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Research Article

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Dynamics for Generalized Incompressible Navier–Stokes Equations in $\mathbb{R}^2$

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Abstract: In this paper, we consider the dynamics for damped generalized incompressible Navier–Stokes equations defined on $\mathbb{R}^2$. The generalized Navier–Stokes equations here refer to the equations obtained by replacing the Laplacian in the classical Navier–Stokes equations by the more general operator $(-\Delta)^\alpha$ with $\alpha \in (\frac{1}{2}, 1)$. We prove that the rate of dissipation of enstrophy vanishes as $\nu \to 0^+$, where ν is the viscosity parameter. Moreover, we prove the existence and finite dimensionality of a global attractor in $(H^1(\mathbb{R}^2))^2$ as $\nu > 0$ is kept fixed for the generalized Navier–Stokes equations.

Keywords: Generalized Navier–Stokes Equation, Inviscid Limit, Global Attractor

MSC 2010: 35Q30, 76D60, 35B41

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1 Introduction

We consider the damped and driven generalized incompressible Navier–Stokes (GNS) equations in $\mathbb{R}^2$

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + u \cdot \nabla u + \gamma u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $\gamma > 0$ is a fixed damping coefficient, and the coefficient $\nu > 0$ is a parameter that we will let vary. The force f is given and time-independent, and the initial velocity is divergence-free and belongs to $(H^1(\mathbb{R}^2))^2$. The fractional Laplacian $\Lambda^{2\alpha} = (-\Delta)^\alpha$ is defined in terms of the Fourier transform

$$\widehat{\Lambda^{2\alpha}\phi}(\xi) = |\xi|^{2\alpha}\widehat{\phi}(\xi),$$

where $\widehat{\phi}(\xi) = \int_{\mathbb{R}^2} \phi(x)e^{-ix\cdot\xi}\,dx$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$.

The fractional Laplacian operator appears in a wide class of physical systems and engineering problems, including Lévy flights, stochastic interfaces and anomalous diffusion problems. In fluid mechanics, the fractional Laplacian is often applied to describe many complicated phenomenons via partial differential equations. Equations (1.1) are generalizations of the classical Navier–Stokes equations.

When $\alpha = 1$, the GNS equations (1.1) reduce to the usual Navier–Stokes equations. One advantage of working with the GNS equations is that they allow simultaneous consideration of their solutions corresponding to a range of α's. The well-posedness and regularity about the GNS equations in different spaces have been studied by many authors, see, for example, Wu [22, 23], Li–Zhai [15], Wu–Fan [24], Zhang–Fang [25].

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In this paper, we mainly consider the dynamics of the solutions of (1.1). More precisely, we accomplish two major goals. First, as $\nu \to 0^+$, in spirit of [5, 12], we consider the zero viscosity limit of long time averages of vorticity. Second, as $\nu > 0$ is kept fixed, we prove the existence and finite dimensionality of a compact global attractor for the dynamical systems generated by (1.1) in $(H^1(\mathbb{R}^2))^2$.

In Section 2, we recall the notation and properties that we will use throughout this paper. Especially, we give a simple proof about the existence and uniqueness of the solution for the GNS equations (1.1), see Lemma 2.1.

Principal substantive questions related to turbulence have been raised since the beginning of the twentieth century, a large number of empirical and heuristical results were derived, among them the works of Lamb [14] on addressing idealized inviscid flows, and Taylor [19] on viscous flows. Anomalous dissipation is important in revealing the turbulence, see [5, 6, 12, 13] for the details. Anomalous dissipation of energy in three-dimensional turbulence is one of the basic statements of physical theory, which is open mostly in mathematics. Recently, Constantin et al. have proved in [5] the absence of anomalous dissipation of enstrophy for two-dimensional forced damped Navier–Stokes equations, and in [6] the absence of anomalous dissipation of energy for forced surface quasi-geostrophic equations. In Section 3, following the blueprint in Constantin–Ramos [5], we prove the absence of anomalous dissipation of enstrophy for the two-dimensional GNS equations (1.1), that is, the following theorem.

Theorem 1.1. *Let $u_0 \in (L^2(\mathbb{R}^2))^2$ be divergence-free and $\nabla^\perp \cdot u_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $f \in W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$. Let $S^{(\nu)}(t, \omega_0)$ be the vorticity of the solution of the damped and driven generalized Navier–Stokes equations. Then*

$$\lim_{\nu \to 0} \nu \Big(\limsup_{t \to \infty} \frac{1}{t} \int_0^t \|\Lambda^\alpha S^{(\nu)}(s + t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)} ds \Big) = 0$$

for any $t_0 > 0$.

Theorem 1.1 shows that a same result as that in [5] holds when we relax the exponent $\alpha = 1$ to $\alpha \in (\frac{1}{2}, 1)$. However, it is unknown for the case $\alpha \le \frac{1}{2}$.

The long time average of solutions considered in Theorem 1.1 corresponds to some invariant measures. Following the idea of [5, 10–12], we will use the so-called stationary statistical solution, which is a measure in the phase space and is a natural extension of the notion of invariant measure for deterministic finite dimensional dynamical systems to infinite dimensional case; we refer the readers to [12] for more details about the stationary statistical solution.

On the other hand, from the pioneering works of [10–12], we know that the stationary statistical solutions are supported in the corresponding global attractor. Hence, in Section 4, we will concentrate on studying the asymptotic behaviors of solutions for system (1.1). More precisely, we consider the existence and finite dimensionality of a compact global attractor in $(H^1(\mathbb{R}^2))^2$ as $\nu > 0$ is kept fixed for the GNS equations (1.1). The main result in this section is the following theorem.

Theorem 1.2. *Let $\alpha \in (\frac{1}{2}, 1)$, $\nu, \gamma > 0$ and $f \in (W^{1,2}(\mathbb{R}^2))^2$ be time-independent. Then the semigroup $\{S(t)\}_{t \ge 0}$ generated by the solutions of (1.1) has a unique global attractor $\mathscr{A}$ in V; that is, $\mathscr{A}$ is compact, invariant in V and attracts every bounded subset of V in the V-norm. Moreover, the fractal dimension of $\mathscr{A}$ is finite in V.*

Since $\nu > 0$ is kept fixed and (1.1) has certain smoothing effect, we can get easily some regularity estimates, which, combined with the *tail estimate* technique introduced in Wang [21], allow us to deduce the existence of a compact global attractor directly. Nevertheless, the finite dimensionality requires much more than just regularity, especially for the systems defined on unbounded domains that lack compact embeddings. In this part, we apply a criterion originally introduced in Chueshov–Lasiecka [3] for the hyperbolic equation, but with a combination of the idea of *l-trajectory* in Malek–Prazak [17], see Theorem 4.8 below. Note that such method allows us to deduce the finite dimensionality of the attractor only via the energy estimates and tail estimates that are already established for the existence of the attractor.

2 Preliminaries

2.1 Notation and Solution

Set

$$\mathcal{V} = \{u \in (C_c^\infty(\mathbb{R}^2))^2 \mid \operatorname{div} u = 0\}.$$

For each $s \in [0, 2]$, define H^s as the completion of $\mathcal{V}$ with respect to the norm $\|\cdot\|_s$, where, for any $u \in \mathcal{V}$,

$$\|u\|_s := \Big(\int_{\mathbb{R}^2} (1 + |\xi|^s)|\hat{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$

Especially, to simplify the notation we denote

$$H = H^0, \quad \|\cdot\| = \|\cdot\|_0 \text{ with inner products } (u, v) = \int_{\mathbb{R}^2} u \cdot v dx, \; u, v \in H^0,$$
$$V = H^1, \quad V' = \text{dual space of } V.$$

Then the mathematical framework of (1.1) is now classical, and we consider the following weak formulation of (1.1): find

$$u \in L^\infty(0, T; V) \cap L^2(0, T; H^{2\alpha}) \quad \text{for all } T > 0 \tag{2.1}$$

such that

$$\frac{d}{dt}(u, v) + \nu(\Lambda^\alpha u, \Lambda^\alpha v) + \gamma(u, v) + b(u, u, v) = (f, v) \quad \text{for all } v \in V, \; t > 0 \tag{2.2}$$

and

$$u(x, 0) = u_0, \tag{2.3}$$

where $b : V \times V \times V \to \mathbb{R}$ is given by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

Lemma 2.1. *Let $\nu, \gamma > 0$, $\alpha \in (\frac{1}{2}, 1)$ and $f \in (W^{1,2}(\mathbb{R}^2))^2$. Then for each initial data $u_0 \in V$ and any $T > 0$, there exists a unique $u \in L^\infty(0, T; V)$ such that* (2.2) *and* (2.3) *hold. Moreover, $u' \in L^2(0, T; H)$ and $u \in C([0, T); V)$.*

This existence and uniqueness indeed can be seen as a special result of Wu [22, 23]. Here, since we can use the vorticity equation in $\mathbb{R}^2$, we give a simple proof as follows. Moreover, since our initial data and forcing term are nice, the definition of the solution given in (2.1)–(2.3) is different, e.g., compared with [22, 23].

Proof. The existence can be obtained by the well-known viscosity solution method (see [16]): consider the following equation, $\varepsilon > 0$:

$$\begin{cases} \partial_t u_\varepsilon + \nu(-\Delta)^\alpha u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \gamma u_\varepsilon + \nabla p_\varepsilon - \varepsilon \Delta u_\varepsilon = f, \\ \nabla \cdot u_\varepsilon = 0, \\ u_\varepsilon(x, 0) = u_0(x). \end{cases} \tag{2.4}$$

By the classical result for Navier–Stokes equations we know that for each $\varepsilon > 0$ and $u_0 \in V$, there exists a unique solution $u_\varepsilon \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\mathbb{R}^2))$ which satisfies $(2.4)_1$ almost everywhere in $\mathbb{R}^2 \times (0, T)$.

We will get a solution for the weak formulation (2.2) by taking $\varepsilon \to 0^+$.

For this, we need to show that $\{u_\varepsilon\}$ is bounded in V with the bounds independent of ε, which will imply $u \in L^\infty(0, T; V)$. Indeed, this can be done by considering the corresponding vorticity ω_ε (the curl of the incompressible two-dimensional velocity):

$$\omega_\varepsilon = \nabla^\perp \cdot u_\varepsilon = \partial_1 u_{\varepsilon 2} - \partial_2 u_{\varepsilon 1},$$

which obeys

$$\partial_t \omega_\varepsilon + u_\varepsilon \cdot \nabla \omega_\varepsilon + \gamma \omega_\varepsilon + \nu \Lambda^{2\alpha} \omega_\varepsilon - \varepsilon \Delta \omega_\varepsilon = g$$

with $g \in L^2(\mathbb{R}^2)$, $g = \nabla^\perp \cdot f$. It is easy to see that $\|\omega_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^2)}$ can be controlled by the bounds which depend only on $\|\omega_\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^2)}$ and the coefficients in the equations. Then the uniformly (with respect to ε) boundedness of u_ε in V can be deduced immediately by noting $u_\varepsilon = K \star \omega_\varepsilon$ (see, e.g., Lemma 3.4; where $K = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$ is the Biot–Savart kernel). Note that here we used crucially the fact that $u_0 \in V$ and $f \in (W^{1,2}(\mathbb{R}^2))^2$.

To see $u \in L^2(0, T; H^{2\alpha})$, we multiply (1.1) (this can be justified by multiplying (2.4) with $\Lambda^{2\alpha} u_\varepsilon$ and then taking the limitation) by $\Lambda^{2\alpha} u$ and integrate in space to deduce that

$$\frac{d}{dt} \|\Lambda^\alpha u\|^2 + 2\gamma \|\Lambda^\alpha u\|^2 + 2\nu \|\Lambda^{2\alpha} u\|^2 \leq 2 \int_{\mathbb{R}^2} |u \cdot \nabla u| \, |\Lambda^{2\alpha} u| dx.$$

We will use the fact $u \in L^\infty(0, T; V)$ to deal with the nonlinear term as follows:

$$\int_{\mathbb{R}^2} |u \cdot \nabla u| \, |\Lambda^{2\alpha} u| dx \leq \|\Lambda^{2\alpha} u\| \cdot \|\nabla u\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2)}, \tag{2.5}$$

where $p \in (2, \infty)$ is large enough (e.g., $H^{(2\alpha-1)^-} \hookrightarrow L^{\frac{2p}{p-2}}(\mathbb{R}^2)$ since $\alpha \in (\frac{1}{2}, 1)$) such that

$$\|\nabla u\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \leq C_\eta \|u\| + \eta \|\Lambda^{2\alpha} u\| \tag{2.6}$$

for any $\eta > 0$; where $(2\alpha - 1)^-$ denotes the number smaller than $2\alpha - 1$ but approaching $2\alpha - 1$. Consequently, combining with $H^1 \hookrightarrow L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$, we have

$$\begin{aligned} &\frac{d}{dt} \|\Lambda^\alpha u(t)\|^2 + 2\gamma \|\Lambda^\alpha u(t)\|^2 + 2\nu \|\Lambda^{2\alpha} u(t)\|^2 \\ &\quad \leq 2 \|\Lambda^{2\alpha} u(t)\| (C_\eta \|u(t)\| + \eta \|\Lambda^{2\alpha} u(t)\|) \|u\|_{L^\infty(0,T;V)} \\ &\quad \leq (3\eta \|\Lambda^{2\alpha} u(t)\|^2 + C'_\eta \|u(t)\|^2) \|u\|_{L^\infty(0,T;V)} \quad \text{for a.e. } t \in (0, T). \end{aligned}$$

Hence, by taking η small enough such that $3\eta \|u\|_{L^\infty(0,T;V)} \leq \nu$, we integrate the above inequality over $(0, T)$ and obtain that

$$\nu \int_0^T \|\Lambda^{2\alpha} u(s)\|^2 ds \leq \|\Lambda^\alpha u(0)\|^2 + C'_\eta \|u\|_{L^\infty(0,T;V)} \int_0^T \|u(s)\|^2 ds,$$

which, combined with the fact $u(0) \in V$ and the a priori bounds about $\|u\|_{L^\infty(0,T;V)}$ again, implies that $u \in L^2(0, T; H^{2\alpha})$.

We can see from (1.1) that $u' \in L^2(0, T; H)$, directly after we obtained $u \in L^\infty(0, T; V) \cap L^2(0, T; H^{2\alpha})$. Then $u \in C([0, T); V)$ follows immediately.

The uniqueness is obvious by noticing the fact that the solution belongs to $L^\infty(0, T; V)$: let (u^i, p^i) be the solutions corresponding to initial data u_0^i, $i = 1, 2$, and write $w = u^1 - u^2$, then w solves the following equation:

$$\begin{cases} \partial_t w + \nu(-\Delta)^\alpha w + u^1 \cdot \nabla u^1 - u^2 \cdot \nabla u^2 + \gamma w + \nabla p^1 - \nabla p^2 = 0, \\ \nabla \cdot w = 0, \\ w(x, 0) = u_0^1 - u_0^2. \end{cases} \tag{2.7}$$

Multiplying (2.7) by w and integrating in space, we get

$$\frac{d}{dt} \|w(t)\|^2 + 2 \min\{\gamma, \nu\} \|w(t)\|_{H^\alpha}^2 \leq 2|b(w, u^1, w)| \leq 2 \|w(t)\|_{L^4(\mathbb{R}^2)}^2 \|u^1(t)\|_V. \tag{2.8}$$

Then, noting that $u^1 \in L^\infty(0, T; V)$ and using the fact (since $\alpha > \frac{1}{2}$)

$$\|w(t)\|_{L^4(\mathbb{R}^2)} \leq C \|w(t)\|^r \|w(t)\|_{H^\alpha}^{1-r}$$

for some constant $r \in (0,1)$, we can deduce from (2.8) that

$$\frac{d}{dt}\|w(t)\|^2 \le C_1\|w(t)\|^2,$$

where the constant C_1 depends on $\|u^1\|_{L^\infty(0,T;V)}$ and γ, ν; which immediately implies that

$$\|w(t)\|^2 \le e^{C_1 t}\|u_0^1 - u_0^2\|^2 \quad \text{for all } t \in [0,T). \tag{2.9}$$

This finishes the proof. □

2.2 Useful Properties

We will frequently use the following properties. For the proofs we refer to [7, 9].

Proposition 2.2. *Let $0 < \alpha < 2$, $x \in \mathbb{R}^2$ and $\theta \in \mathcal{S}$, the Schwartz class. Then*

$$\Lambda^\alpha \theta(x) = C_\alpha\, P.V. \int_{\mathbb{R}^2} \frac{[\theta(x) - \theta(y)]}{|x-y|^{2+\alpha}}\, dy,$$

where $C_\alpha > 0$.

Proposition 2.3 (Pointwise identity). *Let $0 \le \alpha \le 2$, $x \in \mathbb{R}^2$ and $\theta \in \mathcal{S}$. Then*

$$2\theta\Lambda^\alpha\theta(x) = \Lambda^\alpha(\theta^2(x)) + D_\alpha[\theta](x),$$

where

$$D_\alpha[\theta](x) = P.V. \int_{\mathbb{R}^2} \frac{[\theta(y) - \theta(x)]^2}{|x-y|^{2+\alpha}}\, dy \ge 0.$$

Proposition 2.4 (Positivity lemma). *Let $0 \le \alpha \le 2$, $x \in \mathbb{R}^2$ and $\theta, \Lambda^\alpha\theta \in L^p$ with $1 \le p < \infty$. Then*

$$\int_{\mathbb{R}^2} |\theta|^{p-2}\theta\Lambda^\alpha\theta\, dx \ge 0.$$

3 Inviscid Limit

Throughout the section, we assume that $\alpha \in (\frac{1}{2}, 1)$ and $\gamma > 0$ are kept fixed, and that the force $f \in (W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2))^2$ is given and time-independent. The results follow the idea in Constantin–Ramos [5], see also Foias et al. [10–12] for more details.

3.1 Preliminaries

We start with some properties of the solutions, which are the same as that in [5] for the usual damped and driven Navier–Stokes equation.

Theorem 3.1. *Let $u_0 \in V$. Then the solution of* (1.1) *with initial datum u_0 exists for all time, is unique and satisfies the energy equation*

$$\frac{d}{2dt}\int_{\mathbb{R}^2} |u|^2 dx + \gamma \int_{\mathbb{R}^2} |u|^2 dx + \nu \int_{\mathbb{R}^2} |\Lambda^\alpha u|^2 dx = \int_{\mathbb{R}^2} f \cdot u\, dx. \tag{3.1}$$

The kinetic energy is bounded uniformly in time, with bounds independent of viscosity ν:

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le e^{-\gamma t}\Big\{\|u_0\|_{L^2(\mathbb{R}^2)} - \frac{1}{\gamma^2}\|f\|_{L^2(\mathbb{R}^2)}\Big\} + \frac{1}{\gamma^2}\|f\|_{L^2(\mathbb{R}^2)}. \tag{3.2}$$

The vorticity

$$\omega = \nabla^{\perp} \cdot u = \partial_1 u_2 - \partial_2 u_1$$

obeys

$$\partial_t \omega + u \cdot \nabla \omega + \gamma \omega + \nu \Lambda^{2\alpha} \omega = g \tag{3.3}$$

with $g \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $g = \nabla^\perp \cdot f$. *The map*

$$[0, +\infty) \to L^2(\mathbb{R}^2), \quad t \mapsto \omega(t)$$

is continuous. If the initial vorticity ω_0 *is in* $L^p(\mathbb{R}^2)$, $p > 1$, *then the p-enstrophy is bounded uniformly in time, with bounds independent of viscosity:*

$$\|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} \le e^{-\gamma t} \Big\{ \|\omega_0\|_{L^p(\mathbb{R}^2)} - \frac{1}{\gamma^2} \|g\|_{L^p(\mathbb{R}^2)} \Big\} + \frac{1}{\gamma^2} \|g\|_{L^p(\mathbb{R}^2)} \tag{3.4}$$

for $p \ge 1$. *Moreover, the positive semi-orbit*

$$O_+(\omega) = \{\omega = \omega(\cdot, t) \mid t \ge 0\} \subset L^2(\mathbb{R}^2)$$

is equi-integrable in $L^2(\mathbb{R}^2)$*: for every* $\epsilon > 0$, *there exists* $R > 0$ *such that*

$$\int_{|x| \ge R} |\omega(x, t)|^2 \, dx \le \epsilon \tag{3.5}$$

for all $t \ge 0$, *where the radius R depends on the coefficients* γ, ν, α *and* $\omega(x, 0)$.

Proof. See Lemma 2.1 for the existence and uniqueness of solutions.

The energy equation (3.1) follows from the incompressibility of u and integration by parts. The bounds (3.2) and (3.4) follow from the positivity lemma and an application of the Gronwall inequality; see, for example, Section 4.

In the following, we only prove the equi-integrability (3.5). As in [5], we consider the function

$$Y_R(t) = \int_{\mathbb{R}^2} \chi\Big(\frac{x}{R}\Big) \omega^2(x, t) \, dx,$$

where $\chi(\cdot)$ is a nonnegative smooth function supported in $\{x \in \mathbb{R}^2 : |x| \ge \frac{1}{2}\}$ and identically equal to 1 for $|x| \ge 1$. We multiply equation (3.3) by $2\chi(\frac{x}{R})\omega(x, t)$ and integrate in space. The only challenging term we encounter is

$$2\nu \int_{\mathbb{R}^2} \Lambda^{2\alpha} \omega(x) \chi\Big(\frac{x}{R}\Big) \omega(x, t) \, dx.$$

Using Proposition 2.3, we have

$$\begin{aligned} 2\nu \int_{\mathbb{R}^2} \Lambda^{2\alpha} \omega(x, t) \chi\Big(\frac{x}{R}\Big) \omega(x, t) \, dx &\ge \nu \int_{\mathbb{R}^2} \Lambda^{2\alpha} (\omega^2(x, t)) \Big(1 - \Big(1 - \chi\Big(\frac{x}{R}\Big)\Big)\Big) dx \\ &= -\nu \int_{\mathbb{R}^2} \omega^2(x, t) \Lambda^{2\alpha} \Big(1 - \chi\Big(\frac{x}{R}\Big)\Big) dx, \end{aligned}$$

where we have used

$$\int_{\mathbb{R}^2} \Lambda^{2\alpha} (\omega^2) \, dx = \widehat{\Lambda^{2\alpha}(\omega^2)}(0) = 0$$

since $\alpha > 0$, and the fact that $\Lambda^{2\alpha}(1 - \chi(\frac{x}{R}))$ is well defined because $1 - \chi(\frac{x}{R}) \in C_c^\infty(\mathbb{R}^2)$. Moreover, it is easy to see that

$$1 - \chi\Big(\frac{x}{R}\Big) =: \phi\Big(\frac{x}{R}\Big).$$

In view of

$$\Lambda^{2\alpha}\Big(\phi\Big(\frac{x}{R}\Big)\Big)=\frac{1}{R^{2\alpha}}(\Lambda^{2\alpha}\phi)\Big(\frac{x}{R}\Big)$$

and

$$|(\Lambda^{2\alpha}\phi)(x)|\le\int_{\mathbb{R}^2}|\widehat{\Lambda^{2\alpha}\phi}(\xi)|d\xi=\int_{\mathbb{R}^2}|\xi|^{2\alpha}|\widehat{\phi}(\xi)|d\xi\le C_0,$$

we have

$$2\nu\int_{\mathbb{R}^2}\Lambda^{2\alpha}\omega(x,t)\chi\Big(\frac{x}{R}\Big)\omega(x,t)dx\ge-\frac{C_0\nu}{R^{2\alpha}}\|\omega(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}. \tag{3.6}$$

So we can obtain that

$$\frac{d}{dt}Y_R(t)+2\gamma Y_R(t)\le\frac{C}{R}\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|\omega(\cdot,t)\|^2_{L^4(\mathbb{R}^2)}+\frac{C_0\nu}{R^{2\alpha}}\|\omega(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}+C\Big(Y_R(t)\int_{|x|\ge\frac{R}{2}}|g(x)|^2dx\Big)^{\frac12},$$

that is,

$$\begin{aligned}Y_R(t)-e^{-\gamma t}Y_R(0)\le{}&\frac{C}{R}\max_{s\in[0,t]}\{\|u(\cdot,s)\|_{L^2(\mathbb{R}^2)}\}\cdot\int_0^t e^{\gamma(s-t)}\|\omega(\cdot,s)\|^2_{L^4(\mathbb{R}^2)}ds\\&+\frac{C_0\nu}{\gamma R^{2\alpha}}\max_{s\in[0,t]}\{\|\omega(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}\}+\frac{C}{\gamma^2}\int_{|x|\ge\frac{R}{2}}|g(x)|^2dx.\end{aligned} \tag{3.7}$$

On the other hand, multiplying (3.3) by $\omega(x,t)$ and integrating in space, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\omega(x,t)|^2dx+2\gamma\int_{\mathbb{R}^2}|\omega(x,t)|^2dx+2\nu\int_{\mathbb{R}^2}|\Lambda^\alpha\omega(x,t)|^2dx\le2\int_{\mathbb{R}^2}|\omega(x,t)|\,|g(x)|dx,$$

which implies that, for any $t\ge0$,

$$\|\omega(x,t)\|^2_{L^2(\mathbb{R}^2)}+2\nu\int_0^t e^{\gamma(s-t)}\|\Lambda^\alpha\omega(x,s)\|^2_{L^2(\mathbb{R}^2)}ds\le F<\infty$$

with a positive constant F which is bounded in terms of γ, $\|g\|^2_{L^2(\mathbb{R}^2)}$ and $\|\omega(x,0)\|^2_{L^2(\mathbb{R}^2)}$ (but is independent of t). Hence, combining with the embedding $H^\alpha\hookrightarrow L^4(\mathbb{R}^2)$ for $\alpha>\frac12$, we have

$$\int_0^t e^{\gamma(s-t)}\|\omega(x,s)\|^2_{L^4(\mathbb{R}^2)}ds\le C_{F,\nu}<\infty\quad\text{for any } t\ge0. \tag{3.8}$$

Therefore, from (3.2), (3.4), (3.8) and the fact that $|g|^2$ is integrable, the right-hand side of (3.7) will be arbitrary small if we take R large enough. Then the equi-integrability (3.5) follows from the fact that $Y_R(0)$ is small for large R. □

Remark 3.2. The vorticity equation (3.3) with a different draft term is a special case of the viscous surface quasi-geostrophic equation, which has been studied by many authors (e.g., see [1, 6–8] and the references therein). Especially, in [8], Dlotko, Kania and Sun obtained some approximation estimates that may be helpful in considering the inviscid limit problem (as what we do in this section for (1.1)) for the surface quasi-geostrophic equation.

In the following, let $\gamma,\nu>0$ be fixed. We give some a priori estimates about the vorticity $\omega(x,t)$.

We write $S^{(\nu)}(t,\omega_0)$ for the solution of the vorticity equation (3.3) at time $t\ge0$ which started at time $t=0$ from the initial data ω_0.

Theorem 3.3. *Let $\omega_0\in L^2(\mathbb{R}^2)$, $g\in L^2(\mathbb{R}^2)$ and $u\in L^\infty(0,\infty;V)$. Then for any $t_0>0$, the positive semi-orbit*

$$O_+(t_0,\omega_0)=\{\omega(\cdot,t)\mid t\ge t_0\}$$

is relatively compact in $L^2(\mathbb{R}^2)$.

The proof of this theorem follows from the equi-integrability (3.5) and Lemma 3.5 below.

Lemma 3.4. *Let $s \geq 0$ and $\Lambda^s \omega \in L^2(\mathbb{R}^2)$. Then for $u = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \star \omega$, we have $\Lambda^{1+s}u \in (L^2(\mathbb{R}^2))^2$ and*

$$\|\Lambda^{1+s}u\|_{(L^2(\mathbb{R}^2))^2} \leq C\|\Lambda^s \omega\|_{L^2(\mathbb{R}^2)},$$

Proof. Note that $u = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \star \omega = \Lambda^{-1}\mathcal{R}^\perp \omega$, hence

$$\|\Lambda^{1+s}u\|_{(L^2(\mathbb{R}^2))^2} = \|\Lambda^s \mathcal{R}^\perp \omega\|_{(L^2(\mathbb{R}^2))^2} \leq C\|\Lambda^s \omega\|_{L^2(\mathbb{R}^2)},$$

where the constant C depends only on the L^2-bounds of the Riesz transform $\mathcal{R}$. □

Lemma 3.5. *Let $\omega_0 \in L^2(\mathbb{R}^2)$, $g \in L^2(\mathbb{R}^2)$ and $u \in L^\infty(0,\infty;V)$. Then the solution $\omega(\cdot,t)$ of (3.3) is uniformly (with respect to $t \geq t_0$) bounded in $W^{\alpha,2}(\mathbb{R}^2)$ for any $t_0 > 0$.*

Proof. At first, we multiply (3.3) by ω to deduce that

$$\int_0^{t_0}\int_{\mathbb{R}^2} |\Lambda^\alpha \omega(x,\tau)|^2 dx d\tau \leq \widetilde{M}_0, \tag{3.9}$$

where the constant $\widetilde{M}_0$ depends only on ν, $\|g\|_{L^2(\mathbb{R}^2)}$ and $\|\omega(\cdot,0)\|_{L^2(\mathbb{R}^2)}$.

Secondly, we multiply again the vorticity equation (3.3) by $\Lambda^{2\alpha}\omega$ and integrate in space, to get

$$\frac{d}{2dt}\int_{\mathbb{R}^2} |\Lambda^\alpha \omega|^2 dx + \int_{\mathbb{R}^2}(u\cdot\nabla\omega)\Lambda^{2\alpha}\omega dx + \gamma\int_{\mathbb{R}^2}|\Lambda^\alpha\omega|^2 dx + \nu\int_{\mathbb{R}^2}|\Lambda^{2\alpha}\omega|^2 dx = \int_{\mathbb{R}^2} g\Lambda^{2\alpha}\omega dx.$$

The nonlinear term $\int_{\mathbb{R}^2}(u\cdot\nabla\omega)\Lambda^{2\alpha}\omega dx$ can be estimated as (2.5) and (2.6):

$$\left|\int_{\mathbb{R}^2}(u\cdot\nabla\omega)\Lambda^{2\alpha}\omega dx\right| \leq C\|u\|_V \cdot \|\omega\|_{L^2(\mathbb{R}^2)}^{1-r} \cdot \|\Lambda^{2\alpha}\omega\|_{L^2(\mathbb{R}^2)}^{1+r}$$

with some positive constant $r \in (0,1)$. Then, applying the Young inequality, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\Lambda^\alpha\omega|^2 dx + 2\gamma\int_{\mathbb{R}^2}|\Lambda^\alpha\omega|^2 dx \leq \widetilde{M}_1, \tag{3.10}$$

where the constant $\widetilde{M}_1$ depends on $\|u(\cdot,t)\|_V$, $\|\omega(\cdot,t)\|_{L^2(\mathbb{R}^2)}$, $\|g\|_{L^2(\mathbb{R}^2)}$ and ν.

Then, combining with (3.4), (3.9), (3.10) and Lemma 3.4, we can finish the proof by applying a Gronwall's inequality. □

Moreover, we have the following a priori estimates.

Lemma 3.6. *Let $\omega_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $g \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $u \in L^\infty(0,\infty;V)$. Then the solution $\omega(\cdot,t)$ of (3.3) satisfies*

$$\omega \in L^\infty(\mathbb{R}^2 \times (t_0,\infty))$$

for any $t_0 > 0$.

Proof. The proof is based on an idea of Caffarelli and Vasseur [1]. The details for our case are exactly as in [2, Lemmas 2.2 and 2.3]. Note that our assumptions here are stronger than that required in [2]. □

3.2 Stationary Statistical Solutions and Enstrophy Balance

We introduce first the notation of stationary statistical solution for damped and driven generalized incompressible Navier–Stokes equations in the vorticity phase space, in spirit of [5, 10–12]. The solution is a Borel probability measure in $L^2(\mathbb{R}^2)$.

Definition 3.7. A stationary statistical solution of damped and driven generalized incompressible Navier–Stokes equations in the vorticity phase space is a Borel probability measure $\mu^{(\nu)}$ on $L^2(\mathbb{R}^2)$ such that

$$\int_{L^2(\mathbb{R}^2)} \|\omega\|^2_{H^\alpha(\mathbb{R}^2)} d\mu^{(\nu)}(\omega) < \infty \tag{3.11}$$

and

$$\int_{L^2(\mathbb{R}^2)} \langle u \cdot \nabla\omega + \gamma\omega - g, \Psi'(\omega)\rangle + \nu\langle\Lambda^{2\alpha}\omega, \Psi'(\omega)\rangle d\mu^{(\nu)}(\omega) = 0 \tag{3.12}$$

for any test functional $\Psi \in \mathcal{T}$, with $u = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \star \omega$, where we used the notation $\langle v, w\rangle = \int_{\mathbb{R}^2} v(x)w(x)dx$; and

$$\int_{E_1 \le \|\omega\|_{L^2(\mathbb{R}^2)} \le E_2} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} + \nu\|\Lambda^\alpha\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle) d\mu^{(\nu)}(\omega) \le 0 \tag{3.13}$$

for any $0 \le E_1 \le E_2$.

The class $\mathcal{T}$ of cylindrical test functions is given as follows.

Definition 3.8 ([5, 12]). The class $\mathcal{T}$ of test functions is the set of functions $\Psi : L^2(\mathbb{R}^2) \to \mathbb{R}$ of the form

$$\Psi(\omega) := \Psi_I(\omega) = \psi(\langle\omega, \mathbf{w}_1\rangle, \ldots, \langle\omega, \mathbf{w}_m\rangle)$$

or

$$\Psi(\omega) := \Psi_\varepsilon(\omega) = \psi(\langle\alpha_\epsilon(\omega), \mathbf{w}_1\rangle, \ldots, \langle\alpha_\epsilon(\omega), \mathbf{w}_m\rangle),$$

where ψ is a C^1 scalar valued function defined on $\mathbb{R}^m$, $m \in \mathbb{N}$, $\mathbf{w}_1, \ldots, \mathbf{w}_m$ belong to $C_0^2(\mathbb{R}^2)$ and

$$\alpha_\epsilon(\omega) = J_\epsilon\beta(J_\epsilon(\omega)),$$

where $\beta \in C^3$ is a compactly supported function of one real variable, and J_ϵ is the convolution operator

$$J_\epsilon(\omega) = j_\epsilon \star \omega,$$

with $j \ge 0$ a fixed smooth, even function supported in $|z| \le 1$ and with $\int_{\mathbb{R}^2} j(z)dz = 1$.

Remark 3.9. We make mathematical sense of the conditions (3.11) and (3.13) in Definition 3.7: the function $\omega \mapsto \|\omega\|^2_{H^\alpha(\mathbb{R}^2)}$ is Borel measurable in $L^2(\mathbb{R}^2)$ because it is everywhere the limit of a sequence of continuous functions $\omega \mapsto \|J_\epsilon\omega\|^2_{H^\alpha(\mathbb{R}^2)}$.

Remark 3.10. The support of any stationary statistical solution of damped and driven generalized incompressible Navier–Stokes equations is included in the ball:

$$\operatorname{supp}\mu^{(\nu)} \subset \left\{\omega \in L^2(\mathbb{R}^2) : \|\omega\|_{L^2(\mathbb{R}^2)} \le \frac{1}{\gamma}\|g\|_{L^2(\mathbb{R}^2)}\right\}.$$

Indeed (note that the following proof differs slightly from that in [5, 12]), set

$$E = \{\omega \in L^2(\mathbb{R}^2) : E_1^2 \le \|\omega\|^2_{L^2(\mathbb{R}^2)} \le E_2^2\}.$$

Then from (3.13) we have

$$\begin{aligned}\gamma\int_E \|\omega\|^2_{L^2(\mathbb{R}^2)} d\mu^\nu(\omega) &\le \|g\|_{L^2(\mathbb{R}^2)} \int_E \|\omega\|_{L^2(\mathbb{R}^2)} d\mu^\nu(\omega)\\ &\le \|g\|_{L^2(\mathbb{R}^2)}\Big(\int_E \|\omega\|^2_{L^2(\mathbb{R}^2)} d\mu^\nu(\omega)\Big)^{\frac{1}{2}} \cdot (\operatorname{mes}(E))^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\int_E \|\omega\|^2_{L^2(\mathbb{R}^2)} d\mu^\nu(\omega) \le \frac{\|g\|^2_{L^2(\mathbb{R}^2)}}{\gamma^2} \operatorname{mes}(E),$$

which implies immediately that

$$\int_E \Big(\|\omega\|^2_{L^2(\mathbb{R}^2)} - \frac{\|g\|^2_{L^2(\mathbb{R}^2)}}{\gamma^2} \Big) d\mu^\nu(\omega) \leq 0.$$

Thus, we can deduce the support of μ^ν by taking $E_1 = \frac{\|g\|_{L^2(\mathbb{R}^2)}}{\gamma}$ in (3.13).

Remark 3.11. The test functions Ψ in Definition 3.8 are locally bounded and weakly sequentially continuous in $L^2(\mathbb{R}^2)$.

We can compute Ψ' for test functions $\Psi \in \mathcal{T}$ as follows (see [5]):

$$\Psi'_I(\omega) = \sum_{j=1}^m \partial_j \psi(\langle \omega, \mathbf{w}_1 \rangle, \dots, \langle \omega, \mathbf{w}_m \rangle) \mathbf{w}_j$$

and

$$\Psi'_\epsilon(\omega) = \sum_{j=1}^m \partial_j \psi(\langle \alpha_\epsilon(\omega), \mathbf{w}_1 \rangle, \dots, \langle \alpha_\epsilon(\omega), \mathbf{w}_m \rangle)(\beta'(\omega_\epsilon)\mathbf{w}_{j_\epsilon})_\epsilon.$$

Next, we state some important properties.

Lemma 3.12 ([5]). *Let $\Psi \in \mathcal{T}$ and $\omega \in L^2(\mathbb{R}^2)$. Then $\Psi'(\omega) \in C_0^2(\mathbb{R}^2)$. Consider $F_i : L^2(\mathbb{R}^2) \to \mathbb{R}$, $i = 1, 2, 3$, given by*

$$\begin{aligned} F_1(\omega) &= \langle \gamma\omega - g, \Psi'(\omega) \rangle, \\ F_2(\omega) &= \langle \Lambda^\alpha \omega, \Lambda^\alpha \Psi'(\omega) \rangle, \\ F_3(\omega) &= \langle u \cdot \nabla\omega, \Psi'(\omega) \rangle, \qquad u = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \star \omega. \end{aligned}$$

Then these three maps are well defined for $\omega \in L^2(\mathbb{R}^2)$, locally bounded and weakly continuous in $L^2(\mathbb{R}^2)$.

Proof. The maps F_1 and F_3 are exactly as in [5]. For F_2, we just need to follow the idea of [5]:

The fact that $F_2(\omega)$ is well defined follows from that $\Lambda^{2\alpha}\Psi'(\omega)$ is well defined (since $\Psi'(\omega) \in C_0^2(\mathbb{R}^2)$ by the fact that $\mathbf{w}_j \in C_0^2(\mathbb{R}^2)$) and

$$F_2(\omega) = \langle \Lambda^{2\alpha}\Psi'(\omega), \omega \rangle.$$

Concerning the weak continuity of F_2, for Ψ_I, we have

$$\langle \Lambda^{2\alpha}\Psi'_I(\omega), \omega \rangle = \sum_{j=1}^m \partial_j \psi(\langle \omega, \mathbf{w}_1 \rangle, \dots, \langle \omega, \mathbf{w}_m \rangle)\langle \Lambda^{2\alpha}\mathbf{w}_j, \omega \rangle,$$

which is obviously a weakly continuous and locally bounded function of $\omega \in L^2(\mathbb{R}^2)$. In the case of Ψ_ϵ, we have

$$\begin{aligned} \langle \Lambda^{2\alpha}\Psi'_\epsilon(\omega), \omega \rangle &= \sum_{j=1}^m \partial_j \psi(\langle \alpha_\epsilon(\omega), \mathbf{w}_1 \rangle, \dots, \langle \alpha_\epsilon(\omega), \mathbf{w}_m \rangle)\langle \Lambda^{2\alpha}(\beta'(\omega_\epsilon)\mathbf{w}_{j_\epsilon})_\epsilon, \omega \rangle \\ &= \sum_{j=1}^m \partial_j \psi(\langle \alpha_\epsilon(\omega), \mathbf{w}_1 \rangle, \dots, \langle \alpha_\epsilon(\omega), \mathbf{w}_m \rangle)\langle \mathbf{w}_{j_\epsilon}, \beta'(\omega_\epsilon)\Lambda^{2\alpha}\omega_\epsilon \rangle. \end{aligned}$$

If $\omega^i \rightharpoonup \omega$ in $L^2(\mathbb{R}^2)$, then $\omega^i_\epsilon \to \omega_\epsilon$ and $\Lambda^{2\alpha}\omega^i_\epsilon \to \Lambda^{2\alpha}\omega_\epsilon$ converge pointwise, and they are bounded. Consequently, $\beta'(\omega^i_\epsilon)\Lambda^{2\alpha}\omega^i_\epsilon$ converges pointwise and is uniformly bounded. Therefore we use the Lebesgue dominated convergence theorem and obtain that $F_2(\omega)$ is weakly continuous. It is also clear that

$$\|\beta'(\omega_\epsilon)\Lambda^{2\alpha}\omega_\epsilon\|_{L^2(\mathbb{R}^2)} \leq C_\epsilon \|\omega\|_{L^2(\mathbb{R}^2)}.$$

Thus, $F_2(\cdot)$ is locally bounded in $L^2(\mathbb{R}^2)$. □

We define the notation of a renormalized stationary statistical solution of the Euler equation.

Definition 3.13 ([5]). A Borel probability measure μ^0 on $L^2(\mathbb{R}^2)$ is a renormalized stationary statistical solution of the damped and driven Euler equation if

$$\int\limits_{L^2(\mathbb{R}^2)} \langle u\cdot\nabla\omega + \gamma\omega - g, \Psi'(\omega)\rangle d\mu^0(\omega) = 0 \tag{3.14}$$

for any test functional $\Psi \in \mathcal{T}$, where $u = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \star \omega$.

We say that a renormalized stationary statistical solution μ^0 of the Euler equation satisfies the enstrophy balance if

$$\int\limits_{L^2(\mathbb{R}^2)} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle) d\mu^0(\omega) = 0. \tag{3.15}$$

Theorem 3.14. *Let $\mu^{(\nu)}$ be a sequence of stationary statistical solutions of damped and driven generalized incompressible Navier–Stokes equations in vorticity phase space, with $\nu \to 0$. There exist a subsequence, denoted also $\mu^{(\nu)}$, and a Borel probability measure μ^0 on $L^2(\mathbb{R}^2)$ such that*

$$\lim_{\nu\to 0} \int\limits_{L^2(\mathbb{R}^2)} \Phi(\omega) d\mu^{(\nu)}(\omega) = \int\limits_{L^2(\mathbb{R}^2)} \Phi(\omega) d\mu^0(\omega)$$

for all weakly continuous, locally bounded real-valued functions Φ. Furthermore the weak limit measure μ^0 is a renormalized stationary statistical solution of the damped and driven Euler equation.

Proof. By Remark 3.10, the support of $\mu^{(\nu)}$ is included in

$$B = \left\{\omega \in L^2(\mathbb{R}^2) : \|\omega\|_{L^2(\mathbb{R}^2)} \le \frac{1}{\gamma}\|g\|_{L^2(\mathbb{R}^2)}\right\}.$$

The set B endowed with the weak $L^2(\mathbb{R}^2)$ topology is a separable metrizable compact space. We apply Prokhorov's theorem. There exists a subsequence of $\mu^{(\nu)}$ that converges weakly to a Borel probability measure μ^0 in B. So the weak limit μ^0 is a Borel probability measure on B. Because B is convex and so weakly closed in $L^2(\mathbb{R}^2)$, we can extend the measure μ^0 to $L^2(\mathbb{R}^2)$ by setting $\mu^0(X) = \mu^0(X \cap B)$. We claim that μ^0 is a renormalized stationary statistical solution of the damped and driven Euler equation. In order to verify that μ^0 satisfies (3.14), take $\Psi \in \mathcal{T}$. Then for each $i = 1, 2, 3$, noting that $\operatorname{supp}\mu^\nu \subset B$, we have

$$\lim_{\nu\to 0} \int\limits_{L^2(\mathbb{R}^2)} F_i(\omega) d\mu^{(\nu)}(\omega) = \int\limits_{L^2(\mathbb{R}^2)} F_i(\omega) d\mu^0(\omega)$$

in view of Lemma 3.12 (here the limitation $\lim_{\nu\to\infty}$ is taken for the subsequence which is weak convergence). In particular, the sequence $\int_{L^2(\mathbb{R}^2)} F_2(\omega) d\mu^{(\nu)}(\omega)$ is bounded and so

$$\lim_{\nu\to 0} \nu \int\limits_{L^2(\mathbb{R}^2)} F_2(\omega) d\mu^{(\nu)}(\omega) = 0.$$

Because $\mu^{(\nu)}$ are stationary statistical solutions of damped and driven generalized incompressible Navier–Stokes equations, using (3.12), we have

$$\int\limits_{L^2(\mathbb{R}^2)} (F_1(\omega) + F_3(\omega)) d\mu^{(\nu)}(\omega) = -\nu \int\limits_{L^2(\mathbb{R}^2)} F_2(\omega) d\mu^{(\nu)}(\omega).$$

Passing to the limit $\nu \to 0$, we obtain

$$\int\limits_{L^2(\mathbb{R}^2)} (F_1(\omega) + F_3(\omega)) d\mu^0(\omega) = 0.$$

That is, measure μ^0 satisfies condition (3.14), and therefore is a renormalized stationary statistical solution of the damped and driven Euler equation. □

We consider the sets

$$B_p^\infty(r) = \{\omega \in B \mid \|\omega\|_{L^p(\mathbb{R}^2)} \le r,\ \|\omega\|_{L^\infty(\mathbb{R}^2)} \le r\}$$

defined for $r > 0$, $1 \le p < 2$.

In exactly the same way as the proof of [5, Theorem 4.7], we can check that if we proved that the limitation μ^0 is a renormalized stationary statistical solution of the damped and driven Euler equation and μ^0 is supported in some bounded subset, then μ^0 must satisfy the enstrophy balance (3.15). Hence, similar to [5, Theorem 4.7], from Theorem 3.14 we also have the following result.

Theorem 3.15. *Let $\mu^{(\nu)}$ be a sequence of stationary statistical solutions of damped and driven generalized incompressible Navier–Stokes equations in vorticity phase space, with $\nu \to 0$. Assume that there exist $1 < p < 2$ and $r > 0$ such that*

$$\operatorname{supp} \mu^{(\nu)} \subset B_p^\infty(r).$$

Then the limit μ^0 of any weakly convergent subsequence is a renormalized stationary statistical solution of the damped and driven Euler equation that is supported in set $B_p^\infty(r)$ and satisfies the enstrophy balance (3.15).

3.3 Long Time Averages and the Inviscid Limit

In this subsection we consider the stationary statistical solutions obtained as generalized (Banach) limits of long time averages of functionals of determined solutions of the damped and driven generalized incompressible Navier–Stokes equations. These stationary statistical solutions have enough properties to pass to the inviscid limit and are used to prove that the time averaged enstrophy dissipation vanishes in the zero viscosity limit. We start by recalling the concept of the generalized (Banach) limit (see for example [12]).

Definition 3.16. A generalized limit (Banach limit) is a linear continuous functional

$$\operatorname*{Lim}_{t\to\infty} : \mathcal{BC}([0,\infty)) \to \mathbb{R}$$

such that

(i) $\operatorname{Lim}_{t\to\infty}(g) \ge 0$ for all $g \in \mathcal{BC}([0,\infty))$ with $g(s) \ge 0$ for all $s \ge 0$,

(ii) $\operatorname{Lim}_{t\to\infty}(g) = \lim_{t\to\infty} g(t)$, whenever the usual limit exists.

The space $\mathcal{BC}([0,\infty))$ is the Banach space of all bounded continuous real valued functions defined on $[0,\infty)$ endowed with the sup norm.

Remark 3.17 ([12]). It can be shown that any generalized limit satisfies

$$\liminf_{t\to\infty} g(t) \le \operatorname*{Lim}_{t\to\infty}(g) \le \limsup_{t\to\infty} g(t) \quad \text{for all } g \in \mathcal{BC}([0,\infty)).$$

Remark 3.18 ([12]). Given a fixed $g_0 \in \mathcal{BC}([0,\infty))$ and a sequence $t_j \to \infty$ for which $\lim_{j\to\infty} g_0(t_j) = l$ exists, we can construct a generalized limit $\operatorname{Lim}_{t\to\infty}$ satisfying $\operatorname{Lim}_{t\to\infty}(g_0) = l$. This implies that one can choose a functional $\operatorname{Lim}_{t\to\infty}$ so that $\operatorname{Lim}_{t\to\infty} g_0 = \limsup_{t\to\infty} g_0(t)$.

We now state the result about long time averages of the damped and driven generalized incompressible Navier–Stokes equations.

Theorem 3.19. *Let $u_0 \in (L^2(\mathbb{R}^2))^2$, $\nabla^\perp \cdot u_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $f \in W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, and $\operatorname{Lim}_{t\to\infty}$ be a Banach limit. Given $t_0 > 0$. Then μ^ν, defined by*

$$\int_{L^2(\mathbb{R}^2)} \Phi(\omega)\, d\mu^{(\nu)}(\omega) = \operatorname*{Lim}_{t\to\infty} \frac{1}{t} \int_0^t \Phi(S^{(\nu)}(s + t_0, \omega_0))\, ds, \tag{3.16}$$

is a stationary statistical solution of the damped and driven generalized incompressible Navier–Stokes equation in the vorticity phase space, where Φ is a continuous real functional on $L^2(\mathbb{R}^2)$. For any $p > 1$ there exists r

depending only on γ, f, ω_0 but not on ν nor t_0 such that

$$\operatorname{supp}\mu^{(\nu)} \subset B_p^{\infty}(r).$$

The following inequality holds:

$$\nu \int_{L^2(\mathbb{R}^2)} \|\Lambda^\alpha \omega\|_{L^2(\mathbb{R}^2)}^2 d\mu^{(\nu)}(\omega) \le \int_{L^2(\mathbb{R}^2)} [\langle g, \omega\rangle - \gamma\|\omega\|_{L^2(\mathbb{R}^2)}^2] d\mu^{(\nu)}(\omega). \tag{3.17}$$

Proof. From Theorem 3.3, the positive semi-orbit

$$O_+(t_0, \omega_0) = \{\omega(\cdot, t) = S^{(\mu)}(t, \omega_0) \mid t \ge t_0\}$$

is relatively compact in $L^2(\mathbb{R}^2)$. For any $\Phi \in C(L^2(\mathbb{R}^2))$, we have $\Phi \in C(\overline{O_+(t_0, \omega_0)})$ and the function $s \mapsto \Phi(S^{(\nu)}(s + t_0, \omega_0))$ is a continuous bounded function on $[0, \infty)$ and so is its time average on $[0, t]$. Thus we may apply the generalized limit $\operatorname{Lim}_{t\to\infty}$ to it and define the functional

$$\Phi \mapsto \operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t \Phi(S^{(\nu)}(s + t_0, \omega_0))ds.$$

This functional is linear and nonnegative. Hence, applying the Riesz representation theorem on compact spaces, it follows that there exists a Borel measure $\mu^{(\nu)}$ representing it, that is, (3.16) holds. The measure $\mu^{(\nu)}$ is supported on $\overline{O_+(t_0, \omega_0)}$. Extend $\mu^{(\nu)}$ to $L^2(\mathbb{R}^2)$ given by $\mu^{(\nu)}(X) = \mu^{(\nu)}(X \cap \overline{O_+(t_0, \omega_0)})$, for any X Borelian in $L^2(\mathbb{R}^2)$. It follows Theorem 3.1 that the measure $\mu^{(\nu)}$ is supported in the set $B_p^{\infty}(r)$.

Take a test function $\Psi \in \mathcal{T}$. Then, noticing that $O_+(t_0, \omega_0)$ is precompact in $L^2(\mathbb{R}^2)$, we can calculate directly that

$$\int_{L^2(\mathbb{R}^2)} \langle u \cdot \nabla\omega + \gamma\omega + \nu\Lambda^{2\alpha}\omega, \Psi'(\omega)\rangle d\mu^{(\nu)}(\omega) = \operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t \frac{d}{ds}\Psi(S^{(\nu)}(s + t_0, \omega_0))ds = 0,$$

where the second equality is due to the boundedness of Ψ on $O_+(t_0, \omega_0)$. This verifies (3.12) of Definition 3.7.

In order to verify conditions (3.11) and (3.13) we take solution $\omega(t) = S^{(\nu)}(t, \omega_0)$ and mollify it,

$$\omega_\epsilon(t) = J_\epsilon(\omega(t)).$$

We obtain from (3.3) that

$$\frac{d}{2dt}\|\omega_\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 + \gamma\|\omega_\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu\|\Lambda^\alpha\omega_\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 - \langle J_\epsilon g, \omega_\epsilon(t)\rangle = (\rho_\epsilon(u(t), \omega(t)), \nabla\omega_\epsilon(t)),$$

where we have used the identity in [4] with

$$\rho_\epsilon(u, \omega) = \int_{\mathbb{R}^2} j(z)(u(x - \epsilon z) - u(x))(\omega(x - \epsilon z) - \omega(x))dz - (u - u_\epsilon)(\omega - \omega_\epsilon).$$

Integrating in time we deduce

$$\begin{aligned}\frac{1}{t}\int_0^t &[\gamma\|\omega_\epsilon(s + t_0)\|_{L^2(\mathbb{R}^2)}^2 + \nu\|\Lambda^\alpha\omega_\epsilon(s + t_0)\|_{L^2(\mathbb{R}^2)}^2 - \langle J_\epsilon g, \omega_\epsilon(s + t_0)\rangle]ds \\ &= \frac{1}{2t}[\|\omega_\epsilon(t_0)\|_{L^2(\mathbb{R}^2)}^2 - \|\omega_\epsilon(t + t_0)\|_{L^2(\mathbb{R}^2)}^2] + \frac{1}{t}\int_0^t (\rho_\epsilon(u(s + t_0), \omega(s + t_0)), \nabla\omega_\epsilon(s + t_0))ds.\end{aligned}$$

We apply $\operatorname{Lim}_{t\to\infty}$ and from (3.16) we have

$$\begin{aligned}\int_{L^2(\mathbb{R}^2)} &[\gamma\|\omega_\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \nu\|\Lambda^\alpha\omega_\epsilon\|_{L^2(\mathbb{R}^2)}^2 - \langle J_\epsilon g, \omega_\epsilon\rangle] d\mu^{(\nu)}(\omega) \\ &= \operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t (\rho_\epsilon(u(s + t_0), \omega(s + t_0)), \nabla\omega_\epsilon(s + t_0))ds.\end{aligned}$$

Similar to [5], noting that $\partial_i \omega_\epsilon(x) = \int_{\mathbb{R}^2} \partial_i j_\epsilon(z-x)\omega(z)dz$ $(i = 1, 2)$, we have

$$\|\nabla \omega_\epsilon(t)\|_{L^\infty(\mathbb{R}^2)} \le C_j \frac{1}{\epsilon}\|\omega(t)\|_{L^\infty(\mathbb{R}^2)}$$

with the constant C_j depending only on the mollifier $j(\cdot)$. For $u = (u_1, u_2) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \star \omega$, we have

$$\|u(\cdot - \epsilon z) - u(\cdot)\|_H \le c_1\epsilon|z|(\|\nabla u_1\|_H + \|\nabla u_2\|_H) \le c_2\epsilon|z|\|\omega\|_{L^2(\mathbb{R}^2)},$$

and obviously

$$u(x) - u_\epsilon(x) = \int_{\mathbb{R}^2} j(z)(u(x) - u(x-\epsilon z))dz.$$

Therefore, we have

$$\begin{aligned}
&\big|(\rho_\epsilon(u(s+t_0), \omega(s+t_0)), \nabla\omega_\epsilon(s+t_0))\big| \\
&\quad\le C_j\frac{1}{\epsilon}\|\omega(s+t_0)\|_{L^\infty(\mathbb{R}^2)}\Big(\int_{\mathbb{R}^2} j(z)\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}\cdot c_2\epsilon|z|\|\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}dz \\
&\qquad + \|\omega(s+t_0) - \omega_\epsilon(s+t_0)\|_{L^2(\mathbb{R}^2)}\int_{\mathbb{R}^2} j(z)c_2\epsilon|z|\|\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}dz\Big) \\
&\quad\le M'\Big(\int_{\mathbb{R}^2} j(z)\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}|z|dz + \|\omega(s+t_0) - \omega_\epsilon(s+t_0)\|_{L^2(\mathbb{R}^2)}\Big) \\
&\quad\le 2M'\int_{\mathbb{R}^2} j(z)\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}dz,
\end{aligned}$$

where $\delta_h\omega(x) = \omega(x) - \omega(x-h)$, and M' is a constant depending only on $\|\omega(s+t_0)\|_{L^\infty(\mathbb{R}^2)}\|\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}$. Consequently, we have

$$\begin{aligned}
&\Big|\operatorname*{Lim}_{t\to\infty}\frac{1}{t}\int_0^t(\rho_\epsilon(u(s+t_0), \omega(s+t_0)), \nabla\omega_\epsilon(s+t_0))ds\Big| \\
&\qquad\le M\operatorname*{Lim}_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{R}^2} j(z)\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}dzds,
\end{aligned} \tag{3.18}$$

where M is a bound on $\sup_{s\ge0}\|\omega(s+t_0)\|_{L^\infty(\mathbb{R}^2)}\|\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}$.

Note that $\overline{O_+(t_0, \omega_0)}$ is compact in $L^2(\mathbb{R}^2)$. Then for every small number $h > 0$ there exists $\epsilon > 0$ such that

$$\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)} \le h$$

for all $s \ge 0$ and all z in the compact support of j. Therefore we have

$$\int_{L^2(\mathbb{R}^2)} [\gamma\|\omega_\epsilon\|^2_{L^2(\mathbb{R}^2)} + \nu\|\Lambda^\alpha\omega_\epsilon\|^2_{L^2(\mathbb{R}^2)} - \langle J_\epsilon g, \omega_\epsilon\rangle]d\mu^{(\nu)}(\omega) \le h(\epsilon), \tag{3.19}$$

where $0 \le h(\epsilon)$, a function satisfying $\lim_{\epsilon\to0} h(\epsilon) = 0$. We remove the mollifier. First we note that

$$\int_{L^2(\mathbb{R}^2)} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle)d\mu^{(\nu)}(\omega) = \lim_{\epsilon\to0}\int_{L^2(\mathbb{R}^2)} (\gamma\|\omega_\epsilon\|^2_{L^2(\mathbb{R}^2)} - \langle J_\epsilon g, \omega_\epsilon\rangle)d\mu^{(\nu)}(\omega)$$

holds trivially. This, together with (3.19), implies that

$$\nu\limsup_{\epsilon\to0}\int_{L^2(\mathbb{R}^2)} \|\Lambda^\alpha\omega_\epsilon\|^2_{L^2(\mathbb{R}^2)}d\mu^{(\nu)}(\omega) \le -\int_{L^2(\mathbb{R}^2)} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle)d\mu^{(\nu)}(\omega).$$

By Fatou's lemma, we have

$$\nu \int_{L^2(\mathbb{R}^2)} \|\Lambda^\alpha \omega\|^2_{L^2(\mathbb{R}^2)} d\mu^{(\nu)}(\omega) \le - \int_{L^2(\mathbb{R}^2)} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle) d\mu^{(\nu)}(\omega),$$

which proves (3.11) and (3.17).

To verify (3.13), we take, similarly to [5], a smooth, nonnegative, compactly supported function $\chi'(y)$ defined for $y \ge 0$. Then $\chi(y) = \int_0^y \chi'(x)dx$ is bounded on $\mathbb{R}_+$ and

$$\frac{d}{dt}\chi(\|\omega_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}) = \chi'(\|\omega_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)})\frac{d}{dt}\|\omega_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}.$$

We proceed as above by taking time average and long time limit to obtain

$$\begin{aligned}\frac{1}{t}\int_0^t \chi'(\|\omega_\epsilon(t_0+s)\|^2_{L^2(\mathbb{R}^2)})[\gamma\|\omega_\epsilon(s+t_0)\|^2_{L^2(\mathbb{R}^2)} + \nu\|\Lambda^\alpha\omega_\epsilon(s+t_0)\|^2_{L^2(\mathbb{R}^2)} - \langle J_\epsilon g, \omega_\epsilon(s+t_0)\rangle]ds \\ = \frac{1}{2t}[\|\chi(\omega_\epsilon(t_0))\|^2_{L^2(\mathbb{R}^2)} - \|\chi(\omega_\epsilon(t+t_0))\|^2_{L^2(\mathbb{R}^2)}] \\ + \frac{1}{t}\int_0^t \chi'(\|\omega_\epsilon(t_0+s)\|^2_{L^2(\mathbb{R}^2)})(\rho_\epsilon(u(s+t_0), \omega(s+t_0)), \nabla\omega_\epsilon(s+t_0))ds.\end{aligned}$$

Noting that $\chi'(\cdot)$ is bounded, we have

$$\begin{aligned}&\Big|\operatorname*{Lim}_{t\to\infty}\frac{1}{t}\int_0^t \chi'(\|\omega_\epsilon(t_0+s)\|^2_{L^2(\mathbb{R}^2)})(\rho_\epsilon(u(s+t_0), \omega(s+t_0)), \nabla\omega_\epsilon(s+t_0))ds\Big| \\ &\le c_1 M \operatorname*{Lim}_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{R}^2} j(z)\|\delta_{\epsilon z}\omega(s+t_0)\|_{L^2(\mathbb{R}^2)}dzds,\end{aligned}$$

where the constant c_1 is the bound of $\chi'(\cdot)$, and the constant M is the same as that in (3.18).

Hence, we can remove the mollifier as above and obtain

$$\int_{L^2(\mathbb{R}^2)} \chi'(\|\omega\|^2_{L^2(\mathbb{R}^2)})(\nu\|\Lambda^\alpha\omega\|^2_{L^2(\mathbb{R}^2)} + \gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g, \omega\rangle)d\mu^{(\nu)}(\omega) \le 0.$$

Taking $\chi'(y) \to \mathbf{1}_{[E_1^2, E_2^2]}$ pointwise with $0 \le \chi'(y) \le 2$ and using Fatou's lemma, we can deduce (3.13) of Definition 3.7. This concludes the proof of Theorem 3.19. □

We are now ready to prove our first main result about the inviscid limit.

Proof of Theorem 1.1. We argue by contradiction and assume the conclusion were false. Then there exist $\delta > 0$ and a sequence $\nu_k \to 0$, and for each ν_k, there exists a sequence of time $t_j \to \infty$ such that

$$\frac{\nu_k}{t_j}\int_0^{t_j} \|\Lambda^\alpha S^{(\nu_k)}(s+t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)}ds \ge \delta$$

for all t_j. From the energy estimates of (3.3), we have

$$\begin{aligned}\delta &\le \frac{\nu_k}{t_j}\int_0^{t_j} \|\Lambda^\alpha S^{(\nu_k)}(s+t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)}ds \\ &\le \frac{1}{t_j}\int_0^{t_j}[-\gamma\|S^{(\nu_k)}(s+t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g, S^{(\nu_k)}(s+t_0, \omega_0)\rangle]ds \\ &\quad + \frac{1}{2t_j}[\|S^{(\nu_k)}(t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)} - \|S^{(\nu_k)}(t+t_0, \omega_0)\|^2_{L^2(\mathbb{R}^2)}].\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t [-\gamma\|S^{(\nu_k)}(s+t_0,\omega_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g, S^{(\nu_k)}(s+t_0,\omega_0)\rangle]\,ds \geq \delta.$$

By Remark 3.18, we can choose a generalized limit $\operatorname{Lim}_{t\to\infty}$ such that

$$\begin{aligned}\operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t &[-\gamma\|S^{(\nu_k)}(s+t_0,\omega_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g, S^{(\nu_k)}(s+t_0,\omega_0)\rangle]\,ds\\ &= \limsup_{t\to\infty} \frac{1}{t}\int_0^t [-\gamma\|S^{(\nu_k)}(s+t_0,\omega_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g, S^{(\nu_k)}(s+t_0,\omega_0)\rangle]\,ds.\end{aligned}$$

Now, by Theorem 3.19, there exists a stationary statistical solution $\mu^{(\nu_k)}$ supported in $B_p^\infty(r)$ such that

$$\int_{L^2(\mathbb{R}^2)} (-\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} + \langle g,\omega\rangle)\,d\mu^{(\nu_k)}(\omega) \geq \delta > 0. \tag{3.20}$$

Passing to a weakly convergent subsequence, denoted again $\mu^{(\nu_k)}$, using Theorems 3.14 and 3.15, we find a renormalized stationary statistical solution μ^0 of the damped and driven Euler equation that satisfies enstrophy balance (3.15).

Because the function $\omega \mapsto \langle g,\omega\rangle$ is weakly continuous, we have

$$\lim_{k\to\infty} \int_{L^2(\mathbb{R}^2)} \langle g,\omega\rangle\,d\mu^{(\nu_k)}(\omega) = \int_{L^2(\mathbb{R}^2)} \langle g,\omega\rangle\,d\mu^0(\omega).$$

On the other hand, by Fatou's lemma,

$$\gamma \int_{L^2(\mathbb{R}^2)} \|\omega\|^2_{L^2(\mathbb{R}^2)}\,d\mu^0(\omega) \leq \gamma \liminf_{k\to\infty} \int_{L^2(\mathbb{R}^2)} \|\omega\|^2_{L^2(\mathbb{R}^2)}\,d\mu^{(\nu_k)}(\omega).$$

From (3.20) we obtain

$$\int_{L^2(\mathbb{R}^2)} (\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} - \langle g,\omega\rangle)\,d\mu^0(\omega) \leq -\delta < 0,$$

contradicting energy dissipation balance (3.15). This concludes the proof of Theorem 1.1. □

4 Global Attractor for the GNS Equations

Throughout this section, we assume that $\alpha \in (\frac{1}{2}, 1)$, $\nu, \gamma > 0$ are kept fixed, the force $f \in (W^{1,2}(\mathbb{R}^2))^2$ is fixed and time-independent.

4.1 Notation

We first recall in this subsection the notation about global attractor that we will use later; see [18, 20] for more details.

We consider a semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X, i.e. a family of mappings $S(t) : X \to X$, such that

$$S(0) = I_X \quad \text{and} \quad S(t+s) = S(t)S(s) \quad \text{for all } s, t \in [0,\infty) \text{ and } x \in X.$$

Definition 4.1. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A subset $\mathscr{A} \subset X$ is called a global attractor for the semigroup if $\mathscr{A}$ is compact in X and enjoys the following properties:

(i) $\mathscr{A}$ is an invariant set, i.e., $S(t)\mathscr{A} = \mathscr{A}$ for any $t \geq 0$;
(ii) $\mathscr{A}$ attracts all bounded sets of X, i.e., for any bounded subset B of X,

$$\operatorname{dist}(S(t)B, \mathscr{A}) \to 0, \quad \text{as } t \to \infty,$$

where $\operatorname{dist}(A, B)$ is the Hausdorff semidistance of two sets A and B:

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_X.$$

4.2 Global Attractor

According to Lemma 2.1, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ on V as follows:

$$S(t)u_0 : \mathbb{R}^+ \times V \to V, \quad S(t)u_0 = u(t), \tag{4.1}$$

where $u(t)$ is the unique solution of (1.1) corresponding to the initial data $u_0 \in V$.

The main result of this subsection is to prove that the semigroup $\{S(t)\}_{t\geq 0}$ defined by (4.1) has a global attractor in the phase space V, that is:

Theorem 4.2 (Existence). *Let $\alpha \in (\frac{1}{2}, 1)$, $\nu, \gamma > 0$ and $f \in (W^{1,2}(\mathbb{R}^2))^2$ be time-independent. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by the solutions of* (1.1) *has a unique global attractor $\mathscr{A}$ in V.*

To prove Theorem 4.2, we need some dissipation estimates.

Lemma 4.3. *Under the assumptions of Theorem 4.2, there exists a subset $\mathscr{B} \subset V$, which is bounded in $H^{1+\alpha}$ and satisfies the following: for any bounded subset B of V, there exists a $t_B > 0$ which depends only on $\|B\|_V$ such that*

$$S(t)B \subset \mathscr{B} \quad \textit{for all } t \geq t_B.$$

Proof. We divide our proof into three steps.

Step 1. Multiplying equation (1.1) by u and integrating in space, we can deduce that

$$\frac{d}{dt}\|u\|^2 + 2\gamma\|u\|^2 + 2\nu\|\Lambda^\alpha u\|^2 \leq \|f\|\|u\|,$$

which implies that

$$\|u(t)\|^2 + 2\nu \int_0^t e^{\gamma(s-t)}\|\Lambda^\alpha u(s)\|^2 ds \leq e^{-\gamma t}\|u(0)\|^2 + \frac{1}{\gamma^2}\|f\|^2.$$

Hence, there exists a constant t_1 which depends only on $\|u(0)\|$ such that

$$\|u(t)\|^2 + 2\nu \int_0^t e^{\gamma(s-t)}\|\Lambda^\alpha u(s)\|^2 ds \leq \frac{1}{\gamma^2}\|f\|^2 + 1 := M_1 \quad \text{for all } t \geq t_1. \tag{4.2}$$

Step 2. Multiplying the vorticity equation (3.3) by ω and integrating over $\mathbb{R}^2$, we have

$$\frac{d}{dt}\|\omega\|^2 + 2\gamma\|\omega\|^2 + 2\nu\|\Lambda^\alpha \omega\|^2 \leq \|g\|\|\omega\|,$$

which, similarly, implies that there exists a constant t_2 which depends only on $\|\omega(0)\|$ and so $\|u(0)\|_V$ such that

$$\|\omega(t)\|^2 + 2\nu \int_0^t e^{\gamma(s-t)}\|\Lambda^\alpha \omega(s)\|^2 ds \leq \frac{1}{\gamma^2}\|g\|^2 + 1 := M_2 \quad \text{for all } t \geq t_2. \tag{4.3}$$

It follows immediately from (4.2), (4.3) and Lemma 3.4 that

$$\|u(t)\|_V^2 = \|u(t)\|^2 + \|\Lambda u(t)\|^2 \leq M_1 + CM_2 := M_3 \quad \text{for all } t \geq t_1 + t_2,$$

where the constant C comes from Lemma 3.4. Moreover,

$$\int_t^{t+1} \|\Lambda^\alpha \omega(s)\|^2 ds \le \frac{M_2}{2e\nu} \quad \text{for all } t \ge t_2. \tag{4.4}$$

Step 3. Multiplying the vorticity equation (3.3) by $\Lambda^{2\alpha}\omega$ and integrating over $\mathbb{R}^2$, we obtain, in much the same way as in the proof of Lemma 3.5, that

$$\frac{d}{dt}\|\Lambda^\alpha \omega(t)\|^2 + 2\gamma\|\Lambda^\alpha \omega(t)\|^2 \le M_4 \quad \text{for all } t \ge t_1 + t_2,$$

where the positive constant M_4 depends only on M_2, M_3, ν and $\|g\|$. Combined with (4.4), this implies that

$$\|\Lambda^\alpha \omega(t)\|^2 \le \frac{M_4}{2\gamma} + 1 \quad \text{for all } t \ge t_1 + t_2 + \frac{1}{2\gamma}\ln\frac{M_2}{2e\nu}.$$

Therefore, applying Lemma 3.4 again, we can finish our proof by setting

$$\mathscr{B} := \Big\{u \in H^{1+\alpha} : \|u\|^2_{H^{1+\alpha}} \le M_1 + C^2\Big(\frac{M_4}{2\gamma} + 1\Big)\Big\} \quad \text{and} \quad t_B = t_1 + t_2 + \frac{1}{2\gamma}\ln\frac{M_2}{2e\nu}.$$ □

Since the embedding $H^{1+\alpha} \hookrightarrow V$ is not compact, to deduce the necessary asymptotic compactness, we will use the *tail estimates* (see Wang [21]).

Lemma 4.4. *Under the assumptions of Theorem 4.2, for any $\varepsilon > 0$ and any bounded subset $B \subset V$, there exist $T_B > 0$ and $K_B > 0$, such that*

$$\int_{|x|\ge K_B} \|S(t)u_0\|^2 dx \le \varepsilon$$

for any $t \ge T_B$ and $u_0 \in B$.

Proof. The proof is standard and similar to the one of Theorem 3.1 (or see Lemma 4.7 below): taking $\chi(\cdot)$ to be a proper nonnegative smooth cut-off function and multiplying equation (1.1) by $2\chi(\frac{x}{R})u(x,t)$, we can finish the proof by applying the Gronwall inequality. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Lemma 4.3 implies that $\{S(t)\}_{t\ge0}$ has a bounded absorbing set $\mathscr{B}$ in V; Lemmas 4.3 and 4.4 imply that $\{S(t)\}_{t\ge0}$ is asymptotical compact in V. The continuity with respect to initial data in $\mathscr{B}$ follows from (2.9) and interpolation. Hence, Theorem 4.2 follows from the standard criterion in [18, 20]. □

4.3 Finite Dimensionality of the Attractor

In this subsection we prove that the fractal dimension of the global attractor $\mathscr{A}$ obtained in Theorem 4.2 is finite in H^1. We recall that the fractal dimension $\dim_F(Z; X)$ of a compact set Z in space (topology) X is given by

$$\dim_F(Z; X) = \limsup_{r\to0} \frac{\ln N_Z(r; X)}{-\ln r},$$

where $N_Z(r; X)$ is the minimal number of balls in X of radius r needed to cover Z.

From Lemma 4.3, we know that $\mathscr{A}$ is bounded in $H^{1+\alpha}$, consequently, $\mathscr{A}$ is compact in H^s for any $s \in [0, 1]$, especially, $\mathscr{A}$ is closed in H^α.

For convenience, we denote by M the $H^{1+\alpha}$-bounds of $\mathscr{A}$:

$$M = \|\mathscr{A}\|^2_{H^{1+\alpha}} = \sup_{y\in\mathscr{A}} \|y\|^2_{H^{1+\alpha}} < \infty. \tag{4.5}$$

Let $u_0, v_0 \in \mathscr{A}$, and $u(t) = S(t)u_0$, $v(t) = S(t)v_0$ the corresponding solutions of (1.1). Set $w(t) = u(t) - v(t)$. Then $w(t)$ solves the equation

$$\begin{cases} \partial_t w + \nu(-\Delta)^\alpha w + w \cdot \nabla u + v \cdot \nabla w + \gamma w + \nabla p_u - \nabla p_v = 0, \\ \nabla \cdot w = 0, \\ w(x,0) = u_0 - v_0. \end{cases} \tag{4.6}$$

Lemma 4.5. *There exists a positive constant l_1, which depends only on γ, ν and M, such that*

$$\|w(t)\|^2 \le e^{l_1 t}\|u_0 - v_0\|^2 \quad \text{for all } t \ge 0. \tag{4.7}$$

Proof. Multiplying (4.6) by w and integrating in space, we have

$$\frac{d}{dt}\|w(t)\|^2 + 2\gamma\|w(t)\|^2 + 2\nu\|\Lambda^\alpha\|^2 \le 2\int_{\mathbb{R}^2} |w|^2|\nabla u|dx,$$

in which

$$\int_{\mathbb{R}^2} |w|^2|\nabla u|dx \le \|\nabla u\| \cdot \|w\|^2_{L^4(\mathbb{R}^2)} \le M\|w\|^{2r}\|w\|^{2-2r}_{H^\alpha} \le \frac{l_1}{2}\|w\|^2 + \frac{\min\{\gamma,\nu\}}{2}\|w\|^2_{H^\alpha},$$

where $r \in (0,1)$. We have used the invariance of $\mathscr{A}$ and the fact that $\alpha \in (\frac{1}{2}, 1)$. The constant l_1 depends only on γ, ν and M. Then we have

$$\frac{d}{dt}\|w(t)\|^2 + \gamma\|w(t)\|^2 + \nu\|\Lambda^\alpha\|^2 \le l_1\|w(t)\|^2. \tag{4.8}$$

Thus (4.7) follows by an application of the Gronwall inequality. □

Lemma 4.6. *There exists a positive constant l_2, which depends only on ν, γ and M, such that*

$$\|w(t)\|^2_{H^\alpha} + \nu\int_0^t e^{\gamma(s-t)}\|w(s)\|^2_{H^{2\alpha}}ds \le e^{-\gamma t}\|w(0)\|^2_{H^\alpha} + l_2\int_0^t e^{\gamma(s-t)}\|w(s)\|^2 ds \quad \text{for all } t \ge 0. \tag{4.9}$$

Proof. Multiplying (4.6) by $\Lambda^{2\alpha}w$ and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha w(t)\|^2 + \gamma\|\Lambda^\alpha w(t)\|^2 + \nu\|\Lambda^{2\alpha}w\|^2 \le \int_{\mathbb{R}^2} |w|\,|\nabla u|\,|\Lambda^{2\alpha}w|dx + \int_{\mathbb{R}^2} |v|\,|\nabla w|\,|\Lambda^{2\alpha}w|dx. \tag{4.10}$$

Applying the embedding $H^{1+\alpha} \hookrightarrow L^\infty(\mathbb{R}^2)$ and the interpolation, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |v|\,|\nabla w|\,|\Lambda^{2\alpha}w|dx &\le M\|\nabla w\| \cdot \|\Lambda^{2\alpha}w\| \le M\|w\|^r \cdot \|w\|^{2-r}_{H^{2\alpha}} \\ &\le C'_{M,\nu,\gamma}\|w\|^2 + \frac{\min\{\nu,\gamma\}}{4}\|w\|^2_{H^{2\alpha}}. \end{aligned} \tag{4.11}$$

Similarly, applying the embedding $H^\alpha \hookrightarrow L^4(\mathbb{R}^2)$ and the interpolation, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |w|\,|\nabla u|\,|\Lambda^{2\alpha}w|dx &\le \|w\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^4(\mathbb{R}^2)}\|\Lambda^{2\alpha}w\| \le M\|w\|_{L^4(\mathbb{R}^2)}\|\Lambda^{2\alpha}w\| \\ &\le C''_{M,\nu,\gamma}\|w\|^2 + \frac{\min\{\nu,\gamma\}}{4}\|w\|^2_{H^{2\alpha}}. \end{aligned} \tag{4.12}$$

Inserting (4.11) and (4.12) into (4.10), and combining with (4.8), we obtain that

$$\frac{d}{dt}\|w(t)\|^2_{H^\alpha} + \gamma\|w(t)\|^2_{H^\alpha} + \nu\|w(t)\|^2_{H^{2\alpha}} \le l_2\|w\|^2,$$

which, applying the Gronwall inequality, implies (4.9) immediately. Here the constant l_2 depends only on l_1, ν, γ, $C'_{M,\nu,\gamma}$ and $C''_{M,\nu,\gamma}$. □

We also need the following a priori estimates to overcome the difficulty arising from the unboundedness of the spatial domain.

Lemma 4.7. *There exist $t^* > 0$ and $k^* \gg 1$, such that*

$$\|w(t)\|_{H^\alpha}^2 + \int_0^t e^{\gamma(s-t)}\|w_t(s)\|^2 ds + \nu \int_0^t e^{\gamma(s-t)}\|w(s)\|_{H^{2\alpha}}^2 ds \le a(t)\|w(0)\|_{H^\alpha}^2 + E_w(t, k^*) \quad \text{for any } t \ge 0, \tag{4.13}$$

where $a(\cdot) : [0, \infty) \to [0, \infty)$ is continuous and satisfies

$$a(t^*) + a(2t^*) \le \frac{1}{2}, \tag{4.14}$$

and

$$E_w(t, k) = l_2\Big(1 + \frac{l_3}{\nu}\Big)\int_0^t e^{\gamma(s-t)} \int_{|x|\le 2k} |w(x, s)|^2 dxds, \tag{4.15}$$

where the constant l_2 comes from Lemma 4.6, and l_3 is a constant that depends only on M, ν, γ.

Proof. We divide our proof into three steps.

Step 1. Take $\chi(\cdot)$ to be a nonnegative smooth function supported in $\{x \in \mathbb{R}^2 : |x| \ge 1\}$ and identically equal to 1 for $|x| \ge 2$.

Multiplying equation (4.6) by $2\chi(\frac{x}{k})w(x, t)$ and integrating in space, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^2} \chi\Big(\frac{x}{k}\Big)|w|^2 dx + 2\gamma \int_{\mathbb{R}^2} \chi\Big(\frac{x}{k}\Big)|w|^2 dx \le \frac{C_0\nu}{k^{2\alpha}}\|w\|^2 + 2\int_{\mathbb{R}^2} \chi\Big(\frac{x}{k}\Big)|w|^2|\nabla u| dx, \tag{4.16}$$

where we have used an estimate similar to (3.6) for dealing with the fractional term $\Lambda^{2\alpha}w$ and the fact $b(v, w, \chi(\frac{x}{k})w) = b(\chi(\frac{x}{k})v, w, w) = 0$. At the same time, using again that $\alpha > \frac{1}{2}$, we have

$$\int_{\mathbb{R}^2} \chi\Big(\frac{x}{k}\Big)|w|^2|\nabla u| dx \le \Big(\int_{|x|\ge k} |\nabla u|^2 dx\Big)^{\frac{1}{2}} \|w\|_{L^4(\mathbb{R}^2)}^2 \le c_0\Big(\int_{|x|\ge k} |\nabla u|^2 dx\Big)^{\frac{1}{2}} \|w\|_{H^\alpha}^2, \tag{4.17}$$

where the constant c_0 is the embedding constant for $H^\alpha \hookrightarrow L^4(\mathbb{R}^2)$.

Consequently, from (4.16) and (4.17), as k is large enough such that $k^{2\alpha} \ge \frac{C_0\nu}{\gamma}$, by applying the Gronwall inequality, we have

$$\int_{|x|\ge 2k} |w(t)|^2 dx \le \int_{\mathbb{R}^2} \chi\Big(\frac{x}{k}\Big)|w(t)|^2 dx \le e^{-\gamma t}\|w(0)\|^2 + 2c_0 \int_0^t e^{\gamma(s-t)}\Big(\int_{|x|\ge k} |\nabla u(s)|^2 dx\Big)^{\frac{1}{2}} \|w(s)\|_{H^\alpha}^2 ds. \tag{4.18}$$

On the other hand, from (4.7) and (4.9), we have

$$\|w(t)\|_{H^\alpha}^2 \le e^{-\gamma t}\|w(0)\|_{H^\alpha}^2 + \frac{l_2}{l_1}e^{l_1 t}\|w(0)\|^2 \le \Big(1 + \frac{l_2}{l_1}e^{l_1 t}\Big)\|w(0)\|_{H^\alpha}^2. \tag{4.19}$$

We denote

$$I_k := \max_{v\in\mathscr{A}} \int_{|x|\ge k} |\nabla v(x)|^2 dx.$$

Then, combining with the invariance of $\mathscr{A}$, as $k^{2\alpha} \ge \frac{C_0\nu}{\gamma}$, from (4.7), (4.18) and (4.19), we have

$$\int_{|x|\ge 2k} |w(t)|^2 dx \le e^{-\gamma t}\|w(0)\|^2 + \frac{2c_0}{\gamma}\Big(1 + \frac{l_2}{l_1}e^{l_1 t}\Big)I_k\|w(0)\|_{H^\alpha}^2. \tag{4.20}$$

Step 2. Denote by $\langle\cdot,\cdot\rangle$ the dual product between H and H' $(= H)$. Then, for any $\varphi \in H$, from equation (4.6) we deduce that

$$|\langle w_t(s), \varphi\rangle| \le (\gamma\|w(s)\| + \nu\|\Lambda^{2\alpha}w(s)\|)\|\varphi\| + (\|w(s)\|_{L^4(\mathbb{R}^2)}\|\nabla u(s)\|_{L^4(\mathbb{R}^2)} + \|v\|_{L^\infty(\mathbb{R}^2)}\|\nabla w(s)\|)\|\varphi\|,$$

as that for (4.11) and (4.12), we can deduce that

$$|\langle w_t(s), \varphi\rangle| \le (\gamma\|w(s)\| + \nu\|\Lambda^{2\alpha}w(s)\|)\|\varphi\| + l_3'\|w(s)\|_{H^{2\alpha}}\|\varphi\|$$

for some constant l_3' that depends only on M. Then, combining with (4.9), we have

$$\int_0^t e^{\gamma(s-t)}\|w_t(s)\|^2 ds \le C_{\gamma,\nu,l_3'} \int_0^t e^{\gamma(s-t)}\|w(s)\|^2_{H^{2\alpha}} ds := l_3 \int_0^t e^{\gamma(s-t)}\|w(s)\|^2_{H^{2\alpha}} ds. \tag{4.21}$$

Step 3. Now, returning to (4.9), we have

$$\begin{aligned}&\|w(t)\|^2_{H^\alpha} + \nu\int_0^t e^{\gamma(s-t)}\|w(s)\|^2_{H^{2\alpha}} ds\\ &\quad\le e^{-\gamma t}\|w(0)\|^2_{H^\alpha} + l_2\int_0^t e^{\gamma(s-t)}\int_{|x|\le 2k}|w(x,s)|^2 dxds + l_2\int_0^t e^{\gamma(s-t)}\int_{|x|\ge 2k}|w(x,s)|^2 dxds.\end{aligned} \tag{4.22}$$

We set

$$E'(t,k) = e^{-\gamma t} + l_2 t e^{-\gamma t} + \frac{2l_2c_0t}{\gamma}\Big(1 + \frac{l_2}{l_1}e^{l_1t}\Big)I_k \tag{4.23}$$

and

$$E(t,k) = \Big(1 + \frac{l_3}{\nu}\Big)E'(t,k).$$

Then, from (4.21), (4.22) and (4.20), we have

$$\|w(t)\|^2_{H^\alpha} + \int_0^t e^{\gamma(s-t)}\|w_t(s)\|^2 ds + \nu\int_0^t e^{\gamma(s-t)}\|w(s)\|^2_{H^{2\alpha}} ds \le E(t,k)\|w(0)\|^2_{H^\alpha} + E_w(t,k) \tag{4.24}$$

for all $t \ge 0$.

Since $\mathscr{A}$ is compact in V, we have $I_k \to 0$ as $k \to \infty$; thus, in (4.23), we can first take t large enough, e.g., $t = t^* > 0$, and then take k large enough, e.g., $k = k^* \gg 1$, such that

$$E(t^*, k^*) \le \frac{1}{4} \quad\text{and}\quad E(2t^*, k^*) \le \frac{1}{4}.$$

Thus, we can finish the proof of (4.24) by taking $a(s) = E(s, k^*)$. □

We are now ready to state and prove the main result of this subsection.

Theorem 4.8 (Finite dimensionality). *The fractal dimension of the global attractor $\mathscr{A}$ (obtained in Theorem 4.2) is finite in H^1.*

Proof. The proof is based on the idea of *l-trajectories*, see [17]; here we use also a criterion given in [3], namely [3, Theorem 2.15].

Let t^* and k^* be the constants given in Lemma 4.7.

Define the space $W \subset L^2(0, t^*; (L^2(\mathbb{R}^2))^2)$ as follows:

$$W = \Big\{\phi \in L^2(0, t^*; (L^2(\mathbb{R}^2))^2) : \int_0^{t^*} e^{\gamma(s-t^*)}\int_{\mathbb{R}^2}(|\Lambda^{2\alpha}\phi(x,s)|^2 + |\phi_t(x,s)|^2)dxds < \infty\Big\},$$

endowed with the norm

$$\|\phi\|^2_W = \int_0^{t^*} e^{\gamma(s-t^*)}(\nu\|\phi(s)\|^2_{H^{2\alpha}} + \|\phi_t(s)\|^2_{(L^2(\mathbb{R}^2))^2})ds.$$

Then $(W, \|\cdot\|_W)$ is a Banach space.

Define

$$X = H^{\alpha} \times W,$$

endowed with the norm

$$\|(y, z)\|_X^2 = \|y\|_{H^\alpha}^2 + \|z\|_W^2 \quad \text{for all } (y, z) \in X.$$

For the global attractor $\mathscr{A}$ given in Theorem 4.2, define

$$\mathscr{A}_{t^*} = \{(u_0, l(u_0)) : u_0 \in \mathscr{A},\ l(u_0) = \{S(s)u_0 : s \in [0, t^*]\}\};$$

and define the mapping $\mathcal{L}$ on $\mathscr{A}_{t^*}$ as follows:

$$\mathcal{L} : \mathscr{A}_{t^*} \mapsto X, \quad \mathcal{L}(u_0, l(u_0)) = (S(t^*)u_0, l(S(t^*)u_0)) \quad \text{for all } (u_0, l(u_0)) \in \mathscr{A}_{t^*}.$$

From the invariance and compactness of $\mathscr{A}$ and Lemma 4.7, we have that $\mathscr{A}_{t^*}$ is a closed in X and $\mathcal{L}\mathscr{A}_{t^*} = \mathscr{A}_{t^*}$.

For any $(u_0, l(u_0)), (v_0, l(v_0)) \in \mathscr{A}_{t^*}$, we have

$$\begin{aligned}
&\|\mathcal{L}(u_0, l(u_0)) - \mathcal{L}(v_0, l(v_0))\|_X^2 \\
&\quad = \|S(t^*)u_0 - S(t^*)v_0\|_{H^\alpha}^2 \\
&\qquad + \int_0^{t^*} e^{\gamma(s-t^*)}(\nu\|S(s+t^*)u_0 - S(s+t^*)v_0\|_{H^{2\alpha}}^2 + \|(S(s+t^*)u_0 - S(s+t^*)v_0)_t\|^2)ds \\
&\quad = \|S(t^*)u_0 - S(t^*)v_0\|_{H^\alpha}^2 + \int_{t^*}^{2t^*} e^{\gamma(s-2t^*)}(\nu\|S(s)u_0 - S(s)v_0\|_{H^{2\alpha}}^2 + \|(S(s)u_0 - S(s)v_0)_t\|^2)ds \\
&\quad = I_1 + I_2,
\end{aligned}$$

where, from Lemma 4.7, we have

$$I_1 \le a(t^*)\|u_0 - v_0\|_{H^\alpha}^2 + l_2\Big(1 + \frac{l_3}{\nu}\Big)\int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\le 2k^*} |S(s)u_0 - S(s)v_0|^2 dxds$$

and

$$\begin{aligned}
I_2 &\le \int_0^{2t^*} e^{\gamma(s-2t^*)}(\nu\|S(s)u_0 - S(s)v_0\|_{H^{2\alpha}}^2 + \|(S(s)u_0 - S(s)v_0)_t\|^2)ds \\
&\le a(2t^*)\|u_0 - v_0\|_{H^\alpha}^2 + l_2\Big(1 + \frac{l_3}{\nu}\Big)\int_0^{2t^*} e^{\gamma(s-2t^*)} \int_{|x|\le 2k^*} |S(s)u_0 - S(s)v_0|^2 dxds \\
&\le a(2t^*)\|u_0 - v_0\|_{H^\alpha}^2 + l_2\Big(1 + \frac{l_3}{\nu}\Big)e^{-\gamma t^*}\int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\le 2k^*} |S(s)u_0 - S(s)v_0|^2 dxds \\
&\qquad + l_2\Big(1 + \frac{l_3}{\nu}\Big)\int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\le 2k^*} |S(t^*+s)u_0 - S(t^*+s)v_0|^2 dxds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{L}(u_0, l(u_0)) - \mathcal{L}(v_0, l(v_0))\|_X^2 &\le (a(t^*) + a(2t^*))\|u_0 - v_0\|_{H^\alpha}^2 \\
&\quad + l_2\Big(1 + \frac{l_3}{\nu}\Big)(1 + e^{-\gamma t^*})\int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\le 2k^*} |S(s)u_0 - S(s)v_0|^2 dxds \\
&\quad + l_2\Big(1 + \frac{l_3}{\nu}\Big)\int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\le 2k^*} |S(t^*+s)u_0 - S(t^*+s)v_0|^2 dxds
\end{aligned}$$

$$\begin{aligned}
&\leq (a(t^*) + a(2t^*))\|(u_0, l(u_0)) - (v_0, l(v_0))\|_X^2 \\
&\quad + l_2\Big(1 + \frac{l_3}{\nu}\Big)(1 + e^{-\gamma t^*})\|(u_0, l(u_0)) - (v_0, l(v_0))\|_c^2 \\
&\quad + l_2\Big(1 + \frac{l_3}{\nu}\Big)\|\mathcal{L}(u_0, l(u_0)) - \mathcal{L}(v_0, l(v_0))\|_c^2,
\end{aligned} \tag{4.25}$$

where $\|\cdot\|_c$ is the compact seminorm on X defined as follows (a seminorm $\|\cdot\|_c$ on X is said to be compact iff for any bounded set $B \subset X$ there exists a sequence $\{\psi_n\} \subset B$ such that $\|\psi_n - \psi_m\|_c \to 0$ as $m, n \to \infty$):

$$\|(v_0, z(\cdot))\|_c^2 := \int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\leq 2k^*} |z(x,s)|^2\,dx\,ds \quad \text{for all } (v_0, z(\cdot)) \in X.$$

At the same time, combining (4.7) with the first inequality of (4.25), it is easy to see the following Lipschitz continuity:

$$\begin{aligned}
\|\mathcal{L}(u_0, l(u_0)) - \mathcal{L}(v_0, l(v_0))\|_X^2 &\leq (a(t^*) + a(2t^*))\|u_0 - v_0\|_{H^\alpha}^2 \\
&\quad + l_2\Big(1 + \frac{l_3}{\nu}\Big)(1 + e^{-\gamma t^*} + e^{l_1 t^*}) \int_0^{t^*} e^{\gamma(s-t^*)} \int_{|x|\leq 2k^*} |S(s)u_0 - S(s)v_0|^2\,dx\,ds \\
&\leq L\|(u_0, l(u_0)) - (v_0, l(v_0))\|_X^2,
\end{aligned} \tag{4.26}$$

where the constant L depends only on l_i, γ, ν and t^*.

Note that $a(t^*) + a(2t^*) \leq \frac{1}{2}$. By (4.25) and (4.26), we have verified all conditions in [3, Theorem 2.15] for the mapping $\mathcal{L}$ on $\mathscr{A}_{t^*}$. Thus $\mathscr{A}_{t^*}$ is compact in X and has finite fractal dimension.

Finally, note that the canonical projection $P : X \to H^\alpha$ defined as $P(y,z) = y$ is obviously Lipschitz with Lipschitz constant 1 and $\mathscr{A} = P\mathscr{A}_{t^*}$. We thus have

$$\dim_F(\mathscr{A}; H^\alpha) \leq \dim_F(\mathscr{A}_{t^*}; X) < \infty,$$

and the finite dimensionality in H^1 follows immediately by (4.5) and interpolation. □

Proof of Theorem 1.2. It is a direct result of Theorems 4.2 and 4.8. □

5 Some Remarks

For the vorticity equation (3.3), as the viscosity parameter $\nu > 0$ is kept fixed, under the same assumptions about the forcing term and initial data as in Section 3, i.e., $g \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\omega_0 \in L^2(\mathbb{R}^2)$, and proceeding as in Section 4, we can also obtain a compact global attractor $\mathscr{A}^{(\nu)} \subset L^2(\mathbb{R}^2)$ for the corresponding semigroup $\{S^{(\nu)}(t)\}_{t\geq 0}$ in the phase space $L^2(\mathbb{R}^2)$. The necessary asymptotical compactness follows from Lemma 3.5 and the fact that the estimates in (3.7) depends only on the $L^2(\mathbb{R}^2)$-size of initial data (which will deduce the *tail estimate* for $\{S^{(\nu)}(t)\}_{t\geq 0}$ in $L^2(\mathbb{R}^2)$). Moreover, the stationary statistical solution $\mu^{(\nu)}$ obtained in Theorem 3.19 will support in $\mathscr{A}^{(\nu)}$ for each $\nu > 0$.

Concerning the fractal dimension of $\mathscr{A}$ in Theorem 4.8, we indeed (e.g., by checking the general criteria presented in [3, 17]) can give an upper bound which depends explicitly on the parameters in (4.13)–(4.15) (which depend on γ, ν and the size of the forcing term f).

The assumption $\alpha > \frac{1}{2}$ (used, e.g., in Lemma 2.1 and subsequently) is essential in this paper. Hence, how to obtain the same results as that in Sections 3 and 4 for the case $\alpha = \frac{1}{2}$ would be more interesting and challenging.

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