Research Article

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Nontrivial Solutions for Potential Systems Involving the Mean Curvature Operator in Minkowski Space

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Abstract: In this paper, we use the critical point theory for convex, lower semicontinuous perturbations of C^1-functionals to obtain the existence of multiple nontrivial solutions for one parameter potential systems involving the operator $u \mapsto \operatorname{div}(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}})$. The solvability of a general non-potential system is also established.

Keywords: Minkowski Operator, Potential System, Multiple Nontrivial Solutions, Critical Point, Mountain Pass Theorem, Schauder's Fixed Point Theorem

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$ of class C^2 and assume the following hypothesis for $F\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$.

Hypothesis 1.1. (i) $F(\cdot, u, v) : \Omega \to \mathbb{R}$ is measurable for all $(u, v) \in \mathbb{R}^2$ and $F(\cdot, 0, 0) = 0$.
(ii) $F(x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 on $\mathbb{R}^2$ for a.e. $x \in \Omega$.
(iii) For each $\rho > 0$, there exists $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|\nabla F(x, u, v)| \leq \alpha_\rho(x) \quad \text{for a.e. } x \in \Omega \text{ and all } (u, v) \in \mathbb{R}^2 \text{ with } |(u, v)| \leq \rho,$$

where $\nabla F(x, u, v)$ stands for the gradient of $F(x, \cdot, \cdot)$ with respect to $(u, v) \in \mathbb{R}^2$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$.

In this paper we deal with systems of the form

$$\begin{cases} \mathcal{M}(u) = F_u(x, u, v), & x \in \Omega, \\ \mathcal{M}(v) = F_v(x, u, v), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{1.1}$$

where $\mathcal{M}$ stands for the mean curvature operator in Minkowski space:

$$\mathcal{M}(u) = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\Big).$$

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By a *solution* of (1.1) we mean a pair of functions $(u, v) \in W^{2,p}(\Omega) \times W^{2,q}(\Omega)$, with $p, q > N$, such that $\|\nabla u\|_\infty < 1$, $\|\nabla v\|_\infty < 1$ and which satisfies the equations a.e. in Ω and vanishes on $\partial\Omega$. Note that, since $W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$, the evaluation at $\partial\Omega$ is understood in the usual pointwise sense. Here and below, $\|\cdot\|_\infty$ denotes the usual sup-norm on $C(\overline{\Omega})$.

Problems involving the operator $\mathcal{M}$ are originated in differential geometry and theory of relativity. Geometrically, these are related to maximal and constant mean curvature spacelike hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1} := \{(x, t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ endowed with the Lorentzian metric $\sum_{j=1}^N (dx_j)^2 - (dt)^2$, where (x, t) are the canonical coordinates in $\mathbb{R}^{N+1}$) having the property that the trace of the extrinsic curvature is zero, respectively, constant. Also, their geometric interest is motivated by the fact that the spacelike hypersurfaces in the Minkowski space exhibit nice Bernstein type properties (see [3, 4, 17, 18]). On the other hand, assuming that a spacelike hypersurface in $\mathbb{L}^{N+1}$ is the graph of a smooth function $v\colon \Omega \to \mathbb{R}$ with Ω a domain in $\{(x, t) : x \in \mathbb{R}^N, t = 0\} \simeq \mathbb{R}^N$, the (strictly) spacelike condition implies $|\nabla v| < 1$ and v satisfies an equation of type

$$\mathcal{M}(v) = H(x, v) \quad \text{in } \Omega,$$

where H is a prescribed mean curvature function. If H is bounded, then it has been shown by Bartnik and Simon [7] that the above equation subjected to a Dirichlet condition has at least one solution. More recently, the existence of additional solutions, such as of mountain pass type, was obtained in [10], and the existence of Filippov type solutions for discontinuous Dirichlet problems involving the operator $\mathcal{M}$ was established in [12]. The Dirichlet problem in a more general spacetime was considered by Gerhardt [29]. For other recent various existence results concerning such problems with the operator $\mathcal{M}$, we refer the reader to [5, 6, 14, 15, 21, 22] and the references therein.

The importance of spacelike hypersurfaces in general relativity was emphasized in the paper of Lichnerowicz [34], where it was shown that they are convenient as initial data for solving the Cauchy problem for the Einstein equations. Concerning these equations, an interesting use of the spacelike hypersurfaces with constant mean curvature can be found in [4, 19]. Other reasons justifying their importance in the study of different aspects (singularities, gravitational radiation, positivity of mass) from general relativity can be found in [20, 35, 36, 38].

On the other hand, in recent time, based on a variational approach, many papers deal with the existence of solutions for quasilinear elliptic systems of the type

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = G_u(x, u, v), & x \in \Omega, \\ -\operatorname{div}(|\nabla v|^{q-2}\nabla v) = G_v(x, u, v), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{1.2}$$

with $G \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ satisfying various growth conditions. In this view, we refer to the works of Boccardo and de Figueiredo [16], Costa and Magalhães [23] (in the case $p = q = 2$), Felmer, Manásevich and de Thélin [28], de Figueiredo [24], de Nápoli and Mariani [26], and Vélin and de Thélin [39]. For other discussions and some historical details we refer the reader to the paper of de Figueiredo [25]. When Ω is a strip-like domain, multiplicity of solutions for the λ-parametrized problem of the form (1.2) was obtained in [32] by using an abstract critical points result of Ricceri. Also, the existence of multiple solutions for some particular systems defined on an arbitrary domain (bounded or unbounded) was proved by Drábek, Stavrakakis and Zographopoulos [27], and for variational inclusions and hemivariational inequalities systems, we send the reader to the papers of Kristály [31, 33]. We mention also that, in the recent years, much attention has been paid to more general systems of the form

$$\begin{cases} -\operatorname{div}(h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = G_u(x, u, v), & x \in \Omega, \\ -\operatorname{div}(h_2(|\nabla v|^p)|\nabla v|^{p-2}\nabla v) = G_v(x, u, v), & x \in \Omega, \end{cases}$$

subjected to Dirichlet boundary conditions, with $h_1, h_2 \in C(\mathbb{R}, \mathbb{R})$. In this direction, existence and nonexistence of weak solutions were established in [2] for $p \geq 2$, and in [1, 40] for $p = 2$.

In the light of the above and to the best of our knowledge, this is the first study of systems involving the mean curvature operator $\mathcal{M}$ in a general bounded domain Ω; in the particular case when Ω is a ball or an annular domain and the data have a radial structure, such type of systems were considered in [8].

First, we give a variational formulation for (1.1) and we show that (1.1) has a least energy type solution, a result which extends the one obtained in [10] for the problem

$$\mathcal{M}(u) = f(x, u) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0,$$

with f an L^∞ Carathéodory function, to the case of the potential system (1.1).

Next, we employ a technique introduced in [10] (also see [9, 13] for Neumann and periodic problems) to obtain existence of multiple nontrivial (i.e., $\not\equiv (0, 0)$) solutions for the Dirichlet problem

$$\begin{cases} \mathcal{M}(u) = \mu_1(x)|u|^{q_1-1}u - \lambda F_u(x, u, v), & x \in \Omega, \\ \mathcal{M}(v) = \mu_2(x)|v|^{q_2-1}v - \lambda F_v(x, u, v), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{1.3}$$

where $q_1, q_2 > 0$ are fixed, $\mu_1, \mu_2 \in L^\infty(\Omega)$ are given *positive* (i.e., ≥ 0 a.e. in Ω) functions and $\lambda > 0$ is a real parameter. The solutions we provide appear as global minima and as mountain pass critical points of some suitable truncated functionals for sufficiently large values of λ.

For the sake of the completeness, we conclude the paper by providing an "universal" existence result for general *non-potential* systems of the form

$$\begin{cases} \mathcal{M}(u) = f_1(x, u, v, \nabla u, \nabla v), & x \in \Omega, \\ \mathcal{M}(v) = f_2(x, u, v, \nabla u, \nabla v), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{1.4}$$

with $f_1, f_2 \colon \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ Carathéodory functions satisfying the following hypothesis.

Hypothesis 1.2. For each $\rho > 0$, there exists $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|f_i(x, \overline{u}, \overline{\xi})| \leq \alpha_\rho(x) \quad (i = 1, 2)$$

for a.e. $x \in \Omega$ and all $(\overline{u}, \overline{\xi}) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$ with $|(\overline{u}, \overline{\xi})| \leq \rho$; this time, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2 \times \mathbb{R}^{2N}$. The notion of a *solution* for (1.4) is understood similarly to the one for system (1.1).

The rest of the paper is organized as follows. In Section 2 we give the variational formulation for system (1.1). The multiplicity of solutions for problems of type (1.3) is discussed in Section 3. The obtained general multiplicity results are illustrated by concrete examples. Section 4 is devoted to the solvability of system (1.4).

2 The Variational Setting

We reduce system (1.1) to a variational inequality. To this end, let

$$K_0 := \{u \in W^{1,\infty}(\Omega) : \|\nabla u\|_\infty \leq 1,\ u|_{\partial\Omega} = 0\}.$$

We know that the convex set K_0 is compact in $C(\overline{\Omega})$ (see [10]). Also, notice that the functions in K_0 are Lipschitz continuous with uniformly bounded Lipschitz constant. This, together with the fact that they vanish on $\partial\Omega$, implies

$$\|u\|_\infty \leq c(\Omega) \quad \text{for all } u \in K_0, \tag{2.1}$$

with $c(\Omega)$ being a positive constant.

Next, we introduce the functional $\Psi \colon C(\overline{\Omega}) \to (-\infty, +\infty]$ given by

$$\Psi(u) = \begin{cases} \displaystyle\int_\Omega \left[1 - \sqrt{1 - |\nabla u|^2}\right] & \text{for } u \in K_0, \\ +\infty & \text{for } u \in C(\overline{\Omega}) \setminus K_0. \end{cases}$$

This is convex and lower semicontinuous (see [10, Lemma 2.4]). Then, in view of Hypothesis 1.1, we can define $\mathcal{F}\colon C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}$ by

$$\mathcal{F}(u, v) = \int_\Omega F(x, u, v).$$

Using the fact that the Nemytskii operators

$$(u, v) \mapsto F_u(\cdot\,, u(\cdot), v(\cdot)) \quad \text{and} \quad (u, v) \mapsto F_v(\cdot\,, u(\cdot), v(\cdot))$$

are continuous from the product space $C(\overline{\Omega}) \times C(\overline{\Omega})$ to $L^1(\Omega)$, and map bounded sets from $C(\overline{\Omega}) \times C(\overline{\Omega})$ into bounded sets in $L^\infty(\Omega)$, it is easy to check that $\mathcal{F}$ is of class C^1 on $C(\overline{\Omega}) \times C(\overline{\Omega})$ and its Fréchet derivative is given by

$$\langle \mathcal{F}'(u, v), (w_1, w_2)\rangle = \int_\Omega \left[F_u(x, u, v)w_1 + F_v(x, u, v)w_2\right] \quad \text{for all } u, v, w_1, w_2 \in C(\overline{\Omega}).$$

Now, with $\psi\colon C(\overline{\Omega}) \times C(\overline{\Omega}) \to (-\infty, +\infty]$ defined by

$$\psi(u, v) = \Psi(u) + \Psi(v) \quad \text{for all } u, v \in C(\overline{\Omega}),$$

the energy functional associated with problem (1.1) is

$$I := \psi + \mathcal{F}.$$

On account of the above, the functional I has the structure required by Szulkin's critical point theory [37]. Accordingly, $(u, v) \in K_0 \times K_0$ is a *critical point* of I if it is a solution of the variational inequality

$$\psi(w_1, w_2) - \psi(u, v) + \int_\Omega \left[F_u(x, u, v)(w_1 - u) + F_v(x, u, v)(w_2 - v)\right] \geq 0 \tag{2.2}$$

for all $w_1, w_2 \in K_0$. Also, we note that the compactness of $K_0 \subset C(\overline{\Omega})$ implies that I satisfies *the Palais–Smale condition* ((PS) condition for short), i.e., a sequence $\{(u_n, v_n)\} \subset K_0 \times K_0$ such that $I(u_n, v_n) \to c \in \mathbb{R}$ and

$$\psi(w_1, w_2) - \psi(u_n, v_n) + \left\langle \mathcal{F}'(u_n, v_n), (w_1 - u_n, w_2 - v_n)\right\rangle \geq -\varepsilon_n\left|(\|w_1 - u_n\|_\infty, \|w_2 - v_n\|_\infty)\right|$$

for all $w_1, w_2 \in K_0$, where $\varepsilon_n \to 0_+$, contains a convergent subsequence.

Theorem 2.1. *If $(u, v) \in K_0 \times K_0$ is a critical point of I, then it is a solution of problem* (1.1). *Moreover, I is bounded from below and attains its infimum at some $(u_0, v_0) \in K_0 \times K_0$, which solves problem* (1.1).

Proof. Let (u, v) be a critical point of I. Taking in (2.2) $w_2 = v$, one gets

$$\Psi(w_1) - \Psi(u) + \int_\Omega F_u(x, u, v)(w_1 - u) \geq 0,$$

i.e., $u \in K_0$ is a critical point of $\Psi(\cdot) + \mathcal{F}(\cdot\,, v)$, which by virtue of [10, Theorem 2.1] it satisfies

$$\mathcal{M}(u) = F_u(x, u, v), \quad x \in \Omega, \qquad u|_{\partial\Omega} = 0.$$

Analogously, for $w_1 = u$ in (2.2), we have that $v \in K_0$ is a critical point of $\Psi(\cdot) + \mathcal{F}(u, \cdot)$ and verifies

$$\mathcal{M}(v) = F_v(x, u, v), \quad x \in \Omega, \qquad v|_{\partial\Omega} = 0.$$

Next, with constant $c(\Omega)$ entering in (2.1), using Hypothesis 1.1 and the mean value theorem, we obtain that for $\rho = \sqrt{2}c(\Omega)$, there exists $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|F(x, u(x), v(x))| \leq \alpha_\rho(x)\rho \quad \text{for a.e. } x \in \Omega \text{ and all } u, v \in K_0,$$

which yields

$$|\mathcal{F}(u, v)| \leq \int_\Omega |F(x, u, v)| \leq \rho\|\alpha_\rho\|_\infty|\Omega|.$$

Hence, we deduce that I is bounded from below on $C(\overline{\Omega}) \times C(\overline{\Omega})$. As I satisfies the (PS) condition, the result follows by virtue of [37, Theorem 1.7]. □

Remark 2.2. The main features of Theorem 2.1 is that it reduces the search of solutions of (1.1) to finding critical points of I, and states that the potential system (1.1) is always solved by a least energy solution. We emphasize this because the solvability part is actually a consequence of the more general Theorem 4.5 below.

3 Nontrivial Solutions

Under Hypothesis 1.1, in this section we study the existence and multiplicity of nontrivial solutions for the Dirichlet problem (1.3). Associated with (1.3) is the functional $I_\lambda\colon C(\overline{\Omega})\times C(\overline{\Omega})\to(-\infty,+\infty]$ given by

$$I_\lambda(u,v)=\begin{cases}\psi(u,v)+\dfrac{1}{q_1+1}\displaystyle\int_\Omega \mu_1(x)|u|^{q_1+1}+\dfrac{1}{q_2+1}\int_\Omega \mu_2(x)|v|^{q_2+1}-\lambda\int_\Omega F(x,u,v) & \text{for } (u,v)\in K_0\times K_0,\\ +\infty & \text{otherwise.}\end{cases}$$

Theorem 3.1. (i) *If there exists $R_1>0$ such that*

$$F_u(x,u,v)>(\geq)\,0,\quad F_v(x,u,v)\geq(>)\,0\quad \text{for a.e. } x\in\Omega \text{ and all } u,v\in(0,R_1),\tag{3.1}$$

then there exists $\Lambda_1>0$ such that for all $\lambda>\Lambda_1$, one has

$$\inf_{C(\overline{\Omega})\times C(\overline{\Omega})} I_\lambda<0,\tag{3.2}$$

and (1.3) *has a nontrivial solution which is a minimizer of the functional I_λ.*

(ii) *If, in addition, one has either*

$$\lim_{|(u,v)|\to 0}\frac{F(x,u,v)}{u^2+v^2}=0\quad \text{uniformly with } x\in\Omega,\tag{3.3}$$

or

$$\begin{cases}\mu_1^m:=\operatorname{ess\,inf}_\Omega \mu_1>0,\quad \mu_2^m:=\operatorname{ess\,inf}_\Omega \mu_2>0,\\ \displaystyle\lim_{|(u,v)|\to 0}\frac{F(x,u,v)}{|u|^{q_1+1}+|v|^{q_2+1}}=0\quad \text{uniformly with } x\in\Omega,\end{cases}\tag{3.4}$$

then problem (1.3) *has at least two nontrivial solutions for all $\lambda>\Lambda_1$.*

Proof. (i) Let $x_0\in\Omega$ and $r\in(0,R_1)$ be such that $\overline{B}_r(x_0)\subset\Omega$. Setting

$$\eta(x):=\begin{cases}\exp\Big(\dfrac{r^2}{|x-x_0|^2-r^2}\Big) & \text{for } x\in B_r(x_0),\\ 0 & \text{for } x\in\Omega\setminus B_r(x_0),\end{cases}$$

we define

$$\eta_0(x):=\min\{r,\|\nabla\eta\|_\infty^{-1}\}\eta(x)\quad (x\in\Omega).\tag{3.5}$$

As $\eta_0\in K_0$ and $0\leq\eta_0<R_1$, by (3.1) and the mean value theorem, we have

$$\begin{aligned}\int_\Omega F(x,\eta_0,\eta_0)&=\int_{B_r(x_0)}[F(x,\eta_0,\eta_0)-F(x,0,0)]\\ &=\int_{B_r(x_0)}[F_u(x,\theta\eta_0,\theta\eta_0)+F_v(x,\theta\eta_0,\theta\eta_0)]\eta_0\,dx>0,\end{aligned}$$

with $\theta=\theta(x)\in(0,1)$. Then, taking

$$\Lambda_1>\max\left\{0,\ \frac{2\Psi(\eta_0)+\frac{1}{q_1+1}\int_\Omega\mu_1(x)|\eta_0|^{q_1+1}+\frac{1}{q_2+1}\int_\Omega\mu_2(x)|\eta_0|^{q_2+1}}{\int_\Omega F(x,\eta_0,\eta_0)}\right\}$$

and noticing that for all $\lambda>\Lambda_1$, one has $I_\lambda(\eta_0,\eta_0)<0$, the conclusion follows from $I_\lambda(0,0)=0$ and Theorem 2.1.

(ii) From (i), we can find $\Lambda_1 > 0$ such that for all $\lambda > \Lambda_1$, the functional I_λ has a first nontrivial critical point $(u_{\lambda,1}, v_{\lambda,1}) \in K_0 \times K_0$ with

$$I_\lambda(u_{\lambda,1}, v_{\lambda,1}) < 0. \tag{3.6}$$

We shall produce a second nontrivial critical point of I_λ by the Mountain Pass Theorem [37, Theorem 3.2]. To do this, let $\lambda > \Lambda_1$ be fixed. Taking into account (3.6) and the fact that I_λ satisfies the (PS)-condition, it suffices to prove that there exist $\alpha > 0$ and $0 < \rho < |(\|u_{\lambda,1}\|_\infty, \|v_{\lambda,1}\|_\infty)|$ such that

$$I_\lambda(u, v) \geq \alpha \quad \text{for all } u, v \in K_0, \text{ with } |(\|u\|_\infty, \|v\|_\infty)| = \rho. \tag{3.7}$$

First, we assume that (3.3) is satisfied. From the Poincaré inequality and since

$$1 - \sqrt{1 - s^2} \geq \frac{s^2}{2} \quad \text{for all } s \in [0, 1],$$

one has that

$$\Psi(u) = \int_\Omega \left[1 - \sqrt{1 - |\nabla u|^2}\right] \geq c_1 \int_\Omega |u|^2 \quad (u \in K_0), \tag{3.8}$$

with c_1 being a positive constant. Using (3.3), we can find $\delta > 0$ such that

$$F(x, s_1, s_2) \leq \frac{c_1}{2\lambda} |(s_1, s_2)|^2 \tag{3.9}$$

for a.e. $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$ with $|(s_1, s_2)| \leq \delta$. Then, using (3.8) and (3.9), for all $u, v \in K_0$ such that $|(\|u\|_\infty, \|v\|_\infty)| \leq \delta$, we obtain

$$I_\lambda(u, v) \geq \frac{c_1}{2} \int_\Omega [|u|^2 + |v|^2].$$

Let $\rho \in (0, \min\{\delta, |(\|u_{\lambda,1}\|_\infty, \|v_{\lambda,1}\|_\infty)|\})$ and denote

$$\mathcal{K} := \{(u, v) \in K_0 \times K_0 : |(\|u\|_\infty, \|v\|_\infty)| = \rho\}.$$

Since the continuous function $(u, v) \mapsto \int_\Omega [|u|^2 + |v|^2]$ is strictly positive on the compact set $\mathcal{K}$, it follows that

$$\alpha_1 := \inf_{\mathcal{K}} \int_\Omega [|u|^2 + |v|^2] > 0,$$

which implies (3.7) with $\alpha = \alpha_1 c_1/2$.

Next, assume that (3.4) holds true. As above, there exists $\delta > 0$ such that

$$\begin{aligned} F(x, s_1, s_2) &\leq \left(\frac{1}{2\lambda} \min\left\{\frac{\mu_1^m}{q_1 + 1}, \frac{\mu_2^m}{q_2 + 1}\right\}\right)(|s_1|^{q_1+1} + |s_2|^{q_2+1}) \\ &\leq \frac{\mu_1^m}{2\lambda(q_1 + 1)} |s_1|^{q_1+1} + \frac{\mu_2^m}{2\lambda(q_2 + 1)} |s_2|^{q_2+1} \end{aligned}$$

for a.e. $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$ with $|(s_1, s_2)| \leq \delta$. Then, for $u, v \in C(\overline{\Omega})$ with $|(\|u\|_\infty, \|v\|_\infty)| \leq \delta$, one obtains

$$\begin{aligned} I_\lambda(u, v) &\geq \frac{1}{q_1 + 1} \int_\Omega \mu_1(x)|u|^{q_1+1} + \frac{1}{q_2 + 1} \int_\Omega \mu_2(x)|v|^{q_2+1} - \lambda \int_\Omega F(x, u, v) \\ &\geq \frac{\mu_1^m}{2(q_1 + 1)} \int_\Omega |u|^{q_1+1} + \frac{\mu_2^m}{2(q_2 + 1)} \int_\Omega |v|^{q_2+1}, \end{aligned}$$

and the proof follows now analogously as in the case when (3.3) is fulfilled.

Therefore, under assumption (3.3) or (3.4), by the Mountain Pass Theorem, there exists a nontrivial critical point $(u_{\lambda,2}, v_{\lambda,2})$ of I_λ with $I_\lambda(u_{\lambda,2}, v_{\lambda,2}) > 0$. Clearly, $(u_{\lambda,2}, v_{\lambda,2}) \neq (u_{\lambda,1}, v_{\lambda,1})$ and the proof is complete. □

Using similar arguments (adequate handling of the bump functions $\pm\eta_0$ defined in (3.5)), we can prove the following theorem.

Theorem 3.2. (i) *If there exists $R_2 > 0$ such that*

$$F_u(x,u,v) <(\leq)\, 0, \quad F_v(x,u,v) \leq(<)\, 0 \quad \text{for a.e. } x \in \Omega \text{ and all } u, v \in (-R_2, 0), \tag{3.10}$$

then there exists $\Lambda_2 > 0$ such that (3.2) *holds true for all $\lambda > \Lambda_2$, and problem* (1.3) *has a nontrivial solution which is a minimizer of I_λ.*

(ii) *If there exists $R_3 > 0$ such that*

$$F_u(x,u,v) <(\leq)\, 0, \quad F_v(x,u,v) \geq(>)\, 0 \quad \text{for a.e. } x \in \Omega, \text{ all } u \in (-R_3, 0) \text{ and all } v \in (0, R_3), \tag{3.11}$$

then there exists $\Lambda_3 > 0$ such that (3.2) *is satisfied for all $\lambda > \Lambda_3$, and* (1.3) *has a nontrivial solution which is a minimizer of I_λ.*

(iii) *If there exists $R_4 > 0$ such that*

$$F_u(x,u,v) >(\geq)\, 0, \quad F_v(x,u,v) \leq(<)\, 0 \quad \text{for a.e. } x \in \Omega, \text{ all } u \in (0, R_4) \text{ and all } v \in (-R_4, 0), \tag{3.12}$$

then there exists $\Lambda_4 > 0$ such that for all $\lambda > \Lambda_4$, (3.2) *holds true and problem* (1.3) *has a nontrivial solution which is a minimizer of the functional I_λ.*

(iv) *Assume that one of the above condition* (3.10), (3.11) *or* (3.12) *holds true. If, in addition, either* (3.3) *or* (3.4) *is satisfied, then problem* (1.1) *has at least two nontrivial solutions for all $\lambda > 0$ sufficiently large.*

Example 3.3. Let $q_1, q_2 > 0$ and $\mu_1, \mu_2 \in L^\infty(\Omega)$ be positive. Then, there exists $\Lambda > 0$ such that the system

$$\begin{cases} \mathcal{M}(u) + \lambda(3u^2v^2 + v) = \mu_1(x)|u|^{q_1-1}u, & x \in \Omega, \\ \mathcal{M}(v) + \lambda(2u^3v + u) = \mu_2(x)|v|^{q_2-1}v, & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega} \end{cases}$$

has at least one nontrivial solution, while

$$\begin{cases} \mathcal{M}(u) + 3\lambda u^2v^2 = \mu_1(x)|u|^{q_1-1}u, & x \in \Omega, \\ \mathcal{M}(v) + 2\lambda u^3v = \mu_2(x)|v|^{q_2-1}v, & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega} \end{cases} \tag{3.13}$$

has at least two nontrivial solutions whenever $\lambda > \Lambda$.

Denoting by $\mathcal{I}_1$ and $\mathcal{I}_2$ one of the intervals $[0,\infty)$ or $(-\infty, 0]$, next we deal with system (1.3), this time with $F\colon \Omega \times \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ satisfying the following hypothesis.

Hypothesis 3.4. (i) $F(\cdot, u, v) : \Omega \to \mathbb{R}$ is measurable for all $(u, v) \in \mathcal{I}_1 \times \mathcal{I}_2$.

(ii) For a.e. $x \in \Omega$, $F(x, \cdot, \cdot) \in C^1(\mathcal{I}_1 \times \mathcal{I}_2)$.

(iii) For each $\rho > 0$, there exists $\alpha_\rho \in L^\infty(\Omega)$ such that $|\nabla F(x,u,v)| \leq \alpha_\rho(x)$ for a.e. $x \in \Omega$ and all $(u,v) \in \mathcal{I}_1 \times \mathcal{I}_2$ with $|(u,v)| \leq \rho$.

(iv) $F(x,0,v) = F_u(x,0,v) = 0 = F(x,u,0) = F_v(x,u,0)$ for a.e. $x \in \Omega$ and all $(u,v) \in \mathcal{I}_1 \times \mathcal{I}_2$.

Problem (1.3) with F verifying Hypothesis 3.4 will be referred as $(1.3)^+$ when $\mathcal{I}_1 = \mathcal{I}_2 = [0,\infty)$, $(1.3)^-$ when $\mathcal{I}_1 = \mathcal{I}_2 = (-\infty, 0]$, $(1.3)^\pm$ when $\mathcal{I}_1 = [0,\infty)$ and $\mathcal{I}_2 = (-\infty, 0]$ and, respectively, $(1.3)^\mp$ when $\mathcal{I}_1 = (-\infty, 0]$ and $\mathcal{I}_2 = [0,\infty)$. Below, when we refer to (3.3) or (3.4) for F as in Hypothesis 3.4, we shall implicitly mean that $(s_1, s_2) \in \mathcal{I}_1 \times \mathcal{I}_2$. As usually, we denote $u^+ = \max\{0, u\}$ and $u^- = \min\{0, u\}$.

Remark 3.5. If F verifies Hypothesis 3.4 with $\mathcal{I}_1 = \mathcal{I}_2 = [0,\infty)$, then the problem

$$\begin{cases} \mathcal{M}(u) = \mu_1(x)|u|^{q_1-1}u - \lambda F_u(x, u^+, v^+), & x \in \Omega, \\ \mathcal{M}(v) = \mu_2(x)|v|^{q_2-1}v - \lambda F_v(x, u^+, v^+), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega} \end{cases} \tag{3.14}$$

has at least one solution. Indeed, setting $\widetilde{F}(x,u,v) = F(x,u^+,v^+)$ $(x \in \Omega, u, v \in \mathbb{R})$, clearly $\widetilde{F}$ satisfies Hypothesis 1.1 and

$$\nabla\widetilde{F}(x,u,v) = \begin{cases} \nabla F(x,u,v), & (u,v) \in \mathcal{I}_1 \times \mathcal{I}_2, \\ 0, & \text{otherwise } (x \in \Omega), \end{cases}$$

thus Theorem 2.1 applies. Similar conclusions hold true for the other three remaining cases.

Lemma 3.6. *Assume Hypothesis 3.4. If (u,v) is a solution of (3.14), then $u \geq 0$, $v \geq 0$ in Ω and (u,v) solves problem $(1.3)^+$.*

Proof. This follows the outline of the proof of [11, Lemma 3.1] (see also [13]). However, for the sake of completeness, we give it below for one of the components, say u; similar argument can be used for v.

Let $w \in W^{1,\infty}(\Omega)$, with $w|_{\partial\Omega} = 0$, be arbitrarily chosen. Then, multiplying the first equation in (3.14) with w, using [10, Remark 2.1] and the integration by parts formula, one gets

$$-\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1-|\nabla u|^2}} = \int_\Omega \mu_1(x)|u|^{q_1-1}uw - \lambda\int_\Omega F_u(x,u^+,v^+)w. \tag{3.15}$$

Setting $\mathcal{O}_u = \{x \in \Omega : u(x) < 0\}$, from [30, Theorem A.1], we have $u^- \in W^{1,\infty}(\Omega)$, $\nabla u^- = \nabla u$ in $\mathcal{O}_u$ and $\nabla u^- = 0_{\mathbb{R}^N}$ in $\Omega \setminus \mathcal{O}_u$. So, taking $w = u^-$ in (3.15) and using Hypothesis 3.4 (iv), we have

$$-\int_{\mathcal{O}_u} \frac{|\nabla u^-|^2}{\sqrt{1-|\nabla u^-|^2}} = \int_{\mathcal{O}_u} \mu_1(x)|u^-|^{q_1+1} \geq 0. \tag{3.16}$$

Setting $\mathcal{O}_u^\nabla = \{x \in \mathcal{O}_u : |\nabla u^-(x)| > 0\}$ and assuming that meas $\mathcal{O}_u^\nabla > 0$, from (3.16), we obtain the contradiction

$$0 > -\int_{\mathcal{O}_u^\nabla} \frac{|\nabla u^-|^2}{\sqrt{1-|\nabla u^-|^2}} \geq 0.$$

Hence, meas $\mathcal{O}_u^\nabla = 0$ and, as $\nabla u^- = 0_{\mathbb{R}^N}$ in $\Omega \setminus \mathcal{O}_u$, we get that $\nabla u^- = 0_{\mathbb{R}^N}$ a.e. on Ω. As Ω is a domain, we infer that u^- is constant, so $u^- \equiv 0$ in Ω. But, this means $u \geq 0$ in Ω. As also $v \geq 0$, clearly (u,v) solves $(1.3)^+$. This completes the proof. □

Analogously, one obtains the following lemma.

Lemma 3.7. *Assume Hypothesis 3.4.*

(i) *If (u,v) is a solution of*

$$\begin{cases} \mathcal{M}(u) = \mu_1(x)|u|^{q_1-1}u - \lambda F_u(x,u^-,v^-), & x \in \Omega, \\ \mathcal{M}(v) = \mu_2(x)|v|^{q_2-1}v - \lambda F_v(x,u^-,v^-), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases}$$

then $u \leq 0$, $v \leq 0$ in Ω, and (u,v) solves problem $(1.3)^-$.

(ii) *If (u,v) is a solution of*

$$\begin{cases} \mathcal{M}(u) = \mu_1(x)|u|^{q_1-1}u - \lambda F_u(x,u^+,v^-), & x \in \Omega, \\ \mathcal{M}(v) = \mu_2(x)|v|^{q_2-1}v - \lambda F_v(x,u^+,v^-), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases}$$

then $u \geq 0$, $v \leq 0$ in Ω, and (u,v) solves $(1.3)^\pm$.

(iii) *If (u,v) is a solution of*

$$\begin{cases} \mathcal{M}(u) = \mu_1(x)|u|^{q_1-1}u - \lambda F_u(x,u^-,v^+), & x \in \Omega, \\ \mathcal{M}(v) = \mu_2(x)|v|^{q_2-1}v - \lambda F_v(x,u^-,v^+), & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases}$$

then $u \leq 0$, $v \geq 0$ in Ω and (u,v) solves problem $(1.3)^\mp$.

We say that a solution (u,v) is *positive* if $u, v \geq 0$ in Ω. In a similar way, (u,v) will be called *negative* if $u, v \leq 0$, *positive-negative* if $u \geq 0$, $v \leq 0$ and *negative-positive* if $u \leq 0$, $v \geq 0$.

Corollary 3.8. *Assume Hypothesis 3.4 and that* (3.1) *is fulfilled. If either* (3.3) *or* (3.4) *is satisfied, then there exists* $\Lambda > 0$ *such that problem* $(1.3)^+$ *has at least two nontrivial positive solutions for all* $\lambda > \Lambda$.

Proof. This is immediate from Theorem 3.1 and Lemma 3.6. □

Similarly, from Theorem 3.2 and Lemma 3.7, we get the following corollary.

Corollary 3.9. (i) *Assume that Hypothesis 3.4 and* (3.10) *hold true. If either* (3.3) *or* (3.4) *is fulfilled, then there exists* $\Lambda > 0$ *such that for all* $\lambda > \Lambda$, *problem* $(1.3)^-$ *has at least two nontrivial negative solutions.*

(ii) *If Hypothesis 3.4 and* (3.11) *hold true, and either* (3.3) *or* (3.4) *is satisfied, then there exists* $\Lambda > 0$ *such that* $(1.3)^\mp$ *has at least two nontrivial positive-negative solutions for all* $\lambda > \Lambda$.

(iii) *Assume that Hypothesis 3.4 and* (3.12) *hold true. If either* (3.3) *or* (3.4) *holds true, then there exists* $\Lambda > 0$ *such that problem* $(1.3)^\mp$ *has at least two nontrivial negative-positive solutions for all* $\lambda > \Lambda$.

Example 3.10. From Corollary 3.8, with $F\colon \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$F(x,u,v) = \frac{2}{3}\sqrt{u^3v^3},$$

we have that there exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, the system

$$\begin{cases} \mathcal{M}(u) + \lambda\sqrt{uv^3} = \mu_1(x)u^{1/4}, & x \in \Omega, \\ \mathcal{M}(v) + \lambda\sqrt{u^3v} = \mu_2(x)v^{1/4}, & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega} \end{cases}$$

has at least two nontrivial positive solutions provided that both of μ_1^m and μ_2^m are positive. Also, note that, in view of Corollaries 3.8 and 3.9 (iii), system (3.13) from Example 3.3 has actually four nontrivial solutions, two positive and two positive-negative.

By virtue of the above results one has the following theorem.

Theorem 3.11. *Assume that Hypothesis 1.1,* (3.1), (3.10), (3.11) *and* (3.12) *are satisfied. If, in addition,*

$$\begin{cases} F(x,0,v) = F_u(x,0,v) = F_u(x,u,0) = 0, \\ F(x,u,0) = F_v(x,u,0) = F_v(x,0,v) = 0 \quad \text{for all } u,v \in \mathbb{R}, \end{cases} \tag{3.17}$$

and either (3.3) *or* (3.4) *holds true, then there exists* $\Lambda > 0$ *such that for all* $\lambda > \Lambda$, *problem* (1.3) *has at least eight solutions with each component nontrivial – two of them are positive, two are negative, two are positive-negative and the other two are negative-positive.*

Proof. The existence of solutions follows from Corollaries 3.8 and 3.9. Now, if (u,v) is a nontrivial solution of (1.3) and suppose that, e.g., $u \equiv 0$, then multiplying the second equation in (1.3) by v, using (3.17) and the integration by parts formula, we get

$$0 \geq -\int_\Omega \frac{|\nabla v|^2}{\sqrt{1-|\nabla v|^2}} = \int_\Omega \mu_2(x)|v|^{q_2+1} \geq 0,$$

which yields $v \equiv 0$, a contradiction. Similar reasoning works if we start by supposing that $v \equiv 0$. This implies that the obtained eight solutions have each component nontrivial and clearly they are also pairwise distinct. This completes the proof. □

Example 3.12. Let $\gamma(u,v) \in \{1,\dots,4\}$ denote the number of the quadrant to which $(u,v) \in \mathbb{R}^2$ belongs, when $u \neq 0 \neq v$; if one of the components u or v is zero, then we may take $\gamma(u,v) = 0$. Then, setting $F(x,u,v) = \gamma(u,v)u^2v^2/2$, we get that there exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, the system

$$\begin{cases} \mathcal{M}(u) + \lambda\gamma(u,v)uv^2 = 0, & x \in \Omega, \\ \mathcal{M}(v) + \lambda\gamma(u,v)u^2v = 0, & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega} \end{cases}$$

has at least eight solutions, having each component nontrivial – two positive, two negative, two positive-negative and the other two negative-positive. Notice that the system has no oddness property, and none of the solutions can be obtained from another one by a symmetrization.

4 Solvability of System (1.4)

For the proof of the main result of this section, we need some preliminary technical results. The following is proved in [22, Lemma 2.2].

Lemma 4.1. *If $h \in L^\infty(\Omega)$, then the Dirichlet problem*

$$\mathcal{M}(u) = h \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{4.1}$$

has a unique solution $u_h \in W^{2,p}(\Omega)$ for all $1 \le p < \infty$. Moreover, for any given $\Lambda > 0$ and $r > N$, there exists a constant $c = c(\Omega, \Lambda, r) > 0$ such that $\|u_h\|_{W^{2,r}} \le c\|h\|_\infty$ for every $h \in L^\infty(\Omega)$ with $\|h\|_\infty \le \Lambda$.

By means of Lemma 4.1, we can define the *solution operator* $S\colon L^\infty(\Omega) \to W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$ ($r > N$ fixed), which maps any $h \in L^\infty(\Omega)$ into the unique solution $S(h) := u_h \in W^{2,r}(\Omega)$ of problem (4.1).

Lemma 4.2 ([10, Lemma 2.3]). *If $h \in L^\infty(\Omega)$, then $S(h)$ is the unique solution in K_0 of the variational inequality*

$$\int_\Omega \Big[\sqrt{1 - |\nabla u|^2} - \sqrt{1 - |\nabla v|^2} + h(v - u) \Big] \ge 0 \quad \text{for all } v \in K_0.$$

Lemma 4.3. *If $\{h_n\}$ is a bounded sequence in $L^\infty(\Omega)$, $h \in L^\infty(\Omega)$ and $h_n \to h$ in $L^1(\Omega)$, then $S(h_n) \to S(h)$ in $C^1(\overline{\Omega})$.*

Proof. Following an idea from [22], we will show that any subsequence of $\{S(h_n)\}$, still denoted by $\{S(h_n)\}$, has a subsequence $\{S(h_{n_k})\}$ with $S(h_{n_k}) \to S(h)$ in $C^1(\overline{\Omega})$ as $k \to \infty$. This will accomplish the proof.

As $\Lambda := \sup\{\|h_n\|_\infty : n \in \mathbb{N}\} < +\infty$, from Lemma 4.1 we have that

$$\|S(h_n)\|_{W^{2,r}} \le c\|h_n\|_\infty \le c\Lambda \quad \text{for all } n \in \mathbb{N}, \tag{4.2}$$

hence, the whole sequence $\{S(h_n)\}$ is bounded in $W^{2,r}(\Omega)$. Then, by the compactness of the embedding $W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$, the subsequence $\{S(h_n)\}$ has at its turn a subsequence $\{S(h_{n_k})\}$ which (strongly) converges in $C^1(\overline{\Omega})$ to some $w \in W^{2,r}(\Omega)$ as $k \to \infty$. It remains to prove that $w = S(h)$. By virtue of Lemma 4.2, this reduces to showing that for all $v \in K_0$, we have

$$\int_\Omega \Big[\sqrt{1 - |\nabla w|^2} - \sqrt{1 - |\nabla v|^2} + h(v - w) \Big] \ge 0. \tag{4.3}$$

Let $v \in K_0$ be arbitrarily chosen. From $h_{n_k} \to h$ in $L^1(\Omega)$ it is straightforward to see that

$$\int_\Omega h_{n_k}(v - S(h_{n_k})) \to \int_\Omega h(v - w) \quad \text{as } k \to \infty. \tag{4.4}$$

Then, since $|\nabla S(h_{n_k})| \to |\nabla w|$ in $C(\overline{\Omega})$, using (4.4) and the fact that

$$\int_\Omega \Big[\sqrt{1 - |\nabla S(h_{n_k})|^2} - \sqrt{1 - |\nabla v|^2} + h_{n_k}(v - S(h_{n_k})) \Big] \ge 0,$$

we get (4.3) by letting $k \to \infty$. □

Remark 4.4. If $\{h_n\}$ is a bounded sequence in $L^\infty(\Omega)$, then, as above, from (4.2) the sequence $\{S(h_n)\}$ has a subsequence which is convergent in $C^1(\overline{\Omega})$. This together with Lemma 4.3 shows that $S\colon L^\infty(\Omega) \to C^1(\overline{\Omega})$ is completely continuous (see also [22, Lemma 2.3].

Now, we can state the following "universal" existence result.

Theorem 4.5. *If the Carathéodory functions* $f_1, f_2 : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ *satisfy Hypothesis 1.2, then system* (1.4) *has at least one solution.*

Proof. From Hypothesis 1.2, we have that the Nemytskii operators

$$N_i(u, v) = f_i(\,\cdot\,, u(\,\cdot\,), v(\,\cdot\,), \nabla u(\,\cdot\,), \nabla v(\,\cdot\,)) \quad (i = 1, 2)$$

are continuous from the product space $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ to $L^1(\Omega)$, and map bounded sets from $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ into bounded sets in $L^\infty(\Omega)$. From Lemma 4.3 and Remark 4.4, we infer that the operator

$$T : C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

defined by

$$T(u, v) = (S \circ N_1(u, v), S \circ N_2(u, v)) \quad \text{for all } (u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

is completely continuous. Since the convex set

$$U := \{u \in C^1(\overline{\Omega}) : \|\nabla u\|_\infty < 1,\ u|_{\partial\Omega} = 0\} \subset K_0$$

is bounded in $C^1(\overline{\Omega})$ and $T(\overline{U} \times \overline{U}) \subset U \times U$, from Schauder's theorem, the operator T has a fixed point which is a solution of (1.4). □

Remark 4.6. Theorem 4.5 is an extension to the non-potential system (1.4) of the ones obtained in [7, 10, 21, 22, 29] for a single potential and/or non-potential equation.

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