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Elliptic Systems of Schrödinger Type in the Spirit of Benci-Fortunato

Dedicated to V.-D. Benci-Fortunato on his " 130th " birthday

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Abstract

We study the existence of solutions of elliptic systems of the type

$$u,\ \varphi \geq 0 : \quad \begin{cases} u \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)Du) + \varphi(x)|u|^{r-2}u \ = f(x) \geq 0, \\ \\ \varphi \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)D\varphi) = |u|^r, \end{cases}$$

and we examine how the structure of the system gives rise to a regularizing effect on the summability of the solutions; following the theory of elliptic (single) equations, the regularizing effect is unexpected.

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1 Introduction

1.1 Starting blocks

In the first page of [1][1], Vieri Benci and Dino[2] Fortunato write: "we are reduced to study the existence of real numbers ω and real functions u, ϕ satisfying the system

$$\begin{cases} -\frac{1}{2}\Delta u - \phi\, u = \omega\, u, \\ \Delta\phi = 4\pi\, u^2, \end{cases} \tag{1.1}$$

with the boundary and normalizing conditions

$$u(x) = 0, \quad \phi(x) = g \ \text{ on } \partial\Omega, \quad \|u\|_{L^2} = 1.\text{"}$$

Moreover, in section 3, they write: "Let $g = 0$ and define $\varphi = -\phi$. Then the above system reads

$$\begin{cases} -\frac{1}{2}\Delta u + \varphi(x)\, u = \omega\, u, & \text{in } \Omega; \\ -\Delta\varphi = 4\pi\, u^2, & \text{in } \Omega; \\ u = 0, \ \ \varphi = 0, & \text{on } \partial\Omega; \ \|u\|_{L^2} = 1.\text{"} \end{cases} \tag{1.2}$$

Our starting point is the above system; we study the related Dirichlet problem: the difference concerns only the presence of the source $f(x)$. We examine how the structure of the system gives rise to a regularizing effect on the summability of the solutions of the boundary value problem (1.3); following the theory of elliptic (single) equations, the regularizing effect is unexpected.

We consider the semilinear (the nonlinearity of the system does not involve the gradients) system

$$\begin{cases} u \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)Du) + a(x)\varphi(x)g(u) = f(x), \\ \varphi \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)D\varphi) = a(x)G(u), \end{cases} \tag{1.3}$$

where Ω is a bounded, open subset of $\boldsymbol{R}^N$, with $N > 2$, $M(x)$ is a symmetric measurable matrix such that, for $\alpha, \beta \in \boldsymbol{R}^+$,

$$\alpha|\xi|^2 \le M(x)\xi\xi, \quad |M(x)| \le \beta. \tag{1.4}$$

Moreover on the functions $a(x)$, $f(x)$ we assume that

$$0 \le a(x) \in L^{\frac{2N}{N+2}}(\Omega), \tag{1.5}$$

$$0 \le f(x) \in L^{\frac{2N}{N+2}}(\Omega). \tag{1.6}$$

Then we assume that the real functions $g(s)$ and $G(s)$ are continuous, increasing, positive and verifiying $g(0) = 0 = G(0)$ and

$$\text{there exist } \Sigma > 0, \ \lambda > 0 \ \text{ such that } \ s \ge \Sigma \ \text{ implies } \ G(s) \le \lambda\, g(s)s. \tag{1.7}$$

We point out that we prove that $a(x)\varphi(x)g(u) \in L^1(\Omega)$, $a(x)G(u) \in L^1(\Omega)$ without any assumption on the growth assumption on the real functions $g(t)$ and $G(t)$.

[1] paper dedicated to Jürgen Moser
[2] in english is Donato

1.2 Related papers

In the papers below it is possible to find developments of the paper [1] by V. Benci and D. Fortunato: [6], [8], [9], [10], [11], [12], [13], [14], [15], [17], [18], [7]. In general, they deal with problems in $\boldsymbol{R}^3$; always, $g(s) = |s|^2$ and the right hand side of the first equation it is not a given function (source) $f(x)$, but a nonlinear (with polynomial growth) function of u.

1.3 Setting

First of all, we recall some classic results on solutions of elliptic problems.

The positivity of the right hand side in the second equation in (1.3) implies the positivity of its solution, so that, in the first equation, the lower order term does not destroy the coercivity (see [5]) and all the summability or boundedness results on u still hold; of course, if u and φ exist.

1.4 Polynomial nonlinearities

Let ($s \geq 0$)

$$\begin{cases} g(s) = \lambda\, s^{r-1}, \quad \lambda > 0, \\ G(s) = s^r, \\ a(x) = 1. \end{cases} \tag{1.8}$$

As consequence of the above remarks, we make use of the results by Guido Stampacchia, our common "maestro", in [16] (see also [4], [2]) and we can say that if $f(x)$ belongs to $L^m(\Omega)$, then

- if $m > \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, which implies, for every $r > 0$, that $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;
- if $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, which implies that φ belongs to $W_0^{1,2}(\Omega)$, if $\frac{m^{**}}{r} \geq \frac{2N}{N+2}$; that is $m \geq \frac{2rN}{N+2+4r}$;
- if $1 < m < \frac{2N}{N+2}$, then $u \in W_0^{1,m^*}(\Omega)$, which implies that φ belongs to $W_0^{1,2}(\Omega)$, if $\frac{m^{**}}{r} \geq \frac{2N}{N+2}$; that is again $m \geq \frac{2rN}{N+2+4r}$.

Thus it is interesting **to find finite energy solution of the second equation if**

$$m \leq \frac{2rN}{N+2+4r}.$$

That is if

$$\frac{2N}{N+2} \leq m \leq \frac{2rN}{N+2+4r}, \qquad \frac{N+2}{N-2} \leq r. \tag{1.9}$$

Note that the inequality $r \geq \frac{N+2}{N-2}$ assures that the interval $[\frac{2N}{N+2}, \frac{2rN}{N+2+4r}]$ is not empty.

Moreover a solution (u, φ) of

$$\begin{cases} u \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)Du) + r\,\varphi(x)|u|^{r-2}u \ = f(x), \\ \varphi \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)D\varphi) = |u|^r \end{cases} \tag{1.10}$$

is a critical point of the indefinite functional

$$J(v,w) = \frac{1}{2}\int_\Omega M(x)DvDv - \frac{1}{2}\int_\Omega M(x)DwDw + \int_\Omega w|v|^r - \int_\Omega f(x)v$$

with v, $w \in W_0^{1,2}(\Omega)$. Note that the functional J is well defined in $W_0^{1,2}(\Omega)$ if

$$\frac{1}{2^*} + \frac{r}{2^*} \le 1;$$

that is if

$$r \le 2^* - 1 = \frac{N+2}{N-2}. \tag{1.11}$$

Thus our interest in the study of the (non variational) system (1.3) is motivated by the interest in the variational (and particular) case (1.10).

2 Regularizing effect on existence of finite energy solutions (second solution)

In this section we study the unexpected regularizing effect on existence of finite energy solutions of second solution, if we work in the framework of the Remark 1.4.

The first existence result is concerned with the case of a bounded datum.

Proposition 2.1 *Let $F \in L^\infty(\Omega)$. Then there exists a weak solution (z, φ) of the system*

$$\begin{cases} z \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)Dz) + a(x)\,\varphi(x)\,g(z) = F, \\ \varphi \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)D\varphi) = a(x)G(z). \end{cases} \tag{2.12}$$

Moreover, if $F \ge 0$, then $z \ge 0$; of course $\varphi \ge 0$.

Proof. Fix $\psi \in W_0^{1,2}(\Omega)$ and consider the Dirichlet problem

$$z \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)Dz) + a(x)|\psi(x)|g(z) = F.$$

The solution z exists and it is unique and bounded. Thus there exists a unique, positive solution Ψ of the Dirichlet problem

$$\Psi \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)D\Psi) = a(x)G(z).$$

We denote by $C_i > 0$ positive constants depending only on the data. Then (see [16]) there exists C_1, $C_2 > 0$ such that

$$\|z\|_{W_0^{1,2}(\Omega)} \le C_1\|F\|_{L^\infty(\Omega)}, \quad \|z\|_{L^\infty(\Omega)} \le C_2\|F\|_{L^\infty(\Omega)}$$

$$\|\Psi\|_{W_0^{1,2}(\Omega)} \leq C_{(\|z\|_{L^\infty(\Omega)},a)} \leq C_{F,a}.$$

Thus the map $S : \psi \to \Psi$ is well defined and the ball $B(0, C_F)$ is invariant for S. Moreover it is easy to prove that S is strongly continuous. Then the Schauder fixed point theorem yields the existence of a fixed point φ (which is a positive, bounded function).

Corollary 2.1 *Define*

$$f_n = \frac{f}{1 + \frac{1}{n}f}.$$

There exists a solution (u_n, φ_n) *of the system*

$$\begin{cases} 0 \leq u_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)Du_n) + a(x)\varphi_n(x)g(u_n) = f_n, \\ 0 \leq \varphi_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)D\varphi_n) = a(x)G(u_n), \end{cases} \tag{2.13}$$

Theorem 2.1 (Regularizing effect on φ) *Assume (1.4), (1.5), (1.6), (1.7). Then there exist* (u, φ) *weak solution of (1.3), with* $u \in W_0^{1,2}(\Omega)$, $\varphi \in W_0^{1,2}(\Omega)$, $a(x)\varphi(x)g(u) \in L^1(\Omega)$, $a(x)G(u) \in L^1(\Omega)$, *in the following sense*

$$\begin{cases} \displaystyle\int_\Omega M(x)DuDv + \int_\Omega a(x)\varphi(x)g(u)\, v = \int_\Omega f(x)v, \ \forall\ v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega M(x)D\varphi Dw = \int_\Omega a(x)G(u)w, \quad \forall\ w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{2.14}$$

Proof. Step 1 - Note that every u_n, φ_n is a positive function. The sequences $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega)$. Indeed, the use of u_n and $\frac{1}{2\lambda}\varphi_n$ as a test functions in (2.13) yields

$$\begin{cases} \alpha \displaystyle\int_\Omega |Du_n|^2 + \int_\Omega a(x)\varphi_n(x)g(u_n)u_n \leq \int_\Omega f u_n \leq C_3 \|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \left[\int_\Omega |Du_n|^2 \right]^{\frac{1}{2}} \\ \dfrac{\alpha}{2\lambda} \displaystyle\int_\Omega |D\varphi_n|^2 \leq \frac{1}{2\lambda} \int_\Omega a(x)G(u_n)\varphi_n \leq \frac{1}{2} \int_\Omega a(x)g(u_n)u_n\varphi_n + \frac{G(\Sigma)}{2\lambda} \displaystyle\int_\Omega a(x)\varphi_n. \end{cases} \tag{2.15}$$

Note that $\varphi_n \geq 0$, since the right hand side of the second equation is positive. Then we have

$$\int_\Omega |Du_n|^2 + \int_\Omega |D\varphi_n|^2 + \int_\Omega a(x)G(u_n)\varphi_n + \int_\Omega a(x)g(u_n)u_n\varphi_n \leq C_4. \tag{2.16}$$

Thus there exist $u \in W_0^{1,2}(\Omega)$, $\varphi \in W_0^{1,2}(\Omega)$ and two subsequences (not relabelled) such that

$$\begin{cases} u_n \text{ converges weakly in } W_0^{1,2}(\Omega) \text{ and a.e. to } u, \\ \varphi_n \text{ converges weakly in } W_0^{1,2}(\Omega), \text{ strongly in } L^2(\Omega), \text{ and a.e. to } \varphi. \end{cases} \tag{2.17}$$

Thanks to the weak convergence and the Fatou Lemma and (2.16) we have

$$\int_\Omega |Du|^2, \quad \int_\Omega |D\varphi|^2, \quad \int_\Omega a(x)G(u)\varphi, \quad \int_\Omega a(x)g(u)u\varphi \leq C_4. \tag{2.18}$$

Step 2 - Now we prove that

$$a(x)\varphi_n(x)g(u_n) \text{ strongly converges in } L^1(\Omega) \text{ to } a(x)\varphi(x)g(u). \tag{2.19}$$

We point out that the above convergence is related with some techniques of [5] and [2].

Indeed, since $\varphi_n(x)$ converges a.e. to $\varphi(x)$ and $u_n(x)$ converges a.e. to $u(x)$, we prove the $L^1(\Omega)$ convergence thanks to the Vitali theorem.

We use as a test function in the first equation

$$\begin{cases} 0, & \text{if } 0 \le u_n(x) < k; \\ \frac{u_n - k}{\epsilon}, & \text{if } k \le u_n(x) < k + \epsilon; \\ 1, & \text{if } u_n(x) \ge k + \epsilon; \end{cases}$$

and we have, dropping a positive term and taking the limit as $\epsilon \to 0$,

$$\int_{k \le u_n} a(x)\varphi_n(x)g(u_n) \le \int_{k \le u_n} f. \tag{2.20}$$

We fix $\sigma > 0$. Since $f \in L^1(\Omega)$ and the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$, there exists $k_\sigma \in \boldsymbol{N}$ such that for $k \ge k_\sigma$

$$\int_{k \le u_n} f \le \sigma.$$

Thus, for every measurable subset E, we have

$$\int_E a(x)\varphi_n(x)g(u_n) \le g(k) \int_{E \cap \{u_n \le k\}} a(x)\varphi_n(x) + \int_{k \le u_n} a(x)\varphi_n(x)g(u_n)$$

$$\le g(k)\Big[\int_E a(x)^{\frac{2N}{N+2}} \Big]^{\frac{N+2}{2N}} \Big[\int_\Omega \varphi_n(x)^{2^*} \Big]^{\frac{1}{2^*}} + \int_{k \le u_n} f,$$

$$\le C_5 g(k)\Big[\int_E a(x)^{\frac{2N}{N+2}} \Big]^{\frac{N+2}{2N}} + \sigma$$

which implies that (recall that $a(x) \in L^{\frac{2N}{N+2}}(\Omega)$)

$$\lim_{\text{meas}(E) \to 0} \int_E a(x)\varphi_n(x)g(u_n) \le \sigma, \quad \text{uniformly with respect to } n,$$

and then $a(x)\varphi_n(x)g(u_n)$ converges strongly in $L^1(\Omega)$ to $a(x)\varphi(x)g(u)$.

Step 3 - Now we prove that

$$\text{the sequence } \{a(x)G(u_n)\} \text{ converges in } L^1(\Omega) \text{ to } a(x)G(u). \tag{2.21}$$

The use of φ_n in the first equation and the use of u_n in the second equation of (2.13) give

$$\int_\Omega a(x)G(u_n)u_n + \int_\Omega \varphi_n^2 a(x)g(u_n) = \int_\Omega f_n\varphi_n,$$

which implies

$$\int_\Omega a(x)G(u_n)u_n \le \|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \|\varphi_n\|_{L^{2^*}(\Omega)} \le C_f. \tag{2.22}$$

Then

$$\int_E a(x)G(u_n) \le G(k)\int_E a(x) + \int_{k\le u_n} a(x)G(u_n)\frac{u_n}{k} \le G(k)\int_E a(x) + \frac{C_f}{k}$$

As a consequence of this estimate and the a.e. convergence of the sequence $\{u_n(x)\}$ we have (2.21).

Step 4 - Note that the nonlinearity of the system does not involve the gradients, so that we can pass to the limit in the principal parts thanks to the weak convergence of u_n and φ_n in the Sobolev space $W_0^{1,2}(\Omega)$. For the lower order terms we use the convergences (2.19), (2.21). Then we have, for every $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $w \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$,

$$\begin{cases} \displaystyle\int_\Omega M(x)Du_nDv + \int_\Omega a(x)\varphi_n(x)g(u_n)\,v = \int_\Omega f_n v, \\ \displaystyle\int_\Omega M(x)D\varphi_nDw = \int_\Omega a(x)G(u_n)w, \end{cases}$$

and then

$$\begin{cases} \displaystyle\int_\Omega M(x)DuDv + \int_\Omega a(x)\varphi(x)\,g(u)\,v = \int_\Omega fv, \\ \displaystyle\int_\Omega M(x)D\varphi Dw = \int_\Omega a(x)G(u)w. \end{cases}$$

Remark 2.1 We point out that we proved that $a(x)\varphi(x)g(u) \in L^1(\Omega)$, $a(x)G(u) \in L^1(\Omega)$ without any assumption on the growth assumption on the real functions $g(t)$ and $G(t)$.

Remark 2.2 (Test functions) In the above weak formulation of (2.14), we take $v = T_k(u)$ in the first equation and we have

$$\int_\Omega M(x)DuDT_k(u) + \int_\Omega a(x)\varphi(x)g(u)\,T_k(u) = \int_\Omega fT_k(u).$$

Then the Levi theorem (as $k \to \infty$) gets

$$\int_\Omega M(x)DuDu + \int_\Omega a(x)\varphi(x)g(u)u = \int_\Omega fu, \tag{2.23}$$

which implies that it is possible to use u as test function (despite his unboundedness) and that

$$a(x)\varphi(x)g(u)u \in L^1(\Omega). \tag{2.24}$$

Remark 2.3 In the above weak formulation of (2.14), we take $w = T_k(\varphi)$ in the second equation and we have

$$\int_\Omega M(x)D\varphi DT_k(\varphi) = \int_\Omega a(x)G(u)T_k(\varphi).$$

Then the Levi theorem, used two times as $k \to \infty$, gets

$$\int_\Omega M(x)D\varphi D\varphi = \int_\Omega a(x)G(u)\varphi(x), \tag{2.25}$$

which again implies $a(x)G(u)\varphi(x) \in L^1(\Omega)$ (compare with (2.24)).

Remark 2.4 We observe that, if we take $v = u_n$ in (2.13) and then we use the weak lower semicontinuity in $W_0^{1,2}(\Omega)$ and the Fatou Lemma then we have

$$\left| \begin{array}{l} \displaystyle\int_\Omega M(x)DuDu + \int_\Omega a(x)\varphi(x)g(u)u \\ \displaystyle\le \liminf_n \int_\Omega M(x)Du_nDu_n + \liminf_n \int_\Omega a(x)\varphi_n\, g(u_n)u_n \\ \\ \displaystyle\le \liminf_n \int_\Omega f_nu_n = \int_\Omega fu = \int_\Omega M(x)DuDu + \int_\Omega a(x)\varphi(x)g(u)u, \end{array} \right.$$

which implies

$$\left\{ \begin{array}{l} \displaystyle\lim_n \int_\Omega M(x)Du_nDu_n = \int_\Omega M(x)DuDu \\ \\ \displaystyle\lim_n \int_\Omega \varphi_n\, a(x)g(u_n)u_n \;= \int_\Omega \varphi\, a(x)g(u)u. \end{array} \right.$$

That is

$$\left\{ \begin{array}{l} \text{the sequence } \{u_n\} \text{ converges strongly in } W_0^{1,2}(\Omega) \text{ to } u; \\ \text{the sequence } \{a(x)\varphi_n\, g(u_n)u_n\} \text{ converges in } L^1(\Omega) \text{ to } a(x)\varphi(x)g(u)u. \end{array} \right.$$

This convergence, the Vitali theorem and the inequality

$$a(x)\varphi_nG(u_n) \le \lambda a(x)\varphi_n g(u_n)u_n + a(x)\varphi_nG(\Sigma)$$

imply that

$$\text{the sequence } \{a(x)\varphi_nG(u_n)\} \text{ converges in } L^1(\Omega) \text{ to } a(x)\varphi(x)G(u). \tag{2.26}$$

On the other hand, if we take $w = \varphi_n$ as a test function in the second equation of (2.13) we have

$$\begin{aligned} \lim_n \int_\Omega M(x)D\varphi_nD\varphi_n = \lim_n \int_\Omega \varphi_n\, a(x)G(u_n) = \int_\Omega \varphi\, a(x)G(u) \\ = \int_\Omega M(x)D\varphi D\varphi(x)\,. \end{aligned}$$

That is

$$\text{the sequence } \{\varphi_n\} \text{ converges strongly in } W_0^{1,2}(\Omega) \text{ to } \varphi.$$

Remark 2.5 Thus, in the case (1.8), if $\frac{m^{**}}{r} < \frac{2N}{N+2}$, that is $m < \frac{2rN}{N+2+4r}$, we are able to prove that the sequence $\{\varphi_n\}$ converges strongly in $W_0^{1,2}(\Omega)$ to φ even if the right hand side of the second equation does not converge strongly in the dual of $W_0^{1,2}(\Omega)$.

3 Regularizing effect on the summability of the first solution

Theorem 3.1 (Regularizing effect on the summability of u) *Assume*

$$a(x) = 1,$$

(1.4), (1.8), with

$$f \in L^m(\Omega), \quad 2 \le m \le \frac{(r-1)N}{2r}, \qquad \frac{N}{N-2} < r. \tag{3.27}$$

Then the solution u of the first equation belongs to $W_0^{1,2}(\Omega) \cap L^{rm}(\Omega)$.

Proof. Note that the inequality $r > \frac{N}{N-2}$ assures that the interval $[2, \frac{(r-1)N}{2r}]$ is not empty.

Let $B(t)$, $t \ge 0$, a C^1 real convex, increasing function, with $B(0) = 0$. In (2.13) we use as test functions

$$\begin{cases} \varphi_n(x)B'(u_n), \\ B(u_n) \end{cases}$$

and we have

$$\begin{cases} \displaystyle\int_\Omega M(x)Du_nD\varphi_n(x)B'(u_n) + \int_\Omega \varphi_n^2 g(u_n)B'(u_n) \le \int_\Omega f\varphi_n(x)B'(u_n), \\ \displaystyle\int_\Omega M(x)D\varphi_nDu_nB'(u_n) = \int_\Omega G(u_n)B(u_n). \end{cases} \tag{3.28}$$

Then

$$\int_\Omega G(u_n)B(u_n) + \int_\Omega \varphi_n^2(x)g(u_n)B'(u_n) \le \int_\Omega f\left[\varphi_n(x)B'(u_n)\right] \tag{3.29}$$

$$\le \int_{fB'(u_n) \le \frac{1}{2}\varphi_n g(u_n)B'(u_n)} f\varphi_n(x)B'(u_n) + \int_{fB'(u_n) > \frac{1}{2}\varphi_n g(u_n)B'(u_n)} f\varphi_n(x)B'(u_n)$$

$$\le \frac{1}{2}\int_{fB'(u_n) \le \frac{1}{2}\varphi_n g(u_n)B'(u_n)} \varphi_n^2(x)g(u_n)B'(u_n) + \int_{\varphi_n < \frac{2fB'(u_n)}{g(u_n)B'(u_n)}} 2f^2\frac{B'(u_n)}{g(u_n)}.$$

Then we have

$$\int_\Omega G(u_n)B(u_n) \le 2\int_\Omega f^2\frac{B'(u_n)}{g(u_n)}.$$

If $m = 2$, with the choice

$$B(s) = s^r$$

we have that

$$\int_\Omega G(u_n)B(u_n) = \int_\Omega (u_n)^{2r}$$

is bounded. If $m > 2$, with the choice

$$B(s) = s^{rm-r}$$

we have

$$\int_\Omega G(u_n)B(u_n) \le 2\int_\Omega f^2\frac{B'(u_n)}{g(u_n)} \le 2\|f\|^2_{L^m(\Omega)}\left[\int_\Omega\left(\frac{B'(u_n)}{g(u_n)}\right)^{\frac{m}{m-2}}\right]^{\frac{m-2}{m}}$$

and again the right hand side is bounded. Thus, for every $m \ge 2$, we have

$$\int_\Omega (u_n)^{rm} \le C_f. \tag{3.30}$$

Thus we proved the boundedness of the sequence in $L^{rm}(\Omega)$ of the sequence $\{u_n\}$. The summability result on u is then a consequence.

Remark 3.1 We are not able to prove the estimate (3.30), if $1 \le m < 2$. We point out that an estimate of the type

$$\left[\int_\Omega (u_n)^{rm}\right]^{\frac{1}{rm}} \le \tilde{C}_f, \quad \text{if } 1+\frac{1}{r} \le m < 2, \tag{3.31}$$

implies

$$\alpha\int_\Omega |Du_n|^2 + \int_\Omega \varphi_n G(u_n) \le \int_\Omega f(u_n) \le \tilde{C}_f\|f\|_{L^{\frac{rm}{rm-1}}(\Omega)}.$$

Note that $\frac{rm}{rm-1} \le m$.

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