

## A Liouville Type Theorem for Poly-harmonic System with Dirichlet Boundary Conditions in a Half Space \*

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### Abstract

In this paper, we consider the following poly-harmonic system with Dirichlet boundary conditions in a half space  $\mathbb{R}_+^n$ :

$$\begin{cases} (-\Delta)^m u(x) = u^{\alpha_1}(x)v^{\beta_1}(x), & x \in \mathbb{R}_+^n, \\ (-\Delta)^m v(x) = u^{\alpha_2}(x)v^{\beta_2}(x), & x \in \mathbb{R}_+^n, \\ u = \frac{\partial u}{\partial x_n} = \frac{\partial^2 u}{\partial x_n^2} = \dots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0, & x \in \partial\mathbb{R}_+^n, \\ v = \frac{\partial v}{\partial x_n} = \frac{\partial^2 v}{\partial x_n^2} = \dots = \frac{\partial^{m-1} v}{\partial x_n^{m-1}} = 0, & x \in \partial\mathbb{R}_+^n, \end{cases} \quad (0.1)$$

where  $\alpha_i + \beta_i = \frac{n+2m}{n-2m} > 2$ ,  $\alpha_i, \beta_i \geq 1$  for  $i = 1, 2$ . First, we show that, under some mild growth conditions, (0.1) is equivalent to the IE system

$$\begin{cases} u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \\ v(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_2}(y) v^{\beta_2}(y) dy, \end{cases} \quad (0.2)$$

where

$$G_\infty^+(x, y) := \frac{c_n}{|x - y|^{n-2m}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz$$

is the Green's function in  $\mathbb{R}_+^n$  with the same Dirichlet boundary conditions. Then, inspired by the work [12] of Y. Fang and W. Chen on the Dirichlet problem for  $(-\Delta)^m u = u^p$  in  $\mathbb{R}_+^n$ , we use method of moving planes in integral forms to prove the nonexistence of nontrivial nonnegative solutions for IE system (0.2), and as a consequence, we derive the nonexistence of nontrivial nonnegative classical solutions for problem (0.1).

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# 1 Introduction

In this paper, we investigate the following system for poly-harmonic operators with Dirichlet boundary conditions in the half space  $\mathbb{R}_+^n$ :

$$\begin{cases} (-\Delta)^m u(x) = u^{\alpha_1}(x)v^{\beta_1}(x), & x \in \mathbb{R}_+^n, \\ (-\Delta)^m v(x) = u^{\alpha_2}(x)v^{\beta_2}(x), & x \in \mathbb{R}_+^n, \\ u = \frac{\partial u}{\partial x_n} = \frac{\partial^2 u}{\partial x_n^2} = \dots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0, & x \in \partial\mathbb{R}_+^n, \\ v = \frac{\partial v}{\partial x_n} = \frac{\partial^2 v}{\partial x_n^2} = \dots = \frac{\partial^{m-1} v}{\partial x_n^{m-1}} = 0, & x \in \partial\mathbb{R}_+^n, \end{cases} \tag{1.1}$$

where  $m$  is an arbitrary positive integer,  $2m < n < 6m$ ,  $\alpha_i, \beta_i \geq 1$  for  $i = 1, 2$  and  $\mathbb{R}_+^n$  is the  $n$ -dimensional upper half Euclidean space defined by

$$\mathbb{R}_+^n := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

In particular, if we set  $u = v$  and  $\alpha_i + \beta_i = q \in (1, \frac{n+2m}{n-2m}]$  for  $i = 1, 2$ , the system (1.1) will degenerate into a Dirichlet problem of one single poly-harmonic equation

$$\begin{cases} (-\Delta)^m u(x) = u^q(x), & x \in \mathbb{R}_+^n, \\ u = \frac{\partial u}{\partial x_n} = \dots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0, & x \in \partial\mathbb{R}_+^n. \end{cases} \tag{1.2}$$

It is well known that Liouville type theorems (nonexistence of nontrivial nonnegative solutions) are very important in deriving a priori estimates for the solutions to the corresponding family of equations or systems either on domains in Euclidean spaces or on Riemannian manifolds with boundaries. The Liouville type theorems for PDE or IE problems associated with higher order or fractional order Laplacian similar to (1.2) have been extensively studied by many authors (see [1, 2, 7, 12, 13, 14, 17, 23, 24, 25, 28, 30] and the references therein). When  $m = 1$ , B. Gidas and J. Spruck [17] proved the nonexistence of positive solutions for (1.2) in the subcritical cases, their proof works for critical case  $q = \frac{n+2m}{n-2m}$  as well. In [24], under some global integrability conditions, G. Lu and J. Zhu first used the method of moving plane in integral form to derive axial symmetry and proved that the solutions of the equivalent integral equation of (1.2) with general nonlinearity  $f(u)$  instead of  $u^q$  are axially symmetric with respect to some line parallel to the  $x_n$ -axis and nondecreasing in the  $x_n$  direction, which further implies the nonexistence of solutions. G. Lu and J. Zhu [25] also established Liouville type theorems and decay estimates for viscosity solutions to a class of fully nonlinear elliptic equations or systems in half spaces. By using the Green representation formula in a half space, W. Reichel and T. Weth proved in [28] that there are no bounded classical positive solutions to problem (1.2), they also obtained the following equivalence between Dirichlet problem (1.2) and its corresponding IE problem under some boundedness assumptions.

**Theorem 1.1** ([28]) *Suppose that  $u \in C^{2m-1}(\overline{\mathbb{R}_+^n}) \cap W_{loc}^{2m,q}(\mathbb{R}_+^n)$ ,  $q > \frac{n}{2m}$  is a function with the following properties:*

- (i)  $u$  and all partial derivatives of  $u$  of order less than or equal to  $2m - 1$  are bounded,
- (ii)  $u$  satisfies the equation and Dirichlet boundary conditions in (1.2),
- (iii)  $(-\Delta)^m u \in L_{loc}^q(\mathbb{R}_+^n)$  is nonnegative in  $\mathbb{R}_+^n$ .

Then

$$u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y)(-\Delta)^m u(y) dy, \quad x \in \mathbb{R}_+^n,$$

where Green's function in  $\mathbb{R}_+^n$  with the same Dirichlet boundary conditions is given by

$$G_\infty^+(x, y) := \frac{c_n}{|x - y|^{n-2m}} \int_0^{\frac{4xy_n}{|x-y|^2}} \frac{z^{m-1}}{(z + 1)^{\frac{n}{2}}} dz.$$

In [29], W. Reichel and T. Weth were able to remove the boundedness assumptions on  $u$  in the subcritical case by using a doubling lemma.

In [12], under some weaker growth conditions, Y. Fang and W. Chen proved the Liouville type theorems for Dirichlet problem (1.1) and its equivalent IE problem in both subcritical and critical cases. One of the ingredients in their proof is that, in Theorem 1 of [12], they were able to remove Reichel and Weth's boundedness assumptions on  $u$  and all its partial derivatives up to the order  $2m - 1$  and obtain the same results on equivalence as W. Reichel and T. Weth [28] under much weaker assumptions, which require only the  $(m - 1)$ -th partial derivatives of  $u$  grow slower than the linear way, more precisely, for multi-indices  $|\alpha| = m - 1, m \geq 2$ ,

$$|D^\alpha u| = O(|y|^a), \quad \text{for large } |y|, \quad \text{and for some } 0 < a < 1. \tag{1.3}$$

For more related results involving the quantitative and qualitative properties of solutions for PDE or IE problems associated with higher order or fractional order Laplacian, we refer to the works, e.g., [4, 5, 7, 9, 10, 15, 16, 17, 18, 20, 21, 23, 24, 26, 29] and the references therein.

One can observe that a modification of the proof of Theorem 1 in [12] shows that the following similar results for corresponding IE and PDE systems (0.2), (1.1) also hold.

**Theorem 1.2** *Suppose that  $u \in C^{2m-1}(\overline{\mathbb{R}_+^n}) \cap W_{loc}^{2m,q}(\mathbb{R}_+^n)$  and  $v \in C^{2m-1}(\overline{\mathbb{R}_+^n}) \cap W_{loc}^{2m,q}(\mathbb{R}_+^n)$ ,  $q > \frac{n}{2m}$  are a pair of functions with the following properties:*

(i) For  $|\alpha| = m - 1, m \geq 2$ ,

$$|D^\alpha u| = O(|y|^a), \quad |D^\alpha v| = O(|y|^a), \quad \text{for large } |y|, \quad \text{and for some } 0 < a < 1, \tag{1.4}$$

(ii)  $(u, v)$  satisfy the equations and Dirichlet boundary conditions in (1.1),

(iii)  $(-\Delta)^m u \in L_{loc}^q(\mathbb{R}_+^n)$  and  $(-\Delta)^m v \in L_{loc}^q(\mathbb{R}_+^n)$  is non-negative in  $\mathbb{R}_+^n$ .

Then for every  $x \in \mathbb{R}_+^n$ ,  $(u, v)$  satisfy

$$\begin{cases} u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \\ v(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_2}(y) v^{\beta_2}(y) dy. \end{cases}$$

As an immediate consequence of Theorem 1.2, we have

**Corollary 1.1** *If  $(u, v)$  is a pair of nonnegative classical solutions of the Dirichlet problem (1.1) satisfying (1.4), then  $(u, v)$  satisfies IE system (0.2). The growth condition (1.4) is not needed when  $m = 1$ .*

We can also prove easily the following result.

**Theorem 1.3** *If  $(u, v) \in C^{2m}(\overline{\mathbb{R}_+^n}) \times C^{2m}(\overline{\mathbb{R}_+^n})$  is a pair of solutions of IE system (0.2), then  $(u, v)$  satisfies the Dirichlet problem (1.1).*

Since there is no corresponding maximal principles available for differential equations involving higher or fractional order Laplacian, we need first to exploit global properties of the integral equations and apply the method of moving planes in integral forms to prove the Liouville type theorems for the equivalent IE problems. Then, thanks to the equivalence between the IE system (0.2) and Dirichlet problem (1.1) obtained in the above Corollary 1.1 and Theorem 1.3, we can deduce the Liouville type theorems for (1.1) from the corresponding results of IE system (0.2).

In this paper, inspired by the work [12] of Y. Fang and W. Chen on the Dirichlet problem for  $(-\Delta)^m u = u^p$  in  $\mathbb{R}_+^n$ , we use method of moving planes in integral forms presented by W. Chen, C. Li and B. Ou in [9] to prove the nonexistence of nontrivial nonnegative solutions for IE system (0.2), as a consequence, we derive the Liouville type theorem for classical solutions of the Dirichlet problem (1.1). For more related results about the method of moving planes in integral forms, see [5, 6, 9, 8, 20, 24, 32] and the references therein.

Our main results are the following Theorem 1.4 and Corollary 1.2.

**Theorem 1.4** For  $\alpha_i + \beta_i = \frac{n+2m}{n-2m} > 2$ ,  $\alpha_i, \beta_i \geq 1$ ,  $i = 1, 2$ , and  $p = \frac{2n}{n-2m}$ , if  $(u, v)$  satisfying  $u \in L_{loc}^p(\mathbb{R}_+^n)$  and  $v \in L_{loc}^p(\mathbb{R}_+^n)$  is a pair of non-negative solutions of IE system (0.2), then  $(u, v) \equiv (0, 0)$ .

Combining Theorem 1.4 with Corollary 1.1 and Theorem 1.3, we conclude the following corollary.

**Corollary 1.2** For  $\alpha_i + \beta_i = \frac{n+2m}{n-2m} > 2$ ,  $\alpha_i, \beta_i \geq 1$ ,  $i = 1, 2$ , if  $(u, v)$  is a pair of non-negative classical solutions of the Dirichlet problem (1.1) satisfying (1.4), then  $(u, v) \equiv (0, 0)$ . For  $m = 1$ , the growth condition (1.4) is not needed for the conclusion to be valid.

For more related results on PDE or IE systems, see [3, 6, 7, 8, 11, 19, 22, 25, 27, 31, 32, 33] and the references therein.

This paper is organized as follows. In Section 2, We prove Theorem 1.3, combining with Corollary 1.1, we derive the equivalence between the PDE system (1.1) and the IE system (0.2). In Section 3, we will use the method of moving planes in integral forms and Kelvin transforms to prove Theorem 1.4, and thus obtain the Liouville type theorem for PDE system (1.1).

## 2 Equivalence between PDE system and IE system

*Proof.* (of the Theorem 1.3). Since

$$\begin{cases} (-\Delta)^m G_\infty^+(x, y) = \delta(x - y) & \text{in } \mathbb{R}_+^n, \\ G_\infty^+ = \frac{\partial G_\infty^+}{\partial x_n} = \dots = \frac{\partial^{m-1} G_\infty^+}{\partial x_n^{m-1}} = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

by (0.2), we have

$$\begin{aligned} (-\Delta)^m u(x) &= \int_{\mathbb{R}_+^n} (-\Delta)^m G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &= \int_{\mathbb{R}_+^n} \delta(x - y) u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &= u^{\alpha_1}(x) v^{\beta_1}(x) \end{aligned}$$

and

$$u = \frac{\partial u}{\partial x_n} = \dots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0 \quad \text{on } \partial\mathbb{R}_+^n.$$

Similarly, we derive from (0.2) and the above properties of Green's function  $G_\infty^+$  that

$$(-\Delta)^m v(x) = u^{\alpha_2}(x)v^{\beta_2}(x),$$

and

$$v = \frac{\partial v}{\partial x_n} = \dots = \frac{\partial^{m-1} v}{\partial x_n^{m-1}} = 0 \quad \text{on } \partial\mathbb{R}_+^n.$$

This completes the proof of Theorem 1.3. ■

Combining Theorem 1.3 with Corollary 1.1, we obtain the equivalence between PDE system (1.1) and IE system (0.2).

### 3 Non-existence of the positive solutions of IE system

*Proof.* (of the Theorem 1.4). In this section, we will prove Theorem 1.4 by using the method of moving plane in integral forms. To this end, let  $\lambda$  be a positive real number and let the moving plane be

$$T_\lambda = \{x \in \mathbb{R}_+^n : x_n = \lambda\}.$$

We define

$$\Sigma_\lambda := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n : 0 < x_n < \lambda\},$$

and let

$$x^\lambda = (x_1, x_2, \dots, 2\lambda - x_n)$$

be the reflection of the point  $x = (x_1, x_2, \dots, x_n)$  about the plane  $T_\lambda$ , and

$$\Sigma_\lambda^c := \mathbb{R}_+^n \setminus \Sigma_\lambda, \quad \tilde{\Sigma}_\lambda := \{x^\lambda : x \in \Sigma_\lambda\},$$

$$u_\lambda(x) := u(x^\lambda), \quad v_\lambda(x) := v(x^\lambda).$$

To prove the Theorem 1.4, we need the following lemma about properties of the Green's function proved in [13] and [24] independently.

**Lemma 3.1** ([13, 24]) (i) For any  $x, y \in \Sigma_\lambda$ ,  $x \neq y$ , we have

$$G_\infty^+(x^\lambda, y^\lambda) > \max\{G_\infty^+(x^\lambda, y), G_\infty^+(x, y^\lambda)\}$$

and

$$G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y) > |G_\infty^+(x^\lambda, y) - G_\infty^+(x, y^\lambda)|.$$

(ii) For any  $x \in \Sigma_\lambda$ ,  $y \in \Sigma_\lambda^c$ , it holds

$$G_\infty^+(x^\lambda, y) > G_\infty^+(x, y).$$

**Lemma 3.2** For any  $x \in \Sigma_\lambda$ , it holds

$$u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][u^{\alpha_1}(y)v^{\beta_1}(y) - u_\lambda^{\alpha_1}(y)v_\lambda^{\beta_1}(y)]dy,$$

$$v(x) - v_\lambda(x) \leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][u^{\alpha_2}(y)v^{\beta_2}(y) - u_\lambda^{\alpha_2}(y)v_\lambda^{\beta_2}(y)]dy.$$

*Proof.* (of Lemma 3.2). Since

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy + \int_{\Sigma_\lambda} G_\infty^+(x, y^\lambda) u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y) dy \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \\ u(x^\lambda) &= \int_{\Sigma_\lambda} G_\infty^+(x^\lambda, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy + \int_{\Sigma_\lambda} G_\infty^+(x^\lambda, y^\lambda) u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y) dy \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} G_\infty^+(x^\lambda, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \end{aligned}$$

by Lemma 3.1, we have

$$\begin{aligned} u(x) - u_\lambda(x) &= \int_{\Sigma_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &\quad + \int_{\Sigma_\lambda} [G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda)] u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y) dy \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &\leq \int_{\Sigma_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &\quad + \int_{\Sigma_\lambda} [G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda)] u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y) dy \\ &\leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)] u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &\quad - \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)] u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y) dy \\ &= \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)] [u^{\alpha_1}(y) v^{\beta_1}(y) - u_\lambda^{\alpha_1}(y) v_\lambda^{\beta_1}(y)] dy. \end{aligned}$$

Similarly, we obtain

$$v(x) - v_\lambda(x) \leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)] [u^{\alpha_2}(y) v^{\beta_2}(y) - u_\lambda^{\alpha_2}(y) v_\lambda^{\beta_2}(y)] dy.$$

This ends the proof of Lemma 3.2. ■

Since we don't assume global integrability on the pair of solutions  $(u, v)$ , in order to apply the method of moving planes in integral forms, we need to properly use the Kelvin transforms. For  $z^0 \in \partial\mathbb{R}_+^n$ , let

$$\bar{u}(x) = \frac{1}{|x - z^0|^{n-2m}} u\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right), \quad \bar{v}(x) = \frac{1}{|x - z^0|^{n-2m}} v\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right)$$

be the Kelvin transforms of  $u$  and  $v$  centered at point  $z^0$ . We will consider two different possible cases and derive a contradiction in both cases.

Case 1. If there is a  $z^0 = (z_1^0, \dots, z_{n-1}^0, 0) \in \partial\mathbb{R}_+^n$  such that both  $\bar{u}$  and  $\bar{v}$  are not singular at  $z^0$ , then we can deduce

$$u(x) = O\left(\frac{1}{|x - z^0|^{n-2m}}\right), \quad v(x) = O\left(\frac{1}{|x - z^0|^{n-2m}}\right) \quad \text{for } |x| \text{ large.} \tag{3.1}$$

Since  $u \in L^p_{loc}(\mathbb{R}_+^n)$  and  $v \in L^p_{loc}(\mathbb{R}_+^n)$  and  $p = \frac{2n}{n-2m}$ , we infer from (3.1) that

$$\int_{\mathbb{R}_+^n} u^p(y)dy < \infty, \quad \int_{\mathbb{R}_+^n} v^p(y)dy < \infty. \tag{3.2}$$

In this case,  $u$  and  $v$  are globally integrable, we will move the planes in the direction of  $x_n$ -axis to show that  $u$  and  $v$  are monotone increasing in  $x_n$  and thus obtain a contradiction with (3.1). The proof consists of two steps.

Step 1. Define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda : u_\lambda(x) - u(x) < 0\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda : v_\lambda(x) - v(x) < 0\}.$$

For positive  $\lambda$  sufficiently small, we will show that the measure of  $\Sigma_\lambda^u$  and  $\Sigma_\lambda^v$  must be zero. Indeed, for any  $x \in \Sigma_\lambda^u$ , by the mean value theorem and Lemma 3.2, we obtain

$$\begin{aligned} u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][u^{\alpha_1}(y)v^{\beta_1}(y) - u_\lambda^{\alpha_1}(y)v_\lambda^{\beta_1}(y)]dy \\ &\leq \int_{\Sigma_\lambda^u} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]\{u^{\alpha_1}(y)[v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)] \\ &\qquad\qquad\qquad + v_\lambda^{\beta_1}(y)[u^{\alpha_1}(y) - u_\lambda^{\alpha_1}(y)]\}dy \\ &\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^u} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]\{u^{\alpha_1}(y)[v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)]\}dy \\ &= \int_{\Sigma_\lambda^u} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]v_\lambda^{\beta_1}(y)[u^{\alpha_1}(y) - u_\lambda^{\alpha_1}(y)]dy \\ &\quad + \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]u^{\alpha_1}(y)[v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)]dy \\ &\leq \int_{\Sigma_\lambda^u} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]v_\lambda^{\beta_1}(y)[u^{\alpha_1}(y) - u_\lambda^{\alpha_1}(y)]dy \\ &\quad + \int_{\Sigma_\lambda^v} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)]u^{\alpha_1}(y)[v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)]dy \\ &\leq \int_{\Sigma_\lambda^u} G_\infty^+(x^\lambda, y^\lambda)v_\lambda^{\beta_1}(y)[u^{\alpha_1}(y) - u_\lambda^{\alpha_1}(y)]dy \\ &\quad + \int_{\Sigma_\lambda^v} G_\infty^+(x^\lambda, y^\lambda)u^{\alpha_1}(y)[v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)]dy. \end{aligned} \tag{3.3}$$

For  $0 < 2m < n < 6m$ , we have

$$\begin{aligned} G_\infty^+(x^\lambda, y^\lambda) &= \frac{c_n}{|x^\lambda - y^\lambda|^{n-2m}} \int_0^{\frac{4(2\lambda - y_n)(2\lambda - y_n)}{|x^\lambda - y^\lambda|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz \\ &\leq \frac{c}{|x - y|^{n-2m}}. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\begin{aligned}
 u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda^u} \frac{c}{|x-y|^{n-2m}} v_\lambda^{\beta_1}(y) [u^{\alpha_1}(y) - u_\lambda^{\alpha_1}(y)] dy \\
 &\quad + \int_{\Sigma_\lambda^v} \frac{c}{|x-y|^{n-2m}} u^{\alpha_1}(y) [v^{\beta_1}(y) - v_\lambda^{\beta_1}(y)] dy \\
 &= \int_{\Sigma_\lambda^u} \frac{c\alpha_1}{|x-y|^{n-2m}} v_\lambda^{\beta_1}(y) \varphi_\lambda^{\alpha_1-1}(y) [u(y) - u_\lambda(y)] dy \\
 &\quad + \int_{\Sigma_\lambda^v} \frac{c\beta_1}{|x-y|^{n-2m}} u^{\alpha_1}(y) \psi_\lambda^{\beta_1-1}(y) [v(y) - v_\lambda(y)] dy \\
 &\leq \int_{\Sigma_\lambda^u} \frac{c\alpha_1}{|x-y|^{n-2m}} v_\lambda^{\beta_1}(y) u^{\alpha_1-1}(y) [u(y) - u_\lambda(y)] dy \\
 &\quad + \int_{\Sigma_\lambda^v} \frac{c\beta_1}{|x-y|^{n-2m}} u^{\alpha_1}(y) v^{\beta_1-1}(y) [v(y) - v_\lambda(y)] dy,
 \end{aligned} \tag{3.5}$$

where  $\varphi_\lambda(y)$  is valued between  $u_\lambda(y)$  and  $u(y)$ , and  $\psi_\lambda(y)$  is valued between  $v_\lambda(y)$  and  $v(y)$ , therefore on  $\Sigma_\lambda^u$  and  $\Sigma_\lambda^v$ , we have

$$0 \leq u_\lambda(y) \leq \varphi_\lambda(y) \leq u(y), \quad 0 \leq v_\lambda(y) \leq \psi_\lambda(y) \leq v(y).$$

Now, for  $\alpha_i + \beta_i = \frac{n+2m}{n-2m} > 2$ ,  $\alpha_i, \beta_i \geq 1, i = 1, 2$ , we apply Hardy-Littlewood-Sobolev and Hölder inequalities to (3.5) and obtain

$$\begin{aligned}
 \|u - u_\lambda\|_{L^p(\Sigma_\lambda^u)} &\leq C_1 \|u^{\alpha_1} v^{\beta_1-1} (v - v_\lambda)\|_{L^{\frac{np}{n+2mp}}(\Sigma_\lambda^v)} \\
 &\quad + C_2 \|v_\lambda^{\beta_1} u^{\alpha_1-1} (u - u_\lambda)\|_{L^{\frac{np}{n+2mp}}(\Sigma_\lambda^u)} \\
 &\leq C_1 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_1} \|v\|_{L^p(\Sigma_\lambda^v)}^{\beta_1-1} \|v - v_\lambda\|_{L^p(\Sigma_\lambda^v)} \\
 &\quad + C_2 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_1-1} \|v_\lambda\|_{L^p(\Sigma_\lambda^v)}^{\beta_1} \|u - u_\lambda\|_{L^p(\Sigma_\lambda^u)}.
 \end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned}
 \|v - v_\lambda\|_{L^p(\Sigma_\lambda^v)} &\leq C_3 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_2} \|v\|_{L^p(\Sigma_\lambda^v)}^{\beta_2-1} \|v - v_\lambda\|_{L^p(\Sigma_\lambda^v)} \\
 &\quad + C_4 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_2-1} \|v_\lambda\|_{L^p(\Sigma_\lambda^v)}^{\beta_2} \|u - u_\lambda\|_{L^p(\Sigma_\lambda^u)}.
 \end{aligned} \tag{3.7}$$

By (3.2), we can choose sufficiently small positive  $\lambda$  such that

$$\begin{aligned}
 C_1 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_1} \|v\|_{L^p(\Sigma_\lambda^v)}^{\beta_1-1} &\leq \frac{1}{4}, \quad C_2 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_1-1} \|v_\lambda\|_{L^p(\Sigma_\lambda^v)}^{\beta_1} \leq \frac{1}{4}, \\
 C_3 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_2} \|v\|_{L^p(\Sigma_\lambda^v)}^{\beta_2-1} &\leq \frac{1}{4}, \quad C_4 \|u\|_{L^p(\Sigma_\lambda^u)}^{\alpha_2-1} \|v_\lambda\|_{L^p(\Sigma_\lambda^v)}^{\beta_2} \leq \frac{1}{4}.
 \end{aligned} \tag{3.8}$$

By inequalities (3.6), (3.7) and (3.8), we obtain

$$\|u - u_\lambda\|_{L^p(\Sigma_\lambda^u)} = 0, \quad \|v - v_\lambda\|_{L^p(\Sigma_\lambda^v)} = 0,$$

and therefore  $\Sigma_\lambda^u$  and  $\Sigma_\lambda^v$  must be measure zero. Therefore, for positive  $\lambda$  sufficiently small, we must have

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x), \quad \forall x \in \Sigma_\lambda. \tag{3.9}$$

*Step 2.* Inequality (3.9) provides a starting point to move the plane  $T_\lambda = \{x \in \mathbb{R}_+^n : x_n = \lambda\}$ . Now we start from the neighborhood of  $x_n = 0$  and move the plane up as long as (3.9) holds.

Define

$$\lambda_0 = \sup\{\lambda : u_\mu(x) \geq u(x), v_\mu(x) \geq v(x), \mu \leq \lambda, \forall x \in \Sigma_\mu\}.$$

We will prove

$$\lambda_0 = +\infty. \tag{3.10}$$

Suppose on the contrary that  $\lambda_0 < \infty$ , we will show that  $u$  and  $v$  are symmetric about the plane  $T_{\lambda_0}$ , that is

$$u_{\lambda_0}(x) \equiv u(x), v_{\lambda_0}(x) \equiv v(x), \forall x \in \Sigma_{\lambda_0}. \tag{3.11}$$

Otherwise, on  $\Sigma_{\lambda_0}$ ,

$$u_{\lambda_0}(x) \geq u(x), v_{\lambda_0}(x) \geq v(x), \text{ but } u_{\lambda_0}(x) \not\equiv u(x) \text{ and } v_{\lambda_0}(x) \not\equiv v(x). \tag{3.12}$$

We show that the plane can be moved further upward. More precisely, there exists an  $\varepsilon > 0$  such that for any  $\lambda$  in  $[\lambda_0, \lambda_0 + \varepsilon)$ ,

$$u_\lambda(x) \geq u(x), v_\lambda(x) \geq v(x), \forall x \in \Sigma_\lambda. \tag{3.13}$$

Indeed, by the integrability conditions, we can choose  $\varepsilon$  sufficiently small so that for all  $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$ ,

$$\begin{aligned} C_1 \|u\|_{L^p(\Sigma_\lambda)}^{\alpha_1} \|v\|_{L^p(\Sigma_\lambda)}^{\beta_1-1} &\leq \frac{1}{4}, & C_2 \|u\|_{L^p(\Sigma_\lambda)}^{\alpha_1-1} \|v_\lambda\|_{L^p(\Sigma_\lambda)}^{\beta_1} &\leq \frac{1}{4}, \\ C_3 \|u\|_{L^p(\Sigma_\lambda)}^{\alpha_2} \|v\|_{L^p(\Sigma_\lambda)}^{\beta_2-1} &\leq \frac{1}{4}, & C_4 \|u\|_{L^p(\Sigma_\lambda)}^{\alpha_2-1} \|v_\lambda\|_{L^p(\Sigma_\lambda)}^{\beta_2} &\leq \frac{1}{4}. \end{aligned} \tag{3.14}$$

For the continuity of our work, let us postpone the proof of (3.14). Now together with (3.6) and (3.7), we arrive at

$$\|u - u_\lambda\|_{L^p(\Sigma_\lambda)} = 0, \quad \|v - v_\lambda\|_{L^p(\Sigma_\lambda)} = 0,$$

and therefore  $\Sigma_\lambda^u$  and  $\Sigma_\lambda^v$  must be measure zero. Hence, for  $\lambda > \lambda_0$  and sufficiently close to  $\lambda_0$  we have

$$u_\lambda(x) \geq u(x), v_\lambda(x) \geq v(x), \quad \forall x \in \Sigma_\lambda.$$

This contradicts with the definition of  $\lambda_0$ , thus (3.11) must hold.

By (3.11), we derive that the plane  $x_n = 2\lambda_0$  is the symmetric image of the boundary  $\partial\mathbb{R}_+^n$  with respect to the plane  $T_{\lambda_0}$ , and hence  $(u(x), v(x)) = (0, 0)$  when  $x$  is on the plane  $x_n = 2\lambda_0$ . This contradicts with our assumption  $u(x) > 0$  and  $v(x) > 0$ . Therefore, (3.10) must hold.

Now, we prove inequality (3.14). For any small  $\delta > 0$ , we can choose  $R$  large enough so that

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\alpha_1} \|v\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\beta_1-1} &< \delta, & \|u\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\alpha_1-1} \|v_\lambda\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\beta_1} &< \delta, \\ \|u\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\alpha_2} \|v\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\beta_2-1} &< \delta, & \|u\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\alpha_2-1} \|v_\lambda\|_{L^p(\mathbb{R}_+^n \setminus B_R(0))}^{\beta_2} &< \delta. \end{aligned} \tag{3.15}$$

We fix  $R$  and then show that the measure of  $\Sigma_\lambda^u \cap B_R(0)$  and  $\Sigma_\lambda^v \cap B_R(0)$  can be arbitrarily small as  $\lambda$  tends to  $\lambda_0$ . First, for any  $x \in \Sigma_{\lambda_0}$ , we have

$$u_{\lambda_0}(x) - u(x) > 0, \quad v_{\lambda_0}(x) - v(x) > 0. \tag{3.16}$$

Indeed, from Lemma 3.1 and the proof of Lemma 3.2, we have

$$\begin{aligned}
 u_{\lambda_0}(x) - u(x) &\geq \int_{\Sigma_{\lambda_0}} [G_{\infty}^+(x^{\lambda_0}, y^{\lambda_0}) - G_{\infty}^+(x, y^{\lambda_0})][u_{\lambda_0}^{\alpha_1}(y)v_{\lambda_0}^{\beta_1}(y) - u^{\alpha_1}(y)v^{\beta_1}(y)]dy \\
 &\quad + \int_{\Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}} [G_{\infty}^+(x^{\lambda_0}, y) - G_{\infty}^+(x, y)]u^{\alpha_1}(y)v^{\beta_1}(y)dy \\
 &\geq \int_{\Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}} [G_{\infty}^+(x^{\lambda_0}, y) - G_{\infty}^+(x, y)]u^{\alpha_1}(y)v^{\beta_1}(y)dy.
 \end{aligned} \tag{3.17}$$

Similarly,

$$v_{\lambda_0}(x) - v(x) \geq \int_{\Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}} [G_{\infty}^+(x^{\lambda_0}, y) - G_{\infty}^+(x, y)]u^{\alpha_2}(y)v^{\beta_2}(y)dy. \tag{3.18}$$

If we suppose that inequality (3.16) does not hold, then there exists some point  $x^0 \in \Sigma_{\lambda_0}$  such that

$$u_{\lambda_0}(x^0) = u(x^0) \text{ or } v_{\lambda_0}(x^0) = v(x^0).$$

Combining this with (3.17) and (3.18), for any  $y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}$ , we have

$$u^{\alpha_1}(y)v^{\beta_1}(y) = 0 \text{ or } u^{\alpha_2}(y)v^{\beta_2}(y) = 0.$$

Thus we obtain

$$u(y) = 0 \text{ or } v(y) = 0, \quad \forall y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}.$$

This is a contraction with our assumption that  $u > 0$  and  $v > 0$ . Therefore, (3.16) must hold.

For any  $\eta > 0$ , we define

$$E_{\eta} = \{x \in \Sigma_{\lambda_0} \cap B_R(0) : u_{\lambda_0}(x) - u(x) > \eta\}$$

and

$$F_{\eta} = \{\Sigma_{\lambda_0} \cap B_R(0)\} \setminus E_{\eta}.$$

Obviously, (3.16) yields that

$$\lim_{\eta \rightarrow 0} \mu(F_{\eta}) = 0.$$

For  $\lambda > \lambda_0$ , let

$$D_{\lambda} = (\Sigma_{\lambda} \setminus \Sigma_{\lambda_0}) \cap B_R(0).$$

Then it is easy to prove that

$$\{\Sigma_{\lambda}^u \cap B_R(0)\} \subset (\Sigma_{\lambda}^u \cap E_{\eta}) \cup F_{\eta} \cup D_{\lambda}. \tag{3.19}$$

Apparently, the measure of  $D_{\lambda}$  is small when  $\lambda$  is close to  $\lambda_0$ . We will show that the measure of  $\Sigma_{\lambda}^u \cap E_{\eta}$  can be arbitrarily small as  $\lambda$  tends to  $\lambda_0$ . Actually, for any  $x \in \Sigma_{\lambda}^u \cap E_{\eta}$ , we have

$$u_{\lambda}(x) - u(x) = u_{\lambda}(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) < 0.$$

Therefore,

$$u_{\lambda_0}(x) - u_{\lambda}(x) > u_{\lambda_0}(x) - u(x) > \eta.$$

So, we obtain

$$(\Sigma_{\lambda}^u \cap E_{\eta}) \subset G_{\eta} := \{x \in B_R(0) : u_{\lambda_0}(x) - u_{\lambda}(x) > \eta\}. \tag{3.20}$$

By the well-known Chebyshev inequality, we have

$$\mu(G_\eta) \leq \frac{1}{\eta^{p+1}} \int_{G_\eta} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx \leq \frac{1}{\eta^{p+1}} \int_{B_R(0)} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx. \quad (3.21)$$

For each fixed  $\eta$ , the right hand side of inequality (3.21) can be sufficiently small as  $\lambda$  close to  $\lambda_0$ , and our conclusion for  $\Sigma_\lambda^u \cap E_\eta$  follows from (3.20) and (3.21). Therefore, by (3.19) and (3.20), the measure of  $\Sigma_\lambda^u \cap B_R(0)$  can be made sufficiently small if  $\lambda$  is close enough to  $\lambda_0$ . Similarly, the measure of  $\Sigma_\lambda^v \cap B_R(0)$  can also be made sufficiently small if  $\lambda$  is close enough to  $\lambda_0$ . Combining this with (3.15), we derive (3.14).

Now, by (3.10),  $u(x)$  and  $v(x)$  are monotone increasing with respect to  $x_n$ . This contradicts with (3.1). So Case 1 is impossible, and what remains is the following.

*Case 2.* For all  $z^0 = (z_1^0, \dots, z_{n-1}^0, 0) \in \partial\mathbb{R}_+^n$ , at least one of  $\bar{u}$  and  $\bar{v}$  is singular at  $z^0$ . Without loss of generality, we fix an arbitrary point  $z^0$  and assume that both  $\bar{u}$  and  $\bar{v}$  are singular at  $z^0$ . We will prove that  $(\bar{u}, \bar{v})$  is rotationally symmetric about the line passing through  $z^0$  and parallel to the  $x_n$ -axis. By straightforward calculations, we have

$$\begin{aligned} \bar{u}(x) &= \frac{1}{|x - z^0|^{n-2m}} u\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right) \\ &= \frac{1}{|x - z^0|^{n-2m}} \int_{\mathbb{R}_+^n} G_\infty^+\left(\frac{x - z^0}{|x - z^0|^2} + z^0, y\right) u^{\alpha_1}(y) v^{\beta_1}(y) dy \\ &= \frac{1}{|x - z^0|^{n-2m}} \int_{\mathbb{R}_+^n} \frac{G_\infty^+\left(\frac{x - z^0}{|x - z^0|^2} + z^0, \frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right)}{|\tilde{y} - z^0|^{2n}} \\ &\quad \times u^{\alpha_1}\left(\frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right) v^{\beta_1}\left(\frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right) d\tilde{y} \\ &= \int_{\mathbb{R}_+^n} \frac{G_\infty^+\left(\frac{x - z^0}{|x - z^0|^2} + z^0, \frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right)}{|x - z^0|^{n-2m} |\tilde{y} - z^0|^{n-2m}} \left[ \frac{1}{|\tilde{y} - z^0|^{n-2m}} u\left(\frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right) \right]^{\alpha_1} \\ &\quad \cdot \left[ \frac{1}{|\tilde{y} - z^0|^{n-2m}} v\left(\frac{\tilde{y} - z^0}{|\tilde{y} - z^0|^2} + z^0\right) \right]^{\beta_1} d\tilde{y} \\ &= \int_{\mathbb{R}_+^n} G_\infty^+(x, \tilde{y}) \bar{u}^{\alpha_1}(\tilde{y}) \bar{v}^{\beta_1}(\tilde{y}) d\tilde{y}. \end{aligned}$$

Similarly,

$$\bar{v}(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) \bar{u}^{\alpha_2}(y) \bar{v}^{\beta_2}(y) dy.$$

Therefore, if  $(u, v)$  is a pair of solutions of

$$\begin{cases} u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \\ v(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_2}(y) v^{\beta_2}(y) dy, \end{cases} \quad (3.22)$$

then  $(\bar{u}, \bar{v})$  is also a pair of solutions of (3.22). Since  $u \in L_{loc}^p(\mathbb{R}_+^n)$  and  $v \in L_{loc}^p(\mathbb{R}_+^n)$ , for any domain  $\Omega$  that is a positive distance away from  $z_0$ , we have

$$\int_\Omega \bar{u}^p(x) dx < \infty, \quad \int_\Omega \bar{v}^p(x) dx < \infty. \quad (3.23)$$

Now, we apply method of moving planes to  $(\bar{u}, \bar{v})$ .

In this case, for a given real number  $\lambda$ , we redefine

$$\hat{\Sigma}_\lambda = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n : x_1 < \lambda\}, \quad \hat{T}_\lambda = \{x \in \mathbb{R}_+^n : x_1 = \lambda\},$$

and let

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n).$$

For  $x, y \in \hat{\Sigma}_\lambda, x \neq y$ , we have

$$G_\infty^+(x, y) = G_\infty^+(x^\lambda, y^\lambda) > G_\infty^+(x, y^\lambda) = G_\infty^+(x^\lambda, y). \tag{3.24}$$

Obviously, we have

$$\bar{u}(x) = \int_{\hat{\Sigma}_\lambda} G_\infty^+(x, y) \bar{u}^{\alpha_1}(y) \bar{v}^{\beta_1}(y) dy + \int_{\hat{\Sigma}_\lambda} G_\infty^+(x, y^\lambda) \bar{u}_\lambda^{\alpha_1}(y) \bar{v}_\lambda^{\beta_1}(y) dy, \tag{3.25}$$

$$\bar{u}(x^\lambda) = \int_{\hat{\Sigma}_\lambda} G_\infty^+(x^\lambda, y) \bar{u}^{\alpha_1}(y) \bar{v}^{\beta_1}(y) dy + \int_{\hat{\Sigma}_\lambda} G_\infty^+(x^\lambda, y^\lambda) \bar{u}_\lambda^{\alpha_1}(y) \bar{v}_\lambda^{\beta_1}(y) dy. \tag{3.26}$$

From (3.24), (3.25) and (3.26), one can easily derive that

$$\begin{aligned} \bar{u}(x) - \bar{u}(x^\lambda) &= \int_{\hat{\Sigma}_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] \bar{u}^{\alpha_1}(y) \bar{v}^{\beta_1}(y) dy \\ &\quad + \int_{\hat{\Sigma}_\lambda} [G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda)] \bar{u}_\lambda^{\alpha_1}(y) \bar{v}_\lambda^{\beta_1}(y) dy \\ &= \int_{\hat{\Sigma}_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] [\bar{u}^{\alpha_1}(y) \bar{v}^{\beta_1}(y) - \bar{u}_\lambda^{\alpha_1}(y) \bar{v}_\lambda^{\beta_1}(y)] dy. \end{aligned} \tag{3.27}$$

Similarly, we have

$$\bar{v}(x) - \bar{v}(x^\lambda) = \int_{\hat{\Sigma}_\lambda} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)] [\bar{u}^{\alpha_2}(y) \bar{v}^{\beta_2}(y) - \bar{u}_\lambda^{\alpha_2}(y) \bar{v}_\lambda^{\beta_2}(y)] dy.$$

The proof consists of two steps. First, we will move the plane  $\hat{T}_\lambda$  along the direction of  $x_1$ -axis until  $\lambda = z_1^0$ . Then, we can show that the solutions  $\bar{u}(x)$  and  $\bar{v}(x)$  are rotationally symmetric about the line passing through  $z^0$  and parallel to  $x_n$ -axis.

*Step 1.* In this step, in order to provide a starting point to move the plane  $\hat{T}_\lambda$  along the direction of  $x_1$ -axis, we will show that for  $\lambda$  sufficiently negative, and  $\varepsilon > 0$  arbitrarily small,

$$\bar{u}_\lambda(x) - \bar{u}(x) \geq 0, \quad \bar{v}_\lambda(x) - \bar{v}(x) \geq 0, \quad \forall x \in \hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda). \tag{3.28}$$

First, let us define the sets

$$\hat{\Sigma}_\lambda^u := \{x \in \hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda) : \bar{u}_\lambda(x) - \bar{u}(x) < 0\}$$

and

$$\hat{\Sigma}_\lambda^v := \{x \in \hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda) : \bar{v}_\lambda(x) - \bar{v}(x) < 0\}.$$

We only need to prove that for  $\lambda$  sufficiently negative, the sets  $\hat{\Sigma}_\lambda^u$  and  $\hat{\Sigma}_\lambda^v$  must be measure zero. Indeed, for  $x \in \hat{\Sigma}_\lambda^u$ , by (3.27) and the mean value theorem, we obtain

$$\begin{aligned} \bar{u}(x) - \bar{u}_\lambda(x) &= \int_{\hat{\Sigma}_\lambda^u} [G_\infty^+(x, y) - G_\infty^+(x^\lambda, y)][\bar{u}^{\alpha_1}(y)\bar{v}^{\beta_1}(y) - \bar{u}_\lambda^{\alpha_1}(y)\bar{v}_\lambda^{\beta_1}(y)]dy \\ &\leq \int_{\hat{\Sigma}_\lambda^u} G_\infty^+(x, y)\bar{v}_\lambda^{\beta_1}(y)[\bar{u}^{\alpha_1}(y) - \bar{u}_\lambda^{\alpha_1}(y)]dy \\ &\quad + \int_{\hat{\Sigma}_\lambda^v} G_\infty^+(x, y)\bar{u}^{\alpha_1}(y)[\bar{v}^{\beta_1}(y) - \bar{v}_\lambda^{\beta_1}(y)]dy \\ &\leq \int_{\hat{\Sigma}_\lambda^u} \frac{c\alpha_1}{|x - y|^{n-2m}} \bar{v}_\lambda^{\beta_1}(y)\bar{u}^{\alpha_1-1}(y)[\bar{u}(y) - \bar{u}_\lambda(y)]dy \\ &\quad + \int_{\hat{\Sigma}_\lambda^v} \frac{c\beta_1}{|x - y|^{n-2m}} \bar{u}^{\alpha_1}(y)\bar{v}^{\beta_1-1}(y)[\bar{v}(y) - \bar{v}_\lambda(y)]dy. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \bar{v}(x) - \bar{v}_\lambda(x) &\leq \int_{\hat{\Sigma}_\lambda^u} \frac{c\alpha_2}{|x - y|^{n-2m}} \bar{v}^{\beta_2}(y)\bar{u}^{\alpha_2-1}(y)[\bar{u}(y) - \bar{u}_\lambda(y)]dy \\ &\quad + \int_{\hat{\Sigma}_\lambda^v} \frac{c\beta_2}{|x - y|^{n-2m}} \bar{u}^{\alpha_2}(y)\bar{v}^{\beta_2-1}(y)[\bar{v}(y) - \bar{v}_\lambda(y)]dy. \end{aligned}$$

We apply the Hardy-Littlewood-Sobolev inequality and Hölder inequality to the above two inequalities and get

$$\begin{aligned} \|\bar{u} - \bar{u}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} &\leq C_1 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\alpha_1} \|\bar{v}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_1-1} \|\bar{v} - \bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} \\ &\quad + C_2 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\alpha_1-1} \|\bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\beta_1} \|\bar{u} - \bar{u}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \|\bar{v} - \bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} &\leq C_3 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\alpha_2} \|\bar{v}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_2-1} \|\bar{v} - \bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} \\ &\quad + C_4 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\alpha_2-1} \|\bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_2} \|\bar{u} - \bar{u}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)}. \end{aligned} \tag{3.30}$$

By (3.23), we can choose  $N$  sufficiently large such that for  $\lambda \leq -N$ ,

$$\begin{aligned} C_1 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\alpha_1} \|\bar{v}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_1-1} &\leq \frac{1}{4}, \quad C_2 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\alpha_1-1} \|\bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\beta_1} \leq \frac{1}{4}, \\ C_3 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\alpha_2} \|\bar{v}\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_2-1} &\leq \frac{1}{4}, \quad C_4 \|\bar{u}\|_{L^p(\hat{\Sigma}_\lambda^u)}^{\alpha_2-1} \|\bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)}^{\beta_2} \leq \frac{1}{4}. \end{aligned} \tag{3.31}$$

Now inequalities (3.29), (3.30) and (3.31) imply that

$$\|\bar{u} - \bar{u}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} = 0, \quad \|\bar{v} - \bar{v}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} = 0,$$

and hence  $\hat{\Sigma}_\lambda^u$  and  $\hat{\Sigma}_\lambda^v$  must be measure zero.

*Step 2.* We now move the plane  $\hat{T}_\lambda$  continuously toward the right as long as inequality (3.28) holds to its limiting position. Define

$$\lambda_0 = \sup\{\lambda \leq z_1^0 : \bar{u}_\mu(x) \geq \bar{u}(x), \bar{v}_\mu(x) \geq \bar{v}(x), \mu \leq \lambda, \forall x \in \hat{\Sigma}_\mu\}.$$

We will prove that  $\lambda_0 = z_1^0$ . On the contrary, if we suppose that  $\lambda_0 < z_1^0$ . We will show that  $\bar{u}$  and  $\bar{v}$  are rotationally symmetric about  $\hat{T}_{\lambda_0}$ , that is,

$$\bar{u}_{\lambda_0}(x) \equiv \bar{u}(x), \quad \bar{v}_{\lambda_0}(x) \equiv \bar{v}(x), \quad \forall x \in \hat{\Sigma}_{\lambda_0} \setminus B_\varepsilon((z^0)^{\lambda_0}). \quad (3.32)$$

Otherwise, on  $\hat{\Sigma}_{\lambda_0} \setminus B_\varepsilon((z^0)^{\lambda_0})$ ,

$$\bar{u}_{\lambda_0}(x) \geq \bar{u}(x), \quad \bar{v}_{\lambda_0}(x) \geq \bar{v}(x), \quad \text{but } \bar{u}_{\lambda_0}(x) \not\equiv \bar{u}(x) \text{ and } \bar{v}_{\lambda_0}(x) \not\equiv \bar{v}(x).$$

We will show that the plane can be moved further to the right. More precisely, there exists an  $\delta > 0$  such that for any  $\lambda$  in  $[\lambda_0, \lambda_0 + \delta)$ ,

$$\bar{u}_\lambda(x) \geq \bar{u}(x), \quad \bar{v}_\lambda(x) \geq \bar{v}(x), \quad \forall x \in \hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda). \quad (3.33)$$

The proof is similar to *Step 2* in *Case 1*. We only need to use  $\hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda)$  instead of  $\Sigma_\lambda$  and  $\hat{\Sigma}_{\lambda_0} \setminus B_\varepsilon((z^0)^{\lambda_0})$  instead of  $\Sigma_{\lambda_0}$  therein. Obviously, (3.33) contradicts the definition of  $\lambda_0$ . Therefore, (3.32) must hold. That is, if  $\lambda_0 < z_1^0$ , for any  $\varepsilon > 0$ ,

$$\bar{u}_{\lambda_0}(x) \equiv \bar{u}(x), \quad \bar{v}_{\lambda_0}(x) \equiv \bar{v}(x), \quad \forall x \in \hat{\Sigma}_{\lambda_0} \setminus B_\varepsilon((z^0)^{\lambda_0}).$$

Since both  $\bar{u}$  and  $\bar{v}$  are singular at  $z^0$ ,  $\bar{u}$  and  $\bar{v}$  must also be singular at  $(z^0)^{\lambda_0}$ . This is impossible. Thus it is easy to see that  $\lambda_0 = z_1^0$ .

Now we consider two arbitrary points  $X^1$  and  $X^2$ , where  $X^i = (x'_i, x_n) \in \mathbb{R}^{n-1} \times [0, \infty)$ ,  $i = 1, 2$ . Let  $z^0$  be the projection of the midpoint  $X^0 = \frac{X^1 + X^2}{2}$  on  $\partial\mathbb{R}_+^n$ . Set  $Y^i = \frac{X^i - z^0}{|X^i - z^0|^2} + z^0$ ,  $i = 1, 2$ . From the above process of moving plane in  $x_1$  direction, notice that the direction of  $x_1$ -axis can be chosen arbitrarily, it is easy to see that  $\bar{u}(Y^1) = \bar{u}(Y^2)$ , and hence  $u(X^1) = u(X^2)$ . This implies that  $u(x)$  only depend on the  $x_n$ -variable. Similarly, we can deduce that  $v(x)$  only depend on the  $x_n$ -variable.

In the situation of *Case 2*, we aim to prove that  $(u, v) \equiv (0, 0)$ . For  $x = (x', x_n), y = (y', y_n) \in \mathbb{R}^{n-1} \times [0, +\infty)$ , we assume  $(u(x), v(x)) = (u(x_n), v(x_n))$  is a pair of solutions of

$$\begin{cases} u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy, \\ v(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^{\alpha_2}(y) v^{\beta_2}(y) dy, \end{cases} \quad (3.34)$$

where

$$G_\infty^+(x, y) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\frac{4xy}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz$$

is the Green's function in  $\mathbb{R}_+^n$ .

For each fixed  $x \in \mathbb{R}_+^n$ , we set  $|x_n - y_n|^2 =: h^2$  and  $|x' - y'|^2 =: r^2$ . We have the following estimate:

$$\begin{aligned}
 +\infty > u(x) &= u(x_n) \\
 &= \int_{\mathbb{R}_+^n} \frac{c_n}{|x - y|^{n-2m}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} dz u^{\alpha_1}(y) v^{\beta_1}(y) dy \\
 &\gtrsim \int_{\{y \in \mathbb{R}_+^n : |y' - x'| \geq \sqrt{4x_n y_n}\}} \frac{1}{|x - y|^{n-2m}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} z^{m-1} dz u^{\alpha_1}(y) v^{\beta_1}(y) dy \\
 &\gtrsim \int_{\{y \in \mathbb{R}_+^n : |y' - x'| \geq \sqrt{4x_n y_n}\}} \frac{y_n^m}{|x - y|^n} u^{\alpha_1}(y_n) v^{\beta_1}(y_n) dy \\
 &\gtrsim \int_{100x_n}^{\infty} u^{\alpha_1}(y_n) v^{\beta_1}(y_n) y_n^m \\
 &\quad \times \int_{\{y' \in \mathbb{R}^{n-1} : |y' - x'| \geq \sqrt{4x_n y_n}\}} \frac{1}{(|y' - x'|^2 + |y_n - x_n|^2)^{\frac{n}{2}}} dy' dy_n \tag{3.35} \\
 &\gtrsim \int_{100x_n}^{\infty} u^{\alpha_1}(y_n) v^{\beta_1}(y_n) y_n^m \int_{\sqrt{4x_n y_n}}^{\infty} \frac{r^{n-2}}{(r^2 + h^2)^{\frac{n}{2}}} dr dy_n \\
 &\gtrsim \int_{100x_n}^{\infty} \frac{u^{\alpha_1}(y_n) v^{\beta_1}(y_n) y_n^m}{|y_n - x_n|} \int_{\frac{\sqrt{4x_n y_n}}{|y_n - x_n|}}^{\infty} \frac{\gamma^{n-2}}{(1 + \gamma^2)^{\frac{n}{2}}} d\gamma dy_n \\
 &\gtrsim \int_{100x_n}^{\infty} \frac{u^{\alpha_1}(y_n) v^{\beta_1}(y_n) y_n^m}{|y_n - x_n|} \int_1^{\infty} \frac{\gamma^{n-2}}{(1 + \gamma^2)^{\frac{n}{2}}} d\gamma dy_n \\
 &\gtrsim \int_{100x_n}^{\infty} \frac{u^{\alpha_1}(y_n) v^{\beta_1}(y_n) y_n^m}{|x_n - y_n|} dy_n.
 \end{aligned}$$

It follows that there exists a sequence  $\{y_n^i\}$  such that

$$u^{\alpha_1}(y_n^i) v^{\beta_1}(y_n^i) (y_n^i)^m \rightarrow 0, \text{ as } y_n^i \rightarrow \infty.$$

Therefore, we must have

$$u(y_n^i) \rightarrow 0 \text{ or } v(y_n^i) \rightarrow 0, \text{ as } y_n^i \rightarrow \infty.$$

Without loss of generality, we may assume

$$u(y_n^i) \rightarrow 0, \text{ as } y_n^i \rightarrow \infty. \tag{3.36}$$

For simplicity, we set  $u(x) = u(x_n) = u(t)$  and  $v(x) = v(x_n) = v(t)$ . If one of  $u(x)$  and  $v(x)$  is zero, by (3.34), we can infer that both of them are zero. From now on, we may assume that  $u(x) \not\equiv 0$  and  $v(x) \not\equiv 0$ . Therefore, there exists a  $t_0 > 0$  such that  $u(t_0) > 0$  and  $v(t_0) > 0$ . By (3.34), we can deduce further that  $u(t) > 0$  and  $v(t) > 0$  for all  $t \in (0, \infty)$ . In the following, we will consider two different cases.

Case (a).  $m = 2k$  ( $k = 1, 2, \dots$ ) is an arbitrary even integer. From (3.34), we get

$$u^{(2m)}(t) = (-\Delta)^m u(x) = \int_{\mathbb{R}_+^n} (-\Delta)^m G_{\infty}^+(x, y) u^{\alpha_1}(y) v^{\beta_1}(y) dy = u^{\alpha_1}(x) v^{\beta_1}(x) > 0, \tag{3.37}$$

which implies

$$u^{(2m-1)}(t) \text{ is strictly monotone increasing.} \tag{3.38}$$

Now we claim that

$$u^{(2m-1)}(t) \leq 0. \tag{3.39}$$

If not, there exists a  $t_1 > 0$  such that  $u^{(2m-1)}(t_1) > 0$ . By (3.38), we have

$$u^{(2m-1)}(t) \geq u^{(2m-1)}(t_1) > 0 \text{ for } t \geq t_1 > 0.$$

By integrating several times, and let  $t \rightarrow \infty$ , we have  $u(t) \rightarrow \infty$ . This contradicts (3.36). Thus our claim (3.39) holds true. Now (3.39) implies

$$u^{(2m-2)}(t) \text{ is nonincreasing.} \tag{3.40}$$

Moreover, we will prove that

$$u^{(2m-2)}(t) \geq 0.$$

If not, there exists a  $t_2 > 0$ , such that  $u^{(2m-2)}(t_2) < 0$ , by (3.40), we have

$$u^{(2m-2)}(t) \leq u^{(2m-2)}(t_2) < 0 \text{ for } t \geq t_2 > 0.$$

By integrating several times, and let  $t \rightarrow \infty$ , we have  $u(t) \rightarrow -\infty$ . This contradicts  $u(t) > 0$  for  $t \in (0, \infty)$ . Continuing this way, we derive that

$$u(t) \text{ is nonincreasing.} \tag{3.41}$$

Since  $u(0) = 0$  and  $u$  is a non-negative solution, by (3.41), we have  $u \equiv 0$ . Combining this fact with (3.34), we obtain  $v \equiv 0$ .

*Case (b).* For any odd integer  $m = 2k - 1$  ( $k = 1, 2, \dots$ ), one can infer from (3.34) that  $u^{(2m)}(t) < 0$ , then by a similar process as above, we can obtain that

$$u(t) \text{ is nondecreasing.}$$

Now by (3.35), we have for each fixed point  $x \in \mathbb{R}_+^n$ ,

$$\begin{aligned} +\infty > u(x) &= u(x_n) \\ &\geq \int_{100x_n}^{\infty} \frac{u^{\alpha_1}(y_n)v^{\beta_1}(y_n)y_n^m}{|x_n - y_n|} dy_n \\ &\geq u^{\alpha_1}(100x_n) \int_{100x_n}^{\infty} \frac{v^{\beta_1}(y_n)y_n^m}{|x_n - y_n|} dy_n. \end{aligned} \tag{3.42}$$

From (3.42), we can infer that there exists a sequence  $\{y_n^j\}$  such that

$$v^{\beta_1}(y_n^j)(y_n^j)^m \rightarrow 0, \text{ as } y_n^j \rightarrow \infty.$$

Absolutely, we have

$$v^{\beta_1}(y_n^j) \rightarrow 0, \text{ as } y_n^j \rightarrow \infty. \tag{3.43}$$

From (3.34), we obtain

$$-v^{(2m)}(t) = (-\Delta)^m v(x) = \int_{\mathbb{R}_+^n} (-\Delta)^m G_{\infty}^+(x, y) u^{\alpha_2}(y) v^{\beta_2}(y) dy = u^{\alpha_2}(x) v^{\beta_2}(x) > 0. \tag{3.44}$$

Combining (3.43) with (3.44), one can deduce similarly as above that

$$v(t) \text{ is nondecreasing,} \tag{3.45}$$

from which and the estimate (3.42), we can deduce that, for each fixed point  $x \in \mathbb{R}_+^n$ ,

$$\begin{aligned} +\infty &> u(x) = u(x_n) \\ &\geq u^{\alpha_1}(100x_n) \int_{100x_n}^{\infty} \frac{v^{\beta_1}(y_n)y_n^m}{|x_n - y_n|} dy_n \\ &\geq u^{\alpha_1}(100x_n)v^{\beta_1}(100x_n) \int_{100x_n}^{\infty} \frac{y_n^m}{|x_n - y_n|} dy_n \\ &= +\infty. \end{aligned}$$

This is a contradiction. Therefore, we must have  $(u, v) \equiv (0, 0)$ . This completes the proof of Theorem 1.4. ■

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