Advanced Nonlinear Studies **15** (2015), 951–961

Liouvillian First Integrals for Generalized Riccati Polynomial Differential Systems

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Received in revised form 17 November 2014
Communicated by Yiming Long

Abstract

We study the existence and non–existence of Liouvillian first integrals for the generalized Riccati polynomial differential systems of the form $x' = y$, $y' = a(x)y^2 + b(x)y + c(x)$, where $a(x)$, $b(x)$ and $c(x)$ are polynomials.

2010 Mathematics Subject Classification. Primary 34C05, 34A34, 34C14.

Key words. Liouvillian first integrals, algebraic invariant curves, Riccati polynomial differential equations.

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* The author acknowledges the support of a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014SGR568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

† The author acknowledges the support of Portuguese National Funds through FCT - Fundação para a Ciència e a Tecnologia within the projects PTDC/MAT/117106/2010 and PEst-OE/EEI/LA0009/2013 (CAMGSD)

1 Introduction and statement of the main results

A classical problem in the qualitative theory of planar differential equations depending on parameters is to characterize the existence or non–existence of first integrals in function of their parameters.

Let x and y be complex variables. We consider the polynomial differential system

$$x' = y, \qquad y' = a(x)y^2 + b(x)y + c(x), \tag{1.1}$$

where $a(x)$, $b(x)$ and $c(x)$ are polynomials in the variable x and the prime denotes derivative with respect to the time t that can be either real or complex. In fact, if $a(x)c(x) \not\equiv 0$ these systems are called *generalized Riccati differential systems*, if $a(x) \not\equiv 0$ and $c(x) \equiv 0$ they reduce to *linear differential systems*, and if $a(x) \equiv 0$ they are *generalized Liénard differential systems*. Of course, if in the differential system (1.1) we have $x' = 1$ instead of $x' = y$, then we have the *classical Riccati differential equation*, for more information about the classical Riccati differential equation see for instance [4] and the references quoted there.

Our goal is to study the Liouvillian first integrals of the generalized Riccati polynomial differential systems (1.1).

The vector field associated to system (1.1) is

$$X = y\frac{\partial}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial}{\partial y}.$$

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non–locally constant C^1 function $H\colon U \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of X is defined in U. Clearly H is a first integral of X on U if and only if $XH = 0$ in U.

We recall that a *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [5]. The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [5].

The following is the first main result of this paper.

Theorem 1.1 *The following holds for the generalized Riccati polynomial differential systems* (1.1)*:*

(a) Assume $b(x) \equiv 0$ *then*

$$H = \frac{y^2}{2}\exp\Big(-2\int a(x)\,dx\Big) - \int c(x)\exp\Big(-2\int a(u)\,du\Big)\,dx$$

is a Liouvillian first integral.

(b) Assume $b(x) \not\equiv 0$, $c(x) = \kappa a(x)$ *and* $b(x) = \kappa_1 a(x)$ *with* $\kappa, \kappa_1 \in \mathbb{C}$, *then*

(b.1) *if* $\kappa = \kappa_1^2/4$ *we have that*

$$H = \int a(x)\,dx - \frac{\kappa_1}{\kappa_1 + 2y} - \log(\kappa_1 + 2y)$$

is a Liouvillian first integral;

(b.2) *if* $\kappa \neq \kappa_1^2/4$ *we have that*

$$H = \int a(x)\,dx - \frac{1}{2}\log(y^2 + \kappa_1 y + \kappa) + \frac{\kappa_1}{\sqrt{4\kappa - \kappa_1^2}}\arctan\Big(\frac{\kappa_1 + 2y}{\sqrt{4\kappa - \kappa_1^2}}\Big)$$

is a Liouvillian first integral.

Theorem 1.1 follows by direct computations, i.e. under the assumptions of statements (a), (b.1) and (b.2) we have solved the corresponding Riccati differential system (1.1) and we have obtained the first integrals which are given in those statements. To be sure that we do not do any mistake in the computations of these first integrals, we have checked that the obtained first integrals H of Theorem 1.1 satisfy the equation

$$\frac{\partial H}{\partial x}y + \frac{\partial H}{\partial y}(a(x)y^2 + b(x)y + c(x)) = 0.$$

From now on we consider the case in which $b(x) \not\equiv 0$, $c(x) = \kappa(b(x) - \kappa a(x))$ with $\kappa \in \mathbb{C} \setminus \{0\}$ and $b(x)/a(x) \notin \mathbb{C}$.

Theorem 1.2 *The generalized Riccati polynomial differential systems* (1.1) *with* $b(x) \not\equiv 0$, $c(x) = \kappa(b(x) - \kappa a(x))$ *with* $\kappa \in \mathbb{C} \setminus \{0\}$ *and* $b(x)/a(x) \notin \mathbb{C}$ *have no Liouvillian first integrals.*

The proof of Theorem 1.2 is given in Section 3.

To finish this introduction we want to point out that this result is very different from the following one for the classical Riccati polynomial differential systems of the form

$$x' = 1, \quad y' = a(x)y^2 + b(x)y + c(x)$$

with $c(x) = \kappa(b(x) - \kappa a(x))$ with $\kappa \in \mathbb{C} \setminus \{0\}$. In this case it is easy to prove that this system is always Liouvillian integrable because it has the integrating factor of Darboux type

$$\exp\Big(\int (b(x) - 2\kappa a(x))\,dx\Big)(y + \kappa)^{-2}$$

(for more details see Theorem 2.2).

2 Preliminaries of the Darboux theory of integrability for system (1.1)

For studying the existence of Liouvillian first integrals we need to study the so-called Darboux polynomials and exponential factors of the generalized Riccati polynomial differential systems (1.1).

Let $h = h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $h = 0$ is an *invariant algebraic curve* of the vector field X associated to the Riccati polynomial differential system (1.1) if it satisfies

$$y\frac{\partial h}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial h}{\partial y} = Kh,$$

for some polynomial $K = K(x, y) \in \mathbb{C}[x, y]$. This polynomial is called *the cofactor* of $h = 0$ and has degree at most

$$n = \max\{2 + \deg a(x), 1 + \deg b(x), \deg c(x)\} - 1. \tag{2.2}$$

When $h = 0$ is an invariant algebraic curve we also say that h is a *Darboux polynomial* of the generalized Riccati polynomial differential system (1.1). Note that a *polynomial first integral* is a Darboux polynomial with zero cofactor.

An *exponential factor* E of system (1.1) is a function of the form $E = \exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y]$ satisfying $(g, h) = 1$ and

$$\frac{\partial E}{\partial x}y + \frac{\partial E}{\partial y}(a(x)y^2 + b(x)y + c(x)) = LE, \tag{2.3}$$

for some polynomial $L = L(x, y)$ of degree at most n given in (2.2), called the *cofactor* of E.

The existence of exponential factors $\exp(g/h)$ is due to the fact that the multiplicity of the invariant algebraic curve $h = 0$ is larger than 1, for more details see [2, 3] .

Proposition 2.1 *The following statements hold.*

- *(i) If* $E = \exp(g/h)$ *is an exponential factor for the polynomial system* (1.1) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic curve.*
- *(ii) Eventually* e^g *can be exponential factor, coming from the multiplicity of the infinite invariant straight line.*

For a geometrical meaning of the exponential factors and a proof of Proposition 2.1 see [2].

A non-constant function $R\colon U \to \mathbb{C}$ is an *integrating factor* of the polynomial vector field X on U, if one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \operatorname{div}(RP, RQ) = 0, \quad XR = -R\operatorname{div}(P, Q),$$

on U where $P = y$ and $Q = a(x)y^2 + b(x)y + c(x)$. As usual the *divergence* of the vector field X is given by

$$\operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The next result is proved in [3].

Theorem 2.1 *Suppose that the polynomial vector field X of degree m defined in $\mathbb{C}^2$ admits p invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, \dots, p$ and q exponential factors $E_j = \exp(g_j/h_j)$ with cofactors L_j, for $j = 1, \dots, q$. Then there exist $\lambda_i, \mu_j \in \mathbb{C}$, not all zero, such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}\,(P, Q),$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$$

is an integrating factor of the vector field X.

The proof of the following result is given in [1, 3, 5].

Theorem 2.2 *The polynomial Riccati differential system* (1.1) *has a Liouvillian first integral if and only if it has an integrating factor of Darboux type.*

3 Proof of Theorem 1.2

For proving Theorem 1.2 we first characterize the Darboux polynomials (either with zero or with non–zero cofactor) of the generalized Riccati polynomial differential systems (1.1).

We will do it in several steps. The first one is the following result which states that the generalized Riccati polynomial differential systems (1.1) has no polynomial first integrals, i.e. Darboux polynomials with zero cofactor.

Theorem 3.1 *The generalized Riccati polynomial differential systems* (1.1) *have no polynomial first integrals.*

Proof. We proceed by contradiction. Let H be a polynomial first integral of system (1.1), that is

$$y\frac{\partial H}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial H}{\partial y} = 0. \tag{3.4}$$

We write H as a polynomial in the variable y, i.e.

$$H(x, y) = \sum_{j=0}^{m} h_j(x)y^j, \quad \text{where } h_j(x) \text{ is a polynomial in the variable } x.$$

Without loss of generality we can assume that $h_m(x) \neq 0$. Computing the coefficient of degree $m + 1$ in the variable y in (3.4) we get that

$$h_m'(x) + m\,a(x)h_m(x) = 0, \quad \text{that is} \quad h_m(x) = C\exp\Big(-m\int a(x)\,dx\Big) \neq 0.$$

Since $h_m(x)$ is a polynomial, we must have $m = 0$ because $a(x) \not\equiv 0$. Then $H = h_0(x)$. In view of (3.4) we get that H satisfies

$$H'(x) = 0, \quad \text{that is} \quad H(x) \in \mathbb{C},$$

a contradiction with the fact that H is a polynomial first integral. This completes the proof.

The proof of the following proposition is well-known and can be found in [3].

Proposition 3.1 *We suppose that* $h \in \mathbb{C}[x, y]$ *and let* $h = h_1^{n_1} \cdots h_r^{n_r}$ *be its factorization in irreducible factors over* $\mathbb{C}[x, y]$. *Then for a polynomial system* (1.1) $h = 0$ *is an invariant algebraic curve with cofactor* K_h *if and only if* $h_i = 0$ *is an invariant algebraic curve for each* $i = 1, \ldots, r$ *with cofactor* K_{h_i}. *Moreover* $K_h = n_1 K_{h_1} + \cdots + n_r K_{h_r}$.

In view of Proposition 3.1 to study the Darboux polynomials with non–zero cofactor it is enough to study the irreducible ones.

Lemma 3.1 *Let* $h = h(x, y)$ *be an irreducible Darboux polynomial of the generalized Riccati polynomial differential system* (1.1) *with cofactor* $K \neq 0$. *Then* $K = mb(x) + n'(x) - n(x)a(x) + m\, a(x)y$ *with* m *a non-negative integer and* $n(x) \in \mathbb{C}[x]$.

Proof. The cofactor K of any irreducible Darboux polynomial of the generalized Riccati polynomial differential system (1.1), has degree at most n (see (2.2)). We write it as $K(x, y) = \sum_{j=0}^{n} K_j(x)y^j$, where $K_j = K_j(x)$ is a polynomial in the variable x and has at most degree $n - j$. Since h is a Darboux polynomial of system (1.1) with cofactor K it satisfies

$$y\frac{\partial h}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial h}{\partial y} = \left(\sum_{j=0}^{n} K_j(x)y^j\right) h. \tag{3.5}$$

We write h as a polynomial in the variable y, i.e. $h(x, y) = \sum_{j=0}^{m} h_j(x)y^j$, where each $h_j(x)$ is a polynomial in the variable x. Without loss of generality we can assume that $h_m(x) \neq 0$.

Assume $n \geq 2$. Computing the coefficient of y^{n+m} in (3.5) we get

$$0 = K_n(x)h_m(x) \quad \text{that is} \quad K_n(x) = 0.$$

So $n \in \{0, 1\}$ and consequently $K = K_0(x) + yK_1(x)$.

Now computing the coefficient of y^{m+1} in (3.5) we get

$$h_m'(x) + m\, a(x)h_m(x) = h_m(x)K_1(x),$$

that is $h_m'(x) + (m\, a(x) - K_1(x))h_m(x) = 0$. Hence,

$$h_m(x) = C \exp\Big(- \int (m\, a(x) - K_1(x))\, dx\Big) \quad \text{with } C \in \mathbb{C} \setminus \{0\}.$$

Using that $h_m(x) \neq 0$ and that it must be a polynomial, we have $K_1(x) = m\, a(x)$ with m a non–negative integer and $h_m(x) = C$. Now, computing the coefficient of y^m in (3.5) we get

$$h_{m-1}'(x) + (m - 1)a(x)h_{m-1}(x) + mb(x)h_m(x) = ma(x)h_{m-1}(x) + K_0(x)h_m(x).$$

Then since $h_m(x) = C$,

$$h_{m-1}'(x) - a(x)h_{m-1}(x) = C(K_0(x) - mb(x)).$$

Therefore we have a linear differential system. Solving it we get

$$h_{m-1}(x) = C_1 e^{\int a(x)\,dx} + C e^{\int a(x)\,dx} \int (K_0(x) - m\,b(x)) e^{-\int a(u)\,du}\,dx.$$

Since $h_{m-1}(x) \in \mathbb{C}[x]$, we deduce that $C_1 = 0$ and

$$n(x) = e^{\int a(x)\,dx} \int (K_0(x) - mb(x)) e^{-\int a(u)\,du}\,dx,$$

that is

$$K_0(x) - m\,b(x) = n'(x) - n(x)a(x) \quad \text{and} \quad h_{m-1}(x) = Cn(x).$$

This completes the proof.

Theorem 3.2 *The generalized Riccati polynomial differential systems* (1.1) *with* $b(x) \not\equiv 0$, $c(x) = \kappa(b(x) - \kappa a(x))$ *with* $\kappa \in \mathbb{C} \setminus \{0\}$ *and* $b(x)/a(x) \notin \mathbb{C}$ *has the unique irreducible Darboux polynomial* $y + \kappa$ *with non–zero cofactor* $K = a(x)y + b(x) - \kappa a(x)$.

Proof. We write the generalized Riccati polynomial differential system (1.1) with $c(x) = \kappa(b(x) - \kappa a(x))$ with $\kappa \in \mathbb{C} \setminus \{0\}$ as the differential system

$$x' = y, \quad y' = (y + \kappa)(a(x)y + b(x) - \kappa a(x)). \tag{3.6}$$

It is clear from (3.6) that $y + \kappa$ is an irreducible Darboux polynomial of degree one of system (3.6) with cofactor $K = a(x)y + b(x) - \kappa a(x)$. Now we will prove that this is the only one. Let $h = h(x, y)$ be an irreducible Darboux polynomial of system (3.6) with cofactor $K = mb(x) + n'(x) - n(x)a(x) + m\,a(x)y$ where $n(x) \in \mathbb{C}[x]$, see Lemma 3.1. If we denote by $\bar{h} = h(x, -\kappa)$ the restriction of h to $y = -\kappa$ we get that $\bar{h} \neq 0$ (otherwise h would not be irreducible) and it satisfies

$$-\kappa \frac{d\bar{h}}{dx} = (mb(x) + n'(x) - n(x)a(x) - m\,a(x)\kappa)\bar{h}.$$

Solving this linear differential equation we get

$$\bar{h}(x) = C \exp\Big(-\frac{1}{\kappa} \int (mb(x) + n'(x) - n(x)a(x) - m\,a(x)\kappa)\,dx\Big), \quad C \in \mathbb{C} \setminus \{0\}.$$

Since $\bar{h}(x)$ must be a polynomial we must have

$$m\,b(x) + n'(x) - n(x)a(x) - m\,a(x)\kappa = 0 \quad \text{that is} \quad m\,b(x) + n'(x) - n(x)a(x) = m\,a(x)\kappa.$$

Introducing it in K we obtain

$$K = m\,a(x)\kappa + m\,a(x)y = m\,a(x)(y + \kappa).$$

Hence, if $h = h(x, y)$ is an irreducible Darboux polynomial of the generalized Riccati polynomial differential system (3.6), then the cofactor is $K = m\,a(x)(y + \kappa)$.

We write the generalized Riccati polynomial differential system (3.6) as the differential equation

$$y\frac{dy}{dx} = (y+\kappa)(a(x)y + b(x) - \kappa a(x)). \tag{3.7}$$

Then the Darboux polynomial $h = h(x,y) = h(x,y(x))$ satisfies

$$y\frac{dh}{dx} = y\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}((y+\kappa)(a(x)y + b(x) - \kappa a(x))) = m\, a(x)(y+\kappa)h,$$

where m is a non–negative integer, or equivalently

$$\log h = M + \int \frac{m\, a(x)(y+\kappa)}{y}\, dx, \quad \text{where} \quad M \in \mathbb{C}.$$

Hence

$$h = h(x,y(x)) = C \exp\Big(\int \frac{m\, a(x)(y+\kappa)}{y}\, dx\Big), \quad C = e^M \in \mathbb{C}\setminus\{0\}. \tag{3.8}$$

Now we write from (3.6) that

$$\begin{aligned}\frac{dy}{dx} &= \frac{y+\kappa}{y}(a(x)y + b(x) - \kappa a(x)) = a(x)\frac{y+\kappa}{y}\Big(y + \frac{b(x)-\kappa a(x)}{a(x)}\Big)\\ &= a(x)\frac{y+\kappa}{y}(y + r(x)),\end{aligned}$$

where $r(x) = -\kappa + b(x)/a(x)$. Hence the integral in (3.8) becomes

$$\begin{aligned}\int \frac{m\, a(x)(y+\kappa)}{y}\, dx = \int \frac{m\, dy/dx}{y+r(x)}\, dx = \int \frac{m\,(dy/dx + r'(x) - r'(x))}{y+r(x)}\, dx\\ = m\, \log[(y+r(x))] - m\int \frac{r'(x)}{y+r(x)}\, dx.\end{aligned}$$

Hence,

$$h = h(x,y) = C(y+r(x))^m \exp\Big(-m\int \frac{r'(x)}{y+r(x)}\, dx\Big).$$

Since h must be irreducible we must have $m = 1$. Furthermore, since it is a polynomial we must have $r(x)$ to be a polynomial, and $\exp\big(-m\int \frac{r'(x)}{y+r(x)}\, dx\big)$ to be a polynomial. In order that the last function be a polynomial we must have $r'(x) = 0$ but then $r(x) = \kappa_1 \in \mathbb{C}$ which implies that $b(x)/a(x) = \kappa_1 + \kappa \in \mathbb{C}$ which is not possible. This implies that system (3.6) has no more irreducible Darboux polynomials than $y + \kappa$ with $\kappa \in \mathbb{C}\setminus\{0\}$, and this concludes the proof of the theorem.

Lemma 3.2 *Assume that* $\exp(g_1/h_1), \ldots, \exp(g_r/h_r)$ *are exponential factors of some polynomial differential system*

$$x' = P(x,y), \quad y' = Q(x,y), \quad P, Q \in \mathbb{C}[x,y], \tag{3.9}$$

with cofactors L_j *for* $j = 1, \ldots, r$. *Then* $\exp(G) = \exp(g_1/h_1 + \cdots + g_r/h_r)$ *is also an exponential factor of system* (3.9) *with cofactor* $L = \sum_{j=1}^{r} L_j$.

Proof. Using the fact that for $j = 1, \ldots, r$, $E_j = \exp(g_j/h_j)$ are exponential factors of system (3.9) with cofactors L_j we have

$$\frac{\partial(g_j/h_j)}{\partial x}P(x,y)E_j + \frac{\partial(g_j/h_j)}{\partial y}Q(x,y)E_j = L_jE_j,$$

or equivalently,

$$\frac{\partial(g_j/h_j)}{\partial x}P(x,y) + \frac{\partial(g_j/h_j)}{\partial y}Q(x,y) = L_j.$$

Therefore if we set $G = \sum_{j=1}^{r} g_j/h_j$ we get that

$$\frac{\partial G}{\partial x}P(x,y) + \frac{\partial G}{\partial y}Q(x,y) = \sum_{j=1}^{r} L_j = L,$$

and thus if $E = \exp(G)$ we obtain

$$\frac{\partial G}{\partial x}P(x,y)E + \frac{\partial G}{\partial y}Q(x,y)E = LE.$$

This concludes the proof of the lemma.

Proof of Theorem 1.2. It follows from Theorems 3.1 and 3.2 that the unique irreducible Darboux polynomials of system (3.6) is $y + \kappa$. Hence from Propositions 2.1 and 3.1 it follows that in this case we can have exponential factors of the form either $e^{g(x,y)}$ being $g \in \mathbb{C}[x,y]$ or $e^{g_m(x,y)/(y+\kappa)^m}$ with m a positive integer and $g_m \in \mathbb{C}[x,y]$ coprime with $y + \kappa$. We first show that system (3.6) has no exponential factors of the second form.

Let $E = \exp(g_m/(y+\kappa)^m)$ with m a positive integer and $g_m \in \mathbb{C}[x,y]$ such that $y+\kappa$ does not divide g_m. Applying the definition of exponential factor (2.3), after simplifying by E, we obtain

$$\frac{\partial g_m}{\partial x} + \frac{\partial g_m}{\partial y}(y+\kappa)(a(x)y + b(x) - \kappa a(x)) - m(b(x) + a(x)(y-\kappa))g_m = L(y+\kappa)^m,$$

where L is a polynomial of degree n given in (2.2).

Now let $\bar{g}_m$ be the restriction of g_m to $y = -\kappa$ that is $\bar{g}_m(x) = g_m(x, -\kappa)$. Then $\bar{g}_m$ satisfies

$$\bar{g}_m'(x) = m(b(x) - 2\kappa a(x))\bar{g}_m.$$

Solving this linear differential equation we get

$$\bar{g}_m = Ce^{m\int(b(x)-2\kappa a(x))\,dx}, \quad C \in \mathbb{C}.$$

Since $\bar{g}_m$ must be a polynomial and by hypothesis $b(x) \neq 2\kappa a(x)$, then we must have $C = 0$ that is $\bar{g}_m = 0$. But then g_m is divisible by $y + \kappa$, a contradiction. Hence this case is not possible.

In short, if the generalized Riccati polynomial differential systems (1.1) have exponential factors then they must be of the form $e^{g(x,y)}$ with $g(x,y) \in \mathbb{C}[x,y]$. It follows from

Theorems 2.1, 2.2, 3.1 and 3.2 that in order to have a Liouvillian first integral we must have q exponential factors $E_j = \exp(g_j)$ with cofactors L_j such that

$$\sum_{j=1}^{q} \mu_j L_j + \lambda_1 (b(x) + a(x)(y - \kappa)) = -2a(x)y - b(x),$$

that is,

$$\sum_{j=1}^{q} \mu_j L_j = -2a(x)y - b(x) - \lambda_1 (b(x) + a(x)(y - \kappa)).$$

Let $G = \sum_{j=1}^{q} \mu_j g_j \in \mathbb{C}[x, y]$. Then $E = \exp(G) = \exp(\sum_{j=1}^{q} \mu_j g_j)$, is an exponential factor of system (1.1) with the cofactor $L = \sum_{j=1}^{q} \mu_j L_j$ (see Lemma 3.2) and E satisfies

$$y\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y}((y + \kappa)(a(x)y + b(x) - \kappa a(x)) = LE,$$

that is

$$\begin{aligned} y\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}((y + \kappa)(a(x)y + b(x) - \kappa a(x)) &= L \\ &= -2a(x)y - b(x) - \lambda_1 (b(x) + a(x)(y - \kappa)). \end{aligned} \tag{3.10}$$

We write G as a polynomial in the variable y as follows

$$G = \sum_{j=0}^{m} G_j(x) y^j.$$

Computing the coefficient of y^{m+1} with $m \geq 1$ in (3.10) we get

$$G_m'(x) + m\, a(x) G_m(x) = 0,$$

that is

$$G_m(x) = C_m \exp\Big(- m \int a(x)\, dx\Big), \quad C_m \in \mathbb{C}.$$

Since G_m must be a polymoimal we must have $m = 0$, and thus $G = G_0(x)$. Then introducing it in (3.10) we obtain

$$yG_0'(x) = -2a(x)y - b(x) - \lambda_1 (b(x) + a(x)(y - \kappa)).$$

Since $-b(x)(1 + \lambda_1) - a(x)\kappa \neq 0$ (otherwise $b(x)/a(x) \in \mathbb{C}$) we have a contradiction. This concludes the proof.

References

[1] C. Christopher, *Liouvillian first integrals of second order polynomial differential equations*, Electron J. Differential Equations **49** (1999).

[2] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), 63–117.

[3] F. Dumortier, J. Llibre and J.C. Artés, Qualitative Theory of Planar Differential Systems, Universitext, Springer–Verlag, New York, 2006.

[4] M.R.Mokhtarzadeh, M.R. Pournaki and A.A. Razani, *A note on periodic solutions of Riccati equations*, Nonlinear Dynam. **62** (2010), 119–125.

[5] M.F. Singer, *Liouvillian solutions of nth order homogeneous linear differential equations*, Amer. J. Math. **103** (1981), 661–682.