Advanced Nonlinear Studies **15** (2015), 183–189

The Mean Curvature Equation with Oscillating Nonlinearity

Anderson L. A. de Araujo*

Departamento de Matemática
Universidade Federal de Viçosa,
Av Peter Henry Rolfs, s/n, CEP 36570-900, Viçosa, MG, Brasil
e-mail: anderson.araujo@ufv.br

Marcelo Montenegro†

Departamento de Matemática
Universidade Estadual de Campinas, IMECC,
Rua Sérgio Buarque de Holanda, 651, CEP 13083-859, Campinas, SP, Brasil
e-mail: msm@ime.unicamp.br

Received 18 April 2014
Communicated by Jean Mawhin

Abstract

We find a solution of the Dirichlet problem for the prescribed mean curvature equation $-div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(x,u)$ in Ω with $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$ and $f : \Omega \times [0,\infty) \to \mathbb{R}$ is an unbounded continuous function with oscillatory behavior near the origin.

2000 Mathematics Subject Classification. 35J65; 35J25; 35J20; 35J60; 49Q20; 53A10; 76B45.
Key words. Dirichlet problem; prescribed mean curvature; a priori bounds; existence of solution.

1 Introduction

We study the Dirichlet problem for the prescribed mean curvature equation

$$\begin{cases} -div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

*The author acknowledges the support by FAPESP, Brazil, grant 2013/22328-8.
†The author acknowledges the support of CNPQ

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$ and $f : \Omega \times [0, \infty) \to \mathbb{R}$ is an unbounded continuous function with oscillatory behavior near the origin in the sense that follows. There are two sequences of numbers $\theta_n > 0$ and $\beta_n > 0$ such that

$$\theta_n \to 0^+ \text{ and } \beta_n \to 0^+; \tag{1.2}$$

$$f(x, \theta_n \varphi_1) \leq 0, \text{ a.e. in } \Omega \ \forall n \in \mathbb{N}; \tag{1.3}$$

$$f(x, \theta_n \varphi_1) \to 0 \text{ uniformly in } \Omega \text{ as } n \to \infty; \tag{1.4}$$

$$\liminf_n \frac{f(x, \beta_n \varphi_1)}{\beta_n \varphi_1} > \lambda_1 \text{ uniformly in } \Omega \tag{1.5}$$

and for every $w \in H_0^1(\Omega)$, $w \geq 0$, $w \not\equiv 0$ a.e. in Ω there holds

$$\liminf_n \int_\Omega \frac{f(x, \beta_n \varphi_1)}{\beta_n} w dx > \lambda_1 \int_\Omega \varphi_1 w dx, \tag{1.6}$$

where λ_1 is the first eigenvalue for the Dirichlet problem in Ω, i.e., $\lambda_1 > 0$ and $\varphi_1 \in H_0^1(\Omega)$, $\varphi_1 > 0$ in Ω satisfy

$$-\Delta \varphi_1 = \lambda_1 \varphi_1 \text{ in } \Omega. \tag{1.7}$$

The function below is an example that satisfies the above hypotheses

$$f(x,t) = \begin{cases} \lambda_1 \sqrt{t\varphi_1(x)} \sin\left(\dfrac{\sqrt{\varphi_1(x)}}{\sqrt{t}}\right) & , \quad t > 0 \\ 0 & , \quad t = 0. \end{cases}$$

Define the sequences $\theta_n = \left(\frac{1}{2n\pi + \frac{3\pi}{2}}\right)^2$ and $\beta_n = \left(\frac{1}{2n\pi + \frac{\pi}{2}}\right)^2$. Then we have

$$f(x, \theta_n \varphi_1) = \frac{-\lambda_1 \varphi_1}{2n\pi + \frac{3\pi}{2}} \leq 0$$

and

$$\frac{f(x, \beta_n \varphi_1)}{\beta_n \varphi_1} = \lambda_1 \left(2n\pi + \frac{\pi}{2}\right).$$

Therefore, f, θ_n and β_n defined before satisfy (1.2), (1.3), (1.5) and (1.6).

The Dirichlet problem for the mean curvature equation has been studied in many papers e.g. [3], [6], [8], [10] and [17]. When $f(x, u)$ is replaced by $\lambda f(x, u)$, it has been proved in [10] the existence of a solution for $\lambda > 0$ large enough and with a very general f. One of the crucial steps is to bound the gradient and show that it is small whenever λ is large.

The existence of multiple nontrivial solutions of (1.1) with $f(x, u) = \lambda |u|^{p-2} u$ and p subcritical has been studied in [1] and [9]. The number of solutions increases for $\lambda > 0$ large, see also [5]. A result of different nature is obtained in [2], in fact the authors prove the existence of at least three nontrivial radial classical solutions by variational methods.

The problem in $\mathbb{R}^N$ has also been studied. The existence of ground states has been proved in [16] for $f(x, u) = -\lambda u + u^p$ with subcritical and $\lambda > 0$ small. In [4], existence and nonexistence of

radially symmetric grounds states and compact support solutions are investigated. The existence of positive, not necessarily radially symmetric, solutions in $\mathbb{R}^N$ are found in [12].

Oscillatory nonlinearities $f(x,u)$ have been studied in [14] and [15]. In [14] Theorems 3.15 and 3.16, the authors assume that $\liminf_{s\to 0^+} F(x,s)/s^2 \geq 0$ for a.e. x in Ω. As our example shows, we do not assume such condition. In dimension 1, the author in [11] studied a problem with oscillatory potential of type $\mu \sin(s)$ with $\mu > 0$ being a parameter. Here we can study problem (1.1) with f being an unbounded function, and as our example shows, we are able to treat the case where $f(x,s)$ is not a Lipschitz function in the s variable. Also we do not assume that the nonlinearity f is sublinear or superlinear. Roughly speaking we show the existence of a sequence of ordered sub and supersolutions, and consequently a solution exists between them.

We are here concerned with the existence and the multiplicity of weak positive solutions of (1.1). By a positive solution we mean a function $u \in C^1(\overline{\Omega})$ such that $u \geq 0$ and $u \not\equiv 0$ in Ω, satisfying $u = 0$ on $\partial\Omega$ and

$$\int_\Omega \frac{\nabla u \nabla w}{\sqrt{1+|\nabla u|^2}} dx = \int_\Omega f(x,u) w dx,$$

for every $w \in H_0^1(\Omega)$.

We state the main result.

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$. Assume that $f : \Omega \times [0,1) \to \mathbb{R}$ is a continuous function satisfying (1.2), (1.3), (1.5) and (1.6). Then there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, the problem (1.1) has at least one positive solution u_n. Furthermore, this solution can be chosen in such a way that $\|u_n\|_{C^1(\overline{\Omega})} \to 0$ as $n \to +\infty$.*

2 Proof of the main result

We say that a function $\beta \in C^1(\overline{\Omega})$ is a *supersolution* of (1.8) if

$$\beta(x) \geq 0 \text{ on } \partial\Omega$$

and

$$\int_\Omega a(|\nabla\beta|^2)\nabla\beta\nabla w dx \geq \int_\Omega f(x,\beta) w dx$$

for every $w \in H_0^1(\Omega)$, with $w \geq 0$ a.e. in Ω.

Similarly, $\alpha \in C^1(\overline{\Omega})$ is a *subsolution* of (1.8) if

$$\alpha(x) \leq 0 \text{ on } \partial\Omega$$

and

$$\int_\Omega a(|\nabla\alpha|^2)\nabla\alpha\nabla w dx \leq \int_\Omega f(x,\alpha) w dx$$

for every $w \in H_0^1(\Omega)$, with $w \geq 0$ a.e. in Ω.

To prove Theorem 1.1, define an auxiliary quasilinear problem which is uniformly elliptic

$$\begin{cases} -div\left(a(|\nabla u|^2)\nabla u\right) = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.8}$$

The function $a : [0, \infty) \to \mathbb{R}$ is defined by

$$a(v) = \begin{cases} \frac{1}{\sqrt{1+v}} & \text{in} \quad [0, 1[, \\ b(v-2)^2 + c & \text{in} \quad [1, 2[, \\ c & \text{in} \quad [2, +\infty[\end{cases} \tag{1.9}$$

with $b = \frac{1}{8\sqrt{2}}$ and $c = \frac{7}{8\sqrt{2}}$. This function a has been used in [5]. Note that a satisfies the structure conditions of [7], therefore, every weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.8) actually belongs to $C^{1,\sigma}(\overline{\Omega})$ for some $0 < \sigma < 1$.

We also introduce the functional $J : H_0^1(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_\Omega A(|\nabla u|^2) dx - \int_\Omega F(x, u) dx, \tag{1.10}$$

where $A(v) = \int_0^v a(s) ds$ and $F(x, u) = \int_0^u f(x, s) ds$. The following existence and regularity result can be easily derived by combining [15, Lemma 2.1] and [7], see also [5, Lemma 3.1].

Lemma 2.1 *Let Ω be as in Theorem 1.1, $a : [0, \infty) \to \mathbb{R}$ defined in (1.9) and $f : \Omega \times [0, \infty) \to \mathbb{R}$ is a continuous function. Assume that there exist a subsolution α and an supersolution β of (1.8), with $\alpha(x) \le \beta(x)$ in Ω. Then problem (1.8) has at least one solution u such that $\alpha(x) \le u(x) \le \beta(x)$ in Ω and*

$$J(u) = \min\{J(v) | v \in H_0^1(\Omega) \text{ and } \alpha(x) \le v(x) \le \beta(x) \text{ a.e. } \Omega\}.$$

Moreover, there are constants $\sigma > 0$ and $C > 0$, depending only on $\|\alpha\|_\infty$, $\|\beta\|_\infty$ and $\max\{|f(x, u)| \| x \in \overline{\Omega}, \alpha(x) \le u \le \beta(x)\}$, such that for every solution u of (1.8), with $\alpha(x) \le u(x) \le \beta(x)$ in Ω, thus $u \in C^{1,\sigma}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\sigma}(\overline{\Omega})} \le C.$$

We now prove Theorem 1.1.

Notice that for all $n \in \mathbb{N}$, $\theta_n \varphi_1$ is a supersolution of (1.8). Indeed, $\theta_n \varphi_1 \in C^1(\overline{\Omega})$, $\theta_n \varphi_1 \ge 0$ in $\partial\Omega$ and by (1.2), (1.3) and (1.9)

$$\int_\Omega a(|\nabla \theta_n \varphi_1|^2) \nabla \theta_n \varphi_1 \nabla w dx \ge \int_\Omega f(x, \theta_n \varphi_1) w dx$$

for every $w \in H_0^1(\Omega)$, with $w(x) \ge 0$ a.e. in Ω. Therefore, $\theta_n \varphi_1$ is a supersolution of (1.8).

Now we prove the existence of a subsolution to (1.8). Without loss of generality we assume that $0 < \beta_n < 1/n \ \forall n \in \mathbb{N}$. In fact, we will prove that there is $t_0 > 0$ such that for every $w \in H_0^1(\Omega)$, with $w \ge 0$ a.e. in Ω such that for $0 < t < t_0$ we have

$$\int_\Omega a(|\nabla t\varphi_1|^2) \nabla t\varphi_1 \nabla w dx \le \int_\Omega f(x, t\varphi_1) w dx, \tag{1.11}$$

i.e., to $0 < t < t_0$, $t\varphi_1$ is a subsolution of (1.8).

Suppose that (1.11) is false. Then for every $n \in \mathbb{N}$, there exists $w_1 \in H_0^1(\Omega)$, with $w_1 \ge 1$ a.e. in Ω, $w_1 \ne 0$ a.e. in Ω, such that for $0 < t_n < \frac{1}{n}$ we have

$$\int_\Omega a(|\nabla t_n \varphi_1|^2) \nabla t_n \varphi_1 \nabla w_1 dx > \int_\Omega f(x, t_n \varphi_1) w_1 dx.$$

Let $t_n = \beta_n$. Then

$$\int_\Omega a(|\nabla \beta_n\varphi_1|^2)\nabla\varphi_1\nabla w_1 dx > \int_\Omega \frac{f(x,\beta_n\varphi_1)}{\beta_n} w_1 dx.$$

Note that since $\beta_n \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} a(|\nabla \beta_n\varphi_1|^2) = 1 \text{ a.e. in } \Omega$$

and

$$|a(|\nabla \beta_n\varphi_1|^2)\nabla\varphi_1\nabla w_1| \le C|\nabla\varphi_1||\nabla w_1| \in L^1(\Omega).$$

By Lebesgue Dominated Convergence theorem and (1.7) we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega a(|\nabla \beta_n\varphi_1|^2)\nabla\varphi_1\nabla w_1 dx &= \int_\Omega \nabla\varphi_1\nabla w_1 dx\\ &= \int_\Omega -\Delta\varphi_1 w_1 dx\\ &= \lambda_1\int_\Omega \varphi_1 w_1 dx.\end{aligned}$$

On the other hand, from (1.5) by Fatou's lemma

$$\begin{aligned}\lambda_1\int_\Omega \varphi_1 w_1 dx &\le \int_\Omega \liminf_n \frac{f(x,\beta_n\varphi_1)}{\beta_n\varphi_1}\varphi_1 w_1 dx\\ &\le \liminf_n \int_\Omega \frac{f(x,\beta_n\varphi_1)}{\beta_n\varphi_1}\varphi_1 w_1 dx\\ &\le \limsup_n \int_\Omega \frac{f(x,\beta_n\varphi_1)}{\beta_n\varphi_1}\varphi_1 w_1 dx\\ &\le \limsup_n \int_\Omega a(\|\nabla \beta_n\varphi_1\|^2)\nabla\varphi_1\nabla w_1 dx\\ &= \lambda_1\int_\Omega \varphi_1 w_1 dx.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,\beta_n\varphi_1)}{\beta_n} w_1 dx = \lambda_1\int_\Omega \varphi_1 w_1 dx,$$

which contradicts (1.6), hence (1.11) is true.

The sub and supersolutions α and β can be ordered since for each n there exists $t_n = t_n(\theta_n) > 0$ such that

$$t_n\varphi_1 \le \theta_n\varphi_1$$

and $t_n\varphi_1$ is a subsolution of (1.8).

By Lemma 2.1 for each n the problem (1.8) has at least one solution u_n such that

$$t_n\varphi_1 \le u_n \le \theta_n\varphi_1 \text{ in } \Omega, \tag{1.12}$$

$$J(u_n) = \min\{J(v) | v \in H_0^1(\Omega) \text{ and } t_n\varphi_1(x) \le u_n \le \theta_n\varphi_1(x) \text{ a.e. } \Omega\}$$

and

$$\|u_n\|_{C^{1,\sigma}(\overline{\Omega})} \le C, \tag{1.13}$$

where C is independent of n, because $\theta_n \le \theta_0$ and

$$0 \le t_n\varphi_1 \le u_n \le \theta_n\varphi_1 \le \theta_0\varphi_1,$$

this is,

$$|u_n| \leq \theta_0 \varphi_1 \leq \theta_0 \sup_{\overline{\Omega}} \varphi_1 = M, \ \forall n \in \mathbb{N},$$

and C is independent of n by [7, Theorem 1].

By Arzelà-Ascoli Theorem and (1.13), there is a subsequence such that

$$u_{n_k} \to u \ \text{ in } \ C^1(\overline{\Omega}),$$

with $u = 0$ em $\partial\Omega$. But as $k \to \infty$, we obtain $t_{n_k} \to 0$ and $\theta_{n_k} \to 0$. By (1.12), we conclude that $u_{n_k} \to 0$ uniformly in Ω and therefore $u = 0$.

Hence, by (1.13) the sequence $(u_n)_n$ has the property that every subsequence has a subsequence that converges to zero in $C^1(\overline{\Omega})$. In conclusion,

$$u_n \to 0 \ \text{ in } \ C^1(\overline{\Omega}).$$

From the above results, there exists n_0 such that for $n \geq n_0$ we have

$$t_n \varphi_1 \leq u_n \leq \theta_n \varphi_1 \ \text{ in } \Omega$$

and

$$|\nabla u_n(x)| < 1 \ \text{ in } \Omega.$$

Consequently, for $n \geq n_0$, u_n is a solution of (1.1) and

$$\lim_{n\to\infty} \|u_n\|_{C^1(\overline{\Omega})} = 0.$$

References

[1] D. Bonheure, A. Derlet and S. de Valeriola, *On the multiplicity of nodal solutions of a prescribed mean curvature problem*, Math. Nachr. **286** (2013), 1072–1086.

[2] K. C. Chang and T. Zhang, *Multiple solutions of the prescribed mean curvature equation*, Inspired by S.S. Chern, 113–127, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ, 2006.

[3] C. V. Coffman and W. K. Ziemer, *A prescribed mean curvature problem on domains without radial symmetry*, SIAM J. Math. Anal. **22** (1991), 982–990.

[4] M. Conti and F. Gazzola, *Existence of ground states and free-boundary problems for the prescribed mean-curvature equation*, Adv. Differential Equations **7** (2002), 667–694.

[5] P. Habets and P. Omari, *Positive solutions of an indefinite prescribed mean curvature problem on a general domain*, Adv. Nonlinear Stud. **4** (2004), 1–13.

[6] K. Hayasida and M. Nakatani, *On the Dirichlet problem of prescribed mean curvature equations without h-convexity condition*, Nagoya Math. J. **157** (2000) 177–209.

[7] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. T.M.A. **12** (1988), 1203–1219.

[8] R. Lopéz, *An existence theorem of constant mean curvature graphs in Euclidean space*, Glasgow Math. J. **44** (2002) 455–461.

[9] S. Lorca and M. Montenegro, *Multiple solutions for the mean curvature equation*, Topol. Methods Nonlinear Anal. **35** (2010) 61–68.

[10] S. Lorca and P. Ubilla, *Partial differential equations involving subcritical, critical and supercritical nonlinearities*, Nonlinear Anal. **56** (2004) 119–131.

[11] J. Mawhin, *Radial solutions for Neumann problems involving extrinsing curvature and periodic nonlinearities*, Milan Journal of Mathematics **79** (2011), 95–112.

[12] E. S. Noussair and C. A. Swanson, *Positive solutions of the prescribed mean curvature equation in* $\mathbb{R}^N$, Indiana Univ. Math. J. **42** (1993), 559–570.

[13] F. Obersnel, *Classical and non-classical sign changing solutions of a one-dimensional autonomous prescribed curvature equation*, Adv. Nonlinear Stud. **7** (2007), 1–13.

[14] F. Obersnel and P. Omari, *Existence and multiplicity results for the prescribed mean cur- vature equation via lower and upper solutions*, Differential Integral Equations **22** (2009), 853–880.

[15] P. Omari and F. Zanolin, *Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential*, Comm. Partial Differential Equations **21** (1996), 721–733.

[16] L. A. Peletier and J. Serrin, *Ground states for the prescribed mean curvature equation*, Proc. Amer. Math. Soc. **100** (1987), 694–700.

[17] J. Ripoll, *Some existence results and gradient estimates of solutions of the Dirichlet problem for the constant mean curvature equation in convex domains*, J. Differential Equations **181** (2002) 230–241.