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Research Article

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A Note on Symmetry of Solutions for a Class of Singular Semilinear Elliptic Problems

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Abstract: We prove symmetry and monotonicity properties for positive solutions of the singular semilinear elliptic equation

$$-\Delta u = \frac{g(x)}{u^\gamma} + h(x)f(u)$$

in bounded smooth domains with zero Dirichlet boundary conditions. The well-known moving plane method is applied.

Keywords: Singular Semilinear Equations, Symmetry of Solutions, Moving Plane Method

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1 Introduction

We consider positive solutions of the problem

$$\begin{cases} -\Delta u = \dfrac{g(x)}{u^\gamma} + h(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\gamma > 0$ and Ω is a bounded smooth domain. The following assumptions will be needed throughout the paper:
(H_f) f is a locally Lipschitz continuous function, nondecreasing, $f(s) > 0$ for $s > 0$, and $f(0) \geq 0$,
(H_g) g is a locally Lipschitz continuous function, $g \in L^\infty_{\mathrm{loc}}(\Omega)$, and there exists c such that $g \geq c > 0$ in Ω,
(H_h) h is a locally Lipschitz continuous nonnegative bounded function.

Starting from the pioneering works [13, 27], many authors studied semilinear, quasilinear, and fully nonlinear singular elliptic equations (see also [1, 3–12, 14–24, 26]). Generally, the solution of (1.1) does not belong to $H^1_0(\Omega)$. Even in the case $g \equiv 1$ and $h \equiv 0$, it occurs that solutions of (1.1) are not in $H^1_0(\Omega)$ for $\gamma \geq 3$ (see [23]). In [7], a general approach to the variational characterization of (1.1) with $g \equiv 1$ was developed for any $\gamma > 0$. The technique exploited in [7] works in the same way for (1.1) under our assumptions. In particular, following [7], we can consider the decomposition of the solution $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ of (1.1) as

$$u = u_0 + w, \tag{1.2}$$

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where $w \in H_0^1(\Omega)$ and $u_0 \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ is the (weak) solution of the problem

$$\begin{cases} -\Delta u_0 = \dfrac{g(x)}{u_0{}^\gamma} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

i.e.,

$$\int_\Omega \nabla u_0 \nabla \varphi = \int_\Omega \frac{g(x)\varphi}{u_0{}^\gamma} \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

The solution u_0 is unique (see [10]) and can be found via a sub-supersolution method (see [7]) or as the limit of an increasing sequence of positive solutions of a regularized problem, via a truncation argument (see [5]). By [7] it follows that the solution u_0 is continuous up to the boundary and by [5] it follows that u_0 is bounded away from zero in the interior of Ω, i.e.,

$$\text{for all } \omega \subset\subset \Omega \text{ there exists } c_\omega \text{ such that } u_0 \geq c_\omega > 0 \text{ in } \omega.$$

In [8], symmetry and monotonicity properties of solutions of (1.1) have been proved in the case $g \equiv h \equiv 1$. In this paper, we study qualitative properties of solutions of singular semilinear elliptic problems when the singular term is not precisely of the form $\frac{1}{u^\gamma}$, but has mixed behavior. In particular, we deal with nonautonomous equations of the form

$$-\Delta u = \frac{g(x)}{u^\gamma} + h(x)f(u).$$

We point out that our proofs depend strongly on the monotonicity assumptions on the functions g and h, which follow directly from the decreasing nature of the term $\frac{1}{u^\gamma}$ and by the fact that we assume that f is nondecreasing, as in [8]. Following [8], our result is proved using a modification of the well-known moving plane method (see [25]).

Let us introduce some notation. Let ν be a direction in $\mathbb{R}^N$ with $|\nu| = 1$. Given a real number λ, we set

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\}, \quad \Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\},$$

and

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu,$$

i.e., the reflection of x through the hyperplane T_λ^ν. Moreover, we set $(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu)$. Observe that $(\Omega_\lambda^\nu)'$ may not be contained in Ω. Also, we take

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu.$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$\Lambda_1(\nu) = \{\lambda : (\Omega_t^\nu)' \subset \Omega \text{ for any } a(\nu) < t \leq \lambda\}$$

and

$$\lambda_1(\nu) = \sup \Lambda_1(\nu).$$

Moreover, we set

$$u_\lambda^\nu(x) = u(x_\lambda^\nu)$$

for any $a(\nu) < \lambda \leq \lambda_1(\nu)$. Recalling the decomposition (1.2) of the solutions of (1.1), we set $u_0{}_\lambda^\nu(x) = u_0(x_\lambda^\nu)$ and $w_\lambda^\nu(x) = w(x_\lambda^\nu)$.

Now, we can formulate our main result.

Theorem 1.1. *Let u be a solution of* (1.1). *Assume that the domain Ω is strictly convex with respect to the ν-direction and symmetric with respect to T_0^ν. Moreover, assume that*

$$g(x) \geq g(x_0^\nu) \quad \text{and} \quad h(x) \leq h(x_0^\nu) \quad \text{for all } x \in \Omega_0^\nu.$$

Then, u is symmetric with respect to T_0^ν and nondecreasing with respect to the ν-direction in Ω_0^ν. If Ω is a ball, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

2 Symmetry Properties of u_0

We begin by proving some results on u_0.

Proposition 2.1. *Let u_0 be the solution of* (1.3). *Then, for any $a(\nu) < \lambda < \lambda_1(\nu)$, we have*

$$u_0(x) < u_{0\lambda}^{\nu}(x) \quad \text{for all } x \in \Omega_\lambda^\nu \tag{2.1}$$

and

$$\frac{\partial u_0}{\partial \nu}(x) > 0 \quad \text{for all } x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{2.2}$$

Proof. Let $u_n \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta u_n = \dfrac{g_n(x)}{(u_n + \frac{1}{n})^\gamma} & \text{for } x \in \Omega, \\ u_n > 0 & \text{for } x \in \Omega, \\ u_n = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $g_n(x) = \min(g(x), n)$. The existence of u_n was proved in [5] and the uniqueness follows by [7]. Since the problem is no more singular, by standard elliptic estimates it follows that $u_n \in C^2(\overline{\Omega})$. Therefore, we can use the moving plane technique as in [2, 18, 25] to deduce that

$$u_n(x) < u_{n\lambda}^{\nu}(x) \quad \text{for all } x \in \Omega_\lambda^\nu.$$

By [5] we have that u_n converges to u_0 a.e. as n tends to infinity and, therefore, (2.1) follows by passing to the limit. In the same way, we obtain

$$\frac{\partial u_0}{\partial \nu}(x) \geq 0 \quad \text{for all } x \in \Omega_{\lambda_1(\nu)}^\nu$$

and, therefore, (2.2) follows via the strong maximum principle. □

As an immediate consequence of Proposition 2.1, we easily get the following result.

Proposition 2.2. *Let u_0 be the solution of* (1.3). *Assume that the domain Ω is strictly convex with respect to the ν-direction and symmetric with respect to T_0^ν. Then, u_0 is symmetric with respect to T_0^ν and nondecreasing with respect to the ν-direction in Ω_0^ν. Moreover, if Ω is a ball, then u_0 is radially symmetric with $\frac{\partial u_0}{\partial r}(r) < 0$ for $r \neq 0$.*

3 Comparison Principles

We need the following technical result.

Lemma 3.1 (see [8, Lemma 4]). *Let $\gamma > 0$. Consider the function*

$$h_\gamma(x, y, z, t) := x^\gamma (x+y)^\gamma (z+t)^\gamma + x^\gamma z^\gamma (z+t)^\gamma - z^\gamma (x+y)^\gamma (z+t)^\gamma - x^\gamma z^\gamma (x+y)^\gamma$$

and the domain $D \subset \mathbb{R}^4$ defined by

$$D := \{(x, y, z, t) : 0 \leq x \leq z,\ 0 \leq t \leq y\}.$$

Then, it follows that $h_\gamma \leq 0$ in D.

Lemma 3.2. *Let u be a solution of* (1.1) *with $\gamma > 0$ and let w be given by* (1.2). *Then, it follows that*

$$w > 0 \quad \text{in } \Omega.$$

Proof. Since $u \in C(\overline{\Omega})$ and $u_0 \in C(\overline{\Omega})$, then $w \in C(\overline{\Omega}) \cap H_0^1(\Omega)$. By the hypotheses on f and h it is easy to check that u is a supersolution, in the sense of [7, Definition 2.5], of the equation

$$-\Delta v = \frac{g(x)}{v^\gamma}.$$

Arguing as in [7, Lemma 2.8], we get

$$u \geq u_0 \quad \text{in } \Omega$$

and

$$w \geq 0 \quad \text{in } \Omega.$$

We show that $w > 0$ in the interior of Ω making use of the maximum principle in regions where the problem is not singular. Assume by contradiction that there exists a point $x_0 \in \Omega$ such that $w(x_0) = 0$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \subset\subset \Omega$. We have

$$-\Delta w = -\Delta u + \Delta u_0 = \frac{g(x)}{(u_0 + w)^\gamma} + h(x)f(u) - \frac{g(x)}{u_0^\gamma} \geq g(x)\Big(\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma}\Big)$$

in $B_r(x_0)$. Since $u_0(x_0) > 0$, we can assume that u_0 is positive in $B_r(x_0)$ so that $u_0 + w$ is also positive in $B_r(x_0)$. By (H_g) we have

$$g(x)\Big(\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma}\Big) = k(x)w$$

for some bounded coefficient $k(x)$. Therefore, we can find $\Lambda > 0$ such that

$$g(x)\Big(\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma}\Big) + \Lambda w \geq 0$$

in $B_r(x_0)$, so that

$$-\Delta w + \Lambda w \geq 0 \quad \text{in } B_r(x_0).$$

By the strong maximum principle (see [19]) we have $w \equiv 0$ in $B_r(x_0)$ and by a covering argument we have $w \equiv 0$ in Ω. But $w \equiv 0$ in Ω implies that $f \equiv 0$, and we get a contradiction. □

Proposition 3.3. *Let $a(\nu) < \lambda < \lambda_1(\nu)$ and let Ω' be a connected subdomain of Ω_λ^ν. Assume that*

$$g(x) \geq g(x_\lambda^\nu) \quad \textit{and} \quad h(x) \leq h(x_\lambda^\nu) \quad \textit{for all } x \in \Omega'.$$

Let u be a solution of (1.1) *and let w be given by* (1.2). *If*

$$\frac{\partial w}{\partial \nu} \geq 0 \quad \textit{in } \Omega',$$

then the alternative

$$\frac{\partial w}{\partial \nu} > 0 \quad \textit{in } \Omega'$$

or

$$\frac{\partial w}{\partial \nu} = 0 \quad \textit{in } \Omega'$$

holds.

Proof. We begin by observing that by the monotonicity assumptions on the functions g and h it follows that

$$\frac{\partial g}{\partial \nu} \leq 0 \quad \text{and} \quad \frac{\partial h}{\partial \nu} \geq 0 \quad \text{a.e. in } \Omega'.$$

Set

$$g_\nu := \frac{\partial g}{\partial \nu}, \quad h_\nu := \frac{\partial h}{\partial \nu}, \quad w_\nu := \frac{\partial w}{\partial \nu}, \quad u_{0\nu} := \frac{\partial u_0}{\partial \nu}.$$

Since $f' \geq 0$ a.e. by (H_f), we have $u_{0\nu} \geq 0$ in Ω' by Proposition 2.1, $u \geq u_0$ by Lemma 3.2, and, by differentiating the equation in (1.1), we get that w_ν solves

$$\begin{aligned} -\Delta w_\nu &= -\frac{\gamma g(x)}{u^{\gamma+1}} w_\nu + h_\nu(x) f(u) + h(x) f'(u)(w_\nu + u_{0\nu}) - g_\nu(x)\Big(\frac{1}{u_0^\gamma} - \frac{1}{u^\gamma}\Big) + \gamma g(x)\Big(\frac{1}{u_0^{\gamma+1}} - \frac{1}{u^{\gamma+1}}\Big) u_{0\nu} \\ &\geq -\frac{\gamma g(x)}{u^{\gamma+1}} w_\nu. \end{aligned}$$

We recall now that $g \in L^\infty_{\mathrm{loc}}(\Omega)$ by (H_g) and that u is bounded away from zero in Ω'. Therefore, we can find $\Lambda > 0$ such that

$$-\Delta w_\nu \geq -\frac{\gamma g(x)}{u^{\gamma+1}} w_\nu \geq -\Lambda w_\nu,$$

so that the conclusion follows by the standard strong maximum principle (see [19]). □

Proposition 3.4. *Let $a(\nu) < \lambda < \lambda_1(\nu)$ and let $\Omega' \subseteq \Omega_\lambda^\nu$. Assume that*

$$g(x) \geq g(x_\lambda^\nu) \quad \text{and} \quad h(x) \leq h(x_\lambda^\nu) \quad \text{for all } x \in \Omega'.$$

Let u be a solution of (1.1) and let w be given by (1.2). Assume that

$$w \leq w_\lambda^\nu \quad \text{on } \partial\Omega'.$$

Then, there exists a positive constant $\delta = \delta(u, f)$ such that, if $\mathcal{L}(\Omega') \leq \delta$, then

$$w \leq w_\lambda^\nu \quad \text{in } \Omega'.$$

Proof. We have

$$-\Delta(u_0 + w) = \frac{1}{(u_0 + w)^\gamma} + h(x) f(u_0 + w) \quad \text{in } \Omega$$

and

$$-\Delta(u_{0\lambda}^\nu + w_\lambda^\nu) = \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} + h(x_\lambda^\nu) f(u_{0\lambda}^\nu + w_\lambda^\nu) \quad \text{in } \Omega.$$

Since $(w - w_\lambda^\nu)^+ \in H_0^1(\Omega')$, we can consider a sequence of positive functions ψ_n such that

$$\psi_n \in C_c^\infty(\Omega') \quad \text{and} \quad \psi_n \xrightarrow{H_0^1(\Omega')} (w - w_\lambda^\nu)^+.$$

We can also assume that $\operatorname{supp} \psi_n \subseteq \operatorname{supp}(w - w_\lambda^\nu)^+$. Choosing ψ_n as test function in the weak formulation of the above equations and subtracting, we get

$$\int_{\Omega'} (\nabla(u_0 + w) - \nabla(u_{0\lambda}^\nu + w_\lambda^\nu)) \nabla\psi_n = \int_{\Omega'} \Big(\frac{g(x)}{(u_0 + w)^\gamma} + h(x) f(u_0 + w) - \frac{g(x_\lambda^\nu)}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} - h(x_\lambda^\nu) f(u_{0\lambda}^\nu + w_\lambda^\nu)\Big)\psi_n \, dx.$$

Recall that u_0 solves (1.3). It follows easily that $u_{0\lambda}^\nu$ solves

$$\begin{cases} -\Delta u_{0\lambda}^\nu = \dfrac{g(x_\lambda^\nu)}{(u_{0\lambda}^\nu)^\gamma} & \text{in } \Omega, \\ u_{0\lambda}^\nu > 0 & \text{in } \Omega, \\ u_{0\lambda}^\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, since $g(x) \geq g(x_\lambda^\nu)$ and $h(x) \leq h(x_\lambda^\nu)$ in Ω' by assumption, we get

$$\begin{aligned} \int_{\Omega'} \nabla(w - w_\lambda^\nu) \nabla\psi_n &= \int_{\Omega'} \Big(\frac{g(x_\lambda^\nu)}{(u_{0\lambda}^\nu)^\gamma} - \frac{g(x)}{(u_0)^\gamma} + \frac{g(x)}{(u_0 + w)^\gamma} - \frac{g(x_\lambda^\nu)}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma}\Big)\psi_n \, dx \\ &\quad + \int_{\Omega'} (h(x) f(u_0 + w) - h(x_\lambda^\nu) f(u_{0\lambda}^\nu + w_\lambda^\nu))\psi_n \, dx \\ &\leq \int_{\Omega'} g(x)\Big(\frac{1}{(u_{0\lambda}^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} + \frac{1}{(u_0 + w)^\gamma} - \frac{1}{(u_0)^\gamma}\Big)\psi_n \, dx \\ &\quad + \int_{\Omega'} h(x)(f(u_0 + w) - f(u_{0\lambda}^\nu + w_\lambda^\nu))\psi_n \, dx. \end{aligned}$$

Since $u_0 \le u_{0\lambda}^\nu$ in Ω_λ^ν and $w \ge w_\lambda^\nu$ on supp ψ_n, by applying Lemma 3.1 with $u_0 = x$, $w = y$, $u_{0\lambda}^\nu = z$, and $w_\lambda^\nu = t$, we obtain

$$(u_0)^\gamma(u_0+w)^\gamma(u_{0\lambda}^\nu+w_\lambda^\nu)^\gamma + (u_0)^\gamma(u_{0\lambda}^\nu)^\gamma(u_{0\lambda}^\nu+w_\lambda^\nu)^\gamma - (u_{0\lambda}^\nu)^\gamma(u_0+w)^\gamma(u_{0\lambda}^\nu+w_\lambda^\nu)^\gamma - (u_0)^\gamma(u_{0\lambda}^\nu)^\gamma(u_0+w)^\gamma \le 0,$$

so that

$$g(x)\Big(\frac{1}{(u_{0\lambda}^\nu)^\gamma} - \frac{1}{(u_0)^\gamma} + \frac{1}{(u_0+w)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma}\Big) \le 0$$

since $g(x) > 0$ in Ω. Therefore, by (H_f) and (H_h) there exists $C > 0$ such that

$$\begin{aligned}\int_{\Omega'} \nabla(w - w_\lambda^\nu)\nabla\psi_n &\le \int_{\Omega'} h(x)(f(u_0+w) - f(u_{0\lambda}^\nu + w_\lambda^\nu))\psi_n \\ &\le \int_{\Omega'} h(x)(f(u_{0\lambda}^\nu + w) - f(u_{0\lambda}^\nu + w_\lambda^\nu))\psi_n \\ &\le C\int_{\Omega'} (w - w_\lambda^\nu)\psi_n.\end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$\int_{\Omega'} |\nabla(w - w_\lambda^\nu)^+|^2 \le C\int_{\Omega'} |(w - w_\lambda^\nu)^+|^2$$

and, by the Poincaré inequality, we can find $C'(\Omega') > 0$ such that

$$\int_{\Omega'} |\nabla(w - w_\lambda^\nu)^+|^2 \le C'(\Omega')\int_{\Omega'} |\nabla(w - w_\lambda^\nu)^+|^2.$$

For sufficiently small δ, it follows that $C'(\Omega') < 1$. This shows that $(w - w_\lambda^\nu)^+ = 0$ in Ω', which gives that $w \le w_\lambda^\nu$ in Ω'. □

Lemma 3.5. *Let u be a solution of* (1.1). *Assume that*

$$g(x) \ge g(x_\lambda^\nu) \quad \text{and} \quad h(x) \le h(x_\lambda^\nu) \quad \text{for all } x \in \Omega_\lambda^\nu$$

and for any $a(\nu) < \lambda \le \lambda_1(\Omega)$. *Let w be given by* (1.2) *and assume that*

$$w \le w_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu$$

for some $a(\nu) < \lambda \le \lambda_1(\Omega)$. *Then,* $w < w_\lambda^\nu$ *in* Ω_λ^ν *unless* $w \equiv w_\lambda^\nu$ *in* Ω_λ^ν.

Proof. Let us assume that there exists a point $x_0 \in \Omega_\lambda^\nu$ such that $w(x_0) = w_\lambda^\nu(x_0)$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \subset\subset \Omega_\lambda^\nu$. Since $g \ge c > 0$ in Ω by (H_g), we have $g(x) \ge g(x_\lambda^\nu)$ and $h(x) \le h(x_\lambda^\nu)$ by assumption, $w > 0$ by Lemma 3.2, $w \le w_\lambda^\nu$ in Ω_λ^ν by assumption, and $u_0 \le u_{0\lambda}^\nu$ in Ω_λ^ν by Proposition 2.1. Using the above, in $B_r(x_0)$ we have

$$\begin{aligned}-\Delta(w_\lambda^\nu - w) &= -\Delta(u_\lambda^\nu - u_{0\lambda}^\nu) + \Delta(u - u_0) \\ &= g(x)\Big(\frac{1}{u_0^\gamma} - \frac{1}{(u_0+w)^\gamma}\Big) - g(x_\lambda^\nu)\Big(\frac{1}{(u_{0\lambda}^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma}\Big) + h(x_\lambda^\nu)f(u_{0\lambda}^\nu + w_\lambda^\nu) - h(x)f(u_0+w) \\ &\ge g(x)\Big(\frac{1}{u_0^\gamma} - \frac{1}{(u_0+w)^\gamma} + \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu)^\gamma}\Big) + h(x)\big(f(u_{0\lambda}^\nu + w_\lambda^\nu) - f(u_0+w)\big) \\ &> c\Big(\frac{1}{u_0^\gamma} - \frac{1}{(u_{0\lambda}^\nu)^\gamma} + \frac{1}{(u_{0\lambda}^\nu + w)^\gamma} - \frac{1}{(u_0+w)^\gamma}\Big) + h(x)\big(f(u_{0\lambda}^\nu + w_\lambda^\nu) - f(u_0+w)\big) \\ &\quad + g(x)\Big(\frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w)^\gamma}\Big).\end{aligned}$$

Since f is nondecreasing by (H_f) and h is nonnegative by (H_h), we get

$$h(x)\big(f({u_0}_\lambda^\nu + w_\lambda^\nu) - f(u_0 + w)\big) \geq 0.$$

Moreover, since the function

$$h(t) := a^{-\gamma} - b^{-\gamma} + (b+t)^{-\gamma} - (a+t)^{-\gamma}$$

is increasing in $[0,\infty)$ for $0 < a \leq b$, we also have

$$\frac{1}{u_0^\gamma} - \frac{1}{({u_0}_\lambda^\nu)^\gamma} + \frac{1}{({u_0}_\lambda^\nu + w)^\gamma} - \frac{1}{(u_0 + w)^\gamma} \geq 0.$$

It follows that

$$-\Delta(w_\lambda^\nu - w) \geq g(x)\Big(\frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w)^\gamma}\Big).$$

Since ${u_0}_\lambda^\nu(x_0) > 0$, arguing as in Lemma 3.2, we find $\Lambda > 0$ such that, eventually reducing r, we have

$$g(x)\Big(\frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w)^\gamma}\Big) + \Lambda(w_\lambda^\nu - w) \geq 0$$

in $B_r(x_0)$, so that

$$-\Delta(w_\lambda^\nu - w) + \Lambda(w_\lambda^\nu - w) \geq 0 \quad \text{in } B_r(x_0).$$

By the strong maximum principle (see [19]), we get $w_\lambda^\nu - w \equiv 0$ in $B_r(x_0)$. Using a covering argument, it follows that $w_\lambda^\nu - w \equiv 0$ in Ω_λ^ν and the result follows. □

4 Symmetry

Proposition 4.1. *Let u be a solution of* (1.1) *and let w be given by* (1.2). *Assume that*

$$g(x) \geq g(x_{\lambda_1(\nu)}^\nu) \quad \textit{and} \quad h(x) \leq h(x_{\lambda_1(\nu)}^\nu) \quad \textit{for all } x \in \Omega_{\lambda_1(\nu)}^\nu.$$

Then, for any

$$a(\nu) < \lambda < \lambda_1(\nu),$$

we have

$$w(x) < w_\lambda^\nu(x) \quad \textit{for all } x \in \Omega_\lambda^\nu. \tag{4.1}$$

Moreover,

$$\frac{\partial w}{\partial \nu}(x) > 0 \quad \textit{for all } x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{4.2}$$

Finally, (4.1) *and* (4.2) *hold true replacing w by u.*

Proof. Let $\lambda > a(\nu)$. Since $w > 0$ in Ω, by Lemma 3.2 we have

$$w \leq w_\lambda^\nu \quad \text{on } \partial\Omega_\lambda^\nu.$$

Therefore, using Proposition 3.4, for $\mathcal{L}(\Omega_\lambda^\nu)$ sufficiently small, we obtain

$$w \leq w_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \tag{4.3}$$

and $w < w_\lambda^\nu$ in Ω_λ^ν by Lemma 3.5. Set

$$\Lambda_0 = \{\lambda > a(\nu) : w \leq w_t^\nu \text{ in } \Omega_t^\nu \text{ for all } t \in (a(\nu), \lambda]\},$$

which is not empty thanks to (4.3), and

$$\lambda_0 = \sup \Lambda_0.$$

By the definition of $\lambda_1(\nu)$, to prove our result we have to show that $\lambda_0 = \lambda_1(\nu)$. Assume that $\lambda_0 < \lambda_1(\nu)$ and observe that, by continuity, we obtain $w \leq w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. By Lemma 3.5 it follows that $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ unless $w = w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. Because of the zero Dirichlet boundary conditions, since $w > 0$ in the interior of the domain, the case $w \equiv w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ is not possible. Thus, $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$.

We can now consider δ, given by Proposition 3.4, so that the weak comparison principle holds true in any subdomain Ω' if $\mathcal{L}(\Omega') \le \delta$. Fix a compact set $\mathcal{K} \subset \Omega^\nu_{\lambda_0}$ so that $\mathcal{L}(\Omega^\nu_{\lambda_0} \setminus \mathcal{K}) \le \frac{\delta}{2}$. By compactness we find $\sigma > 0$ such that

$$w^\nu_{\lambda_0} - w \ge 2\sigma > 0 \quad \text{in } \mathcal{K}.$$

Take now $\bar{\varepsilon} > 0$ sufficiently small so that $\lambda_0 + \bar{\varepsilon} < \lambda_1(\nu)$ and, for any $0 < \varepsilon \le \bar{\varepsilon}$,

(a) $w^\nu_{\lambda_0+\varepsilon} - w \ge \sigma > 0$ in $\mathcal{K}$,

(b) $\mathcal{L}(\Omega^\nu_{\lambda_0+\varepsilon} \setminus \mathcal{K}) \le \delta$.

Taking (a) into account, it is now easy to check that, for any $0 < \varepsilon \le \bar{\varepsilon}$, we have $w \le w^\nu_{\lambda_0+\varepsilon}$ on the boundary of $\Omega^\nu_{\lambda_0+\varepsilon} \setminus \mathcal{K}$. Consequently, by (b), we can apply the weak comparison principle (Proposition 3.4) to deduce that

$$w \le w^\nu_{\lambda_0+\varepsilon} \quad \text{in } \Omega^\nu_{\lambda_0+\varepsilon} \setminus \mathcal{K}.$$

Thus, $w \le w^\nu_{\lambda_0+\varepsilon}$ in $\Omega^\nu_{\lambda_0+\varepsilon}$ and, by applying Lemma 3.5, we have $w < w^\nu_{\lambda_0+\varepsilon}$ in $\Omega^\nu_{\lambda_0+\varepsilon}$. We get a contradiction with the definition of λ_0 and this shows that $\lambda_0 = \lambda_1(\nu)$. Thus, (4.1) is proved.

By simple geometric considerations and by (4.1) it follows that w is nondecreasing in $\Omega^\nu_{\lambda_1(\nu)}$ in the ν-direction. This gives

$$\frac{\partial w}{\partial \nu}(x) \ge 0 \quad \text{in } \Omega^\nu_{\lambda_1(\nu)},$$

so it is easy to deduce (4.2) from Proposition 3.3.

To prove that (4.1) and (4.2) hold true replacing w with u, just recall that

$$u = u_0 + w$$

and exploit Proposition 2.1. □

Now, we are able to prove our main result.

Proof of Theorem 1.1. We can prove Theorem 1.1 as a consequence of Proposition 4.1. By assumption we have $\lambda_1(\nu) = 0$. By Proposition 4.1 we obtain

$$u(x) \le u^\nu_0(x) \quad \text{for all } x \in \Omega^\nu_0$$

and, replacing ν with $-\nu$, we get

$$u(x) \ge u^\nu_0(x) \quad \text{for all } x \in \Omega^\nu_0.$$

Then, $u(x) = u^\nu_0(x)$ in Ω. The monotonicity of u follows by (4.2). □

References

[1] D. Arcoya, J. Carmona, T. Leonori, P. J. Martinez-Aparicio, L. Orsina and F. Petitta, Existence and nonexistence of solutions for singular quadratic quasilinear equations, *J. Differential Equations* **246** (2009), no. 10, 4006–4042.

[2] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, *Bull. Braz. Math. Soc. (N.S.)* **22** (1991), no. 1, 1–37.

[3] L. Boccardo, A Dirichlet problem with singular and supercritical nonlinearities, *Nonlinear Anal.* **75** (2012), no. 12, 4436–4440.

[4] L. Boccardo, T. Leonori, L. Orsina and F. Petitta, Quasilinear elliptic equations with singular quadratic growth terms, *Commun. Contemp. Math.* **14** (2011), no. 4, 607–642.

[5] L. Boccardo and L. Orsina, Semilinear elliptic equations with singular nonlinearities, *Calc. Var. Partial Differential Equations* **37** (2010), no. 3–4, 363–380.

[6] A. Canino, Minimax methods for singular elliptic equations with an application to a jumping problem, *J. Differential Equations* **221** (2006), no. 1, 210–223.

[7] A. Canino and M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations, *J. Convex Anal.* **11** (2004), no. 1, 147–162.

[8] A. Canino, M. Grandinetti and B. Sciunzi, Semilinear elliptic equations with singular nonlinearities, *J. Differential Equations* **255** (2013), no. 1, 4437–4447.

[9] A. Canino, M. Grandinetti and B. Sciunzi, A jumping problem for some singular semilinear elliptic equations, *Adv. Nonlinear Stud.* **4** (2014), 1037–1054.

[10] A. Canino and B. Sciunzi, A uniqueness result for some singular semilinear elliptic equations, *Commun. Contemp. Math.* (2015), DOI 10.1142/S0219199715500844.

[11] A. Canino, B. Sciunzi and A. Trombetta, Existence and uniqueness for p-Laplace equations involving singular nonlinearities, *NoDEA Nonlinear Differential Equations Appl.* (2016), DOI 10.1007/s00030-016-0361-6.

[12] F. Charro, L. Montoro and B. Sciunzi, Monotonicity of solutions of fully nonlinear uniformly elliptic equations in the half-plane, *J. Differential Equations* **251** (2011), 1562–1579.

[13] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Partial Differential Equations* **2** (1977), 193–222.

[14] F. Da Lio and B. Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations, *J. Eur. Math. Soc. (JEMS)* **9** (2007), no. 2, 317–330.

[15] J. A. Gatica, V. Oliker and P. Waltman, Singular nonlinear boundary value problems for second order ordinary differential equations, *J. Differential Equations* **79** (1989), 62–78.

[16] D. Giacchetti, F. Petitta and S. Segura De Leon, Elliptic equations having a singular quadratic gradient term and a changing sign datum, *Commun. Pure Appl. Anal.* **11** (2012), no. 5, 1875–1895.

[17] D. Giacchetti, F. Petitta and S. Segura De Leon, A priori estimates for elliptic problems with a strongly singular gradient term and a general datum, *Differential Integral Equations* **26** (2013), no. 9, 913–948.

[18] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[19] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 Ed.*, Springer, Berlin, 2001.

[20] N. Hirano, C. Saccon and N. Shioji, Multiple existence of positive solutions for singular elliptic problems with concave and convex nonlinearities, *Adv. Difference Equ.* **9** (2004), 197–220.

[21] N. Hirano, C. Saccon and N. Shioji, Brezis–Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem, *J. Differential Equations* **245** (2008), no. 8, 1997–2037.

[22] A. V. Lair and A. W. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, *J. Math. Anal. Appl.* **211** (1977), no. 2, 371–385.

[23] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.* **111** (1991), 721–730.

[24] C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains, *Comm. Partial Differential Equations* **16** (1991), no. 4–5, 585–615.

[25] J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.* **43** (1971), no. 4, 304–318.

[26] M. Squassina, Boundary behavior for a singular quasi-linear elliptic equation, *J. Math. Anal. Appl.* **393** (2012), no. 2, 692–696.

[27] C. A. Stuart, Existence and approximation of solutions of nonlinear elliptic equations, *Math. Z.* **147** (1976), 53–63.