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New Asymptotic Stability and Uniqueness Results on Periodic Solutions of Second Order Differential Equations Using Degree Theory

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Abstract

We present new criteria for uniqueness and asymptotic stability of periodic solutions of a second order differential equation based on topological degree theory. As an application, we will study some well known equations and some illustrative examples.

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1 Introduction and main results

The study of periodic equations comes from the analysis of recurrent solutions: $T-$periodic solutions, periodic solutions with minimal period nT ($n \geq 2$), they are called subharmonic solutions with order n, quasi-periodic solutions, ... A relevant special case are the $T-$periodic solutions. In its study we can find various aspects; in particular could be formulated the questions on the existence, number of solutions and properties of stability.

Previously, several numbers of methods have been developed in order to study the existence and stability of periodic solutions, starting with the classical theory of a small parameter. The basic idea of this theory consists in the study of a precise equation using

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perturbation's arguments, by means of equations with known solutions (see [14, p.115]). The implicity Function Theorem and general techniques of local analysis are the main tools of this method. Afterwards, other methods have been used, among others, variational ones (see [15]).

Inspired by the results [17, 18, 19, 20], we will present, using topological techniques, some criteria guaranteeing the uniqueness and asymptotic stability of periodic solutions to Liénard type equations. More precisely, we will study the behaviour of $T-$periodic solutions (which will be called periodic solutions) of

$$x'' + f(x)x' + g(t, x) = 0, \tag{1}$$

where $f : (0, +\infty) \to \mathbb{R}$ is continuously differentiable and $g : [0, T] \times (0, +\infty) \to \mathbb{R}$ satisfies the Caratheodory conditions with certain regularity, i.e., $g(t, \cdot)$ is continuously differentiable for almost every $t \in [0, T]$, $g(\cdot, x), g_x(\cdot, x)$ are medible functions verifying that for each compact set $D \subseteq (0, +\infty)$, the functions

$$\sup_{x\in D} |g(\cdot, x)|, \qquad \sup_{x\in D} |g_x(\cdot, x)|$$

are integrable on $[0, T]$. We will understand by periodic solution of (1) a positive and continuously differentiable function x such that x' is absolutely continuous verifying the boundary periodic conditions $x(0) - x(T) = 0 = x'(0) - x'(T)$. When useful we can write $f \in C^1(0, +\infty)$, $g \in \mathrm{Car}^1$ when dealing with the previous concepts.

Equation (1) can be interpreted as the movement equation of a particle with mass equal to 1 and with one degree of freedom, which is submitted to a restoring force type $-g(t, x)$ and one of friction type.

In the literature, several works about the existence of periodic solutions of (1) can be found (see [3, 4, 5, 6, 7, 8, 9, 10, 13, 21]). However the motions of uniqueness and asymptotic stability are less known (see [12, 17, 18, 20]).

Next, we are going to introduce the main results of this paper. The first one corresponds to a new criterion for asymptotic stability of periodic solutions of (1). We denote $[a]_+$ the positive part of a real number a, i.e., $[a]_+ = \max\{a, 0\}$.

Theorem 1.1 *Let x be an isolated periodic solution of* (1). *Assume that $f((0, +\infty)) \subseteq (0, +\infty)$, and*

$$[g_x(t, x(t))]_+ \leq \left[\frac{\pi}{T \exp(\|f(x)\|_1)}\right]^2$$

for almost everywhere $[0, T]$. Then, if $\gamma_T(x) = 1$ ($\gamma_T(x)$ corresponds to the topological index associated to x), x is asymptotically stable.

This result has been obtained as a consequence of one more general, which is proved using Proposition 2.1 of this work. The reason to present Theorem 1.1 in its easiest version, is simply because in this way one can check the hypothesis with less difficulty. One of the advantages of Theorem 1.1 with respect to the classical criteria [2, 17, 18, 20] is that it does not require that $g_x(\cdot, x)$ to be a bounded function. Moreover, we prove in Section 2 that our result is optimal in some sense (see Counter-example 2.1). However, because of the

previous criteria are also optimal, we have to make mention that our result is independent and therefore they are complementary (see Subsection 4.2).

The second main theorem is a uniqueness criterion. We observe that the uniqueness concept is related to the study when certain family of linear differential equations (which are homogeneous) is non-resonant. In contrast to the results in [16], where the author requires hypothesis on disconjugancy to guarantee that the above-mentioned differential equations have topological index equal to 1 and to prove that 0 is an asymptotically stable solution, our results suggest that the concepts of asymptotic stability of the possible periodic solutions of (1) and its uniqueness are independent (see Subsection 4.1).

Theorem 1.2 *Assume that for every pair of possible periodic solutions* x_1, x_2 *of* (1)*, the equation*

$$y'' + f(x_1)y' + [f'(\eta)x_2' + g_x(t,\xi)]y = 0$$

has as a unique solution, the trivial one, where $\eta, \xi : [0,T] \to \mathbb{R}$ *are continuous functions inside of* $[\min\{x_1, x_2\}, \max\{x_1, x_2\}]$. *Then* (1) *has at most one periodic solution.*

This paper is structured as follows: after the introduction we present some results about linear differential equations. We divide this section in two parts; the first one mainly is used in order to recall some results which can be found in the literature. In the second one, we prove new motions on disconjugancy in these equations. Moreover, we introduce the concept of topological index associated to a linear differential equation. In Section 3, we use the concept of index of the linearized equation associated to a periodic solution of (1) in order to relate the above concepts with the Brouwer degree theory. This allows us to prove our main theorems. Finally, we use the last section to illustrate our main results. Moreover, Theorem 1.1 will be applied in order to study a particular equation where the results of Ortega and Zitan cannot be applied.

On the other hand, obviously, in the same way as in [16], our main results can be applied to some particular models. However, we have seen convenient to omit this type of results as applications to follow a unique line of work.

Let us give some basic notation. Let C be the Banach space of continuous functions on $[0,T]$ endowed with the uniform norm $\|\cdot\|_\infty$, AC the subspace of C formed by the absolutely continuous functions. The set of differentiable functions with differential absolutely continuous on $[0,T]$ will be denoted by AC^1. Given an integrable function $p \in L$, we write $p(t) \prec 0$ if $p(t) \leq 0$ for almost every $t \in [0,T]$ and it is negative in a set of positive measure. Furthermore, we can define

$$\overline{p} := \frac{1}{T}\int_0^T p(s)ds, \quad P_+ := \int_0^T [p(s)]_+ ds, \quad P_- := \int_0^T [p(s)]_- ds,$$

where $[a]_+ = \max\{a, 0\}$, $[a]_- := \max\{-a, 0\}$ with $a \in \mathbb{R}$. In this case, we define $\|p\|_1 := P_+ + P_-$.

The above notation will be used throughout this paper, however, other new concepts could be required, in which case they will be introduced.

2 Linear differential equations

In this section we will study some properties of the solutions of the linear equation

$$y''(t) + (p(t)y(t))' + q(t)y(t) = 0, \tag{2}$$

where $p, q \in L(\mathbb{R}/T\mathbb{Z})$ (as usual $L(\mathbb{R}/T\mathbb{Z})$ denotes the set of $T-$periodic and integrable functions). Understanding by solution of the equation (2) a function $y \in AC^1$ such that (2) holds.

Notice that by uniqueness of the initial boundary problem associated to (2), any solution can be extended to the whole set of $\mathbb{R}$.

Definition 2.1 Let y be a solution of (2), t_0 some zero of y. We define the conjugate point t_0^* of t_0 by

$$t_0^* = \begin{cases} \inf\{t \in (t_0, +\infty) : y(t) = 0\} \\ +\infty \qquad\qquad\qquad\qquad y(t) \neq 0 \qquad \forall \quad t > t_0. \end{cases}$$

We say that (2) is disconjugate if for any solution y, and each $t_0 \in [0, T)$ such that $y(t_0) = 0$, $t_0^* - t_0 > T$.

In the next section we are going to give basic notions and definitions which are well known, however they are very important for entering into the notion of topological index associated to a linear differential equation.

2.1 Definitions

It will be convenient to introduce the classical definition of the characteristic multipliers associated to (2). We will use an easy definition. However other equivalent definitions can be found in the literature (see for instance [1, 11]).

Definition 2.2 $\alpha \in \mathbb{C}$ is called a characteristic multiplier associated to (2) if there exists a non–trivial absolutely continuous function φ defined on $\mathbb{R}$, verifying (2) on almost everywhere and such that

$$\varphi(t + T) = \alpha\varphi(t) \qquad \forall \quad t \in \mathbb{R}.$$

Notice that $\varphi(\mathbb{R})$ may have no empty intersection with $\mathbb{C} \setminus \mathbb{R}$. However, if $\alpha \in \mathbb{R}$, φ can be chosen such that its image is completely contained on $\mathbb{R}$. An immediate consequence of the previous definition is proved in the following assertion.

Lemma 2.1 *Assume that* (2) *is disconjugate. Then it has no negative characteristic multipliers.*

By the application of the Jacobi–Liouville formula and taking into account that the characteristic multipliers associated to (2) are the proper values of its monodromy matrix, we prove

Lemma 2.2 *Let $p \in C^1(\mathbb{R}/T\mathbb{Z})$ ($C^1(\mathbb{R}/T\mathbb{Z})$ denotes the set of continuously differentiable and $T-$periodic functions) be a non-trivial and non-negative function. Then $\alpha_1\alpha_2 < 1$, where α_1 and α_2 are the characteristic multipliers of the equation* (2)*.*

Proof. According to the Jacobi–Liouville formula we know that $\alpha_1\alpha_2 = \exp\left[-\int_0^T p(s)ds\right] < 1$. ■

Consequently it follows the following result:

Lemma 2.3 *Under the previous hypothesis, if* (2) *is disconjugate and*

$$\operatorname{sgn}\left[(1-\alpha_1)(1-\alpha_2)\right] = 1. \tag{3}$$

Then $0 < |\alpha_i| < 1$ for every $i \in \{1,2\}$. In other words, each solution of the equation (2) *is asymptotically stable.*

Proof. If $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \mathbb{R}$, by Lemma 2.2, $|\alpha_1| = |\alpha_2| = \alpha_1\alpha_2 < 1$. On the contrary, by virtue of Lemma 2.1, we observe that $0 < \alpha_1, \alpha_2$. Hence, according to Lemma 2.2 and (3) it is required that $0 < |\alpha_i| < 1$ for every $i \in \{1,2\}$. ■

Definition 2.3 We call topological index associated to (2) to the number

$$\operatorname{sgn}\left[(1-\alpha_1)(1-\alpha_2)\right],$$

where α_1, α_2 are the characteristic multipliers associated to (2). Moreover, if its characteristic multipliers are different from 1, (2) is non-degenerate.

The previous definition will have an important role in order to introduce a very easy definition of topological index associated to a periodic solution of (1).

2.2 Disconjugacy motions on $[0, T]$

Since equation (2) is equivalent to

$$(y' + p(t)y)' + q(t)y = 0 \tag{4}$$

almost everywhere on $[0,T]$; then, a solution of (2) could be understood as a function y such that $y \in AC$ and $y' + py \in AC$, and furthermore, satisfies (4). However, using the transformation

$$\varphi(t) = \int_0^t \exp\left(\int_0^s p(\xi)d\xi\right)ds, \qquad u(\varphi(t)) = y(t)\exp\left(\int_0^t p(s)ds\right) \tag{5}$$

in (4), we arrive to the equation

$$u''(\varphi(t)) = -q(t)\exp\left(-2\int_0^t p(s)ds\right)u(\varphi(t)) \qquad \text{for a. e. } t \in [0,T]. \tag{6}$$

Now, let $x = \varphi(t)$, then $x \in [0, \int_0^T \exp\left(\int_0^t p(s)ds\right)dt]$ and since $\varphi'(t) > 0$ for $t \in [0, T]$, there exists φ^{-1}. Then, from (6) we get

$$u''(x) = h(x)u(x) \qquad \text{for a. e. } x \in [0, T_0], \tag{7}$$

where

$$h(x) = -q(\varphi^{-1}(x)) \exp\left(-2\int_0^{\varphi^{-1}(x)} p(s)ds\right), \qquad T_0 = \int_0^T \exp\left(\int_0^t p(s)ds\right)dt.$$

Moreover, according to (5) it is clear that y has a zero at t_0 if and only if u has a zero at $x_0 = \varphi(t_0)$. Then the question about "disconjugacy on $[0, T]$" to (4) and (7) are equivalent.

Immediately one can prove the following assertion.

Proposition 2.1 *Let y be a solution of* (4) *and let $t_0, t_1 \in [0, +\infty)$, $t_1 > t_0$ be such that $y(t_0) = 0$, $y(t_1) = 0$ and $y(t) \neq 0$ for $t \in (t_0, t_1)$. Let, also*

$$[q(t)]_+ \exp\left(-2\int_{t_0}^t p(s)ds\right) \prec \left(\frac{\pi}{\int_{t_0}^{t_0+T} \exp\left(\int_{t_0}^s p(\xi)d\xi\right)ds}\right)^2 \tag{8}$$

for almost every $t \in [t_0, t_0 + T]$. Then $t_1 - t_0 > T$.

Proof. Let us assume that $t_1 - t_0 \leq T$, then $t_1 \in (t_0, t_0 + T]$. Using the transformation (5), we arrive to (7) (with t_0 instead of 0). By multiplying (7) by u, integrating from 0 to $\int_{t_0}^{t_1} \exp\left(\int_{t_0}^t p(s)ds\right)dt$ and using Wirtinger's inequality, from (8) we get a contradiction. ■

The previous proposition cannot be improved, i.e., it is optimal. This is proved below.

Counter–example 2.1 Let $p \in L(\mathbb{R}/T\mathbb{Z})$ be such that

$$\int_0^T \exp\left(\int_0^t p(s)ds\right)dt = \pi$$

$$q(t) = \exp\left(2\int_0^t p(s)ds\right) \qquad \text{for a. e. } t \in [0, T],$$

and put

$$y(t) = \exp\left(-\int_0^t p(s)ds\right)\sin\left(\int_0^t \exp\left(\int_0^s p(\xi)d\xi\right)ds\right) \qquad \forall \quad t \in [0, T].$$

Then y is a solution of (4) with $y(0) = 0$, $y(T) = 0$. Moreover,

$$[q(t)]_+ \exp\left(-2\int_0^t p(s)ds\right) = \left(\frac{\pi}{\int_0^T \exp\left(\int_0^s p(\xi)d\xi\right)ds}\right)^2 \qquad \text{for a. e. } t \in [0, T].$$

Some corollaries can be obtained from Proposition 2.1.

Corollary 2.1 *Assume that*

$$[q(t)]_+ \leq \left[\frac{\pi}{T\exp(\|p\|_1)}\right]^2 \qquad \text{for a. e. } t \in [0,T],$$

where we write $\prec$ *instead of* $\leq$ *whenever* $p \equiv 0$. *Then* (2) *is disconjugate.*

If, moreover, we assume that $\|p\|_\infty \leq \tilde{M}$, we prove

Corollary 2.2 *Assume that*

$$[q(t)]_+ \leq \left[\frac{\tilde{M}\pi}{\exp(\tilde{M}T)-1}\right]^2 \qquad \text{for a. e. } t \in [0,T],$$

where we write $\prec$ *instead of* $\leq$ *whenever* $p \equiv 0$. *Then* (2) *is disconjugate.*

Taking into account the difficulty to apply Proposition 2.1, sometimes to apply the previous corollaries could be of interest. However, depending on the equation it may be that the previous proposition gives more general effective conditions.

Here we have proved new criteria to study the distance between the zeros of (2) when $p, q \in L(\mathbb{R}/T\mathbb{Z})$, obtaining Corollaries 2.1, 2.2 for suitable p, q. As a complement of these results, one may cite a classical criterion proved by Ortega and Zitan [20, Proposition 1.2.] when p, q are bounded functions (see also [2]).

3 Asymptotic stability and uniqueness of periodic solutions

In this section we present the proofs of our main results. We are going to start giving a definition of topological index associated to a non-degenerate periodic solution of (1). Let x be a fixed periodic solution, we consider the linearized equation associated to x, i.e.,

$$y''(t) + (f(x(t))y(t))' + g_x(t, x(t))y(t) = 0. \tag{9}$$

Notice that (9) corresponds to the equation (2) whenever

$$p(t) = f(x(t)), \qquad q(t) = g_x(t, x(t)) \qquad \forall \quad t \in [0,T].$$

When (9) is a non-degenerate equation, we say that x is a non-degenerate solution of (1). Moreover, in this case, one can define the topological index associated to x coinciding with the topological index of (9). More precisely:

Definition 3.1 Let x be a periodic solution of (1), we define its topological index, denoted by $\gamma_T(x)$, as the topological index of its linearized equation (9).

Under the framework of Definition 3.1 it can be proved that

$$\gamma_T(x) = \deg_B(I - P_T, B, 0),$$

where $\deg_B$ denotes the Brouwer degree, P_T is the well known Poincaré map (P_T is well defined and continuously differentiable because $f \in C^1$ and $g \in \mathrm{Car}^1$), and B is a ball having its center at $(x(0), x'(0))$ which does not contain more fixed points of P_T.

Here we present the proofs of our main results:

Proof of Theorem 1.1. According to Corollary 2.1, defining p, q as above, we deduce that (9) is a disconjugate linear equation. Since its topological index is equal to 1, then $\mathrm{sgn}[(1-\alpha_1)(1-\alpha_2)] = 1$ where α_1, α_2 are the characteristic multipliers associated to (9). Applying Lemma 2.3, and taking into account that (9) is disconjugate we conclude that $0 < |\alpha_i| < 1$, $i = 1, 2$.

If $\alpha_1 \in \mathbb{C}\backslash\mathbb{R}$, the result follows from [19, Corollary 2.3] (because γ_{2T} is well defined, and $\gamma_{2T}(x) = \mathrm{sgn}[(1-\alpha_1^2)(1-\alpha_2^2)] = 1$). On the contrary, if $\alpha_1, \alpha_2 \in \mathbb{R}$, the result is obtained from Lemma 2.1 using [19, Theorem 2.2]. ■

Proof of Theorem 1.2. Let us assume that there exist two periodic solutions of (1). Let x_1, x_2 be such solutions. Defining the difference function $d = x_1 - x_2$ and applying the Mean Value Theorem we prove the existence of $\eta, \xi : [0, T] \to \mathbb{R}$ verifying

$$d'' + p(t)d' + \widetilde{q}(t)d = 0,$$

where $p = f(x_1), \widetilde{q} = f'(\eta)x_2' + g_x(t, \xi)$ and $\eta, \xi \in [\min\{x_1, x_2\}, \max\{x_1, x_2\}]$. Finally, the hypothesis implies that $d \equiv 0$. ■

Remark 3.1 The same results can be obtained in the same way considering equations with non-singular nonlinearities, extending in an obvious way the definition of periodic solution of (1) and the definitions of the sets Car^1 and $C^1(0, +\infty)$ on the whole $\mathbb{R}$.

4 Examples

This section is devoted to the application of our main results. We will present two parts. In the first one, we will study an extensive type of equations exhibiting uniqueness under certain conditions, using Theorem 1.2. Moreover, we illustrate the obtained results with known equations. In the second part, we introduce an equation showing an example where Theorem 1 works, however [20, Proposition 1.2.] cannot be applied.

4.1 Uniqueness results for differential equations with nonlinear term having controlled differential

Theorem 1.1 can be applied easily to the general equation (1) with $f \equiv 0$ and assuming g is defined on a subset of $\mathbb{R}$ (by a previous remark, we can assume that g is defined on the whole $\mathbb{R}$) whenever we have some control on g_x. To prove the results in this section it will be convenient to study the case when Hill equation

$$y'' + a(t)y = 0 \tag{10}$$

has a unique periodic solution, whenever $a : [0, T] \to \mathbb{R}$ is an integrable function. For this propose we prove the following assertions.

Lemma 4.1 *Assume that there exists $n \in \mathbb{N}$ such that*

$$\left(\frac{2\pi}{T}\right)^2 n^2 \prec a(t) \prec \left(\frac{2\pi}{T}\right)^2 (n+1)^2. \tag{11}$$

Then (10) *has a unique periodic solution.*

Proof. Let y be a periodic solution of (10). We change to polar coordinates $(y, y') \mapsto (r, \vartheta)$

$$y = r\sin\vartheta, \qquad y' = r\cos\vartheta.$$

Then from (10) we deduce that

$$\vartheta' = \cos^2\vartheta + a(t)\sin^2\vartheta.$$

Now, using (11) we can obtain a contradiction assuming that the number of rotations of (y, y') around of the origin is an integer. Since the unique constant periodic solution of (10) is $y \equiv 0$ we conclude the proof. ■

Lemma 4.2 *Assume that $a(t) \prec 0$. Then* (10) *has a unique periodic solution.*

Proof. In this case it is well-known that there exists a Green function associated to (10) with negative sign. ■

According to Theorem 1.1, the study of the uniqueness of (1) (to recall that we are assuming $f \equiv 0$) reduces to study the uniqueness of

$$y'' + g_x(t, \xi)y = 0,$$

where $\xi \in [\min\{x_1, x_2\}, \max\{x_1, x_2\}]$ when x_i (i=1,2) is a possible periodic solution of (1). Therefore, if we can guarantee that (1) has a priori estimates, i.e., if there exist constants $r \leq R$ such that, every possible periodic solution of (1) verifies

$$r \leq x(t) \leq R \qquad \forall \quad t \in [0, T];$$

then, according to Lemmas 4.1-4.2, we can state:

Theorem 4.1 *Assume that*

$$\left(\frac{2\pi}{T}\right)^2 n^2 \prec \inf_{z\in[r,R]} g_x(t,z) \leq \sup_{z\in[r,R]} g_x(t,z) \prec \left(\frac{2\pi}{T}\right)^2 (n+1)^2$$

is true for some $n \in \mathbb{N}$, then (1) *has at most one periodic solution.*

Theorem 4.2 *Assume that*

$$\sup_{z\in[r,R]} g_x(t,z) \prec 0.$$

Then (1) *has at most one periodic solution.*

Notice that in the first theorem necessarily $g_x(\cdot, z)$ has to be bounded from above and from below, and, moreover, it must be essentially positive. However, in order to apply Theorem 4.2 it requieres that for every $z \in [r, R]$, the nonlinearity of $g_x(\cdot, z)$ has to be essentially negative, but it can be unbounded from below.

Remark 4.1 If instead of using Lemmas 4.1-4.2 in order to ensure that (10) has at most one periodic solution, we use other results, new versions of the previous theorems could be done.

Example 4.1 It is well-known that, using the arguments of [9, Theorem 2.1.] one could obtain a priori estimates for the Rayleigh-Plesset type equations

$$x'' + \frac{g_1}{x^\delta} - \frac{g_2}{x^\gamma} = h(t)x^\beta$$

whenever $\overline{h} \leq 0$, as well as the existence of a (positive) periodic solution when $g_1, g_2 > 0$, $h \in L^1$ and $\gamma \geq \delta$ ($\gamma \geq 1$), and $\beta \in [0, 1)$. To obtain explicitly a priori estimates in an easy way, it will be convenient to consider the particular equation

$$x'' + \frac{1}{x} - \frac{1}{x^2} = h(t) \tag{12}$$

(when $\overline{h} \leq 0$). In this case we can take

$$R = 1 + \frac{T}{8}\left[H_- + \sqrt{H_- - 16(\overline{h} - 1)}\right], \quad r = \frac{1 - x_1}{\frac{1-x_1}{x_1} + \left(\frac{T}{4} + \|h\|_1\right)\left(\frac{3\|h\|_1}{2} + \frac{T}{8}\right)},$$

where $x_1 = 1$ if $\overline{h} = 0$ or else $x_1 = (1 - \sqrt{1 - 4\overline{h}})/2\overline{h} < 1$; to prove that every nonconstant possible periodic solution x of (12) verifies that $r < x < R$. Therefore, since any constant solution of (12) is less than or equal to 1 and greater or equal than r, we can apply Theorem 4.1 using the previously defined positive constants r, R. For example, in this case, (12) has a unique periodic solution whenever

$$1 + \frac{T}{8}\left[H_- + \sqrt{H_- - 16(\overline{h} - 1)}\right] \leq 2,$$
$$\left(\frac{2\pi}{T}\right)^2 n^2 < g'(R) \leq g'(r) < \left(\frac{2\pi}{T}\right)^2 (n+1)^2$$

for some $n \in \mathbb{N}$, where $g(x) = 1/x - 1/x^2$. As an immediate consequence, one can prove, taking $n = 0$ and sending $T \to 0$, that for any $\overline{h} < 0$ there exists $T_0 > 0$ (depending on $\overline{h}$) such that if $T \leq T_0$, (12) has a unique periodic solution.

As a particular case of (1), we can consider equations with a nonlinear term having bounded differential, i.e., there exist constants $(-\infty \leq) k_1 \leq k_2 (< +\infty)$ such that

$$k_1 \leq g_x(t, z) \leq k_2 \qquad \forall t \in [0, T], \quad z \in \operatorname{Dom} g$$

(as usual Dom g denotes the domain of the function g). In this case we do not require to know the existence of a priori estimates, obtaining as an immediate consequence of the previous results applied to the equations

$$x'' + r(t)x + g(t, x) = 0 \tag{13}$$

the following assertion.

Corollary 4.1 *Under the previous setting, if either there exists $n \in \mathbb{N}$ such that*

$$\left(\frac{2\pi}{T}\right)^2 n^2 < r(t) + k_1 \leq r(t) + k_2 < \left(\frac{2\pi}{T}\right)^2 (n+1)^2$$

or

$$r(t) + k_2 < 0.$$

Then (13) *has at most one periodic solution.*

Example 4.2 The equation

$$x'' + ax + \sin x = 0$$

has a unique $2\pi-$periodic solution whenever either $a \in \cup_{n=1}^{+\infty}(n^2+1, n^2+2n)$ or $a < -1$.

Example 4.3 A well-known type of equations in literature are the equations of Lazer and Solimini with attractive singularity. In this example we will study

$$x'' + r(t)x + \frac{1}{x^\sigma} = h(t), \tag{14}$$

where $\sigma > 0$ and $r, h \in C$. Although we cannot ensure the existence of a priori estimates, applying the second part of the above corollary we can deduce that (14) has at most one periodic solution whenever $r(t) \leq 0$ for almost all $t \in [0, T]$. Moreover, a basic application of the method of lower and upper solutions states that (14) has at least one periodic solution for every $h \in C$ when $\sup_{t\in[0,T]} r(t) < 0$.

On the other hand, with regard to uniqueness the previous result cannot be improved at least when an additional condition is introduced. Indeed, if $\overline{r} \geq 0$, assuming the existence of a periodic solution of (14), x, multiplying in (14) by x^{-1} and taking integrals over $[0, T]$, we arrive to a contradiction whenever $h \leq 0$. On the contrary, if we assume that r changes sign and $\overline{r} < 0$, we can give an example proving that (14) may not have any periodic solution. To do this, we consider the function $r(t) = -\sin t + \cos^2 t$. Now one can prove that

$$\int_0^{2\pi} h(s) \exp(-\sin s) ds > 0$$

is a necessary condition for the existence of a $2\pi-$periodic solution of (14), taking into account that $u(t) = \exp(-\sin t)$ is a solution of $x'' + r(t)x = 0$.

In conclusion, we can guarantee the existence and uniqueness of a periodic solution of (14) whenever $\sup_{t\in[0,T]} r(t) < 0$. However, when $r < 0$, we can only guarantee uniqueness in the case when (14) has a periodic solution, but the problem of existence in such a case remains open. Finally, in both cases, when $\overline{r} \geq 0$ or when r changes sign and $\overline{r} < 0$ one requires some additional conditions in order to guarantee the existence of a periodic solution.

4.2 Application of Theorem 1.1

As it was mentioned previously, Theorem 1.1 is independent of the results in [20]. To prove this, we introduce the following example: we choose $M > 1$ such that

$$\frac{3}{4}\left(\frac{M+1}{2}\right)^2 - M \geq \frac{1}{4}$$

holds. Taking $m = 1$, we can prove that

$$\frac{M-m}{2} \geq \frac{1}{2}\left[1 + \left(\frac{M+m}{2}\right)^2\right]^{\frac{1}{2}}. \tag{15}$$

Let us define the continuously differentiable function $f \in C^1$ in such a way that $f(x) = M$ whenever $x \geq 2$, $f(x) = 1$ whenever $x \leq 1$, and $1 \leq f \leq M$ on $[1, 2]$. Thus, $x(t) = 4/3 + \sin t$ is a 2π periodic solution of

$$x'' + f(x)x' + \left[\frac{1}{2\exp(2\pi M)}\right]^2 x = \left[\frac{1}{2\exp(2\pi M)}\right]^2 \left(\frac{4}{3} + \sin t\right) + f\left(\frac{4}{3} + \sin t\right)\cos t - \sin t. \tag{16}$$

By means of an argument which will be shown below, we deduce that $\gamma_{2\pi}(x) = 1$. According to Theorem 1.1, one can prove that x is an asymptotically stable solution of (16). However, since (15) implies that

$$\alpha - \beta(\gamma + \alpha^{1/2}) < 0,$$

because $\beta \geq \alpha^{1/2}$, being $\beta = (M-m)/2$, $\gamma = (M+m)/2$ and $\alpha = 1/4 + \gamma^2/4$; the criterion proved by Ortega and Zitan does not give information on the stability of x.

Proof of $\gamma_{2\pi}(x) = 1$. Let us consider the linearized equation of (16)

$$y'' + (p(t)y)' + ay = 0, \tag{17}$$

where $p(t) = f(4/3 + \sin t)$ and $a = \left[\frac{1}{2\exp(2\pi M)}\right]^2$. In this case, since (17) is disconjugate, the characteristic multipliers α_1, α_2 are positive and verify $0 < \alpha_1\alpha_2 < 1$. Therefore, only two cases could happen, either $\alpha_1 < 1, \alpha_2 > 1$ or $\alpha_1 < 1, \alpha_2 < 1$. We will prove that the first case does not hold. For this, we are going to prove that $\Delta < 1 + W$, where Δ is the discriminant of (17) and $W = \exp(\int_0^{2\pi} p(s)ds)$ (it will be convenient to recall that α_1, α_2 are the roots of $\mu^2 - \Delta\mu + W = 0$).

We define, for every $\lambda \in [0, 1]$, the functions

$$p_\lambda : [0, 2\pi] \to \mathbb{R}, \qquad p_\lambda(t) = (1-\lambda)p(t) + \lambda\overline{p} > 0,$$

and the equations

$$y'' + (p_\lambda(t)y)' + ay = 0. \tag{18}$$

Notice that for every λ, (18) is disconjugate. Moreover, $W(0) = W$ and $\Delta(\lambda)$ is a continuous function, being $W(\lambda) = \exp(\int_0^{2\pi} p_\lambda(s)ds)$ and $\Delta(\lambda)$ the discriminant associated to (18). Taking into account that $\Delta(1) < 1 + W$, if there exits $\lambda_1 \in [0, 2\pi]$ such that $\Delta(\lambda_1) \geq 1 + W$, then there would exist $\lambda_0 \in [0, 2\pi]$ such that $\Delta(\lambda_0) = 1 + W$. In this case, a non-trivial periodic solution of (18) with $\lambda = \lambda_0$ will be. Denoting by φ such solution, since (18) is disconjugate, we can assume that $\varphi > 0$ on $[0, 2\pi]$. However, integrating on $[0, 2\pi]$ in (18) we obtain a contradiction.

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