Research Article

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Ground States for a Nonlinear Schrödinger System with Sublinear Coupling Terms

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Abstract: We study the existence of ground states for the coupled Schrödinger system

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \mu_i |u_i|^{2q-2} u_i + \sum\limits_{j \neq i} b_{ij} |u_j|^q |u_i|^{q-2} u_i, \\ u_i \in H^1(\mathbb{R}^n), \quad i = 1, \dots, d, \end{cases}$$

$n \geq 1$, for $\lambda_i, \mu_i > 0$, $b_{ij} = b_{ji} > 0$ (the so-called "symmetric attractive case") and $1 < q < n/(n-2)^+$. We prove the existence of a nonnegative ground state $(u_1^*, \dots, u_d^*)$ with u_i^* radially decreasing. Moreover, we show that if in addition $q < 2$, such ground states are positive in all dimensions and for all values of the parameters.

Keywords: Nontrivial Ground States, Coupled Nonlinear Schrödinger Systems, Nehari Manifold

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1 Introduction

In this paper we consider the system of d equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \mu_i |u_i|^{2q-2} u_i + \sum\limits_{j \neq i} b_{ij} |u_j|^q |u_i|^{q-2} u_i, \\ u_i \in H^1(\mathbb{R}^n), \quad i = 1, \dots, d \end{cases} \tag{1.1}$$

with $\lambda_i, \mu_i > 0$, and $b_{ij} = b_{ji} > 0$, which appears in several physical contexts, namely in nonlinear optics (see for instance [1] and the references therein). We also assume that

$$1 < q < \frac{n}{(n-2)^+} := \begin{cases} +\infty & \text{if } n = 1, 2, \\ \frac{n}{n-2} & \text{if } n \geq 3. \end{cases} \tag{1.2}$$

Observe that the system is variational, more precisely of gradient type, since solutions can be obtained as critical points of the C^1-functional

$$I_d : E := (H^1(\mathbb{R}^n))^d \to \mathbb{R}$$

defined by

$$I_d(\mathbf{u}) = I_d(u_1, \dots, u_d) := \frac{1}{2} \sum_{i=1}^d \|u_i\|_{\lambda_i}^2 - \frac{1}{2q} \sum_{i=1}^d \mu_i |u_i|_{2q}^{2q} - \frac{1}{q} \sum_{i,j=1,\, i<j}^d b_{ij} |u_i u_j|_q^q,$$

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where $|\cdot|_q$ denotes the standard $L^q(\mathbb{R}^n)$ norm, while

$$\|u_i\|_{\lambda_i}^2 := \int_{\mathbb{R}^n} (|\nabla u_i|^2 + \lambda_i u_i^2)\, dx, \quad i = 1, \dots, d.$$

We will focus on the existence of *ground state solutions* of (1.1), that is, solutions of the system that achieve the *ground state level*

$$c := \inf\{I_d(\mathbf{u}) : \mathbf{u} \neq \mathbf{0},\ I_d'(\mathbf{u}) = 0\}. \tag{1.3}$$

A very interesting question is whether, when c is achieved, the ground state $\mathbf{u}$ is *nontrivial*, meaning that all its components u_i are nonzero.

This problem has attracted a lot of attention in the last decade, specially in the particular case of $d = 2$ equations:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{2q-2} u_1 + b|u_2|^q |u_1|^{q-2} u_1, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{2q-2} u_2 + b|u_1|^q |u_2|^{q-2} u_2. \end{cases}$$

For $\mu_1 = \mu_2 = 1$, Maia, Montefusco and Pellacci proved in [18] that c is always achieved, while there exists a positive ground state (i.e., $u_1, u_2 > 0$ in $\mathbb{R}^n$) if $b \geq \Lambda$, for a certain $\Lambda > 0$ depending on λ_2/λ_1. The same type of result was proved by Ambrosetti and Colorado [2, 3] for $q = 2$, $n = 2, 3$, and by de Figueiredo and Lopes [11] for $n = 1$. On the other hand, for $q \geq 2$, there are regions where all nonnegative solutions must have a null component, as it was observed for instance by Bartsch and Wang [4], Sirakov [21] and Chen and Zou [7].

The optimal bounds for the existence of nontrivial ground states were found by Mandel [19] for every q as in (1.2). More precisely, in [19, Theorem 1] it is shown that there exists $\bar{b} := b(\lambda_2/\lambda_1, q, n)$ such that for $b < \bar{b}$ all ground states have a trivial component (we will call them *semitrivial* ground states), while for $b > \bar{b}$ all ground states are nontrivial. For $\mu_1 = \mu_2 = 1$ and $\lambda_2/\lambda_1 = \omega^2 \geq 1$, the threshold is given by the expression (see [19, (5)])

$$\bar{b} = \inf\Big\{ \frac{\hat{c}_0^{-2q}(\|u\|_1^2 + \|v\|_{w^2}^2)^q - |u|_{2q}^{2q} - |v|_{2q}^{2q}}{2|uv|_q^q} : u, v \in H^1(\mathbb{R}^n) \Big\},$$

where $\hat{c}_0 := \|u_0\|_1 |u_0|_{2q}^{-1}$ and u_0 is the unique positive radially decreasing solution of $-\Delta u_0 + u_0 = |u_0|^{2q-2} u_0$ in $\mathbb{R}^n$ (for the uniqueness result, see [13]). It is also shown that $\bar{b} = 0$ if $1 < q < 2$ (see [19, Lemmas 1 (i) and 2 (i)]).

Our aim is to generalize this last result for an arbitrary number of equations. In order to state our results, let us first introduce some notations.

We will study the minimization problem

$$\inf\{I_d(u) : u \in \mathcal{N}_d\},$$

where the so-called Nehari manifold $\mathcal{N}_d$ is defined by

$$\mathcal{N}_d := \{\mathbf{u} \in E : \mathbf{u} \neq 0,\ \nabla I_d(\mathbf{u}) \perp \mathbf{u}\},$$

that is, $\mathbf{u} \in \mathcal{N}_d$ if and only if $\mathbf{u} \neq 0$ and

$$\tau_d(\mathbf{u}) := \langle \nabla I_d(\mathbf{u}), \mathbf{u} \rangle_{L^2} = \sum_{i=1}^d \|u_i\|_{\lambda_i}^2 - \Big(\sum_{i=1}^d \mu_i |u_i|_{2q}^{2q} + 2 \sum_{j<i} b_{ij} |u_i u_j|_q^q \Big).$$

Under condition (1.2) it is classical to check that $\mathcal{N}_d$ is a manifold, and that minimizers on the Nehari manifold are ground state solutions.

When dealing with system (1.1) it is often necessary to treat the case $n = 1$ separately due to the lack of compactness of the injection $H_r^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$, $q > 2$, where $H_r^1(\mathbb{R})$ denotes the space of the radially symmetric functions of $H^1(\mathbb{R})$. This lack of compactness is, in a sense, a consequence of the inequality

$$|u(x)| \leq C|x|^{\frac{1-n}{2}} \|u\|_{H^1(\mathbb{R}^n)} \tag{1.4}$$

for $u \in H_r^1(\mathbb{R}^n)$. Indeed, (1.4) gives no decay in the case $n = 1$. However, if u is also radially decreasing, it is easy to establish that

$$|u(x)| \leq C|x|^{-\frac{n}{2}} \|u\|_{L^2(\mathbb{R}^n)},$$

which provides decay in all space dimensions, hence the compactness follows by applying the classical

Strauss' compactness lemma [24]. Hence, putting

$$H^1_{rd}(\mathbb{R}^n) = \{u \in H^1_r(\mathbb{R}^n) : u \text{ is radially decreasing}\},$$

we get the compactness of the injection $H^1_{rd}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for *all* $n \geq 1$ (see the appendix of [5] for more details), a fact that does not seem very well known. We will use this result to present a unified approach for the problem of the energy minimization of (1.1), valid in all space dimensions. In fact, by putting $E_{rd} = (H^1_{rd}(\mathbb{R}^n))^d$, the cone of symmetric radially decreasing nonnegative functions of E, we will prove the following result (see also [4, 17, 18]):

Proposition 1.1. *Let $n \geq 1$ and take q satisfying* (1.2). *Then there exists a minimizing sequence $(u_{1,k}, \dots, u_{d,k})$ in E_{rd} for the minimization problem* (1.3). *Furthermore, $(u_{1,k}, \dots, u_{d,k}) \to (u_1^*, \dots, u_d^*) \in E_{rd}$ as $k \to \infty$, strongly in E. In particular,*

$$I(u_1^*, \dots, u_d^*) = \min_{\mathcal{N}_d} I(u_1, \dots, u_d) = \min_{\mathcal{N}_d \cap E_{rd}} I(u_1, \dots, , u_d) = c.$$

Concerning the existence of ground states with nontrivial components, we will show our main result:

Theorem 1.2. *Let $n \geq 1$, $\lambda_i > 0$, $\mu_i > 0$ and $b_{ij} = b_{ji} > 0$. For $1 < q < \min\{2, 2/(n-2)^+\}$ system* (1.1) *admits a ground state solution $\mathbf{u} = (u_1, \dots, u_d) \in E_{rd}$ with $u_i > 0$ for all i. Moreover, all possible ground state solutions have nontrivial components.*

We recall that such theorem was shown by Mandel for systems with $d = 2$ equations, as a consequence of a more general result, namely the characterization of the optimal threshold $\bar{b}$ defined before. In general, extending results from 2 to 3 or more equations is not straightforward, as systems with $d \geq 3$ equations often present a more complex structure with respect to its $d = 2$ counterpart (see for instance the recent results in [10]). However, by simplifying Mandel's approach, we will be able to prove Theorem 1.2, arguing by induction in the number of equations. Roughly, assuming that a subsystem of (1.1) with $d-1$ equations has a certain ground state solution with nontrivial components, we will construct an element $(U_1, \dots, U_d) \in \mathcal{N}_d$ with lower energy I_d.

For $d \geq 3$ equations, the first results concerning the properties of ground states seem to be the papers by Sirakov [21] (check Theorem 4 (iv) therein) and Liu and Wang [17]. In the latter, a nontrivial ground state is proved to exist in the case $n = 2, 3$, $q = 2$, $\lambda_1 = \dots = \lambda_d$, $\mu_1 = \dots = \mu_d$ and for $b_{ij} = b$ sufficiently large (see also [6] for $d = 3$, or [22, Theorem 1.6 and Remark 3]).

Recently, in [10], the authors joint with S. Correia presented, for $q = 2$, optimal qualitative conditions under which the ground states are nontrivial or, conversely, semitrivial. Theorem 1.2 states that, for $1 < q < 2$, for all values of the parameters, the ground states are nontrivial. This corresponds to an important difference with respect to the case $q \geq 2$, where there are values of the parameters for which all ground states are semitrivial. In such a situation, it becomes an interesting question to study if there exist *least energy nontrivial solutions* of (1.1), that is, solutions minimizing the energy among the set of all nontrivial solutions. This has been done for $d = 2$ equations in [2, 7, 14, 21], and for $d \geq 2$ in [20, 22, 23], among others. For some recent results in this directions concerning a Schrödinger-KdV system, see also [8, 9]. For multiplicity results for (1.1), we refer to [16].

2 Proof of Proposition 1.1

We begin by observing that, for $(u_1, \dots, u_d) \in E$ with $(u_1, \dots, u_d) \neq (0, \dots, 0)$ and $\tau_d(u_1, \dots, u_d) \leq 0$, there exists $t \in]0, 1]$ such that $(tu_1, \dots, tu_d) \in \mathcal{N}_d$. Indeed, if $\tau_d(u_1, \dots, u_d) = 0$, we choose $t = 1$. If $\tau_d(u_1, \dots, u_d) < 0$ we simply notice that

$$\tau(tu_1, \dots, tu_d) = t^2\Big(\sum_{i=1}^d \|u_i\|^2_{\lambda_i} - t^{2q-2}\Big(\sum_{i=1}^d |u_i|^{2q}_{2q} + 2\sum_{i<j} b_{ij}|u_i u_j|^q_q\Big)\Big) := t^2 T_{\mathbf{u}}(t),$$

with $T_{\mathbf{u}}(0) > 0$ and $T_{\mathbf{u}}(1) < 0$.

Also, we notice that if $(u_1, \dots, u_d) \in \mathcal{N}_d$, then

$$I(u_1, \dots, u_d) = \Big(\frac{1}{2} - \frac{1}{2q}\Big) \sum_{i=1}^{d} \|u_i\|_{\lambda_i}^2 = \Big(\frac{1}{2} - \frac{1}{2q}\Big)\Big(\sum_{i=1}^{d} |u_i|_{2q}^{2q} + 2\sum_{i<j} b_{ij}|u_iu_j|_q^q\Big). \tag{2.1}$$

We now take a minimizing sequence $(u_{1,k}, \dots, u_{d,k}) \in \mathcal{N}_d$ for the problem

$$\inf\{I_d(u_1, \dots, u_d) : (u_1, \dots, u_d) \in \mathcal{N}_d\}.$$

From (2.1), it is clear that this infimum is nonnegative, hence $(u_{1,k}, \dots, u_{d,k})$ is a bounded sequence in E.

We put $u_{i,k}^*$ the decreasing radial rearrangements of $|u_{i,k}|$, $i = 1, \dots, d$. It is well known that this rearrangement preserves the L^p norm $(1 \le p \le +\infty)$. Furthermore, the Pólya–Szegö inequality

$$|\nabla f^*|_2 \le \big|\nabla|f|\big|_2$$

in addition with the inequality $|\nabla|f||_2 \le |\nabla f|_2$ (see [15]) shows that

$$\sum_{i=1}^{d} \|u_{i,k}^*\|_{\lambda_i}^2 \le \sum_{i=1}^{d} \|u_i\|_{\lambda_i}^2.$$

On the other hand, the Hardy–Littlewood inequality

$$\int |fg| \le \int f^* g^*$$

combined with the monotonicity of the map $\lambda \mapsto \lambda^q$ (see for instance [12] for details) yields $\|fg\|_q \le \|f^*g^*\|_q$ and, finally (as $b_{ij} > 0$),

$$\tau_d(u_1^*, \dots, u_d^*) \le \tau_d(u_1, \dots, u_n) = 0.$$

Next, let $t_k \in \,]0, 1]$ such that $(t_k u_{1,k}^*, \dots, t_k u_{d,k}^*) \in \mathcal{N}_d$. We obtain

$$I(t_k u_{1,k}^*, \dots, t_k u_{d,k}^*) = t_k^2\Big(\frac{1}{2} - \frac{1}{2q}\Big) \sum_{i=1}^{d} \|u_{i,k}^*\|_{\lambda_i}^2 \le \Big(\frac{1}{2} - \frac{1}{2q}\Big) \sum_{i=1}^{d} \|u_{i,k}\|_{\lambda_i}^2 = I(u_{1,k}, \dots, u_{d,k}).$$

This way, we obtain a minimizing sequence $(t_k u_{1,k}^*, \dots, t_k u_{d,k}^*)$ in E_{rd}, denoted again, in what follows, by $(u_{1,k}, \dots, u_{d,k})$. Since this sequence is bounded in E_{rd}, up to a subsequence, $u_{i,k} \rightharpoonup u_i^*$ in $H^1(\mathbb{R}^n)$ weak. Also, since the injection $E_{rd} \to L^{2q}(\mathbb{R}^n)$ is compact, up to a subsequence, $u_{i,k} \to u_i^*$ in $L^{2q}(\mathbb{R}^n)$ strong, for all $n \ge 1$. Moreover, $(u_1^*, \dots, u_d^*) \ne (0, \dots, 0)$, since (from the definition of $\mathcal{N}_d$ and by Sobolev and Cauchy–Schwarz inequalities)

$$\sum_{i=1}^{d} |u_{i,k}|_{2q}^2 \le C_1 \sum_{i=1}^{d} \|u_{i,k}\|_{\lambda_i}^2 \le C_2 \sum_{i=1}^{d} |u_{i,k}|_{2q}^{2q}.$$

Thus there exists $\delta > 0$, independent from k, such that $\sum_{i=1}^{d} |u_{i,k}|_{2q} \ge \delta$, and by the strong convergence also $\sum_{i=1}^{d} |u_i^*|_{2q} \ge \delta > 0$.

Since

$$\tau(u_1^*, \dots, u_d^*) \le \liminf \tau(u_{1,k}, \dots, u_{d,k}) = 0,$$

once again we can take $t \in \,]0, 1]$ such that $(tu_1^*, \dots, tu_d^*) \in \mathcal{N}_d$. Then,

$$\begin{aligned} \inf_{\mathcal{N}_d} I_d \le I(tu_1^*, \dots, tu_d^*) &= t^2\Big(\frac{1}{2} - \frac{1}{2q}\Big) \sum_{i=1}^{d} \|u_i^*\|_{\lambda_i}^2 \\ &\le \Big(\frac{1}{2} - \frac{1}{2q}\Big) \liminf \sum_{i=1}^{d} \|u_{i,k}\|_{\lambda_i}^2 = \liminf I(u_{1,k}, \dots, u_{d,k}) = \inf_{\mathcal{N}_d} I_d. \end{aligned}$$

This implies that $(tu_1^*, \dots, tu_d^*)$ is a minimizer. In particular, all inequalities above are in fact equalities, thus $t = 1$, $(u_1^*, \dots, u_d^*) \in \mathcal{N}_d$ and $u_{i,k} \to u_i^*$ in $H^1(\mathbb{R}^n)$ strong.

It is then clear that $(u_1^*, \dots, u_d^*)$ is a ground state solution, which concludes the proof of Proposition 1.1. Observe also that $u_i^* \ge 0$, and by the strong maximum principle either $u_i > 0$ or $u_i \equiv 0$.

3 Ground States with Nontrivial Components. Proof of Theorem 1.2

As stated in the introduction, the general result will be obtained by induction on the number of equations d. We begin by considering the case $d = 2$. In this case, the result stated in Theorem 1.2 was recently obtained by Mandel in [19] in the case $\mu_1 = \mu_2 = 1$. Here, we will cover this case by a different (and more direct) method, considering also arbitrary $\mu_1, \mu_2 > 0$. Furthermore, as stated previously, our method will also extend easily to more general systems of $d \geq 3$ equations.

Denote by c_i the energy level of the (unique) positive ground state u_i of

$$-\Delta u + \lambda_i u = \mu_i |u|^{2q-2} u.$$

Without loss of generality, we may assume that $c_1 \leq c_2$. Hence, in order to prove our result, it is sufficient to exhibit $(U_1, U_2) \in \mathcal{N}_2$, with $U_1, U_2 \neq 0$, such that $I_2(U_1, U_2) < I_2(u_1, 0) = c_1$.

For a fixed $w \in H^1(\mathbb{R}^n) \setminus \{0\}$ and for $\theta > 0$ that will be chosen later, we begin by computing $t > 0$ such that $(tu_1, t\theta w) \in \mathcal{N}_2$, that is,

$$\tau_2(tu_1, t\theta w) = t^2 \|u_1\|_{\lambda_1}^2 + t^2\theta^2 \|w\|_{\lambda_2}^2 - t^{2q}(\mu_1 |u_1|_{2q}^{2q} + \mu_2 \theta^{2q} |w|_{2q}^{2q} + 2b_{12}\theta^q |u_1 w|_q^q) = 0,$$

from where we obtain that

$$t^{2q-2} = \frac{\|u_1\|_{\lambda_1}^2 + \theta^2 \|w\|_{\lambda_2}^2}{\mu_1 |u_1|_{2q}^{2q} + \mu_2 \theta^{2q} |w|_{2q}^{2q} + 2b_{12}\theta^q |u_1 w|_q^q}.$$

Since $u_1 \in \mathcal{N}_1$, we have $\|u_1\|_{\lambda_1}^2 = \mu_1 |u_1|_{2q}^{2q}$, and we obtain

$$t^{2q-2} = \frac{1 + \theta^2 C_1}{1 + \mu_2 \theta^{2q} C_2 + 2b_{12}\theta^q C_3}, \tag{3.1}$$

where

$$C_1 = \frac{\|w\|_{\lambda_2}^2}{\|u_1\|_{\lambda_1}^2}, \quad C_2 = \frac{|w|_{2q}^{2q}}{\|u_1\|_{\lambda_1}^2}, \quad C_3 = \frac{|u_1 w|_q^q}{\|u_1\|_{\lambda_1}^2}.$$

Since $(tu_1, t\theta w) \in \mathcal{N}_2$, we have

$$I(tu, t\theta w) = \left(\frac{1}{2} - \frac{1}{2q}\right)(\|tu_1\|_{\lambda_1}^2 + \theta^2 \|tw\|_{\lambda_2}^2) = t^2 \left(\frac{1}{2} - \frac{1}{2q}\right)(1 + C_1\theta^2)\|u_1\|_{\lambda_1}^2,$$

and condition $I_2(tu_1, t\theta w) < I_2(u_1, 0)$ is equivalent to

$$t^2(1 + \theta^2 C_1) < 1,$$

that is, in view of (3.1),

$$\left(\frac{1 + \theta^2 C_1}{1 + \mu_2 \theta^{2q} C_2 + 2b_{12}\theta^q C_3}\right)^{\frac{1}{q-1}} (1 + C_1\theta^2) < 1$$

and

$$\frac{(1 + \theta^2 C_1)^q}{1 + \mu_2 \theta^{2q} C_2 + 2b_{12}\theta^q C_3} < 1.$$

Thus, we obtain

$$\frac{(1 + \theta^2 C_1)^q - 1 - \mu_2 \theta^{2q} C_2}{\theta^q} < 2b_{12} C_3.$$

By noticing that, for $1 < q < 2$,

$$\lim_{\theta \to 0^+} \frac{(1 + \theta^2 C_1)^q - 1}{\theta^q} = 0,$$

we conclude that this condition holds for small θ, which concludes the proof for $d = 2$.

We now consider system (1.1) with $d > 2$ equations. Given $I \subsetneq \{1, 2, \dots, d\}$ denote by c_I the ground state level of the system

$$-\Delta u_i + \lambda_i u_i = \mu_i |u_i|^{2q-2} u_i + \sum_{j \in I,\, j \neq i} b_{ij} |u_j|^q |u_i|^{q-2} u_i, \quad i \in I.$$

Let us now assume, by induction hypothesis, that there exists a ground state level c_I with $\#I = d-1$ and $c_I < c_J$ for all J with $\#J < d-1$. Without loss of generality, we assume that

$$c := c_{\{1,\dots,d-1\}} = \min\{c_I : \#I = d-1\},$$

where c is achieved by the nontrivial ground state $(u_1, \dots, u_{d-1}) \in \mathcal{N}_{d-1}$, solution of

$$-\Delta u_i + \lambda_i u_i = \mu_i |u_i|^{2q-2} u_i + \sum_{j=1,\, j \neq i}^{d-1} b_{ij} |u_j|^q |u_i|^{q-2} u_i, \quad i = 1, \dots, d-1.$$

Noticing that $I_d(u_1, \dots, u_{d-1}, 0) = I_{d-1}(u_1, \dots, u_{d-1})$, we will prove our assertion by exhibiting $(U_1, \dots, U_d) \in \mathcal{N}_d$, $U_i \neq 0$, such that $I_d(U_1, \dots, U_d) < I_d(u_1, \dots, u_{d-1}, 0) = c$, which guarantees that the energy level of $(U_1, \dots, U_d)$ is inferior to the energy level of any solution of (1.1) with trivial components.

In this regard, for fixed $w \in H^1(\mathbb{R}^n)$, $w \neq 0$, and $\theta > 0$, we choose $t > 0$ such that

$$(tu_1, \dots, tu_{d-1}, t\theta w) \in \mathcal{N}_d.$$

This condition is equivalent to $\tau_d(tu_1, \dots, tu_{d-1}, t\theta w) = 0$, that is,

$$t^2 \Big(\sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2 + \theta^2 \|w\|_{\lambda_d}^2 \Big) = t^{2q} \Big(\sum_{i=1}^{d-1} \mu_i |u_i|_{2q}^{2q} + \mu_d \theta^{2q} |w|_{2q}^{2q} + 2 \sum_{i,j=1,\, j<i}^{d-1} b_{ij} |u_i u_j|_q^q + 2 \sum_{i=1}^{d-1} b_{id} \theta^q |u_i w|_q^q \Big).$$

Since $(u_1, \dots, u_{d-1}) \in \mathcal{N}_{d-1}$, we have

$$\sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2 = \sum_{i=1}^{d-1} \mu_i |u_i|_{2q}^{2q} + 2 \sum_{i,j=1,\, j<i}^{d-1} b_{ij} |u_i u_j|_q^q,$$

which yields

$$t^{2q-2} = \frac{1 + \theta^2 C_1}{1 + \mu_d \theta^{2q} C_2 + 2 \sum_{i=1}^{d-1} b_{id} \theta^q D_i}, \tag{3.2}$$

where we have put

$$C_1 = \frac{\|w\|_{\lambda_d}^2}{\sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2}, \quad C_2 = \frac{|w|_{2q}^{2q}}{\sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2}, \quad D_i = \frac{|u_i w|_q^q}{\sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2}.$$

Now, observe that, since $(tu_1, \dots, tu_{d-1}, t\theta w) \in \mathcal{N}_d$, we have

$$I_d(tu_1, \dots, tu_{n-1}, t\theta w) = \Big(\frac{1}{2} - \frac{1}{2q} \Big) \Big(\sum_{i=1}^{d-1} \|tu_i\|_{\lambda_i}^2 + \theta^2 \|tw\|_{\lambda_d}^2 \Big) = t^2 \Big(\frac{1}{2} - \frac{1}{2q} \Big) (1 + C_1 \theta^2) \sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2.$$

Since

$$I_d(u_1, \dots, u_{d-1}, 0) = I_{d-1}(u_1, \dots, u_{d-1}) = \Big(\frac{1}{2} - \frac{1}{2q} \Big) \sum_{i=1}^{d-1} \|u_i\|_{\lambda_i}^2,$$

we obtain that the condition $I_d(U_1, \dots, U_d) < c$ is equivalent to $t^2(1 + C_1 \theta^2) < 1$, and, in view of (3.2), to

$$\frac{(1 + \theta^2 C_1)^q}{1 + \mu_d \theta^{2q} C_2 + 2 \sum_{i=1}^{d-1} b_{id} \theta^q D_i} < 1$$

or

$$\frac{(1 + \theta^2 C_1)^q - 1 - \mu_d \theta^{2q} C_2}{\theta^q} < 2 \sum_{i=1}^{d-1} b_{id} D_i.$$

which holds for θ small enough, and the proof is complete.

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