Research Article

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Construction of Solutions for a Nonlinear Elliptic Problem on Riemannian Manifolds with Boundary

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Abstract: Let (M, g) be a smooth compact n-dimensional Riemannian manifold ($n \geq 2$) with smooth $(n-1)$-dimensional boundary ∂M. We prove that the stable critical points of the mean curvature of the boundary generates $H^1(M)$ solutions for the following singularly perturbed elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = u^{p-1} & \text{in } M, \\ u > 0 & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

when ε is small enough. Here p is subcritical.

Keywords: Riemannian Manifold with Boundary, Nonlinear Elliptic Equations, Neumann Boundary Condition, Mean Curvature, Lyapunov–Schmidt

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1 Introduction

Let (M, g) be a smooth compact n-dimensional Riemannian manifold ($n \geq 2$) with boundary ∂M, which is the union of a finite number of smooth, connected and boundaryless $n-1$ submanifolds embedded in M. Here g denotes the Riemannian metric tensor. By the Nash theorem [17], we can consider (M, g) as a regular submanifold embedded in $\mathbb{R}^N$.

We consider the following Neumann problem:

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = u^{p-1} & \text{in } M, \\ u > 0 & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1.1}$$

where $p > 2$ if $n = 2$ and $2 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$, ν is the external normal to ∂M and ε is a positive parameter.

We are interested in finding solutions $u \in H^1(M)$ to problem (1.1), where

$$H^1(M) = \Big\{ u : M \to \mathbb{R} : \int_M |\nabla u|_g^2 + u^2 \, d\mu_g < \infty \Big\}$$

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and μ_g denotes the volume form on M associated to g. More precisely, we want to show that, for ε sufficiently small, we can construct a solution which has a peak near a stable critical point of the scalar curvature of the boundary, as stated in the following.

Definition 1.1. Let $f \in C^1(N, \mathbb{R})$, where (N, g) is a Riemannian manifold. We say that $K \subset N$ is a C^1-stable critical set of f if $K \subset \{x \in N : \nabla f(x) = 0\}$ and for any $\mu > 0$ there exists $\delta > 0$ such that if $h \in C^1(N, \mathbb{R})$ with

$$\max_{d_g(x,K)\le\mu} |f(x) - h(x)| + |\nabla f(x) - \nabla h(x)| \le \delta,$$

then h has a critical point x_0 with $d_g(x_0, K) \le \mu$. Here d_g denotes the geodesic distance associated to the Riemannian metric g.

Now we can state the main theorem.

Theorem 1.2. *Assume $K \subset \partial M$ is a C^1-stable critical set of the mean curvature of the boundary. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a solution $u_\varepsilon \in H^1(M)$ which concentrates at a point $\xi_0 \in K$ as ε goes to zero.*

Problem (1.1) in a flat domain has a long history. Starting from a problem of pattern formation in biology, Lin, Ni and Takagi [14, 18] showed the existence of a mountain pass solution for problem (1.1) and proved that this solution has exactly one maximum point which lies on the boundary of the domain. Moreover, in [19] Ni and Takagi proved that the maximum point of the solution approaches the maximum point of the mean curvature of the boundary when the perturbation parameter ε goes to zero.

Thenceforth, many papers were devoted to the study of problem (1.1) on flat domains. In particular, in [3, 20] it was proved that any stable critical point of the mean curvature of the boundary generated a single peaked solution whose peak approaches the critical point as ε vanishes. Moreover, in [10, 12, 13, 22] the existence of multipeak solutions whose peaks lies on the boundary was studied. We also mention the series of works [8, 9, 11, 21], in which the existence of solutions which have internal peaks was proved.

For the case of a manifold M, problem (1.1) was first studied in [2], where Byeon and Park proved the existence of a mountain pass solution when the manifold M is closed and when the manifold M has a boundary. They showed that for ε small, such a solution has a spike which approaches – as ε goes to zero – a maximum point of the scalar curvature when M is closed, and a maximum point of the mean curvature of the boundary when M has a boundary.

For the case of a closed manifold M, Benci, Bonanno and Micheletti [1] showed that problem (1.1) has at least $\operatorname{cat} M + 1$ nontrivial positive solutions when ε goes to zero. Here $\operatorname{cat} M$ denotes the Lusternik–Schnirelmann category of M. Moreover, in [15] the effect of the geometry of the manifold (M, g) was examined. In fact, it was shown that positive solution of the problem are generated by stable critical points of the scalar curvature of M.

More recently, for the case of a manifold M with boundary ∂M, we proved in [6] that problem (1.1) has at least $\operatorname{cat} \partial M$ non trivial positive solutions when ε goes to zero. We can compare the result of [6] with Theorem 1.2. In fact, in [7] the authors proved that, generically with respect to the metric g, the mean curvature of the boundary has nondegenerate critical points. More precisely, the set of metrics for which the mean curvature has only nondegenerate critical points is an open dense set among all the C^k metrics on M, $k \ge 3$. Thus, generically with respect to the metric, the mean curvature has $P_1(\partial M)$ nondegenerate (hence stable) critical points, where $P_1(\partial M)$ is the Poincaré polynomial of ∂M, namely $P_t(\partial M)$ evaluated in $t = 1$. So, generically with respect to the metric, problem (1.1) has $P_1(\partial M)$ solution, and we have $P_1(\partial M) \ge \operatorname{cat} \partial M$, where for many cases the strict inequality holds.

The paper is organized as follows. In Section 2 some preliminary notions are introduced, which are necessary for the comprehension of the paper. In Section 3 we study the variational structure of the problem, and we perform the finite dimensional reduction. In Section 4 the proof of Theorem 1.2 is sketched, while the expansion of the reduced functional is carried out in Section 5. Finally, the Appendix collects some technical lemmas.

2 Preliminary Results

In this section we give some general facts preliminary to our work. These results are widely present in the literature. We refer mainly to [2, 4, 5, 15] and the reference therein.

First of all we need to define a suitable coordinate chart on the boundary.

We know that on the tangent bundle of any compact Riemannian manifold $\mathcal{M}$ it is defined the exponential map $\exp\colon T\mathcal{M} \to \mathcal{M}$ which is of class C^∞. Moreover, there exist a constant $R_M > 0$, called radius of injectivity, and a finite number of $x_i \in \mathcal{M}$ such that $\mathcal{M} = \bigcup_{i=1}^{l} B_g(x_i, R_M)$ and $\exp_{x_i}\colon B(0, R_M) \to B_g(x_i, R_M)$ is a diffeomophism for all i. By choosing an orthogonal coordinate system $(y_1, \dots, y_n)$ of $\mathbb{R}^n$ and by identifying $T_{x_0}\mathcal{M}$ with $\mathbb{R}^n$ for $x_0 \in \mathcal{M}$, we can define using the exponential map the so-called normal coordinates. For $x_0 \in \mathcal{M}$, g_{x_0} denotes the metric read through the normal coordinates. In particular, we have $g_{x_0}(0) = \mathrm{Id}$. We set $|g_{x_0}(y)| = \det(g_{x_0}(y))_{ij}$ and $g^{ij}_{x_0}(y) = ((g_{x_0}(y))_{ij})^{-1}$.

Definition 2.1. Assume that q belongs to the boundary ∂M, and let $\bar y = (y_1, \dots, y_{n-1})$ be the Riemannian normal coordinates on the $(n-1)$-manifold ∂M at the point q. For a point $\xi \in M$ close to q, there exists a unique $\bar\xi \in \partial M$ such that $d_g(\xi, \partial M) = d_g(\xi, \bar\xi)$. We set $\bar y(\xi) \in \mathbb{R}^{n-1}$ the normal coordinates for $\bar\xi$ and $y_n(\xi) = d_g(\xi, \partial M)$. Then, we define a chart $\psi_q^\partial\colon \mathbb{R}^n_+ \to M$ such that $(\bar y(\xi), y_n(\xi)) = (\psi_q^\partial)^{-1}(\xi)$. These coordinates are called the *Fermi coordinates* at $q \in \partial M$. The Riemannian metric $g_q(\bar y, y_n)$ read through the Fermi coordinates satisfies $g_q(0) = \mathrm{Id}$.

We denote by d_g^∂ and $\exp^\partial$, respectively, the geodesic distance and the exponential map on by ∂M. By the compactness of ∂M, there exist R^∂ and a finite number of points $q_i \in \partial M$, $i = 1, \dots, k$, such that

$$I_{q_i}(R^\partial, R_M) := \{x \in M : d_g(x, \partial M) = d_g(x, \bar\xi) < R_M,\ d_g^\partial(q_i, \bar\xi) < R^\partial\}$$

form a covering of $(\partial M)_{R^M} := \{x \in M : d_g(x, \partial M) < R^M\}$, and on every I_{q_i} the Fermi coordinates are well defined. In the following, we choose $R = \min\{R^\partial, R_M\}$ in order to have the finite covering

$$M \subset \left\{\bigcup_{i=1}^{k} B(q_i, R)\right\} \cup \left\{\bigcup_{i=k+1}^{l} I_{\xi_i}(R, R)\right\},$$

where $k, l \in \mathbb{N}$, $q_i \in M \setminus \partial M$ and $\xi_i \in \partial M$.

For $p \in \partial M$, consider the projection $\pi_p\colon T_pM \to T_p\partial M$ on the tangent space $T_p\partial M$. For a pair of tangent vectors $X, Y \in T_p\partial M$, we define the second fundamental form $\mathrm{II}_p(X, Y) := \nabla_X Y - \pi_p(\nabla_X Y)$. The mean curvature at the boundary H_p, where $p \in \partial M$, is the trace of the second fundamental form.

If we consider the Fermi coordinates in a neighborhood of p, and denote by $(h_{ij})_{i,j=1,\dots,n-1}$ the matrix of the second fundamental form, then we have the well-known formulas

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y_n + O(|y|^2) \quad \text{for } i, j = 1, \dots, n-1, \tag{2.1}$$

$$g^{in}(y) = \delta_{in}, \tag{2.2}$$

$$\sqrt{g}(y) = 1 - (n-1)H(0)y_n + O(|y|^2), \tag{2.3}$$

where $(y_1, \dots, y_n)$ are the Fermi coordinates and, by definition of h_{ij},

$$H = \frac{1}{n-1}\sum_i^{n-1} h_{ii}. \tag{2.4}$$

Also, by [4, equation (3.2)], we have that

$$\frac{\partial^2}{\partial y_n \partial y_i}\sqrt{g}(y)\bigg|_{y=0} = -(n-1)\frac{\partial H}{\partial y_i}(0) \quad \text{for } i = 1, \dots, n-1. \tag{2.5}$$

It is well known that in $\mathbb{R}^n$ there exists a unique positive radially symmetric function $V(y) \in H^1(\mathbb{R}^n)$ which satisfies

$$-\Delta V + V = V^{p-1} \quad \text{on } \mathbb{R}^n. \tag{2.6}$$

Moreover, the function V as well as its derivative decay exponentially at infinity, that is,

$$\lim_{|y|\to\infty} V(|y|)|y|^{\frac{n-1}{2}}e^{|y|} = c \quad \text{and} \quad \lim_{|y|\to\infty} V'(|y|)|y|^{\frac{n-1}{2}}e^{|y|} = -c$$

for some $c > 0$. On the half space $\mathbb{R}^n_+ = \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n \geq 0\}$, we can define the function

$$U(y) = V|_{y_n \geq 0}.$$

The function U satisfies in $\mathbb{R}^n_+$ the following Neumann problem:

$$\begin{cases} -\Delta U + U = U^{p-1} & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial U}{\partial y_n} = 0 & \text{on } \{y_n = 0\}. \end{cases} \tag{2.7}$$

We set $U_\varepsilon(y) = U(\frac{y}{\varepsilon})$.

Lemma 2.2. *The space solution of the linearized problem*

$$\begin{cases} -\Delta\varphi + \varphi = (p-1)U^{p-2}\varphi & \textit{in } \mathbb{R}^n_+, \\ \dfrac{\partial\varphi}{\partial y_n} = 0 & \textit{on } \{y_n = 0\} \end{cases} \tag{2.8}$$

is generated by a linear combination of

$$\varphi^i = \frac{\partial U}{\partial y_i}(y) \quad \textit{for } i = 1, \dots, n-1.$$

Proof. It is trivial that every linear combination of φ^i is a solution of (2.8). We notice that $\frac{\partial U}{\partial y_n}$ is not a solution of (2.8) because the derivative on $\{y_n = 0\}$ is not zero.

For the converse, suppose that $\bar\varphi(y)$ is a solution of (2.8). Then, by even reflection around y_n, we can construct a solution $\tilde\varphi$ of

$$-\Delta\tilde\varphi + \tilde\varphi = (p-1)U^{p-2}\tilde\varphi \quad \text{in } \mathbb{R}^n \tag{2.9}$$

with $\frac{\partial\tilde\varphi}{\partial y_n} = 0$ on $y_n = 0$. But all solution of (2.9) with zero derivative on $y_n = 0$ are linear combinations of $\frac{\partial V}{\partial y_j}$ with $j = 1, \dots, n-1$. □

We endow $H^1(M)$ with the scalar product

$$\langle u, v\rangle_\varepsilon := \frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla u, \nabla v) + uv\, d\mu_g$$

and the norm $\|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2}$. We call H_ε the space H^1 equipped with the norm $\|\cdot\|_\varepsilon$. We also define L^p_ε as the space $L^p(M)$ endowed with the norm $|u|_{\varepsilon,p} = (\frac{1}{\varepsilon^n}\int_M u^p\, d\mu_g)^{1/p}$.

For any $p \in [2, 2^*)$ if $n \geq 3$ or for all $p \geq 2$ if $n = 2$, the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L_{\varepsilon,p}$ is a compact, continuous map, and $|u|_{\varepsilon,p} \leq c\|u\|_\varepsilon$ holds for some constant c not depending on ε. We define the adjoint operator $i^*_\varepsilon : L_{\varepsilon,p'} \hookrightarrow H_\varepsilon$ as

$$u = i^*_\varepsilon(v) \Leftrightarrow \langle u, \varphi\rangle_\varepsilon = \frac{1}{\varepsilon^n}\int_M v\varphi\, d\mu_g,$$

so we can rewrite problem (1.1) in an equivalent formulation

$$u = i^*_\varepsilon((u^+)^{p-1}).$$

Remark 2.3. We have that $\|i^*_\varepsilon(v)\|_\varepsilon \leq c|v|_{p',\varepsilon}$.

From now on we set, for the sake of simplicity,

$$f(u) = (u^+)^{p-1} \quad \text{and} \quad f'(u) = (p-1)(u^+)^{p-2}.$$

We want to split the space H_ε into a finite dimensional space generated by the solution of (2.8) and its orthogonal complement. For $\xi \in \partial M$ and $R > 0$ fixed, we consider on the manifold the functions

$$Z^i_{\varepsilon,\xi} = \begin{cases} \varphi^i_\varepsilon((\psi^\partial_\xi)^{-1}(x))\chi_R((\psi^\partial_\xi)^{-1}(x)), & x \in I_\xi(R) := I_\xi(R,R), \\ 0, & \text{elsewhere}, \end{cases} \tag{2.10}$$

where $\varphi^i_\varepsilon(y) = \varphi^i(\frac{y}{\varepsilon})$ and $\chi_R \colon B^{n-1}(0,R) \times [0,R) \to \mathbb{R}^+$ is a smooth cut off function such that $\chi_R \equiv 1$ on $B^{n-1}(0,R/2) \times [0,R/2)$ and $|\nabla\chi| \leq 2$.

In the following, for the sake of simplicity, we denote

$$D^+(R) = B^{n-1}(0,R) \times [0,R). \tag{2.11}$$

Let

$$K_{\varepsilon,\xi} := \operatorname{span}\{Z^1_{\varepsilon,\xi}, \dots, Z^{n-1}_{\varepsilon,\xi}\}.$$

We can split H_ε into the sum of the $(n-1)$-dimensional space and its orthogonal complement with respect of $\langle\cdot,\cdot\rangle_\varepsilon$, i.e.,

$$K^\perp_{\varepsilon,\xi} := \{u \in H_\varepsilon : \langle u, Z^i_{\varepsilon,\xi}\rangle_\varepsilon = 0\}.$$

We solve problem (1.1) by a Lyapunov–Schmidt reduction, defined by

$$W_{\varepsilon,\xi}(x) = \begin{cases} U_\varepsilon((\psi^\partial_\xi)^{-1}(x))\chi_R((\psi^\partial_\xi)^{-1}(x)), & x \in I_\xi(R) := I_\xi(R,R), \\ 0, & \text{elsewhere}. \end{cases}$$

We look for a function of the form $W_{\varepsilon,\xi} + \phi$ with $\phi \in K^\perp_{\varepsilon,\xi}$ such that

$$\Pi^\perp_{\varepsilon,\xi}\{W_{\varepsilon,\xi} + \phi - i^*_\varepsilon[f(W_{\varepsilon,\xi} + \phi)]\} = 0, \tag{2.12}$$

$$\Pi_{\varepsilon,\xi}\{W_{\varepsilon,\xi} + \phi - i^*_\varepsilon[f(W_{\varepsilon,\xi} + \phi)]\} = 0, \tag{2.13}$$

where $\Pi_{\varepsilon,\xi} \colon H_\varepsilon \to K_{\varepsilon,\xi}$ and $\Pi^\perp_{\varepsilon,\xi} \colon H_\varepsilon \to K^\perp_{\varepsilon,\xi}$ are, respectively, the projection on $K_{\varepsilon,\xi}$ and $K^\perp_{\varepsilon,\xi}$. We see that $W_{\varepsilon,\xi} + \phi$ is a solution of (1.1) if and only if $W_{\varepsilon,\xi} + \phi$ solves (2.12) and (2.13).

Hereafter, we collect a series of results which will be useful in the paper.

Definition 2.4. Given $\xi_0 \in \partial M$, using the normal coordinates on the sub manifold ∂M, we define

$$\mathcal{E}(y,x) = (\exp^\partial_{\xi(y)})^{-1}(x) = \left(\exp^\partial_{\exp^\partial_{\xi_0} y}\right)^{-1}(\exp^\partial_{\xi_0} \bar\eta) = \tilde{\mathcal{E}}(y,\bar\eta),$$

where $x, \xi(y) \in \partial M$, $y, \bar\eta \in B(0,R) \subset \mathbb{R}^{n-1}$ and $\xi(y) = \exp^\partial_{\xi_0} y$, $x = \exp^\partial_{\xi_0} \bar\eta$. Using the Fermi coordinates around ξ_0, in a similar way, we define

$$\mathcal{H}(y,x) = (\psi^\partial_{\xi(y)})^{-1}(x) = \left(\psi^\partial_{\exp^\partial_{\xi_0} y}\right)^{-1}(\psi^\partial_{\xi_0}(\bar\eta,\eta_n)) = \tilde{\mathcal{H}}(y,\bar\eta,\eta_n) = (\tilde{\mathcal{E}}(y,\bar\eta),\eta_n),$$

where $x \in M$, $\eta = (\bar\eta,\eta_n)$ with $\bar\eta \in B(0,R) \subset \mathbb{R}^{n-1}$ and $0 \leq \eta_n < R$, $\xi(y) = \exp^\partial_{\xi_0} y \in \partial M$ and $x = \psi^\partial_{\xi_0}(\eta)$.

Lemma 2.5. *Let $x = \psi^\partial_{\xi_0}(\varepsilon z)$, where $z = (\bar z, z_n)$ and $\xi(y) = \exp^\partial_{\xi_0}(y)$. Then, for $j = 1, \dots, n-1$, we have*

$$\frac{\partial}{\partial y_j} W_{\varepsilon,\xi(y)}(x)\Big|_{y=0} = \sum_{k=1}^{n-1}\Big[\frac{1}{\varepsilon}\chi_R(\varepsilon z)\frac{\partial}{\partial z_k}U(z) + U(z)\frac{\partial}{\partial z_k}\chi_R(\varepsilon z)\Big]\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\varepsilon\bar z)\Big|_{y=0}.$$

In order to prove Lemma 2.5, we need some preliminaries.

Lemma 2.6. *We have that*

$$\mathcal{E}(0,\bar\eta) = \bar\eta \qquad \text{for } \bar\eta \in \mathbb{R}^{n-1},$$

$$\frac{\partial\tilde{\mathcal{E}}_k}{\partial\eta_j}(0,\bar\eta) = \delta_{jk} \qquad \text{for } y \in \mathbb{R}^{n-1},\ j,k = 1,\dots,n-1,$$

$$\frac{\partial \tilde{\mathcal{E}}_k}{\partial y_j}(0,0) = -\delta_{jk} \quad \textit{for } j,k = 1,\dots,n-1,$$
$$\frac{\partial^2 \tilde{\mathcal{E}}_k}{\partial y_j \partial \eta_h}(0,0) = 0 \quad \textit{for } j,h,k = 1,\dots,n.$$

Proof. We recall that $\tilde{\mathcal{E}}(y,\bar\eta) = (\exp^{\partial}_{\xi(y)})^{-1}(\exp^{\partial}_{\xi_0}\bar\eta)$, so the first claim is obvious. For $y,\bar\eta \in B(0,R) \subset \mathbb{R}^{n-1}$, let us introduce

$$F(y,\bar\eta) = (\exp^{\partial}_{\xi_0})^{-1}(\exp^{\partial}_{\xi(y)}(\bar\eta)) \quad \text{and} \quad \Gamma(y,\bar\eta) = (y, F(y,\bar\eta)).$$

We notice that $\Gamma^{-1}(y,\beta), = (y,\tilde{\mathcal{E}}(y,\beta))$. We can easily compute the derivative of Γ. Given $\hat y,\hat\eta \in \mathbb{R}^{n-1}$, we have

$$\Gamma'(\hat y,\hat\eta)[y,\beta] = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ F'_y(\hat y,\hat\eta) & F'_\eta(\hat y,\hat\eta) \end{pmatrix}\begin{pmatrix} y \\ \beta \end{pmatrix},$$

and thus

$$(\Gamma^{-1})'(\hat y,\hat\eta)[y,\beta] = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ -(F'_\eta(\hat y,\hat\eta))^{-1}F'_y(\hat y,\hat\eta) & (F'_\eta(\hat y,\hat\eta))^{-1} \end{pmatrix}\begin{pmatrix} y \\ \beta \end{pmatrix}.$$

Here $y,\beta \in \mathbb{R}^{n-1}$. Now by direct computation we have that

$$F'_\eta(0,\hat\eta) = \mathrm{Id}_{\mathbb{R}^{n-1}} \quad \text{and} \quad F'_y(\hat y,0) = \mathrm{Id}_{\mathbb{R}^{n-1}},$$

and so $\frac{\partial\tilde{\mathcal{E}}_k}{\partial\eta_j}(0,\hat\eta) = ((F'_\eta(0,\hat\eta))^{-1})_{jk} = \delta_{jk}$ and $\frac{\partial\tilde{\mathcal{E}}_k}{\partial y_j}(0,0) = (-(F'_\eta(0,0))^{-1}F'_y(0,0))_{jk} = -\delta_{jk}$. For the last claim, we refer to [15, Lemma 6.4]. □

Lemma 2.7. *We have that*

$$\tilde{\mathcal{H}}(0,\bar\eta,\eta_n) = (\bar\eta,\eta_n) \quad \textit{for } \bar\eta \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+,$$
$$\frac{\partial\tilde{\mathcal{H}}_k}{\partial y_j}(0,0,\eta_n) = -\delta_{jk} \quad \textit{for } j,k = 1,\dots,n-1, \eta_n \in \mathbb{R}_+,$$
$$\frac{\partial\tilde{\mathcal{H}}_n}{\partial y_j}(y,\bar\eta,\eta_n) = 0 \quad \textit{for } j = 1,\dots,n-1, y,\bar\eta \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+,$$
$$\frac{\partial\tilde{\mathcal{H}}_k}{\partial \eta_n}(y,\bar\eta,\eta_n) = 0 \quad \textit{for } j,k = 1,\dots,n-1, \bar\eta \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+,$$
$$\frac{\partial^2\tilde{\mathcal{H}}_k}{\partial \eta_n\partial y_j}(y,\bar\eta,\eta_n) = 0 \quad \textit{for } j,k = 1,\dots,n-1, \bar\eta \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+.$$

Proof. The first three claims follow immediately by Definition 2.4 and Lemma 2.6. For the last two claims, observe that $\tilde{\mathcal{H}}_k(y,\bar\eta,\eta_n) = \tilde{\mathcal{E}}_k(y,\bar\eta)$, which does not depends on η_n nor its derivatives. □

We now prove the claimed result.

Proof of Lemma 2.5. By Definition 2.4, let $x = \psi^{\partial}_{\xi_0}(\eta) = \psi^{\partial}_{\xi_0}(\bar\eta,\eta_n)$ with $\eta = (\bar\eta,\eta_n) \in \mathbb{R}^n$, and $\xi(y) = \exp^{\partial}_{\xi_0}(y)$ with $y \in \mathbb{R}^{n-1}$. Then, we have that

$$W_{\varepsilon,\xi(y)}(x) = U\Big(\frac{\tilde{\mathcal{H}}(y,\eta)}{\varepsilon}\Big)\chi_R(\tilde{\mathcal{H}}(y,\eta)).$$

For fixed j, by Lemma 2.7, we have

$$\begin{aligned}\frac{\partial}{\partial y_j} W_{\varepsilon,\xi(y)}(x)\Big|_{y=0} &= \sum_{k=1}^{n} \frac{\partial}{\partial v_k}[\chi_R(\tilde{\mathcal{H}}(y,\eta))U_\varepsilon(\tilde{\mathcal{H}}(y,\eta))]\Big|_{\tilde{\mathcal{H}}(0,\eta)} \frac{\partial}{\partial y_i}\tilde{\mathcal{H}}_k(y,\eta)\Big|_{y=0} \\ &= \sum_{k=1}^{n-1} \frac{\partial}{\partial \eta_k}[\chi_R(\eta)U_\varepsilon(\eta)]\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\bar\eta)\Big|_{y=0} \\ &= \sum_{k=1}^{n-1} \frac{\partial}{\partial z_k}[\chi_R(\varepsilon z)U(z)]\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\varepsilon\bar z)\Big|_{y=0}\end{aligned}$$

Because $\tilde{\mathcal{H}}_k(y,\bar\eta,\eta_n) = \mathcal{E}_k(y,\bar\eta)$ for $k = 1,\dots,n-1$, using the change of variables $\eta = \varepsilon z = (\varepsilon\bar z,\varepsilon z_n)$, we get the claim. □

3 Reduction to Finite Dimensional Space

In this section we find a solution for equation (2.12). In particular, we prove that for all $\varepsilon > 0$ and for all $\xi \in \partial M$ there exists $\phi_{\varepsilon,\xi} \in K^{\perp}_{\varepsilon,\xi}$ solving (2.12). Here and hereafter, all the proofs are similar to [15]. So, for the sake of simplicity, we will underline the parts where differences appear, and sketch the rest of the proofs (we will provide precise references for each proof).

We introduce the linear operator $L_{\varepsilon,\xi}\colon K^{\perp}_{\varepsilon,\xi} \to K^{\perp}_{\varepsilon,\xi}$ with

$$L_{\varepsilon,\xi}(\phi) := \Pi^{\perp}_{\varepsilon,\xi}\{\phi - i^*_\varepsilon[f'(W_{\varepsilon,\xi})\phi]\}.$$

Thus, we can rewrite equation (2.12) as follows:

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi},$$

where

$$N_{\varepsilon,\xi} := \Pi^{\perp}_{\varepsilon,\xi}\{i^*_\varepsilon[f(W_{\varepsilon,\xi} + \phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi]\}$$

is the nonlinear term and

$$R_{\varepsilon,\xi} := \Pi^{\perp}_{\varepsilon,\xi}\{i^*_\varepsilon[f(W_{\varepsilon,\xi})] - W_{\varepsilon,\xi}\}$$

is the remainder term. The first step is to prove that the linear term is invertible.

Lemma 3.1. *There exist ε_0 and $c > 0$ such that for any $\xi \in \partial M$ and $\varepsilon \in (0, \varepsilon_0)$, we have that*

$$\|L_{\varepsilon,\xi}\|_\varepsilon \geq c\|\phi\|_\varepsilon \quad \textit{for any } \phi \in K^{\perp}_{\varepsilon,\xi}.$$

The proof of this lemma is given in the Appendix. We estimate now the remainder term $R_{\varepsilon,\xi}$.

Lemma 3.2. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in \partial M$ and for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq c\varepsilon^{1+n/p'}.$$

Proof. We proceed as in [15, Lemma 3.3]. We define on M the function $V_{\varepsilon,\xi}$ such that $W_{\varepsilon,\xi} = i^*_\varepsilon(V_{\varepsilon,\xi})$, and thus $-\varepsilon^2\Delta_g W_{\varepsilon,\xi} + W_{\varepsilon,\xi} = V_{\varepsilon,\xi}$.

It is well known (see [16, p. 134]), by the definition of the Laplace–Beltrami operator, that in a local chart, we have that

$$-\Delta v = -\Delta_g v + (g^{ij}_\xi - \delta_{ij})\frac{\partial^2}{\partial x_i \partial x_j} v - g^{ij}_\xi \Gamma^k_{ij}\frac{\partial}{\partial x_k} v,$$

where Δ is the Euclidean Laplace operator. Thus, if we define

$$\tilde V_{\varepsilon,\xi}(y) = V_{\varepsilon,\xi}(\psi^\partial_\xi(y)), \quad y \in D^+(R),$$

then we have

$$\begin{aligned} \tilde V_{\varepsilon,\xi}(y) &= -\varepsilon^2\Delta_g(U_\varepsilon\chi_R) + U_\varepsilon\chi_R \\ &= U^{p-1}_\varepsilon\chi_R - \varepsilon^2 U_\varepsilon\Delta\chi_R - 2\varepsilon^2\nabla U_\varepsilon\nabla\chi_R - \varepsilon^2(g^{ij}_\xi - \delta_{ij})\frac{\partial^2}{\partial y_i\partial y_j}(U_\varepsilon\chi_R) + \varepsilon^2 g^{ij}_\xi\Gamma^k_{ij}\frac{\partial}{\partial y_k}(U_\varepsilon\chi_R). \end{aligned} \tag{3.1}$$

Also, by Remark 2.3 and by the definition of $R_{\varepsilon,\xi}$,

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq \|i^*_\varepsilon f(W_{\varepsilon,\xi}) - W_{\varepsilon,\xi}\|_\varepsilon \leq c|W^{p-1}_{\varepsilon,\xi} - V_{\varepsilon,\xi}|_{p',\varepsilon}.$$

Finally, by the definition of $W_{\varepsilon,\xi}$ and by (3.1), we get

$$|W^{p-1}_{\varepsilon,\xi} - V_{\varepsilon,\xi}|^{p'}_{p',\varepsilon} = \int_{D^+(R)} |U^{p-1}_\varepsilon(y)\chi^{p-1}_R(y) - \tilde V_{\varepsilon,\xi}(y)|^{p'} |g_\xi(y)|^{1/2}\, dy$$

$$\leq c \int_{D^+(R)} |U_\varepsilon^{p-1}(y)(\chi_R^{p-1}(y) - \chi_R(y))|^{p'} dy$$
$$+ c\varepsilon^{2p'} \int_{D^+(R)} U_\varepsilon^{p'} |\Delta\chi_R|^{p'} dy + c\varepsilon^{2p'} \int_{D^+(R)} |\nabla U_\varepsilon \cdot \nabla\chi_R|^{p'} dy$$
$$+ c\varepsilon^{2p'} \int_{D^+(R)} |g_\xi^{ij}(y) - \delta_{ij}|^{p'} \left|\frac{\partial^2}{\partial y_i \partial y_j}(U_\varepsilon\chi_R)(y)\right|^{p'} dy$$
$$+ c\varepsilon^{2p'} \int_{D^+(R)} |g_\xi^{ij}(y)\Gamma_{ij}^k(y)|^{p'} \left|\frac{\partial}{\partial y_k}(U_\varepsilon\chi_R)(y)\right|^{p'} dy.$$

By the definition of χ_r and the exponential decay, using (2.1) and (2.2), we have

$$\begin{aligned}\varepsilon^{2p'} \int_{D^+(R)} |g_\xi^{ij}(y) - \delta_{ij}|^{p'} \left|\frac{\partial^2}{\partial y_i \partial y_j}(U_\varepsilon\chi_R)(y)\right|^{p'} dy &= \varepsilon^{2p'} \int_{D^+(R)} |g_\xi^{ij}(y) - \delta_{ij}|^{p'} \left|\frac{\partial^2}{\partial y_i \partial y_j}U_\varepsilon(y)\right|^{p'} dy + O(\varepsilon^{n+p'})\\
&\leq \varepsilon^n \int_{\mathbb{R}^n_+} |g_\xi^{ij}(\varepsilon z) - \delta_{ij}|^{p'} \left|\frac{\partial^2}{\partial z_i \partial z_j}U(z)\right|^{p'} dz + O(\varepsilon^{n+p'})\\
&\leq \varepsilon^{n+p'} \int_{\mathbb{R}^n_+} \left|\frac{\partial^2}{\partial z_i \partial z_j}U(z)\right|^{p'} dz + O(\varepsilon^{n+p'}) = O(\varepsilon^{n+p'}).\end{aligned}$$

The other terms can be estimated in a similar way. □

Using the fixed point theorem and the implicit function theorem, we can solve equation (2.12).

Proposition 3.3. *There exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in \partial M$ and for all $\varepsilon \in (0, \varepsilon_0)$, there exists a unique $\phi_{\varepsilon,\xi} = \phi(\varepsilon, \xi) \in K_{\varepsilon,\xi}^\perp$ which solves* (2.12). *Moreover,*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon < c\varepsilon^{1+n/p'}.$$

Finally, $\xi \mapsto \phi_{\varepsilon,\xi}$ is a C^1 map.

Proof. The proof is similar to that of [15, Proposition 3.5], which we refer to for all details. We want to solve (2.12) by a fixed point argument. We define the operator

$$T_{\varepsilon,\xi}\colon K_{\varepsilon,\xi}^\perp \to K_{\varepsilon,\xi}^\perp, \quad T_{\varepsilon,\xi}(\phi) = L_{\varepsilon,\xi}^{-1}(N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}).$$

By Lemma 3.1, $T_{\varepsilon,\xi}$ is well defined and we have

$$\begin{aligned}\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq c(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon),\\
\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon &\leq c(\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon)\end{aligned}$$

for some suitable constant $c > 0$. By the mean value theorem (and the properties of i^*), we get

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq c\big|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})\big|_{p/(p-2),\varepsilon}\|\phi_1 - \phi_2\|_\varepsilon.$$

By [15, Remark 3.4], we have that $|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{p/(p-2),\varepsilon} \ll 1$, provided $\|\phi_1\|_\varepsilon$ and $\|\phi_2\|_\varepsilon$ are small enough. Thus, there exists $0 < C < 1$ such that $\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq C\|\phi_1 - \phi_2\|_\varepsilon$. Also, with the same estimates we get

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq c(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1}).$$

This, combined with Lemma 3.2, gives

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq c(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon) \leq c(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1} + \varepsilon^{1+n/p'}).$$

So, there exists $c > 0$ such that $T_{\varepsilon,\xi}$ maps a ball of center 0 and radius $c\varepsilon^{1+n/p'}$ in $K_{\varepsilon,\xi}^\perp$ into itself and it is a contraction. So there exists a fixed point $\phi_{\varepsilon,\xi}$ with norm $\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq \varepsilon^{1+n/p'}$.

The regularity of $\phi_{\varepsilon,\xi}$ with respect to ξ is proved via the implicit function theorem. Let us define the functional

$$G\colon \partial M \times H_\varepsilon \to \mathbb{R}, \quad G(\xi, u) = \Pi^{\perp}_{\varepsilon,\xi}\{W_{\varepsilon,\xi} + \Pi^{\perp}_{\varepsilon,\xi}u + i^*_\varepsilon[f(W_{\varepsilon,\xi} + \Pi^{\perp}_{\varepsilon,\xi}u)]\} + \Pi_{\varepsilon,\xi}u.$$

We have that $G(\xi, \phi_{\varepsilon,\xi}) = 0$, and that the operator $\frac{\partial}{\partial u}G(\xi, \phi_{\varepsilon,\xi})\colon H_\varepsilon \to H_\varepsilon$ is invertible. This concludes the proof. □

4 Sketch of the Proof of Theorem 1.2

In Proposition 3.3 we found a function $\phi_{\varepsilon,\xi}$ solving (2.12). In order to solve (2.13), we define the functional

$$J_\varepsilon\colon H^1(M) \to \mathbb{R}, \quad J_\varepsilon(u) = \frac{1}{\varepsilon^n}\int_M \frac{1}{2}\varepsilon^2|\nabla u|^2_g + \frac{1}{2}u^2 - \frac{1}{p}(u^+)^p\, d\mu_g.$$

In what follows, we will often use the notation $F(u) = \frac{1}{p}(u^+)^p$.

By J_ε, we define the reduced functional $\tilde{J}_\varepsilon$ on ∂M as

$$\tilde{J}_\varepsilon(\xi) = J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}),$$

where $\phi_{\varepsilon,\xi}$ is uniquely determined by Proposition 3.3.

Remark 4.1. Our goal is to find the critical points for $\tilde{J}_\varepsilon$, since any critical point ξ for $\tilde{J}_\varepsilon$ corresponds to a function $\phi_{\varepsilon,\xi} + W_{\varepsilon,\xi}$ which solves equation (2.13).

At this point we give the expansion for the functional $\tilde{J}_\varepsilon$ with respect to ε. By Lemma 5.1 and Lemma 5.2,

$$\tilde{J}_\varepsilon(\xi) = C - \varepsilon H(\xi) + o(\varepsilon), \tag{4.1}$$

C^1 uniformly with respect to $\xi \in \partial M$ as ε goes to zero. Here $H(\xi)$ is the mean curvature of the boundary ∂M at ξ. If ξ_0 is a C^1-stable critical point for H, then in light of (4.1) and by the definition of C^1-stability, we have that for ε small enough, there exists a critical point ξ_ε for $\tilde{J}_\varepsilon$ close to ξ_0 , and we can prove Theorem 1.2.

5 Asymptotic Expansion of the Reduced Functional

In this section we study the asymptotic expansion of $\tilde{J}_\varepsilon(\xi)$ with respect to ε.

Lemma 5.1. *We have that*

$$\tilde{J}_\varepsilon(\xi) = J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + o(\varepsilon), \tag{5.1}$$

uniformly with respect to $\xi \in \partial M$ as ε goes to zero. Moreover, by setting $\xi(y) = \exp^{\partial}_{\xi}(y)$, $y \in B^{n-1}(0, r)$, we have

$$\begin{aligned}\frac{\partial}{\partial y_h}\tilde{J}_\varepsilon(\xi(y))\Big|_{y=0} &= \frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\Big|_{y=0}\\ &= \frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)})\Big|_{y=0} + o(\varepsilon)\end{aligned} \tag{5.2}$$

Proof. We split the proof into several steps.

Step 1: We prove (5.1). Using (2.12), we get

$$\begin{aligned}\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi}) = \frac{1}{2}\|\phi_{\varepsilon,\xi}\|^2_\varepsilon + \frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla W_{\varepsilon,\xi}, \nabla\phi_{\varepsilon,\xi}) + W_{\varepsilon,\xi}\phi_{\varepsilon,\xi} - f(W_{\varepsilon,\xi})\phi_{\varepsilon,\xi}\, d\mu_g\\ - \frac{1}{\varepsilon^n}\int_M F(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - F(W_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})\phi_{\varepsilon,\xi}\end{aligned}$$

$$= -\frac{1}{2}\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})]\phi_{\varepsilon,\xi}\, d\mu_g$$
$$- \frac{1}{\varepsilon^n}\int_M F(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - F(W_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})\phi_{\varepsilon,\xi}.$$

By the mean value theorem, we obtain that

$$|\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi})| \le \frac{1}{2}\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \left|\frac{1}{\varepsilon^n}\int_M f'(W_{\varepsilon,\xi} + t_1\phi_{\varepsilon,\xi})\phi_{\varepsilon,\xi}^2\right| + \left|\frac{1}{\varepsilon^n}\int_M f'(W_{\varepsilon,\xi} + t_2\phi_{\varepsilon,\xi})\phi_{\varepsilon,\xi}^2\right|$$

for some $t_1, t_2 \in (0,1)$. Now, by the properties of f', we can conclude that

$$|\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi})| \le c(\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \|\phi_{\varepsilon,\xi}\|_\varepsilon^p),$$

and in light of Proposition 3.3, we obtain (5.1).

Step 2: In order to prove (5.2), consider that

$$\begin{aligned}
&\frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - \frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)})\\
&\quad = J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\Big[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\Big] - J_\varepsilon'(W_{\varepsilon,\xi(y)})\Big[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\Big]\\
&\quad = [J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon'(W_{\varepsilon,\xi(y)})]\Big[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\Big] + J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\Big[\frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\Big]\\
&\quad =: L_1 + L_2.
\end{aligned}$$

Step 3: We estimate L_2. By (2.12), we have that

$$J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\Big[\frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\Big] = \sum_{l=1}^{n-1} c_\varepsilon^l \Big\langle Z^l_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon.$$

We prove that

$$\sum_{l=1}^{n-1}|c_\varepsilon^l| = O(\varepsilon). \tag{5.3}$$

Indeed, by (2.12) and (A.10), for some positive constant C, we have

$$J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})[Z^s_{\varepsilon,\xi(y)}] = \sum_{l=1}^{n-1} c_\varepsilon^l\langle Z^l_{\varepsilon,\xi(y)}, Z^s_{\varepsilon,\xi(y)}\rangle_\varepsilon = C\sum_{l=1}^{n-1} c_\varepsilon^l(\delta_{ls} + o(1)). \tag{5.4}$$

Also, since $\phi_{\varepsilon,\xi(y)} \in K^\perp_{\varepsilon,\xi(y)}$, we have

$$J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})[Z^s_{\varepsilon,\xi(y)}] = \frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla W_{\varepsilon,\xi(y)}, \nabla Z^s_{\varepsilon,\xi(y)}) + W_{\varepsilon,\xi(y)}Z^s_{\varepsilon,\xi(y)} - f(W_{\varepsilon,\xi(y)})Z^s_{\varepsilon,\xi(y)}\, d\mu_g$$
$$- \frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)})]Z^s_{\varepsilon,\xi(y)}\, d\mu_g.$$

By (2.1), (2.2) and (2.3), after a change of variables, we have

$$\begin{aligned}
&\frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla W_{\varepsilon,\xi(y)}, \nabla Z^s_{\varepsilon,\xi(y)}) + W_{\varepsilon,\xi(y)}Z^s_{\varepsilon,\xi(y)} - f(W_{\varepsilon,\xi(y)})Z^s_{\varepsilon,\xi(y)}\, d\mu_g\\
&\quad = \int_{\mathbb{R}^n_+} \nabla U\nabla\varphi^l + U\varphi^l - f(U)\varphi^l dz + O(\varepsilon) = O(\varepsilon).
\end{aligned}$$

Besides, by the mean value theorem, for some $t \in (0,1)$,

$$\begin{aligned}\bigg|\frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-f(W_{\varepsilon,\xi(y)})]Z^s_{\varepsilon,\xi(y)}\,d\mu_g\bigg| &= \bigg|\frac{1}{\varepsilon^n}\int_M [f'(W_{\varepsilon,\xi(y)}+t\phi_{\varepsilon,\xi(y)})]Z^s_{\varepsilon,\xi(y)}\phi_{\varepsilon,\xi(y)}\,d\mu_g\bigg|\\
&\le c\frac{1}{\varepsilon^n}\int_M (|W_{\varepsilon,\xi(y)}|^{p-2}+|\phi_{\varepsilon,\xi(y)}|^{p-2})|Z^s_{\varepsilon,\xi(y)}||\phi_{\varepsilon,\xi(y)}|\,d\mu_g\\
&\le c(\|W_{\varepsilon,\xi(y)}\|_\varepsilon^{p-2}+\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon^{p-2})\|Z^s_{\varepsilon,\xi(y)}\|_\varepsilon\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon\\
&= O(\varepsilon^{1+n/p'}) = o(\varepsilon).\end{aligned}$$

Hence, $J'_\varepsilon(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})[Z^s_{\varepsilon,\xi(y)}] = O\varepsilon)$ and, comparing with (5.4), we get (5.3). At this point, by (A.5), (5.3) and Proposition (3.3), we have that

$$\begin{aligned}|L_2| &\le \bigg|\sum_{l=1}^{n-1} c^l_\varepsilon\Big\langle Z^l_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon\bigg|\\
&= \bigg|\sum_{l=1}^{n-1} c^l_\varepsilon\Big\langle \frac{\partial}{\partial y_h}Z^l_{\varepsilon,\xi(y)}, \phi_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon\bigg|\\
&\le \bigg(\sum_{l=1}^{n-1}|c^l_\varepsilon|\bigg)\bigg\|\frac{\partial}{\partial y_h}Z^l_{\varepsilon,\xi(y)}\bigg\|_\varepsilon\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon\\
&\le O(\varepsilon^{1+n/p'}) = o(\varepsilon).\end{aligned}$$

Step 4: We estimate L_1. We have

$$\begin{aligned}L_1 &= \Big\langle \phi_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon - \frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-f(W_{\varepsilon,\xi(y)})]\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\,d\mu_g\\
&= \Big\langle \phi_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}], \frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon\\
&\quad - \frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-f(W_{\varepsilon,\xi(y)})-f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\,d\mu_g\\
&= \Big\langle \phi_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}], \frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}+\frac{1}{\varepsilon}Z^h_{\varepsilon,\xi(y)}\Big\rangle_\varepsilon\\
&\quad - \frac{1}{\varepsilon}\langle \phi_{\varepsilon,\xi(y)}, Z^h_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\rangle_\varepsilon\\
&\quad - \frac{1}{\varepsilon^n}\int_M [f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-f(W_{\varepsilon,\xi(y)})-f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\,d\mu_g\\
&=: A_1 + A_2 + A_3.\end{aligned}$$

For the first term, by (A.7), we have

$$\begin{aligned}A_1 &\le \|\phi_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\|_\varepsilon\bigg\|\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}+\frac{1}{\varepsilon}Z^h_{\varepsilon,\xi(y)}\bigg\|_\varepsilon\\
&\le c\varepsilon\|\phi_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\|_\varepsilon\\
&\le c\varepsilon(\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon + -|f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}|_{p',\varepsilon})\\
&\le \varepsilon\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon = o(\varepsilon).\end{aligned}$$

For the second term, in light of Proposition 3.3 and equation (A.4), we have

$$A_2 \le \frac{1}{\varepsilon}\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon\|Z^h_{\varepsilon,\xi(y)} - i^*_\varepsilon[f'(W_{\varepsilon,\xi(y)})Z^h_{\varepsilon,\xi(y)}]\|_\varepsilon = O(\varepsilon^{1+2n/p'}) = o(\varepsilon).$$

In order to estimate the last term, we have to consider separately the cases where $2 \le p < 3$ and $p \ge 3$.

We recall from [15, Remark 3.4] that

$$|f'(W_{\varepsilon,\xi}+v) - f'(W_{\varepsilon,\xi})| \le \begin{cases} c(p)|v|^{p-2}, & 2 < p < 3,\\ c(p)[W^{p-3}_{\varepsilon,\xi}|v| + |v|^{p-2}], & p \ge 3.\end{cases}$$

For $p \geq 3$, by the growth properties of f and using (A.5), we get

$$\begin{aligned} A_3 &\leq \frac{c}{\varepsilon^n} \int_M [|W_{\varepsilon,\xi(y)}|^{p-3} \phi^2_{\varepsilon,\xi(y)} + |\phi_{\varepsilon,\xi(y)}|^{p-1}] \Big| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big| \, d\mu_g \\ &\leq \|\phi_{\varepsilon,\xi(y)}\|^2_\varepsilon \Big\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big\|_\varepsilon + \|\phi_{\varepsilon,\xi(y)}\|^{p-1}_\varepsilon \Big\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big\|_\varepsilon \\ &\leq O(\varepsilon^{1+2n/p'}) + O(\varepsilon^{p-2+(p-1)n/p'}) = o(\varepsilon), \end{aligned}$$

since $p \geq 3$.

For $2 < p < 3$, in a similar way, we get

$$A_3 \leq \frac{c}{\varepsilon^n} \int_M \phi^{p-1}_{\varepsilon,\xi(y)} \Big| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big| \, d\mu_g \leq \|\phi_{\varepsilon,\xi(y)}\|^{p-1}_\varepsilon \Big\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big\|_\varepsilon = o(\varepsilon),$$

which concludes the proof. □

Lemma 5.2. *We have that*

$$J_\varepsilon(W_{\varepsilon,\xi}) = C - \varepsilon\alpha H(\xi) + o(\varepsilon),$$

C^0-uniformly with respect to ξ as ε goes to zero, where

$$C := \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\,dz \quad \text{and} \quad \alpha := \frac{(n-1)}{2} \int_{\mathbb{R}^n_+} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_n^3\,dz$$

Proof. By the definition of J_ε, we have

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) =& \frac{1}{2} \int_{D^+(R/\varepsilon)} \sum_{i,j=1}^n g^{ij}(\varepsilon z) \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i} \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_j} |g(\varepsilon z)|^{1/2}\,dz \\ &+ \int_{D^+(R/\varepsilon)} \Big[\frac{1}{2}(U(z)\chi_{R/\varepsilon}(z))^2 - \frac{1}{p}(U(z)\chi_{R/\varepsilon}(z))^p\Big] |g(\varepsilon z)|^{1/2}\,dz. \end{aligned}$$

By (2.1), (2.2) and (2.3), we easily get

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) =& \int_{D^+(R/\varepsilon)} \frac{1}{2}\sum_{i=1}^n \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i} \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i}\,dz + \int_{D^+(R/\varepsilon)} \frac{1}{2}(U(z)\chi_{R/\varepsilon}(z))^2 - \frac{1}{p}(U(z)\chi_{R/\varepsilon}(z))^p\,dz \\ &+ \varepsilon \int_{D^+(R/\varepsilon)} \sum_{i,j=1}^{n-1} h^{ij}(0) z_n \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i} \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_j}\,dz \\ &- \frac{n-1}{2}\varepsilon \int_{D^+(R/\varepsilon)} H(\xi) z_n \sum_{i=1}^n \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i} \frac{\partial(U(z)\chi_{R/\varepsilon}(z))}{\partial z_i}\,dz \\ &- \frac{n-1}{2}\varepsilon \int_{D^+(R/\varepsilon)} H(\xi) z_n (U(z)\chi_{R/\varepsilon}(z))^2\,dz + \frac{n-1}{p}\varepsilon \int_{D^+(R/\varepsilon)} H(\xi) z_n (U(z)\chi_{R/\varepsilon}(z))^p\,dz + o(\varepsilon) \\ =& \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\,dz - \varepsilon(n-1)H(\xi) \int_{\mathbb{R}^n_+} z_n\Big(\frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\Big)\,dz \\ &+ \varepsilon \sum_{i,j=1}^{n-1} h^{ij}(0) \int_{\mathbb{R}^n_+} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_i z_j z_n\,dz + o(\varepsilon). \end{aligned}$$

Using Lemma A.1, finally we have

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) =& \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\,dz \\ &- \varepsilon(n-1)H(\xi) \int_{\mathbb{R}^n_+} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_n^3\,dz + \varepsilon \sum_{i,j=1}^{n-1} h^{ij}(0) \int_{\mathbb{R}^n_+} \Big(\frac{U'(|z|)}{|z|}\Big)^2 z_i z_j z_n\,dz + o(\varepsilon). \end{aligned}$$

Now, by symmetry arguments and by (2.4) we have that

$$\begin{aligned}\sum_{i,j=1}^{n-1} h^{ij}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_n\,dz &= \sum_{i,j=1}^{n-1} h^{ij}(0)\delta_{ij}\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_iz_jz_n\,dz\\ &= \sum_{i=1}^{n-1} h^{ii}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_i^2z_n\,dz\\ &= (n-1)H(\xi)\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2z_n\,dz,\end{aligned}$$

and, by simple computations in polar coordinates,

$$\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_1^2z_n\,dz = \frac{1}{2}\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_n^3\,dz. \tag{5.5}$$

Concluding, we get

$$J_\varepsilon(W_{\varepsilon,\xi}) = \int_{\mathbb{R}^n_+}\frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\,dz - \varepsilon H(\xi)\Big[\frac{(n-1)}{2}\int_{\mathbb{R}^n_+}\Big(\frac{U'(|z|)}{|z|}\Big)^2 z_n^3dz\Big] + o(\varepsilon). \qquad \square$$

Lemma 5.3. *Let $\xi(y) = \exp_\xi^\partial(y)$ with $y \in B^{n-1}(0,r)$. Then, we have*

$$\frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)})\Big|_{y=0} = -\varepsilon\alpha\Big(\frac{\partial}{\partial y_h}H(\xi(y))\Big)\Big|_{y=0} + o(\varepsilon),$$

uniformly with respect to ξ as ε goes to zero.

Proof. For simplicity, we prove the claim for $h = 1$. The cases where $h = 2,\dots,n-1$ are straightforward. Let us consider first

$$\frac{\partial}{\partial y_1}\int_{I_\xi(R)}\frac{1}{2\varepsilon^n}W^2_{\varepsilon,\xi(y)}\,d\mu_g\Big|_{y=0} = \int_{I_\xi(R)}\frac{1}{\varepsilon^n}W_{\varepsilon,\xi}\frac{\partial}{\partial y_1}W_{\varepsilon,\xi(y)}\Big|_{y=0}\,d\mu_g.$$

By Lemma 2.5 and by the exponential decay of U, we have

$$\begin{aligned}\frac{\partial}{\partial y_1}\int\frac{1}{2\varepsilon^n}W^2_{\varepsilon,\xi(y)}d\mu_g\Big|_{y=0} &= \int_{\mathbb{R}^n_+}U(z)\chi_R(\varepsilon z)\Big[U(z)\frac{\partial\chi_R}{\partial z_k}(\varepsilon z) + \frac{1}{\varepsilon}\frac{\partial U}{\partial z_k}(z)\chi_R(\varepsilon z)\Big]\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}\,dz\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}U(z)\frac{\partial U}{\partial z_k}(z)\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}\,dz + o(\varepsilon)\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}U(z)\frac{U'(z)}{|z|}z_k\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}\,dz + o(\varepsilon),\end{aligned}$$

where $\tilde{\mathcal{E}}$ is defined in Definition 2.4. Expanding in ε, by Lemma 2.6 and by (2.3), we obtain

$$\begin{aligned}\frac{\partial}{\partial y_1}\int\frac{1}{2\varepsilon^n}W^2_{\varepsilon,\xi(y)}\,d\mu_g\Big|_{y=0} &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}U(z)\frac{U'(z)}{|z|}z_k\Big(-\delta_{1k} + \frac{1}{2}\varepsilon^2E^k_{ij}z_iz_j\Big)\Big(1 - \varepsilon(n-1)Hz_n + \frac{1}{2}\varepsilon^2G_{ls}z_lz_s\Big)\,dz + o(\varepsilon)\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}U(z)\frac{U'(z)}{|z|}\Big[-z_1(1 - \varepsilon(n-1)Hz_n + \frac{1}{2}\varepsilon^2G_{ls}z_lz_s) + \frac{1}{2}z_k\varepsilon^2E^k_{ij}z_iz_j\Big]\,dz + o(\varepsilon),\end{aligned}$$

where

$$E^k_{ij} = \frac{\partial^2}{\partial z_i\partial z_j}\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,z)\Big|_{y=0,z=0} \quad\text{and}\quad G_{ls} = \frac{\partial^2}{\partial z_i\partial z_j}|g(z)|^{1/2}\Big|_{z=0}.$$

By symmetry, the only terms remaining are the ones containing $z_r^2 z_n$, thus

$$\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)}\,d\mu_g\Big|_{y=0} = \varepsilon\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}[-z_1 G_{n1} z_n z_1 + z_k E^k_{kn} z_k z_n]\,dz + o(\varepsilon).$$

By (2.5) we have that $G_{n1}(0) = -(n-1)\frac{\partial H}{\partial z_1}(0)$, and in light of Lemma 2.7 we get $E^k_{kn} = 0$. We conclude that

$$\begin{aligned}\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)}\,d\mu_g\Big|_{y=0} &= \varepsilon\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}(n-1)\Big(\frac{\partial H}{\partial z_1}(0)\Big)z_n z_1^2\,dz + o(\varepsilon)\\ &= \varepsilon\int_{\mathbb{R}^n_+} \frac{\partial}{\partial z_1}\Big(\frac{1}{2}U^2(z)\Big)(n-1)\Big(\frac{\partial H}{\partial z_1}(0)\Big)z_n z_1\,dz + o(\varepsilon)\\ &= -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{2}U^2(z)z_n\,dz + o(\varepsilon).\end{aligned}$$

In the same way, we get that

$$\begin{aligned}\frac{\partial}{\partial y_1}\int \frac{1}{p\varepsilon^n} W^p_{\varepsilon,\xi(y)}\,d\mu_g\Big|_{y=0} &= \varepsilon\int_{\mathbb{R}^n_+} U^{p-1}(z)\frac{U'(z)}{|z|}(n-1)\Big(\frac{\partial}{\partial z_1}H(0)\Big)z_n z_1^2\,dz + o(\varepsilon)\\ &= -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{p}U^p(z)z_n\,dz + o(\varepsilon).\end{aligned}$$

Now we look at the last term

$$I := \frac{\partial}{\partial y_1}\int_{I_\xi(R)} \frac{\varepsilon^2}{2\varepsilon^n}|\nabla W_{\varepsilon,\xi(y)}|_g^2\,d\mu_g\Big|_{y=0} = \int_{I_\xi(R)}\frac{\varepsilon^2}{\varepsilon^n} g\Big(\nabla W_{\varepsilon,\xi}\nabla\frac{\partial}{\partial y_1}W_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\,d\mu_g,$$

and, again using Lemma 2.5 and the decay of U, we have

$$I = \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} g^{ij}(\varepsilon z)\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\Big(\frac{\partial U}{\partial z_k}\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,\varepsilon z)\Big)\Big|_{y=0}|g(\varepsilon z)|^{1/2}\,dz + o(\varepsilon).$$

Recall (2.1) (2.2) and (2.3), and set, with abuse of notation, $h_{in} = h_{nj} = 0$ for all $i, j = 1,\dots,n$. Then, we have

$$\begin{aligned}I = \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}&\Big(\delta_{ij} + 2\varepsilon h_{ij}z_n + \frac{1}{2}\varepsilon^2\gamma^{ij}_{rt}z_r z_t\Big)\Big(1 - \varepsilon(n-1)Hz_n + \frac{1}{2}\varepsilon^2 G_{ls}z_l z_s\Big)\\ &\times\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\Big(\frac{\partial U}{\partial z_k}\Big(-\delta_{1k} + \frac{1}{2}\varepsilon^2 E^k_{vw}z_v z_w\Big)\Big)\,dz + o(\varepsilon),\end{aligned}$$

where

$$E^k_{vw} = \frac{\partial^2}{\partial z_v\partial z_w}\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,z)\Big|_{y=0,z=0},\qquad G_{ls} = \frac{\partial^2}{\partial z_l\partial z_s}|g(z)|^{1/2}\Big|_{z=0}\quad\text{and}\quad \gamma^{ij}_{rt} = \frac{\partial^2}{\partial z_r\partial z_t}g^{ij}(z)\Big|_{z=0}.$$

More explicitly,

$$\begin{aligned}I = &-\frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}\,dz + \int_{\mathbb{R}^n_+}(n-1)Hz_n\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}\,dz - \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}G_{ls}z_l z_s\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}\,dz\\ &- 2\int_{\mathbb{R}^n_+}h_{ij}z_n\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}\,dz + 2\varepsilon\int_{\mathbb{R}^n_+}(n-1)Hh_{ij}z_n^2\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}\,dz - \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{rt}z_r z_t\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}\,dz\\ &+ \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\Big(\frac{\partial U}{\partial z_k}E^k_{vw}z_v z_w\Big)\,dz + o(\varepsilon)\\ =:&\ I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\end{aligned}$$

We easily have

$$I_1 = -\frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} \frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}\,dz = -\frac{1}{2\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial z_1}|\nabla U|^2\,dz = 0,$$

and, in a similar way and by integration by parts,

$$I_2 = (n-1)H(0)\frac{1}{2}\int_{\mathbb{R}^n_+} z_n\frac{\partial}{\partial z_1}|\nabla U|^2\,dz = 0,$$

$$I_3 = -\frac{1}{4}\varepsilon\int_{\mathbb{R}^n_+} G_{ls}(0)z_l z_s\frac{\partial}{\partial z_1}|\nabla U|^2\,dz = \varepsilon\frac{1}{2}\int_{\mathbb{R}^n_+} G_{1s}(0)z_s|\nabla U|^2\,dz.$$

Moreover, by symmetry, the only nonzero contribution comes from the term containing z_n, so by (2.5),

$$I_3 = \varepsilon\frac{1}{2}\int_{\mathbb{R}^n_+} G_{1n}(0)z_n|\nabla U|^2\,dz = -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{2}|\nabla U|^2 z_n\,dz.$$

Since h_{ij} is symmetric, we have

$$I_4 = -\int_{\mathbb{R}^n_+} h_{ij}(0)z_n\frac{\partial}{\partial z_1}\Big(\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_j}\Big)\,dz = 0,$$

and, similarly, by integration by parts, we also obtain that $I_5 = 0$.

For I_7, we have

$$I_7 = \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_i\partial z_k}E^k_{vw}z_v z_w\,dz + \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_k}(E^k_{vi}z_v + E^k_{iw}z_w)\,dz,$$

and, since $\frac{\partial U}{\partial z_i} = \frac{U'(z)}{|z|}z_i$,

$$\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_k}E^k_{vi}z_v = \int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 E^k_{vi}z_i z_k z_v\,dz. \tag{5.6}$$

The only nonzero integrals are the ones of the form $z_r^2 z_n$ and, since $E^n_{rr} = 0$ (Lemma 2.7), all the terms in (5.6) are 0. With a similar argument, since

$$\frac{\partial U}{\partial z_i\partial z_k} = \frac{U'(z)}{|z|}\delta_{ik} + \frac{U''(z)|z| - U'}{|z|^3}z_k z_i,$$

we can conclude that $I_7 = 0$.

Finally, let us consider I_6. Since g^{ij} is symmetric, integrating by parts gives

$$I_6 = -\frac{1}{4}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{rt}z_r z_t\frac{\partial}{\partial z_1}\Big(\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_j}\Big)\,dz = \frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{1t}\Big(\frac{U'(z)}{|z|}\Big)^2 z_i z_j z_t\,dz.$$

By (2.2), we have that $\gamma^{nj}_{1t} = \gamma^{in}_{1t} = 0$ for all $i, j, t = 1, \dots, n$. In addition, by symmetry, only the term which contains $z_r^2 z_n$ gives a nonzero contribution. Thus, by (2.1) and (5.5),

$$\begin{aligned}
I_6 &= \frac{1}{2}\varepsilon\sum_{i=1}^{n-1}\int_{\mathbb{R}^n_+}\gamma^{ii}_{1n}\Big(\frac{U'(z)}{|z|}\Big)^2 z_i^2 z_n\,dz\\
&= \varepsilon\sum_{i=1}^{n-1}\frac{\partial h^{ii}}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_i^2 z_n\,dz\\
&= \varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_1^2 z_n\,dz\\
&= \frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_n^3\,dz.
\end{aligned}$$

Concluding, we have

$$\begin{aligned}\frac{\partial}{\partial y_1} J_\varepsilon(W_{\varepsilon,\xi(y)})\Big|_{y=0} &= -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big[\frac{1}{2}|\nabla U|^2+\frac{1}{2}|U|^2-\frac{1}{p}|U|^p\Big] z_n\,dz\\ &\quad+\frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_n^3\,dz,\end{aligned}$$

and, by Lemma A.1,

$$\frac{\partial}{\partial y_1} J_\varepsilon(W_{\varepsilon,\xi(y)})\Big|_{y=0} = -\frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_n^3\,dz. \qquad \square$$

A Technical Lemmas

Here we collect a series of estimates that we used in the paper as well as the proofs of some lemmas which were previously claimed.

Lemma A.1. *We have that*

$$\begin{aligned}\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n\,dz &= \int_{\mathbb{R}^n_+}\Big(\frac{U'(z)}{|z|}\Big)^2 z_n^3\,dz\\ &= \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2 z_n\,dz+\frac{1}{2}\int_{\mathbb{R}^n_+}U^2 z_n\,dz-\frac{1}{p}\int_{\mathbb{R}^n_+}U^p z_n\,dz.\end{aligned}$$

Proof. We multiply $-\Delta U$ by $z_n^2\partial_{z_n}U$ and integrate by parts over $\mathbb{R}^n_+$, obtaining

$$\begin{aligned}-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U\,dz &= \int_{\mathbb{R}^n_+}(\Delta\partial_{z_n}U)z_n^2 U\,dz+2\int_{\mathbb{R}^n_+}\Delta U z_n U\,dz\\ &= -\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)z_n^2\partial_{z_i}U\,dz-2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)z_n\delta_{in}U\,dz\\ &\quad-2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}U)z_n\partial_{z_i}U\,dz-2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}U)\delta_{in}U\,dz\\ &= -\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)z_n^2\partial_{z_i}U\,dz-2\int_{\mathbb{R}^n_+}(\partial^2_{z_n}U)z_n U\,dz\\ &\quad-2\int_{\mathbb{R}^n_+}|\nabla U|^2 z_n\,dz-2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)U\,dz.\end{aligned}$$

Now, again by integration by parts,

$$\begin{aligned}-\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)z_n^2\partial_{z_i}U\,dz &= \int_{\mathbb{R}^n_+}(\partial_{z_n}U)z_n^2\Delta U\,dz+2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_n}U)\delta_{in}z_n\partial_{z_i}U\,dz\\ &= \int_{\mathbb{R}^n_+}(\partial_{z_n}U)z_n^2\Delta U\,dz+2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n\,dz.\end{aligned}$$

Thus,

$$\begin{aligned}-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U\,dz &= \int_{\mathbb{R}^n_+}(\partial_{z_n}U)z_n^2\Delta U\,dz+2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n\,dz\\ &\quad-2\int_{\mathbb{R}^n_+}(\partial^2_{z_n}U)z_n U\,dz-2\int_{\mathbb{R}^n_+}|\nabla U|^2 z_n\,dz-2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)U\,dz,\end{aligned}$$

that is,

$$-\int_{\mathbb{R}^n_+} \Delta U z_n^2 \partial_{z_n} U\,dz = \int_{\mathbb{R}^n_+} (\partial_{z_n} U)^2 z_n\,dz - \int_{\mathbb{R}^n_+} (\partial^2_{z_n} U) z_n U\,dz - \int_{\mathbb{R}^n_+} |\nabla U|^2 z_n\,dz - \int_{\mathbb{R}^n_+} (\partial_{z_n} U) U\,dz.$$

Now

$$0 = \int_{\mathbb{R}^n_+} \partial_{z_n}(z_n U \partial_{z_n} U)\,dz = \int_{\mathbb{R}^n_+} U\partial_{z_n} U + z_n(\partial_{z_n} U)^2 + z_n U \partial^2_{z_n} U\,dz,$$

and we get

$$-\int_{\mathbb{R}^n_+} \Delta U z_n^2 \partial_{z_n} U\,dz = 2\int_{\mathbb{R}^n_+} (\partial_{z_n} U)^2 z_n\,dz - \int_{\mathbb{R}^n_+} |\nabla U|^2 z_n\,dz. \tag{A.1}$$

In a similar way, we prove that

$$\int_{\mathbb{R}^n_+} U z_n^2 \partial_{z_n} U\,dz = -\int_{\mathbb{R}^n_+} U^2 z_n\,dz, \tag{A.2}$$

$$\int_{\mathbb{R}^n_+} U^{p-1} z_n^2 \partial_{z_n} U\,dz = -\frac{2}{p}\int_{\mathbb{R}^n_+} U^p z_n\,dz. \tag{A.3}$$

Now, multiplying by $z_n^2\partial_{z_n} U$ both terms of (2.7), integrating over $\mathbb{R}^n_+$ and using (A.1)–(A.3), we obtain the claim. □

The following lemma collects several estimates on $Z^j_{\varepsilon,\xi}$.

Lemma A.2. *There exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi_0 \in \partial M$ and for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|Z^h_{\varepsilon,\xi} - i^*[f'(W_{\varepsilon,\xi})Z^h_{\varepsilon,\xi}]\|_\varepsilon \le c\varepsilon^{1+N/p'}, \tag{A.4}$$

$$\Big\|\frac{\partial}{\partial y_h} Z^l_{\varepsilon,\xi(y)}\Big\|_\varepsilon = O\Big(\frac{1}{\varepsilon}\Big), \quad \Big\|\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)}\Big\|_\varepsilon = O\Big(\frac{1}{\varepsilon}\Big), \tag{A.5}$$

$$\Big\langle Z^l_{\varepsilon,\xi_0}, \Big(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)}\Big)_{|y=0}\Big\rangle = -\frac{1}{\varepsilon}c\delta_{lh} + o\Big(\frac{1}{\varepsilon}\Big), \tag{A.6}$$

$$\Big\|\frac{1}{\varepsilon} Z^h_{\varepsilon,\xi_0} + \Big(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)}\Big)_{|y=0}\Big\|_\varepsilon \le c\varepsilon \tag{A.7}$$

for $h = 1, \dots, n-1$, $l = 1, \dots, n$.

Proof. The proof of (A.4) is similar to that of Lemma 3.2, and will be omitted. The other three estimates are similar to [15, Lemmas 6.1–6.3], which we refer to for the proof of the claim. □

Proof of Lemma 3.1. We will prove it by contradiction. We assume that there exist sequences $\varepsilon_k \to 0$, $\xi_k \in \partial M$ with $\xi_k \to \xi \in \partial M$ and $\phi_k \in K^\perp_{\varepsilon_k,\xi_k}$ with $\|\phi\|_{\varepsilon_k} = 1$ such that

$$L_{\varepsilon_k,\xi_k}(\phi_k) = \psi_k \quad \text{with } \|\psi_k\|_{\varepsilon_k} \to 0 \text{ for } k \to +\infty.$$

By the definition of L_{ε_k,ξ_k}, there exists $\zeta_k \in K_{\varepsilon_k,\xi_k}$ such that

$$\phi_k - i^*_{\varepsilon_k}[f'(W_{\varepsilon_k,\xi_k})\phi_k] = \psi_k + \zeta_k. \tag{A.8}$$

We prove that $\|\zeta_k\|_{\varepsilon_k} \to 0$ for $k \to +\infty$. Let $\zeta_k = \sum_{j=1}^{n-1} a^k_j Z^j_{\varepsilon_k,\xi_k}$ with $Z^i_{\varepsilon,\xi}$ being defined in (2.10). By using the fact that $\phi_k, \psi_k \in K^\perp_{\varepsilon_k,\xi_k}$ and (A.8), we have

$$\begin{aligned}\sum_{j=1}^{n-1} a^k_j \langle Z^j_{\varepsilon_k,\xi_k}, Z^h_{\varepsilon_k,\xi_k}\rangle_{\varepsilon_k} &= -\langle i^*_{\varepsilon_k}[f'(W_{\varepsilon_k,\xi_k})\phi_k], Z^h_{\varepsilon_k,\xi_k}\rangle_{\varepsilon_k}\\ &= -\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k Z^h_{\varepsilon_k,\xi_k}\,d\mu_g.\end{aligned} \tag{A.9}$$

By elementary properties of φ^j, we have that

$$\langle Z^j_{\varepsilon_k,\xi_k}, Z^j_{\varepsilon_k,\xi_k}\rangle_{\varepsilon_k} = C\delta_{jh} + o(1) \quad \text{for all } j,h = 1,\dots,n-1, \tag{A.10}$$

where C is a positive constant.

We set

$$\tilde\phi_k := \begin{cases} \phi_k(\psi^\partial_{\xi_k}(\varepsilon_k z))\chi_R(\varepsilon_k z), & z \in D^+(R),\\ 0, & \text{otherwise.}\end{cases}$$

We get easily that $\|\tilde\phi_k\|_{H^1(\mathbb{R}^n)} \le c\|\phi_k\|_{\varepsilon_k} \le c$ for some positive constant c. Thus, there exists $\tilde\phi \in H^1(\mathbb{R}^n)$ such that $\tilde\phi_k \to \tilde\phi$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for all $2 \le p < 2^*$ if $n \ge 3$ or $p \ge 2$ if $n = 2$.

We recall that $\phi_k \in K^\perp_{\varepsilon_k,\xi_k}$, so

$$\begin{aligned}
-\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k Z^h_{\varepsilon_k,\xi_k}\,d\mu_g &= \langle \phi_k, Z^h_{\varepsilon_k,\xi_k}\rangle_\varepsilon - \frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k Z^h_{\varepsilon_k,\xi_k}\,d\mu_g\\
&= \frac{1}{\varepsilon_k^n}\int_M [\varepsilon_k^2\nabla\phi_k\nabla Z^h_{\varepsilon_k,\xi_k} + Z^h_{\varepsilon_k,\xi_k}\phi_k - f'(W_{\varepsilon_k,\xi_k})\phi_k Z^h_{\varepsilon_k,\xi_k}]\,d\mu_g\\
&= \int_{D^+(R/\varepsilon_k)} \Bigg[\sum_{l,m=1}^n g^{lm}(\varepsilon_k z)\frac{\partial\tilde\phi_k}{\partial z_l}\frac{\partial(\varphi^h(z))}{\partial z_m} + \tilde\phi_k\varphi^h(z)\Bigg]|g(\varepsilon_k z)|^{1/2}\,dy\\
&\qquad - \int_{D^+(R/\varepsilon_k)} f'(U(z)\chi_R(\varepsilon_k z))\tilde\phi_k\varphi^h(z)|g(\varepsilon_k z)|^{1/2}\,dz + o(1)\\
&= \int_{\mathbb{R}^n} \nabla\tilde\phi\nabla\varphi^h + \tilde\phi\varphi^h - f'(U)\tilde\phi\varphi^h\,dz + o(1) = o(1),
\end{aligned}$$

because φ^h is a weak solution of the linearized problem (2.8). So we can rewrite (A.9), obtaining

$$ca^k_h + o(1) = \sum_{j=1}^{n-1} a^k_j\langle Z^j_{\varepsilon_k,\xi_k}, Z^h_{\varepsilon_k,\xi_k}\rangle_{\varepsilon_k} = -\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k Z^h_{\varepsilon_k,\xi_k}\,d\mu_g = o(1),$$

so $a^k_h \to 0$ for all h while $k \to +\infty$, and thus $\|\zeta_k\|_{\varepsilon_k} \to 0$ for $k \to +\infty$.

Setting $u_k := \phi_k - \psi_k - \zeta_k$, (A.8) can be read as

$$\begin{cases} -\varepsilon_k^2\Delta_g u_k + u_k = f'(W_{\varepsilon_k,\xi_k})u_k + f'(W_{\varepsilon_k,\xi_k})(\psi_k + \zeta_k) & \text{in } M,\\ \dfrac{\partial u_k}{\partial\nu} = 0 & \text{on } \partial M.\end{cases} \tag{A.11}$$

Multiplying (A.11) by u_k and integrating by parts, we get

$$\|u_k\|_{\varepsilon_k} = \frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})u_k^2 + f'(W_{\varepsilon_k,\xi_k})(\psi_k + \zeta_k)u_k. \tag{A.12}$$

By Hölder's inequality, and recalling that $|u|_{\varepsilon,p} \le c\|u\|_\varepsilon$, we have

$$\begin{aligned}
\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})(\psi_k+\zeta_k)u_k &\le \Big(\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{n/2}\Big)^{2/n}|u_k|^{(n-2)/(2n)}_{\varepsilon_k,2n/(n-2)}|\psi_k+\zeta_k|^{(n-2)/(2n)}_{\varepsilon_k,2n/(n-2)}\\
&\le c\Big(\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{n/2}\Big)^{2/n}\|u_k\|^{(n-2)/(2n)}_{\varepsilon_k}\|\psi_k+\zeta_k\|^{(n-2)/(2n)}_{\varepsilon_k}.
\end{aligned} \tag{A.13}$$

Now,

$$\begin{aligned}
\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{n/2}\,d\mu_g &\le \frac{1}{\varepsilon_k^n}\int_{I_{\xi_k}(R)} (U_\varepsilon((\psi^\partial_{\xi_k})^{-1}(x)))^{n(p-2)/2}\,d\mu_g\\
&\le c\int_{D^+(R/\varepsilon)} (U(z))^{n(p-2)/2}\,dz \le c
\end{aligned} \tag{A.14}$$

for some positive constant c.

Combining (A.12), (A.13) and (A.14), and recalling that $\|u_k\|_{\varepsilon_k} \to 1$ and $\|\psi_k + \zeta_k\|_{\varepsilon_k} \to 0$ while $k \to +\infty$, we get

$$\frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k,\xi_k}) u_k^2 \to 1 \quad \text{while } k \to +\infty. \tag{A.15}$$

We will see how this leads us to a contradiction.

We set

$$\tilde{u}_k(z) := u_k(\psi^{\partial}_{\xi_k}(\varepsilon_k z))\chi_R(\varepsilon_k z) \quad \text{for } z \in \mathbb{R}^n_+,$$

and have

$$\|\tilde{u}_k\|_{H^1(\mathbb{R}^n)} \leq c\|u_k\|_{\varepsilon_k} \leq c.$$

So, up to subsequence, there exists $\tilde{u} \in H^1(\mathbb{R}^n_+)$ such that $\tilde{u}_k \to \tilde{u}$ weakly in $H^1(\mathbb{R}^n_+)$and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n_+)$, $p \in (2, 2^*)$ if $n \geq 3$ or $p > 2$ if $n = 2$. By (A.11) we deduce that

$$\begin{cases} -\Delta\tilde{u} + \tilde{u} = f'(U)\tilde{u} & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial \tilde{u}}{\partial x_n} = 0 & \text{on } \{x_n = 0\}. \end{cases} \tag{A.16}$$

We prove also that

$$\langle \varphi^h, \tilde{u}\rangle_{H^1} = 0 \quad \text{for all } h \in 1, \dots, n-1. \tag{A.17}$$

In fact, since $\phi_k, \psi_k \in K^{\perp}_{\varepsilon,\xi}$ and $\|\zeta_k\|_{\varepsilon_k} \to 0$, we have

$$|\langle Z^h_{\varepsilon_k,\xi_k}, u_k\rangle_{\varepsilon_k}| = |\langle Z^h_{\varepsilon_k,\xi_k}, \zeta_k\rangle_{\varepsilon_k}| \leq \|Z^h_{\varepsilon_k,\xi_k}\|_{\varepsilon_k}\|\zeta_k\|_{\varepsilon_k} = o(1). \tag{A.18}$$

On the other hand, by direct computation, we get

$$\begin{aligned} \langle Z^h_{\varepsilon_k,\xi_k}, u_k\rangle_{\varepsilon_k} &= \frac{1}{\varepsilon_k^n}\int_M \varepsilon_k^2 g(\nabla Z^h_{\varepsilon_k,\xi_k} \nabla u_k) + Z^h_{\varepsilon_k,\xi_k} u_k \\ &= \int_{D^+(R/\varepsilon_k)} \sum_{l,m=1}^n g^{lm}(\varepsilon_k z) \frac{\partial(\varphi^h(z)\chi_R(\varepsilon_k z))}{\partial z_l} \frac{\partial \tilde{u}_k}{\partial z_m} |g(\varepsilon_k z)|^{\frac{1}{2}}\, dz + \int_{D^+(R/\varepsilon_k)} \varphi^h(z)\chi_R(\varepsilon_k z)\tilde{u}_k |g(\varepsilon_k z)|^{\frac{1}{2}}\, dz \\ &= \int_{\mathbb{R}^n_+} (\nabla\varphi^h \nabla\tilde{u} + \varphi^h\tilde{u})\, dz + o(1). \end{aligned} \tag{A.19}$$

So, by (A.18) and (A.19), we obtain (A.17).

Now (A.17) and (A.16) imply that $\tilde{u} = 0$. Thus,

$$\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k}(x))u_k^2(x)\, d\mu_g \leq \frac{1}{\varepsilon_k^n}\int_{I_g(R)} f'(U_\varepsilon((\psi^{\partial}_{\xi_k})^{-1}(x)))u_k^2(x)\, d\mu_g = c\int_{D^+(R/\varepsilon_k)} f'(U(z))\tilde{u}_k^2(z) = o(1),$$

which contradicts (A.15). This concludes the proof. □

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