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A Priori Bounds and Existence of Solutions for Slightly Superlinear Elliptic Problems

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Abstract

We consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = a(x) f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where a is a continuous function which may change sign and f is superlinear but does not satisfy the standard Ambrosetti-Rabinowitz condition. We show that if f is regularly varying of index one at infinity then the above problem has a positive solution, provided a satisfies some additional assumptions. Our proof uses an abstract theorem due to L. Jeanjean on critical points of functionals with mountain-pass structure, and it relies on the obtention of a priori bounds for positive solutions.

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1 Introduction and results

In this paper we are concerned with the existence of positive solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u = a(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 3$), $a \in C(\overline{\Omega})$ and $f \in C[0, +\infty)$. Under different assumptions on the nonlinear term f, this problem has received considerable attention in the past years, especially in the case where a is positive in $\overline{\Omega}$. We shall not assume this condition and allow a to change sign.

One of the most fruitful ways to deal with problem (1.1) is the use of variational techniques, which take into account that weak solutions of (1.1) are critical points in $H_0^1(\Omega)$ of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a(x)F(u),$$

where $F(t) = \int_0^t f(s)ds$. A restriction on the growth of f at infinity is then mandatory to guarantee that I is well-defined and C^1. Therefore it is usually assumed that f grows subcritically, that is, there exists η with $0 < \eta < \frac{N+2}{N-2}$ and $C > 0$ such that $|f(t)| \leq C(1 + t^\eta)$ for $t \geq 0$. Adding some conditions on f often allows to obtain existence of critical points via some standard minimax theorems.

A typical situation arises when f is superlinear at 0 and at infinity, that is,

$$\lim_{t\to 0+} \frac{f(t)}{t} = 0 \tag{1.2}$$

and

$$\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty, \tag{1.3}$$

in which case I has a mountain pass structure. If I satisfies the well-known Palais-Smale condition, then the mountain pass theorem provides a critical point of I (see [6] for the original theorem and also [18], [35] for further related results). Note that under (1.2) we have $f(0) = 0$, so that (1.1) admits the trivial solution $u \equiv 0$.

To ensure that I verifies the Palais-Smale condition, a standard assumption on f was introduced in [6] and has been later frequently used in the literature. Namely, the so called Ambrosetti-Rabinowitz condition ((AR) for short):

$$\text{There exist } \theta > 2 \text{ and } t_0 > 0 \text{ such that } 0 < \theta F(t) \leq tf(t) \quad \forall t \geq t_0.$$

However, when f does not satisfy (AR), the fulfillment of the Palais-Smale condition is harder to check. The main feature of these nonlinearities is that they are *slightly superlinear* at infinity, in the following sense:

$$\lim_{t\to\infty} \frac{f(t)}{t^p} = 0, \quad \text{for all } p > 1.$$

A typical example of a such nonlinearity is

$$f(t) = t(\log(1+t))^q \tag{1.4}$$

where $q > 0$. Several authors have weakened the (AR) condition to encompass these nonlinearities and proved that I satisfies the Palais-Smale condition when $a > 0$ in Ω, cf. [19, 26, 31].

However, in our present setting, the difficulty due to the failure of the (AR) condition is aggravated by the possible change of sign in a. In this case, the functional I still preserves the mountain pass structure but the Palais-Smale condition seems difficult to handle. As a matter of fact, when a changes sign and f is not powerlike, this issue has been addressed in [3] (see also [2]), where the authors assume that f is superlinear and behaves like a power at infinity, i.e.

$$(\mathcal{G}) \qquad \text{there exists } p > 1 \text{ such that } \lim_{t\to\infty} \frac{f(t)}{t^p} = 1.$$

Note that this condition implies (AR). Furthermore, in [3] the authors assume the so called *thick zero set* condition on $a \in C(\overline{\Omega})$:

$$(\mathcal{AT}) \qquad \overline{(\Omega_+)} \cap \overline{(\Omega_-)} = \emptyset,$$

where

$$\Omega_+ := \{x \in \Omega;\ a(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega;\ a(x) < 0\}.$$

In [28], still under $(\mathcal{G})$, the authors were able to remove $(\mathcal{AT})$, but at the expense of some alternative strong conditions on a. To the best of our knowledge, only [21] has dealt with the situation where a changes sign and f does not satisfy (AR), more precisely, for the special case $f(t) = t\log(1+t)$.

We shall use the method from [19] (see also [20]), which is based on embedding the functional I into a one-parameter family of functionals I_λ, where $\lambda \in [\frac{1}{2}, 1]$, say, and $I_1 = I$. If every functional I_λ possesses the mountain pass structure, then for almost every $\lambda \in [\frac{1}{2}, 1]$ a *bounded* Palais-Smale sequence can be obtained for I_λ. The key to succeed in finding a critical point of I is then the obtention of *a priori* bounds for critical points of I_λ for $\lambda \in [\frac{1}{2}, 1]$. We shall obtain these *a priori* bounds by the scaling method introduced in [12], although not in a standard way, since we are not assuming that f behaves like a power at infinity (see [34], where a similar problem has been considered in a fully nonlinear setting).

Instead of focusing on the model (1.4), we shall consider a larger class of nonlinearities. To this end, we recall the following definition: a positive, measurable function defined in $(0, +\infty)$ is called *regularly varying* (at infinity) of index $q \in \mathbb{R}$ if

$$\lim_{t\to+\infty} \frac{f(\lambda t)}{f(t)} = \lambda^q, \qquad \text{for every } \lambda > 0. \tag{1.5}$$

When $q = 0$, f is called *slowly varying*.

In this work, we will be particularly interested in the case $q = 1$, that is, in functions satisfying

$$\lim_{t\to+\infty} \frac{f(\lambda t)}{f(t)} = \lambda, \qquad \text{for every } \lambda > 0. \tag{1.6}$$

Examples of such functions (also satisfying (1.2) and (1.3)) are, besides (1.4),

$$f(t) = t\log(1+\log(1+t))^q \quad \text{and} \quad f(t) = t(e^{(\log(1+t))^q} - 1),$$

where $q > 0$. The first systematic analysis of regularly varying functions seems to be due to J. Karamata in [23, 24]. See also [29, 32] for a comprehensive introduction.

As for the weight function a, we will assume throughout that $a \in C(\overline{\Omega})$, and $\Omega_+ \neq \emptyset$, where $\Omega_+ := \{x \in \Omega : \ a(x) > 0\}$. Moreover we assume either of the following three conditions:

(a) $a > 0$ in $\overline{\Omega}$;

(b) $\overline{\Omega}_+ \subset \Omega$ and $a \geq 0$ in a neighborhood of $\partial\Omega_+$;

(c) $\partial\Omega_+$ is a $(N-1)$-dimensional C^1 manifold contained in Ω and there exist $\gamma, C_1, C_2 > 0$ such that

$$C_1 \leq \frac{|a(x)|}{d(x)^\gamma} \leq C_2, \quad \text{for } x \text{ in a neighborhood of } \partial\Omega^+,$$

where $d(x) = \operatorname{dist}(x, \partial\Omega_+)$.

These conditions arise often in the literature when obtaining *a priori* bounds for solutions, especially (b) and (c) for indefinite type problems (see for instance [5, 7, 9]). Note that if a changes sign then (b) implies ($\mathcal{AT}$). We stress that, to the best of our knowledge, *a priori* bounds for (1.1) with f satisfying (1.6) have not been established even when a is non-negative.

We state now our main result:

Theorem 1.1 *Assume $a \in C(\overline{\Omega})$ satisfies $\Omega_+ \neq \emptyset$ and either* (a)*,* (b) *or* (c) *above. Let $f \in C[0,+\infty)$ be positive and satisfy* (1.2)*,* (1.3) *and* (1.6)*. Then problem* (1.1) *admits at least a positive weak solution $u \in C^{1,\alpha}(\overline{\Omega})$.*

Because of its own interest, we single out a special case of Theorem 1.1 with a logarithmic nonlinearity:

Corollary 1.1 *Assume $a \in C(\overline{\Omega})$ satisfies $\Omega_+ \neq \emptyset$ and either* (a)*,* (b) *or* (c) *above. Then for every $q > 0$ the problem*

$$\begin{cases} -\Delta u = a(x)u(\log(1+u))^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

admits at least a positive weak solution $u \in C^{1,\alpha}(\overline{\Omega})$.

Besides its own interest, nonlinearites which are slightly superlinear are also important as they arise from a class of quasilinear problems with quadratic growth in the gradient. Indeed, it is well-known (cf. [25]) that the change of variables

$$v = e^u - 1$$

reduces the quasilinear problem

$$\begin{cases} -\Delta u = f(x,u) + |\nabla u|^2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

to the semilinear one

$$\begin{cases} -\Delta v = (v+1) f(x, \log(v+1)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This procedure has been used for instance in [1, 21, 22], where (1.8) has been considered with $f(x,u)$ depending linearly on u (see also [14], [15], [16], [17], where (1.8) was analyzed in the context of the p-Laplacian, when f is comparable to a power). We shall then use Theorem 1.1 to deal with (1.8) when $f(x,u)$ is nonlinear on u, namely

$$\begin{cases} -\Delta u = a(x) g(u) + |\nabla u|^2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Our first result on this problem is:

Corollary 1.2 *Assume $a \in C(\overline{\Omega})$ satisfies $\Omega_+ \neq \emptyset$ and either* (a), (b) *or* (c) *above. Let $g \in C[0,+\infty)$ be positive, regularly varying of index $q > 0$, and such that*

$$\lim_{t \to 0+} \frac{g(t)}{t} = 0. \tag{1.10}$$

Then problem (1.9) *admits at least a positive weak solution $u \in C^{1,\alpha}(\overline{\Omega})$.*

Note that Corollary 1.2 applies to the model case $g(t) = t^q$, with $q > 1$. One may then ask what happens if $q \leq 1$ (observe that in this case the condition (1.10) no longer holds). The case $q = 1$ has been treated in previous papers (see for instance [21]), so that we shall deal only with the case $q < 1$. More precisely, since only the condition at zero is relevant, we assume that $g \in RV_q$ for some $q > 0$ and that there exists $r \in (0,1)$ such that

$$\lim_{t \to 0+} \frac{g(t)}{t^r} = 1 \tag{1.11}$$

(of course $r \neq q$ is possible). It turns out that (1.9) has in this case a concave-convex structure, therefore the existence of at least two solutions is expected under a smallness condition on $\|a_+\|_\infty$:

Corollary 1.3 *Assume $a \in C(\overline{\Omega})$ satisfies $\Omega_+ \neq \emptyset$ and either* (a), (b) *or* (c) *above. Let $g \in C[0,+\infty)$ be positive, regularly varying of index $q > 0$, and such that* (1.11) *holds for some $r \in (0,1)$. Then there exists $a_0 > 0$ such that problem* (1.9) *admits at least* two *nonnegative nontrivial weak solutions in $C^{1,\alpha}(\overline{\Omega})$, provided that $\|a_+\|_{L^\infty(\Omega)} \leq a_0$.*

Remark 1.1 Observe that, since $0 < r < 1$ and a changes sign, nonnegative nontrivial solutions of (1.9) are not necessarily strictly positive in Ω.

As a final comment, it is worth mentioning that the proof of Theorem 1.1 can be extended with almost no additional cost to handle functions $f \in RV_q$ with $q > 1$. However, we shall stick to the case $q = 1$ in our presentation, since this seems to be the less studied case in the literature. Furthermore, an extension to equations involving more general operators is also possible. For instance, the problem

$$\begin{cases} -\Delta_p u = a(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $p > 1$ can be analyzed, under the same hypothesis on a, provided that $f \in RV_{p-1}$. The main arguments can be carried out likewise, except that the proof of Lemma 3.1 below requires the use of a Picone identity (cf. [4]) or the results in [11]. Theorems 1 in [33] and Section 1 in [37] (or Theorem 2.4 in [30]) have to be employed instead of Theorems 8.17 and 8.18 in [13], and the classical regularity results have to be replaced by the $C^{1,\alpha}$ bounds of [8], [36] to pass to the limit.

The rest of this paper is organized as follows: in Section 2 we include a short reminder on some properties of functions of regular variation, which will be used in the proofs. Section 3 deals with the obtention of *a priori* bounds for all possible positive solutions of a parametrized version of (1.1) and Section 4 is devoted to the proof of Theorem 1.1 and Corollaries 1.2 and 1.3.

2 A brief reminder on regular variation

We devote this section to recall some properties of regularly varying functions which were introduced and first analyzed in [23], [24]. For proofs and further developments we refer the interested reader to the books [32], [29]. The set of functions of regular variation of index q will be denoted by RV_q. A first important property is the following (cf. Theorem 1.1 in [32]):

Theorem 2.1 *Assume $f \in RV_q$. Then the limit in* (1.5) *is uniform with respect to λ in intervals of the form $[a, b]$, $0 < a < b < +\infty$.*

Another important property of the functions in RV_q is that, for every $\delta > 0$ there holds

$$\lim_{t\to+\infty} f(t)/t^{q+\delta} = 0, \quad \text{and} \quad \lim_{t\to+\infty} f(t)/t^{q-\delta} = +\infty.$$

In particular, if $f \in RV_q$ with $q > 1$ then f satisfies (AR). We shall then focus on functions in RV_1. If $f \in RV_1$ and $f \in C[0, +\infty)$ then we have, for every $\delta > 0$,

$$f(t) \le C(1 + t^{1+\delta}), \quad t \in [0, +\infty), \tag{2.1}$$

where C depends on δ. This is a consequence of the following representation theorem, which will be also useful in the next section (cf. Theorem 1.2 in [32]).

Theorem 2.2 *Assume $f \in RV_q$ is continuous in $[0,+\infty)$. Then there exist a positive, continuous function h with a positive limit at infinity and a continuous function σ with $\lim_{t\to+\infty}\sigma(t)=0$ such that*

$$f(t)=t^q\,h(t)\,\exp\left(\int_1^t\frac{\sigma(\tau)}{\tau}d\tau\right),\quad t>0.$$

Remark 2.1 From Theorem 2.2 it follows that if $f\in RV_q$ with $q>0$ then $\lim_{t\to+\infty}f(t)=+\infty$.

The last statement in this section concerns the composition of slowly varying functions (see Section 1.5, 3º in [32] for its proof). It will be useful in the proof of Corollary 1.2 in Section 4.

Theorem 2.3 *Assume $f\in RV_q$, $g\in RV_0$ and $\lim_{t\to+\infty}g(t)=+\infty$. Then $f\circ g$ is slowly varying.*

Remark 2.2 To be precise, the proof in [32] deals only with the case $q=0$, but it can be easily extended to an arbitrary $q\in\mathbb{R}$.

3 A priori bounds

The objective of this section is to obtain *a priori* bounds of all possible positive weak solutions of (1.1). However, for its use in Section 4, we need to deal with a slightly more general version of (1.1). Set, as usual, $a_-=\min\{0,a\}$, $a_+=\max\{0,a\}$, and for $\lambda\in[\frac12,1]$ consider the problem

$$\begin{cases}-\Delta u=(a_-(x)+\lambda a_+(x))f(u) & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega.\end{cases}\tag{3.1}$$

Then we have:

Theorem 3.1 *Assume $a\in C(\overline{\Omega})$ verifies one of the conditions* (a)*,* (b) *or* (c) *in the Introduction. Let $f\in C[0,+\infty)$ be positive and satisfy* (1.3) *and* (1.6)*. Then there exists $M>0$ such that, for every $\lambda\in[\frac12,1]$ and every positive weak solution $u\in C^{1,\alpha}(\overline{\Omega})$ of* (3.1) *we have*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})}\le M.$$

The proof of this theorem relies on a standard scaling argument, which in turn uses a Liouville theorem for a linear equation. Although the result is well-known, we include its proof for completeness.

Lemma 3.1 *Let $c>0$. Then there are no positive weak solutions $v\in C^1(\mathbb{R}^N_+)$ of the inequality $-\Delta v\ge cv$ in $\mathbb{R}^N_+$.*

Proof. Assume there exists a positive weak solution $v \in C^1(\mathbb{R}^N_+)$ of $-\Delta v \geq cv$ in $\mathbb{R}^N_+$. Let ϕ_R be a positive eigenfunction associated to the Dirichlet principal eigenvalue $\lambda_1(R)$ of $-\Delta$ in the ball $B_R(2Re_N)$. Take $\phi_R \chi_{B_R(2Re_N)}$ as a test function in the weak formulation of this equation to obtain:

$$\begin{aligned} c \int_{B_R(2Re_N)} v\,\phi_R &\leq \int_{B_R(2Re_N)} \nabla v \nabla \phi_R \\ &= -\int_{B_R(2Re_N)} v\,\Delta\phi_R + \int_{\partial B_R(2Re_N)} v\,\frac{\partial \phi_R}{\partial \nu} \\ &\leq \lambda_1(R) \int_{B_R(2Re_N)} v\,\phi_R. \end{aligned}$$

Thus $c \leq \lambda_1(R)$. Since $\lambda_1(R) = \lambda_1(1)/R^2$, letting $R \to +\infty$ we have $c \leq 0$, a contradiction.

We now proceed to the proof of Theorem 3.1. Note that when (a) or (c) holds the proof follows from a standard scaling argument (essentially as in [12]), but when (b) holds the argument is slightly different and uses a weak Harnack inequality to be able to pass to the limit (see also [5], [10], [30] for related proofs).

Proof of Theorem 3.1. The first step is to show that $\|u\|_{L^\infty(\Omega_+)}$ is bounded. Assume this is not true; then there exist sequences $\{\lambda_n\} \subset [\frac{1}{2}, 1]$ and $\{u_n\} \subset C^{1,\alpha}(\overline{\Omega})$ such that u_n is a positive solution of (3.1) with $\lambda = \lambda_n$ and $M_n := \|u_n\|_{L^\infty(\Omega_+)} \to +\infty$. Let $x_n \in \overline{\Omega}_+$ be such that $u_n(x_n) = M_n$. By passing to subsequences we may assume $\lambda_n \to \lambda \in [\frac{1}{2}, 1]$, $x_n \to x_0 \in \overline{\Omega}_+$. From now on, we will consider three different cases, namely:

1. $x_0 \in \Omega_+$.
2. $x_0 \in \partial\Omega_+$ and (b) holds.
3. $x_0 \in \partial\Omega_+$ and (c) holds.

Note that, when a satisfies condition (a), the proof is essentially the same as Case (1) and Case (3) with $\gamma = 0$, so that it will be omitted.

1. Assume first $x_0 \in \Omega_+$. Choose $\mu_n = (f(M_n)/M_n)^{-\frac{1}{2}}$ and observe that $\mu_n \to 0$ by (1.3) and the hypothesis $M_n \to +\infty$. Define the functions

$$v_n(y) = \frac{u_n(x_n + \mu_n y)}{M_n}, \qquad y \in \Omega_n,$$

where $\Omega_n := \{z \in \mathbb{R}^N : \ x_n + \mu_n z \in \Omega_+\}$. It is easy to see that $v_n(0) = 1$, $0 \leq v_n \leq 1$ and

$$-\Delta v_n = \lambda_n a(x_n + \mu_n y)\frac{f(M_n v_n)}{f(M_n)} \quad \text{in } \Omega_n. \tag{3.2}$$

Next observe that, by Theorem 2.2, it follows that there exist a positive and continuous function h with a positive limit at infinity and a continuous function σ with

$\lim_{t\to+\infty}\sigma(t)=0$, such that

$$\begin{aligned}\frac{f(M_nv_n)}{f(M_n)} &= v_n\frac{h(M_nv_n)}{h(M_n)}\exp\left(\int_{M_n}^{M_nv_n}\frac{\sigma(\tau)}{\tau}d\tau\right)\\ &\le v_n\frac{h(M_nv_n)}{h(M_n)}\exp\left(\int_{M_nv_n}^{M_n}\frac{|\sigma(\tau)|}{\tau}d\tau\right).\end{aligned}\tag{3.3}$$

Now choose $\delta\in(0,1)$; there exists $K>0$ such that $|\sigma(t)|\le\delta$ if $t\ge K$. Let $S=\max_{0\le t\le K}f(t)$. If $M_nv_n\le K$ we have $f(M_nv_n)/f(M_n)\le S$ for n large enough. When $M_nv_n\ge K$ we deduce from (3.3) that

$$\frac{f(M_nv_n)}{f(M_n)}\le Cv_n\exp(-\delta\log v_n)=Cv_n^{1-\delta}\le C$$

for some $C>0$ (from now on, we will freely use the letter C to denote different constants). Therefore, the right-hand side of (3.2) is bounded. Since v_n is also bounded, we deduce by standard estimates (cf. [13, Chapter 9]) that $\{v_n\}$ is locally bounded in $W^{2,p}(\mathbb{R}^N)$ for every $p>1$, thus by Sobolev embedding $\{v_n\}$ is locally bounded in $C^{1,\alpha}(\mathbb{R}^N)$ for every $\alpha\in(0,1)$. Hence by a diagonal procedure we obtain a subsequence, still denoted by v_n, such that $v_n\to v$ in $C^1_{\rm loc}(\mathbb{R}^N)$, where $v\ge0$, $v(0)=1$. Observe that v is weakly superharmonic in $\mathbb{R}^N$, therefore by the strong maximum principle $v>0$ in $\mathbb{R}^N$. This implies v_n is bounded away from zero for large n in compact subsets of $\mathbb{R}^N$, so that, passing to the limit in (3.2) and using (1.6) and Theorem 2.1 we obtain that

$$-\Delta v=\lambda a(x_0)v\qquad\text{in }\mathbb{R}^N,$$

which is a contradiction with Lemma 3.1 since $\lambda a(x_0)>0$.

2. Now assume $x_0\in\partial\Omega_+$ and (b) holds. Then there exists $R>0$ such that $a\ge0$ in $B_{4R}(x_0)$. We claim that for every $p\in[1,\frac{N}{N-2})$ there exists a subsequence u_{n_k} such that

$$\int_{B_{2R}(x_0)}u_{n_k}^p\to+\infty.\tag{3.4}$$

Assume this is not true. Then there exist $p\in[1,\frac{N}{N-2})$ and $C>0$ such that

$$\int_{B_{2R}(x_0)}u_n^p\le C.\tag{3.5}$$

By (2.1) for every $\delta>0$ we have $-\Delta u_n\le C|a|_\infty(1+u_n^{1+\delta})$ in $B_{4R(x_0)}$, where C depends on δ. This inequality can be written as $-\Delta u_n+b_nu_n\le C|a|_\infty$ where $b_n=-C|a|_\infty u_n^\delta$. By (3.5), b_n is bounded in $L^{\frac{p}{\delta}}(B_{2R}(x_0))$. Choosing δ small so that $\frac{p}{\delta}>\frac{N}{2}$, we can apply Theorem 8.17 in [13] to obtain

$$\sup_{B_R(x_0)}u_n\le C,$$

which is a contradiction since $u_n(x_n) \to +\infty$ and $x_n \in B_R(x_0)$ for large n. This proves the claim.

For notational simplicity, we continue to denote the subsequence verifying (3.4) by $\{u_n\}$. Now notice that $-\Delta u_n \geq 0$ in $B_{4R}(x_0)$, so that we can apply the weak Harnack inequality [13, Theorem 8.18] to obtain that

$$\inf_{B_R(x_0)} u_n \geq C\left(\int_{B_{2R}(x_0)} u_n^p\right)^{\frac{1}{p}} \to +\infty.$$

Choose an arbitrary $y_0 \in B_R(x_0) \cap \Omega_+$ and $r > 0$ such that $B_r(y_0) \subset \Omega_+$. Define $\theta_n = (f(u_n(y_0))/u_n(y_0))^{-\frac{1}{2}}$ (which converges to zero by (1.3) and since $u_n(y_0) \to +\infty$) and

$$w_n(y) = \frac{u_n(y_0 + \theta_n y)}{u_n(y_0)}, \qquad y \in B_{\frac{r}{\theta_n}}.$$

It can be checked that

$$-\Delta w_n = \lambda_n a(y_0 + \theta_n y)\frac{f(u_n(y_0)w_n)}{f(u_n(y_0))} \qquad \text{in } B_{\frac{r}{\theta_n}}. \tag{3.6}$$

An argument as in Case (1) shows that the right hand side of (3.6) can be bounded by $C(1 + w_n^{1+\delta})$ for some $C > 0$ and small $\delta > 0$. Hence we can apply [13, Theorem 8.18] again to obtain, for every $R < r/(4\theta_n)$:

$$\int_{B_{2R}} w_n^p \leq C,$$

for every $p \in [1, \frac{N}{N-2})$ and some $C = C(p, R) > 0$. If δ is small so that $\frac{p}{\delta} > \frac{N}{2}$, we get from Theorem 8.17 in [13] that

$$\sup_{B_R} w_n \leq C.$$

Hence the right-hand side of (3.6) is locally bounded, so we can pass to the limit as before and obtain that, up to a subsequence, $w_n \to w$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ where $-\Delta w = \lambda a(y_0)w$ in $\mathbb{R}^N$ and $w > 0$, $w(0) = 1$. This contradicts Lemma 3.1.

3. Finally, consider the case where $x_0 \in \partial\Omega_+$ and (c) holds. With no loss of generality, we may assume that the inward unit normal to $\partial\Omega_+$ at x_0 verifies $\nu(x_0) = e_N$, the last vector of the canonical basis of $\mathbb{R}^N$. Let $\eta_n = (f(M_n)/M_n)^{-\frac{1}{2+\gamma}} \to 0$ and define

$$z_n(y) = \frac{u_n(x_n + \eta_n y)}{M_n}, \qquad y \in \widetilde{\Omega}_n,$$

where $\widetilde{\Omega}_n := \{x \in \Omega : \ x_n + \eta_n y \in \Omega\}$. It is easily seen that

$$-\Delta z_n = \eta_n^{-\gamma} c_n(y)\frac{f(M_n z_n)}{f(M_n)} \qquad \text{in } \widetilde{\Omega}_n, \tag{3.7}$$

where $c_n(y) = a_-(x_n+\eta_n y) + \lambda_n a_+(x_n+\eta_n y)$. Let $\delta_n = d(x_n)$ and take points $\xi_n \in \partial\Omega_+$ such that $\delta_n = |x_n - \xi_n|$. If we denote by ν_n the inward pointing normal to $\partial\Omega_+$ at ξ_n, it follows by the regularity of $\partial\Omega_+$ that $d(x_n + \eta_n y) = |\delta_n + \eta_n y \cdot \nu_n + o(\eta_n)|$. Next observe that, due to condition (c), we have in $\widetilde{\Omega}_n$

$$\begin{aligned} \frac{|c_n(y)|}{\eta_n^\gamma} &\le \frac{|a_-(x_n+\eta_n y)| + |a_+(x_n+\eta_n y)|}{\eta_n^\gamma} \\ &\le C\frac{d(x_n+\eta_n y)^\gamma}{\eta_n^\gamma} \le C\left|\frac{\delta_n}{\eta_n} + y\cdot\nu_n + o(1)\right|^\gamma . \end{aligned} \tag{3.8}$$

Assume for the moment that δ_n/η_n is bounded. Passing to a subsequence we may assume $\delta_n/\eta_n \to d \ge 0$. By (3.8) and an argument as in part (1), the right hand side of (3.7) is locally bounded. Therefore, up to a subsequence, $z_n \to z$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ where $z(0) = 1$ and $z > 0$ in $\mathbb{R}^N_+$. Using condition (b) on a and an inequality similar to (3.8), we see that $-\Delta z \ge C(d+y_N)^\gamma z$ in $\mathbb{R}^N_+$. Considering, for example $\bar{z}(y) = z(y+e_N)$, we obtain $-\Delta\bar{z} \ge C\bar{z}$ in $\mathbb{R}^N_+$, which contradicts Lemma 3.1.

Thus only the case $\delta_n/\eta_n \to +\infty$ remains to be considered. Denote $\beta_n = (\eta_n/\delta_n)^{\frac{\gamma}{2}} \to 0$ and perform the change of variables $\tilde{y} = y/\beta_n$, $\tilde{z}_n(\tilde{y}) = z_n(y)$. Similar arguments as before yield $\tilde{z}_n \to \tilde{z}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, with $\tilde{z}$ verifying $-\Delta\tilde{z} \ge C\tilde{z}$ in $\mathbb{R}^N$, $\tilde{z} > 0$, $\tilde{z}(0) = 1$, a contradiction with Lemma 3.1.

Therefore we have shown that there exists a positive constant $C > 0$ such that $\|u\|_{L^\infty(\Omega_+)} \le C$ for every positive solution of (3.1). Let $D = \Omega \setminus \overline{\Omega}_+$ and observe that u verifies $-\Delta u \le 0$ in D and $u \le C$ on ∂D. Hence $u \le C$ in D by the maximum principle.

To summarize, we have $\|u\|_{L^\infty(\Omega)} \le C$ for every weak solution $u \in C^{1,\alpha}(\overline{\Omega})$ and every $\lambda \in [\frac{1}{2}, 1]$. By employing again $W^{2,p}$ estimates and the Sobolev embedding, we also have $\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le M$, where M does not depend on $\lambda \in [\frac{1}{2}, 1]$. This concludes the proof.

4 Proof of the main results

This section is dedicated to the proof of our main results. As already mentioned, the proofs reliy on an abstract result due to [19], which ensures that, after embedding problem (1.1) in the one-parameter family (3.1), one can obtain a bounded Palais-Smale sequence for I_λ for almost every $\lambda \in [\frac{1}{2}, 1]$.

For the benefit of the reader we include the results of [19] that we are going to use, and refer to [19] for the proofs (see also [20] for more details and applications).

Theorem 4.1 *Let X be a Banach space provided with the norm $\|\cdot\|$. Consider a family of C^1 functionals defined on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in \left[\frac{1}{2}, 1\right]$$

where $B(u) \ge 0$ for every $u \in X$ and $A(u) \to +\infty$ when $\|u\| \to +\infty$. Assume there exists $e \in X$ such that, with

$$\Gamma := \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = e\}$$

we have

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(e)\}. \tag{4.1}$$

Then, for almost every $\lambda \in [\frac{1}{2}, 1]$, there exists a bounded Palais-Smale sequence $\{v_n\} \subset X$ at the level c_λ, that is:

(a) $\{v_n\}$ is bounded;

(b) $I_\lambda(v_n) \to c_\lambda$;

(c) $I'_\lambda(v_n) \to 0$ in X'.

Corollary 4.1 *Let X, I_λ be as in Theorem 4.1, and assume that both B and B' map bounded sets into bounded sets. Suppose in addition that for all $\lambda \in [\frac{1}{2}, 1]$, all bounded Palais-Smale sequences admit a convergent subsequence. Then there exists a sequence $\{(\lambda_n, u_n)\} \subset [\frac{1}{2}, 1] \times X$ with $\lambda_n \to 1$, $I_{\lambda_n}(u_n) = c_{\lambda_n}$, $I'(u_{\lambda_n}) = 0$. Moreover, if $\{u_n\}$ is bounded then*

$$I(u_n) \to \lim_{n\to+\infty} c_{\lambda_n} = c_1 \qquad \text{and} \qquad I'(u_n) \to 0.$$

Proof of Theorem 1.1. We will work in the Banach space $H_0^1(\Omega)$, endowed with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$. Define $f(t)$ to be zero for $t < 0$, and for $\lambda \in [\frac{1}{2}, 1]$, consider the family of functionals

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega (a_-(x) + \lambda a_+(x))F(u),$$

where $F(t) = \int_0^t f(s)ds$. Thanks to (2.1), it is well-known that I_λ is C^1 (cf. for instance Proposition B.10 in [27]). Let

$$A(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a_-(x)F(u), \qquad \text{and} \qquad B(u) = \int_\Omega a_+(x)F(u).$$

It is clear that $B(u) \geq 0$ in $H_0^1(\Omega)$ and $A(u) \to +\infty$ when $\|u\| \to +\infty$. Moreover, B and B' map bounded sets into bounded sets (since B' is compact; see Proposition B.10 in [27] again).

Although it is more or less standard by now, let us check that I_λ verifies the conditions in Theorem 4.1 (cf. Theorem 2.15 in [27]). Observe that, by (2.1) and (1.2), for every $\varepsilon, \delta > 0$, there exists a constant $C > 0$ such that $f(t) \leq 2\varepsilon t + C(2+\delta)t^{1+\delta}$ for $t \in \mathbb{R}$. This yields

$$0 \leq F(t) \leq \varepsilon t^2 + Ct^{2+\delta}, \quad t \in \mathbb{R}.$$

Using Poincaré and Sobolev inequalities we obtain

$$\begin{aligned}\int_\Omega (a_-(x) + \lambda a_+(x))F(u) &\leq \|a_+\|_{L^\infty(\Omega)} \int_\Omega \varepsilon u^2 + Cu^{2+\delta} \\ &\leq C\varepsilon \int_\Omega |\nabla u|^2 + C\left(\int_\Omega |\nabla u|^2\right)^{\frac{2+\delta}{2}} \\ &\leq C\|u\|^2(\varepsilon + \|u\|^\delta).\end{aligned}$$

Therefore

$$I_\lambda(u) \geq \left(\frac{1}{2} - C\varepsilon - C\|u\|^\delta\right)\|u\|^2,$$

and it follows that $I_\lambda(u) \geq c > 0$ on $\|u\| = r$ if r is small enough.

On the other hand, observe that (1.3) gives $F(t)/t^2 \to +\infty$ as $t \to +\infty$. Hence for every $K > 0$, there exists $M > 0$ such that $F(t) \geq Kt^2$ if $t \geq M$. Now choose $x_0 \in \Omega_+$ and $r > 0$ such that $B_r(x_0) \subset \Omega_+$. Let $\varphi \in C_0^\infty(B_r(x_0))$ with $\inf_{B_{\frac{r}{2}}(x_0)} \varphi > 0$. For sufficiently large μ it follows that $\mu \inf_{B_{\frac{r}{2}}(x_0)} \varphi \geq M$, so that

$$\begin{aligned}\int_\Omega (a_-(x) + \lambda a_+(x))F(\mu\varphi) &\geq \frac{1}{2}\inf_{B_{\frac{r}{2}}(x_0)} a_+(x) \int_{B_{\frac{r}{2}}(x_0)} F(\mu\varphi) \\ &\geq \frac{K}{2}\inf_{B_{\frac{r}{2}}(x_0)} a_+(x) \int_{B_{\frac{r}{2}}(x_0)} \mu^2\varphi^2 = KC\mu^2.\end{aligned}$$

Thus

$$I_\lambda(\mu\varphi) \leq \mu^2\left(\frac{1}{2}\|\varphi\|^2 - KC\right) < 0$$

if K is large enough.

As a consequence of the previous discussion, taking $e = \mu\varphi$ with μ large enough, we see that condition (4.1) in Theorem 4.1 holds.

Next, it is well-known that bounded Palais-Smale sequences for I_λ admit a convergent subsequence, since f verifies (2.1) (see for instance Proposition B.35 in [27]). Therefore, by Corollary 4.1 there exist sequences $\{\lambda_n\} \subset [\frac{1}{2}, 1]$ and $\{u_n\} \subset H_0^1(\Omega)$ with $\lambda_n \to 1$ and $I_{\lambda_n}(u_n) = c_{\lambda_n}$, $I'_{\lambda_n}(u_n) = 0$. In particular, for each $n \in \mathbb{N}$, u_n is a weak solution of

$$\begin{cases} -\Delta u = (a_-(x) + \lambda_n a_+(x))f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 8.15 in [13] and owing to (2.1) we have $u_n \in L^\infty(\Omega)$, so that standard regularity applies as before and $u_n \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0,1)$. Let us see that u_n is nonnegative. For this sake, let $D_n = \{x \in \Omega : u_n < 0\}$ and assume D_n is nonempty. Since $-\Delta u_n = 0$ in D_n (recall that $f(t) = 0$ for $t < 0$) with $u_n = 0$ on ∂D_n, we have by the maximum principle that $u_n \equiv 0$ in D_n, a contradiction. Hence $u_n \geq 0$ in Ω. Since f is locally Lipschitz at zero by (1.3), it follows from the strong maximum principle that $u_n > 0$ in Ω.

Thus Theorem 3.1 implies $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq M$. That is, the sequence $\{u_n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ and *a fortiori* in $H_0^1(\Omega)$. Moreover, there exists a subsequence, still denoted by $\{u_n\}$ with $u_n \to u$ in $C^1(\overline{\Omega})$ for some $u \in C^1(\overline{\Omega})$. It follows that u is a nonnegative weak solution of (1.1). By Corollary 4.1 again, $I_1(u) = c_1 > 0$, so that u is nontrivial, therefore positive in Ω. Finally, a regularity argument gives $u \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0,1)$. This concludes the proof.

Proof of Corollary 1.2. With the change of variable $v = e^u - 1$, problem (1.9) reads

$$\begin{cases} -\Delta v = a(x)(v+1)g(\log(v+1)) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

Let f be given by $f(t) = (t+1)g(\log(t+1))$ for $t > 0$. Then $\lim_{t\to 0+} \frac{g(t)}{t} = 0$ provides (1.2). Moreover, since g is regularly varying of index $q > 0$, it follows that $\lim_{t\to+\infty} g(t) = +\infty$, so that f satisfies (1.3). Finally, since $\log(t+1)$ is slowly varying, we deduce that $g(\log(t+1))$ is slowly varying as well by Theorem 2.3, so that f is regularly varying of index 1, that is, f satisfies (1.6). Therefore, by Theorem 1.1, problem (4.2) has a positive weak solution $v \in C^{1,\alpha}(\overline{\Omega})$ so that $u = \log(v+1) \in C^{1,\alpha}(\overline{\Omega})$ is a positive weak solution of (1.8).

Proof of Corollary 1.3. The proof is essentially the same as that of Corollary 1.2 based on Theorem 1.1. First, note that with the change of variable $v = e^u - 1$, we deal again with problem (4.2). We consider the parametrized version

$$\begin{cases} -\Delta v = (a_-(x) + \lambda a_+(x))(v+1)g(\log(v+1)) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\lambda \in [\frac{1}{2}, 1]$, together with the family of functionals

$$I_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega (a_-(x) + \lambda a_+(x))H(v),$$

where $H(t) = \int_0^t (s+1)g(\log(s+1))ds$. By (1.11), and since g is regularly varing of index q, we obtain $g(t) \le At^r + Bt^{q+r}$, say, for some positive constants A, B and every $t > 0$. Thus

$$H(t) \le t(t+1)(A(\log(t+1))^r + B(\log(t+1))^{q+r}) \le C(t^{r+1} + t^{2+\delta})$$

for $t > 0$, where $C > 0$ and $\delta > 0$ is as small as desired. This in turn yields

$$\begin{aligned} \int_\Omega (a_-(x) + \lambda a_+(x))H(v) &\le C\|a_+\|_{L^\infty(\Omega)} \int_\Omega v^{r+1} + v^{2+\delta} \\ &\le \|a_+\|_{L^\infty(\Omega)} \left(C\left(\int_\Omega v^2\right)^{\frac{1+r}{2}} + C\|v\|^{2+\delta} \right) \\ &\le C\|a_+\|_{L^\infty(\Omega)}(\|v\|^{1+r} + \|v\|^{2+\delta}), \end{aligned}$$

so that

$$I_\lambda(v) \ge \frac{1}{2}\|v\|^2 - C\|a_+\|_{L^\infty(\Omega)}(\|v\|^{1+r} + \|v\|^{2+\delta}).$$

It follows from here that $I_\lambda \ge c > 0$ on $\|v\| = \eta$, for η sufficiently small, provided $\|a_+\|_{L^\infty(\Omega)}$ is also chosen small.

Now observe that $\inf_{B_\eta} I_1 < 0$ (this is easily seen since $I_1(se) < 0$ for small $s > 0$ and an arbitrary nonnegative and nontrivial $e \in C_0^\infty(\Omega_+)$). Thus by the weak lower semicontinuity of I_1 we deduce that it reaches its minimum in B_η at some $v_1 \ne 0$. Then v_1 is a weak solution of (4.3) with $\lambda = 1$. It follows as before that $v_1 \in C^{1,\alpha}(\overline{\Omega})$ and $v_1 \ge 0$ in Ω. Thus $u_1 = \log(v_1 + 1) \in C^{1,\alpha}(\overline{\Omega})$ is a nonnegative solution of (1.9).

The second solution u_2 is obtained by arguing essentially as in the proof of Theorem 1.1 (remark that $I_1(u_1) < 0 < I_1(u_2)$, so that actually $u_1 \ne u_2$). It is only worth mentioning that the a priori bounds given by Theorem 3.1 are easily seen to be valid also for nonnegative solutions.

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