| \leq |b|_\infty$. If it satisfies (3.3), we use η_{12} to decrease J while keeping those bounds in Theorem 1 and 2. If $\int_I (1+b^2)|I| > \epsilon_0$, then we keep it. The size of such pieces in time is $O(\frac{1}{|b|_\infty})\sqrt{\epsilon_0}$ or larger. Therefore there are finitely many of such almost $|b|_\infty$ pieces.

There might be jumps from $|b|_\infty$ to ν. On such pieces, possibly after rescaling, we introduce $\eta = sgnb\omega$. Using (2.2) we got a vector field to C_β along x, thus a decreasing flow for J. It might destroy our control on $|b|_\infty$ or the one on $\int (|b|-\nu)^+$. We want to avoid this. There are several cases.

If there is no oscillation on such a piece, then starting from $\frac{|b|_\infty}{2}$ we use ω to decrease. This flow might increase $|b|_\infty$, but since we start from an interval where b is less than $\frac{|b|_\infty}{2}$, we could use this flow without actually increasing $|b|_\infty$. Once an almost $|b|_\infty$ piece occurs, we start to use the cancellation flow η_{12}.

There might be oscillations on such pieces. Fix a small numbers ϵ_1. If there is sizable high sharp oscillation, namely $|b|$ goes below the level $\nu + \delta_1$ after a time span ϵ_1 and goes below this level before $2\epsilon_1$ again after moving above $\nu + 2\delta_1$ somewhere in between. we apply the cancellation flow η_{12} to this sharp oscillation to decrease J. Thanks to a modification of estimate 1 of Theorem 1, the peak of this oscillation also decreases. Therefore we can assume that such oscillation does not occur. $|b|$ remains above $\nu + \delta_1$ for time span of the order of ϵ_1 in the sinks, if we are in the zone where $|b|$ is above $\nu + 2\delta_1$. We then introduce the ω flow $\eta = \omega sgnb$ in these time intervals. But the ω flow might increase the bounds on $|b|_\infty$ and $\int (|b|-\nu)^+$ sharply. In order to overcome this problem, we introduce $\eta_2 = \omega sgnb + M\sum_i \eta_{12,i}$, where we require that every oscillation of size at least $2\delta_1$ to contribute an η_{12} to η_2. We claim that only finitely many such oscillations develop. Therefore η_2 is well defined.

Now let us make this precise. Every oscillation of size at least $2\delta_1$ satisfies

$$\int_{I_i} \dot{b}^+ \geq 2\delta.$$

Thanks to Theorem 4.1, on the whole zone I we are considering, the contribution of the cancellation flow $\sum \eta_{12,i}$ towards $\frac{\partial}{\partial s}\int_I \dot{b}^+$ is

$$\frac{\partial}{\partial s}\int_I \dot{b}^+ \leq -C_1(V)\sum_i \frac{\tilde{\bar{\mu}}_i - \mu_{2,i}}{|x_{1,i}^- - x_{2,i}^-|^2} + C_2(V)\sum_i \frac{1}{\tilde{\bar{\mu}}_i - \mu_{2,i}} + C(V)c(1 + \int_I \dot{b}^+). \qquad (6.90)$$

The introduction of $\eta = \omega$ on a time interval K larger than $\frac{1}{|b|_\infty^{10,000}}$ add a term of the form $C(V) \int_K b^4$. Therefore $\int_I \dot{b}^+$ remains bounded on bounded intervals.

We can choose ϵ_1 so that

$$C_1(V)(2\delta_1)^2 \geq C_2(V)\epsilon_1^2.$$

This implies for those oscillation with $|x_{1,i}^- - x_{2,i}^-| \leq \epsilon_1$, the first two terms on the right side of (6.90) is non-positive. Thus we have

$$\frac{\partial}{\partial s} \int_I \dot{b}^+ \leq C(V)c(1 + \int_I \dot{b}^+).$$

Solving this differential equation, we have

$$\int_I \dot{b}(s,t)^+ dt \leq (1 + \int_I \dot{b}(0,t)^+ dt)e^{cC(V)s} - 1.$$

Thus after an finite time s, the number of such oscillations with size of $2\delta_1$ is finite unless the curves are no longer in the chosen neighborhood V.

Observe that the introduction of ω implies:

$$\frac{\partial a}{\partial s} = -\int_0^1 b\eta \leq -\int_K b\omega \leq -\frac{\nu c_1}{|b|_\infty^{10000}}.$$

Given that $|b|_\infty$ is a priori bounded, c can be bounded away from zero by (3.15) .

Now we show that with careful choice of M, then the control on $|b|_\infty$ and $\int (|b| - \nu)^+$ is kept. By (3.25), we have

$$\frac{\partial}{\partial s}(\frac{b}{a}) \leq \frac{1}{a}(\ddot{\omega} + b^2\omega - \dot{\overbrace{\bar{\mu}\omega}}) + \omega(a\tau - b\bar{\mu}_\xi) + M\frac{1}{a^2}(-C_1 \sum_i \frac{\Gamma_i}{|x_{1,i}^- - x_{2,i}^-|^2} + cC_2(1 + |b|_\infty^3)).$$

We can choose $M \geq C_2(V)|b|_\infty^5$ so that

$$\frac{1}{a}(\ddot{\omega} + b^2\omega - \dot{\overbrace{\bar{\mu}\omega}}) + \omega(a\tau - b\bar{\mu}_\xi) - M\frac{1}{a^2}C_1 \sum_i \frac{\Gamma_i}{|x_{1,i}^- - x_{2,i}^-|^2} < 0.$$

Thus

$$\frac{\partial}{\partial s}(\frac{b}{a}) \leq -\frac{C_1(V)}{a}\frac{\partial a}{\partial s}.$$

ω does not destroy the control on $\int (|b| - \nu)^+$, because its support does not touch the region where $|b|$ is less than ν. With the introduction of ω, we get additional terms of the type

$$\sum_{i,j} \int_{t_{i,j}}^{\tilde{t}_{i,j+1}} \omega(a\tau - b\bar{\mu}_\xi) \leq C \int_I (1 + b^2) \leq C(-\frac{\partial a}{\partial s}).$$

Thus estimate 2 in Theorem 3.1 and 4.3 holds as well.

This allows to get rid of large pieces where $|b|$ is essentially less than $\nu - \delta_1$, because b cannot rise to ν on a large size interval. The result is that our curves have to become ν-stretched with, maybe, lots of oscillations above their limit profile, of a very small L^1 contribution.

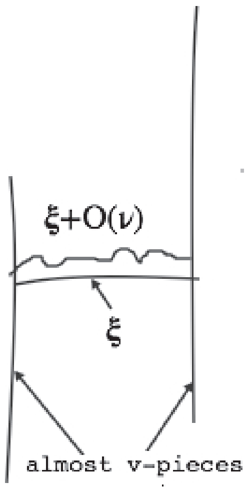

Figure 2: minimizing process

Remark 6.1 In this process, the zeroes of b never increases along a flow-line.

It is obvious that η_{12} and ω never touch the zeroes of b. The only troublesome term is $\eta = cb$. By (2.1) and (2.2), the contribution of this term is

$$\frac{\partial b}{\partial s} = \frac{1}{a}\overbrace{\left(\dot{\eta} + b\Big(\int_0^t b\eta - t\int_0^1 b\eta - \eta\bar{\mu}\Big)\right)}^{\cdot} + a\tau\eta - \bar{\mu}_\xi b\eta = \frac{a}{c}\ddot{b},$$

at t_0 such that $b(t_0) = \dot{b}(t_0) = 0$.

Thus if b tries to cross zero from above at t_0, $\frac{\partial b}{\partial s}$ becomes positive immediately and therefore cannot cross zero. When b approaches 0 from below, it is the same. Thus the zeroes of b never increases along a flow-line.

7 The last step: Arriving at $\bigcup \Gamma_{2k}$

Lemma 7.1 *Let us consider a nearby piece of ξ-orbit between two points x^+ and x^-. The tangent vector reads as $\xi + O(\nu)v$, with v small. Let $\bar{a}$ be the ξ-length of this piece of orbit, $a_c^1(x^-), \cdots, a_c^k(x^-)$ be the characteristic ξ-lengths at $\bar{x}$, where $a_c^k(x^-)$ is the length of the ξ-orbit such that v turns $k\pi$ from x^- to x^+. Assume that $|\bar{a} - a_c^j| \geq \delta(\nu) > 0$, where $\delta(\nu)$ is a function of ν tending to zero with ν. Then, there is a unique ξ-piece of orbit connects them in the neighborhood of this $\xi + O(\nu)v$ piece of orbit.*

Proof. We denote the two consecutive almost v-orbits as O_1 and O_2 and they intersect with the connecting $\xi + O(\nu)v$ piece of orbit at x_1 and x_2 respectively. We also denote the one-parameter group of ξ as ϕ_s. Then $\bigcup_{s\geq 0} \phi_s(O_1)$ is a surface.

Let s_1 be the ξ-length of the $[x_1, x_2]$ piece. We may assume that $D\phi_{s_1}(v(x_2))$ has a nonzero component on $-[\xi, v]$ as long as we are not in a neighborhood of a characteristic piece, i.e., ξ-pieces where the Dirichlet problem $\ddot{\eta} + a^2\tau\eta = 0, \eta(x_1) = \eta(x_2) = 0$ has a nonzero solution.

Thus near $\phi_{s_1}(x_2)$, which is very close to x_2, the tangent plane of $\bigcup_{s\geq 0}\phi_s(O_1)$ is generated by ξ and $D\phi_{s_1}(v + O(\frac{1}{|b|_\infty}))$, hence it is transerval to v if $|b|_\infty$ is large enough. For z close to x_2 on O_2, $\dot{x}(z)$ is equal to $v + O(\frac{1}{|b|_\infty})$. Thus O_2 and $\bigcup_{s\geq 0}\phi_s(O_1)$ are of complemental dimensions in M. It implies that there exists a unique xi-piece connecting O_1 and O_2 in the L^∞ neighborhood of the near ξ-piece $[x_1, x_2]$. Therefore there is a minimizing process that decreases $\int_0^1 a$ from a v-stretched curve to a curve having only ξ and $\pm v$ pieces.

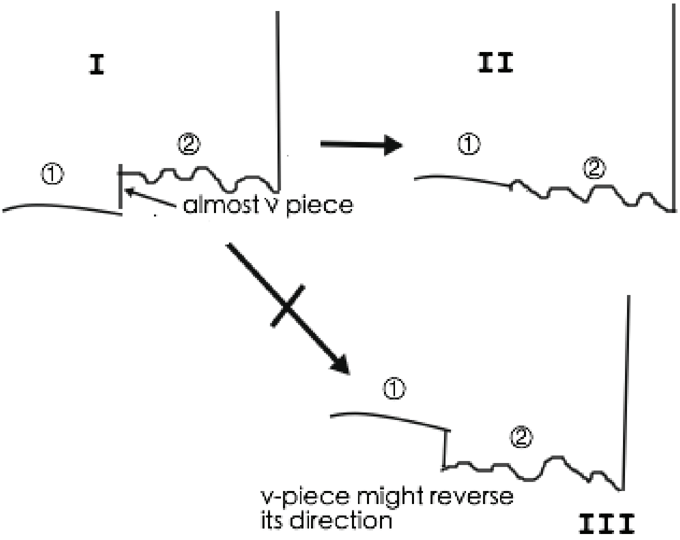

Figure 3: reversed orientation

An observation is in order here: along this minimizing process, an almost v-jump(its tangent vector is $v + \frac{a}{|b|_\infty|}\xi$, with $|b|_\infty$ very large) could reverse orientation if it is large enough, see figure 3.

In that case, we stop the process at phase II. We had an almost ξ-piece ① and another almost ξ-piece ②around a small almost v-jump. Taking together at phase II, they form an almost ξ-piece. The process can be iterated. Of course, if the $\xi + O(v)v$ piece is not small, this process is not minimizing. But such pieces are of finite number, bounded independent of v. Therefore, these curves converge as $v \to 0$ to a curve in $\bigcup \Gamma_{2k}$.

8 Appendix

In order to prove Proposition 2.2, we need the following lemma:

Lemma 8.1 *Let δ be a 1-form and $z(t)$ be a tangent vector along a loop $x(t)$ on a manifold M. Then the first variation of $\delta(\dot{x})$ is given by*

$$\partial_z \delta(\dot{x}) = \overbrace{\delta(z)}^{\cdot} - d\delta(\dot{x}, z). \tag{8.91}$$

This formula can be proved by taking a local chart and then carrying out the computation in the local chart. Now we proceed to prove Proposition 2.

Proof. Thanks to (8.91), we have:

$$\begin{aligned}
\frac{\partial a}{\partial s} &= \partial_z \alpha(\dot{x}) = \overbrace{\alpha(z)}^{\cdot} - d\alpha(\dot{x}, z) \\
&= \overbrace{\lambda + \bar{\mu}}^{\cdot} - (b\eta - c\mu), \\
\frac{\partial c}{\partial s} &= \partial_z \beta(\dot{x}) = \overbrace{\beta(z)}^{\cdot} - d\beta(\dot{x}, z) \\
&= \dot{\eta} - (a\mu - b\lambda) d\beta(\xi, v) = \dot{\eta} - (a\mu - b\lambda),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial b}{\partial s} &= \partial_z \gamma(\dot{x}) = \overbrace{\gamma(z)}^{\cdot} - d\gamma(\dot{x}, z) \\
&= \dot{\mu} - (b\eta - c\mu) - (a\eta - c\lambda) d\gamma(\xi, w) - (b\eta - c\mu) d\gamma(v, w).
\end{aligned}$$

Simple compuatation shows

$$d\gamma(\xi, w) = -\tau, \ d\gamma(v, w) = \bar{\mu}_\xi.$$

Thus we finish the proof of this proposition.

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