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Research Article

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Theory of “Critical Points at Infinity” and a Resonant Singular Liouville-Type Equation

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Abstract: We consider the following Liouville-type equation on domains of $\mathbb{R}^2$ under Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \varrho \dfrac{Ke^u}{\int_\Omega Ke^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varrho \in \mathbb{R}$ and K is a smooth nonnegative function having N zeros $q_1, \dots, q_N$, which takes in a neighborhood of a zero q_j the following form:

$$K(x) = K_j(x)|x - q_j|^{2\gamma_j} \quad \text{with } K_j(x) > 0 \text{ and } \gamma_i \in \mathbb{R} \text{ such that } 0 < \gamma_j := \gamma_j(q_j) \notin \mathbb{N}.$$

Using some dynamical and topological tools from the “critical point theory at infinity” of Bahri, we study the critical points at infinity of the related variational problem. Then we derive from our analysis some existence results in the so-called resonant case, that is, when the parameter ϱ is of the form $\sum_{i=1}^{\sigma} 8\pi(1+\gamma_i) + \sum_{i=\sigma+1}^{m} 8\pi$ for a subset $(q_{i_1}, \dots, q_{i_\sigma})$ of $\Sigma := \{q_1, \dots, q_N\}$. In particular, we provide an Euler–Poincaré-type criterium for existence of solutions.

Keywords: Mean Field Type Equation, Critical Points at Infinity, Infinite Dimensional Morse Theory, Variational and Topological Methods

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Dedicated to the fond memory of Abbas Bahri

1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^2$ be a bounded regular domain. We consider the following singular Liouville-type equation:

$$\begin{cases} -\Delta u = \varrho \dfrac{Ke^u}{\int_\Omega Ke^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_ϱ^K}$$

where $\varrho \in \mathbb{R}$ and K is a smooth nonnegative function having N zeros $q_1, \dots, q_N$, which takes in a neighborhood of a zero q_j the following form:

$$K(x) = K_j(x)|x - q_j|^{2\gamma_j} \quad \text{with } K_j(x) > 0 \text{ and } \gamma_i \in \mathbb{R} \text{ such that } 0 < \gamma_j := \gamma_j(q_j) \notin \mathbb{N}.$$

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Equation (P_ϱ^K) is a generalization of the following singular mean field equation:

$$\begin{cases} -\Delta v = \varrho \dfrac{he^v}{\int_\Omega he^v} - 4\pi \sum\limits_{i=1}^{N} \gamma(q_i)\delta_{q_i} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{SMF}$$

where h is a smooth positive function, δ_{q_i} is the Dirac measure at $q_i \in \Omega$ and $\varrho, \gamma(q_i) \in \mathbb{R}$ with $0 < \gamma(q_i) \notin \mathbb{N}$. Indeed, we denote by $G(q, \cdot)$ Green's function of Δ under Dirichlet boundary conditions with pole at q,

$$\begin{cases} -\Delta G(q, x) = \delta_q & \text{in } \Omega, \\ G(q, x) = 0 & \text{on } \partial\Omega, \end{cases}$$

and set

$$v(x) = u(x) - 4\pi \sum_{j=1}^{N} \gamma_j(q_j) G(q_j, x).$$

Then u satisfies equation (P_ϱ^K) with

$$K(x) := h(x) \exp\Big(-4\pi \sum_{j=1}^{N} \gamma_j(q_j) G(q_j, x)\Big). \tag{1.1}$$

In particular, the study of equation (SMF) is motivated by the study of vortex-type configurations in the electroweak theory of Glashow–Salam–Weinberg and in self-dual Chern–Simons theories as well as the prescribed Gauss-curvature problem on surfaces with conical singularities and Onsager's statistical mechanics description of two-dimensional turbulence in presence of vortex sources.

Problem (P_ϱ^K) has a variational structure. Indeed, its solutions are in a one-to-one correspondence with the critical points of the functional

$$J_\varrho(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \ln\Big(\int_\Omega K e^u \Big),$$

defined on the Sobolev space $H_0^1(\Omega)$. Moreover, the way of finding critical points of J_ϱ and the compactness features of the related variational problem depend strongly on the range of values taken by the parameter ϱ. Indeed, let

$$\Sigma := \{q_1, \ldots, q_N\}$$

be the set of zeros of K with weights γ_j satisfying

$$0 < \gamma_j := \gamma_j(q_j) \notin \mathbb{N}.$$

For $m \in \mathbb{N}$, $0 \le \sigma \le \inf(m, N)$ and $\{q_{i_1}, \ldots, q_{i_\sigma}\} \subset \Sigma$ we set

$$\mathcal{G} := \Big\{ 8\pi(m - \sigma) + \sum_{j=1}^{\sigma} 8\pi(1 + \gamma_j) : m \in \mathbb{N},\ 0 \le \sigma \le N,\ \sigma \le m \Big\}.$$

We will call problem (P_ϱ^K) *resonant* if $\varrho \in \mathcal{G}$, and *non-resonant* if ϱ does not belong to $\mathcal{G}$. The major difference between these two classes of problems lies in the fact that solutions of ($P_{\varrho\pm\varepsilon}^K$) are bounded for all $\varepsilon \ge 0$ in the non-resonant case (see [11, 17]), while blow-up phenomena occur in the resonant one.

The singular mean field equation (SMF) has attracted a lot of attention in the last two decades. But while in the non-resonant case there are several existence results (see [12, 14–17, 22, 25, 26, 39, 42, 46] and the references therein), in the resonant one, besides a work of Bartolucci and Lin [13] regarding the case $\varrho = 8\pi$, there are, up to the authors' knowledge no existence results in the literature. Furthermore, the regular case, that is, when K is a positive function, has been the subject of intensive study starting from the seminal works of Brezis and Merle [19] and Caglioti, Lions, Marchioro, and Pulvirenti [20, 21]; see the works [1, 18, 23, 24, 27–35, 35, 37, 38, 40, 44, 45], the monographs [41, 43] and the references therein.

The goal of this paper is to address the resonant case, that is, when the parameter ϱ belongs to the set $\mathcal{G}$ for some $0 \le \sigma \le N$, $m \ge \sigma$. In contrast to the non-resonant one, in this case *critical points at infinity* occur; these are noncompact orbits of the gradient flow along which the functional remains bounded.

Our strategy goes along the method initiated by Bahri, in the framework of contact form geometry and Yamabe-type equations; see [2, 3, 5, 6, 10] and also in collaboration with Coron [7, 8]. This method consists of identifying the "critical points at infinity" of the associated variational problem, performing a Morse-type reduction around them and computing their contribution to the difference of topology between level sets of the associated Euler–Lagrange functional J_ϱ. Our main existence result is derived by comparing the total index of the critical points at infinity with the Euler characteristic of very negative level sets of J_ϱ.

To state our main results, we introduce the following notation: for $\mathbf{q}_\sigma := (q_1, \dots, q_\sigma)$ and $0 < \sigma < m$ we define the function

$$\mathcal{F}^m_{\mathbf{q}_\sigma} : (\Omega \setminus \Sigma)^{m-\sigma} \setminus \mathbb{F}_{m-\sigma}(\Omega \setminus \Sigma) \to \mathbb{R}$$

by

$$\mathcal{F}^m_{\mathbf{q}_\sigma}(a_{\sigma+1}, \dots, a_m) := \sum_{i=1}^{\sigma} (1+\gamma_i)\Big(\ln(K_i(a_i)) - 4\pi(1+\gamma_i)H(a_i, a_i) + 4\pi \sum_{j\ne i} (1+\gamma_j)G(a_i, a_j)\Big) + \sum_{i=\sigma+1}^{m} \Big(\ln(K(a_i)) - 4\pi H(a_i, a_i) + 4\pi \sum_{j \ne i}(1+\gamma_j)G(a_i, a_j)\Big),$$

where

$$\gamma_i = 0 \quad \text{if } i \ge \sigma + 1, \qquad a_i = q_i \quad \text{if } i \le \sigma,$$

and $\mathbb{F}_{m-\sigma}(\Omega \setminus \Sigma)$ denotes the fat diagonal of $(\Omega \setminus \Sigma)^{m-\sigma}$.

In the case where $m = \sigma$, we denote by $\mathcal{F}^\sigma_{\mathbf{q}_\sigma}$ the constant

$$\mathcal{F}^\sigma_{\mathbf{q}_\sigma} := \sum_{i=1}^{\sigma} (1+\gamma_i)\Big(\ln(K_i(q_i)) - 4\pi(1+\gamma_i)H(q_i, q_i) + 4\pi \sum_{j \ne i}(1+\gamma_j)G(q_i, q_j)\Big).$$

In the case $\sigma = 0$, we define the function $\mathcal{F}^m_0 : \Omega^m \setminus \mathbb{F}_m(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}^m_0(a_1, \dots, a_m) := \sum_{i=1}^{m} \Big(\ln(K(a_i)) - 4\pi H(a_i, a_i) + 4\pi \sum_{j\ne i} G(a_i, a_j)\Big).$$

For a critical point $A^m_0 := (a_1, \dots, a_m)$ of $\mathcal{F}^m_0$ we define in a neighborhood of a_i for $i = 1, \dots, m$ the function

$$\mathcal{F}_{A^m_0, i}(x) := K(x) \exp\Big(-8\pi H(a_i, x) + 8\pi \sum_{j \ne i} G(a_j, x)\Big).$$

Next, for $0 < \sigma < m$ let $z_1, \dots, z_p$ be the critical points of $\mathcal{F}^m_{\mathbf{q}_\sigma}$. It is easy to see that the components of each critical point $z_i := (y_{\sigma+1}, \dots, y_m)$ satisfy

$$|y_j - y_k| \ge c \quad \text{for all } j \ne k,$$
$$d(y_k, \partial\Omega) \ge c \quad \text{for all } k.$$

For $0 < \sigma < m$ we set $\mathbf{a} := (a_{\sigma+1}, \dots, a_m) \in \bigcup_{j=1}^p B(z_j, \eta)$ and

$$A^m_\sigma := (q_1, \dots, q_\sigma, \mathbf{a}) := (a_1, \dots, a_\sigma, \dots, a_m) \in \Omega^m \setminus \mathbb{F}_m(\Omega).$$

For $i = 1, \dots, \sigma$ we define $\mathcal{F}_{A^m_\sigma, i}$ in a neighborhood of q_i by

$$\mathcal{F}_{A^m_\sigma, i}(x) := K_i(x) \exp\Big(-8\pi(1+\gamma_i)H(a_i, x) + 8\pi \sum_{j\ne i}(1+\gamma_j)G(a_j, x)\Big). \tag{1.2}$$

Likewise, for $i \in \{\sigma + 1, \dots, m\}$ we define $\mathcal{F}_{A^m_\sigma, i}$ in a neighborhood of a_i by

$$\mathcal{F}_{\mathbf{A}^m_\sigma, i}(x) := K(x) \exp\Big(-8\pi H(a_i, x) + 8\pi \sum_{j \ne i}(1+\gamma_j)G(a_j, x)\Big). \tag{1.3}$$

Next, we introduce the following non-degeneracy condition.

Condition ($\mathrm{ND}_{m,\sigma}$). We say that the function K satisfies the non-degeneracy condition ($\mathrm{ND}_{m,\sigma}$) if the following conditions hold:

(i) For $\sigma = 0$ all critical points of $\mathcal{F}_0^m$ are non-degenerate and at every critical point $A_0^m := (a_1, \dots, a_m)$ we have that

$$\mathcal{L}(A_0^m) := \sum_{i=1}^{m} \Delta \mathcal{F}_{A_0^m,i}(a_i) \neq 0.$$

(ii) For $0 < \sigma < m$ all critical points of $\mathcal{F}^m_{\mathbf{q}_\sigma}$ are non-degenerate and at every critical point $\mathbf{a} := (a_{\sigma+1}, \dots, a_m)$ we have for $A_\sigma^m := (q_1, \dots, q_\sigma, \mathbf{a}) := (a_1, \dots, a_m)$ that

$$\mathcal{L}(A_\sigma^m) := \sum_{k:\gamma_{i_k}=\gamma_*} (\mathcal{F}_{A_\sigma^m,i_k}(q_{i_k}))^{\frac{1}{1+\gamma_*}} \Delta \ln(\mathcal{F}_{A_\sigma^m,i_k})(q_{i_k}) \neq 0, \quad \text{where } \gamma_* := \max_{1\le j\le\sigma} \gamma_{i_j}.$$

(iii) For $\sigma = m$ we have

$$\mathcal{L}(A_\sigma^\sigma) := \sum_{k:\gamma_{i_k}=\gamma_*} (\mathcal{F}_{A_\sigma^\sigma,i_k}(q_{i_k}))^{\frac{1}{1+\gamma_*}} \Delta \ln(\mathcal{F}_{A_\sigma^\sigma,i_k})(q_{i_k}) \neq 0.$$

We set

$$\begin{aligned}
\mathcal{K}^-_{m,0} &:= \{A_0^m := (a_1, \dots, a_m) \text{ such that } \nabla \mathcal{F}_0^m(a_1, \dots, a_m) = 0 \text{ and } \mathcal{L}(A_0^m) < 0\},\\
\mathcal{K}^-_{m,\sigma} &:= \{A_\sigma^m := (a_1, \dots, a_m) \text{ such that } \nabla \mathcal{F}^m_{\mathbf{q}_\sigma}(a_{\sigma+1}, \dots, a_m) = 0 \text{ and } \mathcal{L}(A_\sigma^m) < 0\} \quad \text{if } 0 < \sigma < m,\\
\mathcal{K}^-_{\sigma,\sigma} &:= \begin{cases} \emptyset & \text{if } \mathcal{L}(A_\sigma^\sigma) > 0,\\ \{A_\sigma^\sigma\} & \text{if } \mathcal{L}(A_\sigma^\sigma) < 0. \end{cases}
\end{aligned}$$

To each point $A_\sigma^m \in \mathcal{K}^-_{m,\sigma}$ we associate an index defined by

$$\mathrm{ind} : \mathcal{K}^-_{m,\sigma} \to \mathbb{N}, \quad A_\sigma^m \mapsto \begin{cases} 3m - 1 - \mathrm{ind}(\mathcal{F}_0^m, A_0^m) & \text{if } \sigma = 0,\\ 3m - 1 - 2\sigma - \mathrm{ind}(\mathcal{F}^m_{\mathbf{q}_\sigma}, \mathbf{a}) + 2\sum_{j=1}^{\sigma}(1 + [\gamma_j]) & \text{if } 0 < \sigma < m,\\ \sigma - 1 + 2\sum_{j=1}^{\sigma}(1 + [\gamma_j]) & \text{if } \sigma = m, \end{cases}$$

where $\mathrm{ind}(\mathcal{F}_0^m, A_0^m)$ (resp. $\mathrm{ind}(\mathcal{F}^m_{\mathbf{q}_\sigma}, \mathbf{a})$) stands for the Morse index of $\mathcal{F}_0^m$ (resp. $\mathcal{F}^m_{\mathbf{q}_\sigma}$) at its critical point A_0^m (resp. $\mathbf{a}$).

Theorem 1.1. *Let $\varrho_* \in \mathcal{G}$ and let $(\mathbf{q}_{\sigma_1}, m_1), \dots, (\mathbf{q}_{\sigma_l}, m_l)$, with $\sigma_i \le m_i$, such that*

$$\varrho_{(\mathbf{q}_{\sigma_i}, m_i)} := 8\pi(m_i - \sigma_i) + 8\pi \sum_{j=1}^{\sigma_i}(1 + \gamma_j) = \varrho_*$$

for $i \le l$. Assume that $0 \le K \in C^2(\Omega)$, defined by (1.1), satisfies the non-degeneracy condition ($\mathrm{ND}_{m,\sigma}$) for each $(\mathbf{q}_\sigma, m)$ such that $\varrho_{(\mathbf{q}_\sigma, m)} = \varrho_*$. If

$$\sum_{i=1}^{l} \sum_{A_{\sigma_i}^{m_i} \in \mathcal{K}^-_{m_i,\sigma_i}} (-1)^{\mathrm{ind}(A_{\sigma_i}^{m_i})} \neq 1 + \sum_{\substack{(\mathbf{q}_\sigma, m):\\ \sigma\le m,\, 8\pi \le \varrho_{(\mathbf{q}_\sigma,m)} < \varrho_*}} (-1)^\sigma \binom{g + N + m - \sigma - 2}{m - \sigma},$$

then problem $(P^K_{\varrho_*})$ has at least one solution, where g denotes the number of holes inside Ω.

Remark 1.2. If for some $(\mathbf{q}_{\sigma_i}, m_i)$ the set $\mathcal{K}^-_{m_i,\sigma_i}$ is empty, then the corresponding sum will be replaced by 0. If $g + N = 1$, then the sum on the right-hand side is 0.

Remark 1.3. It turns out that when the function satisfies the non-degeneracy condition ($\mathrm{ND}_{m,\sigma}$), the set of solutions to problem $(P^K_{\varrho_*})$ is bounded in $C^{2,\alpha}(\overline{\Omega})$ and the Leray–Schauder degree is given by

$$d_{\varrho_*} = 1 - \sum_{i=1}^{l} \sum_{A_{\sigma_i}^{m_i} \in \mathcal{K}^-_{m_i,\sigma_i}} (-1)^{\mathrm{ind}(A_{\sigma_i}^{m_i})} + \sum_{(\mathbf{q}_\sigma, l):\, 8\pi \le \varrho_{(\mathbf{q}_\sigma,l)} < \varrho_*} (-1)^\sigma \binom{g + N + l - \sigma - 2}{l - \sigma}.$$

Hence the assumption of the theorem is equivalent to the fact that the degree is not zero.

Remark 1.4. For generic K and $\varrho_* \in \mathcal{G}$ the number of solutions of $(K^K_{\varrho_*})$ is lower bounded by $|d_{\varrho_*}|$.

As an application of our main result, we consider the following particular cases.

Corollary 1.5. *Let* $\varrho_* = 8\pi(1+\gamma_{\min})$ *and assume that one of the following conditions hold:*
(i) Ω *is not a simply connected domain (which implies that* $g \geq 1$*).*
(ii) *There exists* q_i *such that* $\gamma(q_i) = \gamma_{\min}$ *and* $\mathcal{L}(q_i) > 0$.
(iii) $N \geq 2$ *and either* $\#\{\gamma(q_i) : \gamma(q_i) = \gamma_{\min}\} < N$ *or* $[\gamma_{\min}] \geq 1$.
Then problem $(P^K_{\varrho_*})$ *has at least one solution.*

Corollary 1.6. *We take* $\gamma_i := \gamma(q_i)$ *for* $q_i \in \Sigma$ *with* $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N$ *and assume that* $\gamma_1 + \gamma_2 \notin \mathbb{N}$. *Let* $\varrho_* := 8\pi(2+\gamma_1+\gamma_2)$. *If* $N \geq 3$, *we assume that* $\gamma_3 > \gamma_1 + \gamma_2$. *Furthermore, we assume that one of the following conditions holds:*
(i) $N = 2$, $g = 0$, $[\gamma_1+\gamma_2] - [\gamma_1] - [\gamma_2] = 1$, *and* $\mathcal{L}(A^2_2) < 0$.
(ii) $N = 2$, $g = 0$, $[\gamma_1+\gamma_2] - [\gamma_1] - [\gamma_2] = 0$, *and* $\mathcal{L}(A^2_2) > 0$.
(iii) $N + g = 3$, $[\gamma_1+\gamma_2] = 0$ *and* $\mathcal{L}(A^2_2) < 0$.
(iv) $N + g = 3$, $[\gamma_1+\gamma_2] \neq 0$.
(v) $N + g \geq 4$.
Then problem $(P^K_{\varrho_*})$ *has at least one solution.*

The remainder of this paper is organized as follows: Some notations and known facts are presented in Section 2. In Section 3 we provide some useful refined expansions of the gradient near potential neighborhoods at infinity. In Section 4 we perform a finite-dimensional reduction, while Section 5 is devoted to the characterization of “critical points at infinity”. The proofs of our main results are given in Section 6. Finally, in Appendix A we collect the technical estimates used in the paper.

2 Neighborhood at Infinity and Lack of Compactness

For $a \in \mathbb{R}^2$ and $\lambda > 0$ we define on $\mathbb{R}^2$ the following function:

$$\delta_{a,\lambda}(x) := \ln\Big(\frac{8(\gamma+1)^2\lambda^2}{(1+\lambda^2|x-a|^{2+2\gamma})^2}\Big).$$

This function satisfies

$$-\Delta\delta_{a,\lambda} = |x-a|^{2\gamma}e^{\delta_{a,\lambda}} \text{ in } \mathbb{R}^2 \quad \text{and} \quad \int_{R^2} |x-a|^{2\gamma}e^{\delta_{a,\lambda}} = 8\pi(\gamma+1).$$

For $a \in \Omega$ we introduce the function $P\delta_{a,\lambda}$ which is the unique solution of the equation

$$\begin{cases} -\Delta P\delta_{a,\lambda} = |x-a|^{2\gamma}e^{\delta_{a,\lambda}} & \text{in } \Omega, \\ P\delta_{a,\lambda} = 0 & \text{on } \partial\Omega. \end{cases}$$

For $\mathbf{q}_\sigma := (q_1, \dots, q_\sigma)$ and $1 \leq \sigma \leq m$ let us define

$$V(\mathbf{q}_\sigma, m, \varepsilon) := \Big\{u := \sum_{i=1}^m P\delta_{a_i,\lambda_i} + w : a_i = q_i \text{ for all } i \leq \sigma,\ (a_{\sigma+1}, \dots, a_m) \in \bigcup_{j=1}^p B(z_j, \eta),$$
$$\|w\| \leq c_1\varepsilon,\ \|\nabla J(u)\| \leq c_2\varepsilon,\ \lambda_i \geq c_3\varepsilon^{-1},\ \lambda_i/\lambda_j \leq C\Big\},$$

where the c_i’s are some fixed constants and C is a large positive constant. We notice that if $\sigma = m$, the set $V(\mathbf{q}_\sigma, \sigma, \varepsilon)$ depends only on the variables λ_i and w.

For $A^m_\sigma := (a_1, \dots, a_m)$ and $i \leq \sigma$, that is, $a_i = q_i \in \Sigma$, we define

$$\overline{\psi}_i(x) := -2\frac{\gamma_i+1}{\gamma_i}\frac{1}{1+\lambda_i^2|x-q_i|^{2+2\gamma_i}}\frac{1}{\mathcal{F}_{A^m_\sigma,i}(q_i)}\nabla\mathcal{F}_{A^m_\sigma,i}(q_i)(x-q_i), \quad x \in \mathbb{R}^2. \tag{2.1}$$

It is easy to verify that $\overline{\psi}_i$ satisfies

$$-\Delta\overline{\psi}_i = |x - q_i|^{2\gamma_i} e^{\delta_{q_i,\lambda_i}}\Big(\overline{\psi}_i + \frac{1}{\mathcal{F}_{A^m_\sigma,i}(q_i)}\nabla\mathcal{F}_{A^m_\sigma,i}(q_i)(x - q_i)\Big) \quad \text{in } \mathbb{R}^2.$$

We define the function ψ_i as the projection of $\overline{\psi}_i$ onto $H^1_0(\Omega)$. It satisfies

$$\begin{cases} \Delta\psi_i = \Delta\overline{\psi}_i & \text{in } \Omega, \\ \psi_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We point out that the function ψ_i depends on the variables λ_i and all points a_j.

Finally, we define the function $\widetilde{P\delta}_{q_i,\lambda_i}$ as

$$\begin{aligned} \widetilde{P\delta}_{q_i,\lambda_i} &:= P\delta_{q_i,\lambda_i} + \psi_i & &\text{if } i \le \sigma, \\ \widetilde{P\delta}_{a_i,\lambda_i} &:= P\delta_{a_i,\lambda_i} & &\text{if } i \ge \sigma + 1. \end{aligned}$$

For $i \le \sigma$ it satisfies

$$-\Delta\widetilde{P\delta}_{q_i,\lambda_i} = |x - q_i|^{2\gamma_i} e^{\delta_{q_i,\lambda_i}}\Big(1 + \overline{\psi}_i + \frac{1}{\mathcal{F}_{A^m_\sigma,i}(q_i)}\nabla\mathcal{F}_{A^m_\sigma,i}(q_i)(x - q_i)\Big).$$

Following the ideas of Bahri and Coron in their proof of [8, Proposition 7], we consider the following minimization problem:

$$\min\Big\{\|u - \sum_{i=1}^{m} \alpha_i\widetilde{P\delta}_{a_i,\lambda_i}\| : \alpha_i > 0,\ \lambda_i > 0,\ 1 \le i \le m,\ a_i \in \Omega, \sigma + 1 \le i \le m\Big\}. \tag{2.2}$$

Note that in the case $\sigma = m$ the points a_i are fixed as $a_i = q_i$ for each $i = 1, \dots, \sigma$.

Lemma 2.1. *For $u \in V(\mathbf{q}_\sigma, m, \varepsilon)$ and ε small the previous problem* (2.2) *has, up to permutation, only one solution. The variables α_i satisfy $|\alpha_i - 1| = O(\varepsilon)$.*

Hence every $u \in V(\mathbf{q}_\sigma, m, \varepsilon)$ can be written as

$$u = \sum_{i=1}^{m} \alpha_i\widetilde{P\delta}_{a_i,\lambda_i} + w,$$

where $\|w\| < c\varepsilon$, $|\alpha_i - 1| < c\varepsilon$ for each i and

$$\begin{cases} \langle w, \widetilde{P\delta}_{a_i,\lambda_i}\rangle = \Big\langle w, \dfrac{\partial\widetilde{P\delta}_{a_i,\lambda_i}}{\partial\lambda_i}\Big\rangle = 0 \quad \text{for all } i = 1, \dots, m, \\ \text{if } \sigma < m,\ \text{then } (a_{\sigma+1}, \dots, a_m) \in \bigcup B(z_j, \eta) \text{ and } \Big\langle w, \alpha_i\dfrac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} + \sum_{j\le\sigma}\alpha_j\dfrac{\partial\psi_j}{\partial a_i}\Big\rangle = 0 \text{ for all } i \ge \sigma + 1, \end{cases} \tag{2.3}$$

where the points z_j's are the critical points of $\mathcal{F}^m_{\mathbf{q}_\sigma}$.

In the following, for $a = (a_1, \dots, a_m)$ and $\Lambda = (\lambda_1, \dots, \lambda_m)$, we denote

$$E^m_{a,\Lambda} := \{w \in H^1_0(\Omega) : w \text{ satisfies } (2.3)\}.$$

We point out that the set $V(\mathbf{q}_\sigma, m, \varepsilon)$ plays a crucial role in the description of the lack of compactness occurring in $(P^K_{\varrho_*})$ for $\varrho_* \in \mathcal{G}$. Indeed, using a result of Lucia [36], the functional J_{ϱ_*} has a pseudo-gradient whose endpoints are in one-to-one correspondence with the limit of blowing-up solutions of

$$\begin{cases} -\Delta u = (\varrho_* - \varepsilon)\dfrac{Ke^u}{\int_\Omega Ke^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$P^K_{\varrho_*-\varepsilon}$}$$

Now using the blow-up analysis introduced in [11, 16] and refined in [25], we infer that noncompact orbits of the gradient flow should enter in some $V(\mathbf{q}_\sigma, m, \varepsilon)$ for some $0 \le \sigma \le N$ and $m \ge \sigma$.

3 Expansion of the Gradient in the Neighborhood at Infinity

Proposition 3.1. *Let* $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$ *and let* $A_\sigma^m := (a_1, \dots, a_m)$.

(i) *For* $i = 1, \dots, \sigma$ *we have*

$$\Big\langle \nabla J(u), \lambda_i \frac{\partial \widetilde{P\delta_i}}{\partial \lambda_i} \Big\rangle = 16\pi(1+\gamma_i)\frac{\tau_i}{\alpha_i} + O\Big(|\alpha_i - 1|^2 + \frac{1}{\lambda^2} + \frac{1}{\lambda^{3/(1+\gamma_i)}}\Big)$$
$$- \frac{16\pi(1+\gamma_i)^2}{\lambda_i^{2/(1+\gamma_i)}}\Big(I_3(\gamma_i)\frac{\Delta \mathcal{F}_{A_\sigma^m,i}(a_i)}{\mathcal{F}_{A_\sigma^m,i}(a_i)} + \frac{(1+\gamma_i)}{\gamma_i^2}\frac{|\nabla \mathcal{F}_{A_\sigma^m,i}(a_i)|^2}{(\mathcal{F}_{A_\sigma^m,i}(a_i))^2}(2-\gamma_i)I_4(\gamma_i)\Big),$$

where

$$I_n(\gamma) := \int_0^\infty \frac{r^{3+2\gamma}}{(1+r^{2+2\gamma})^n}\,dr \quad \text{and} \quad \tau_i = 1 - \frac{\varrho_*}{8\pi(1+\gamma_i)} \frac{\pi \lambda_i^{4\alpha_i - 2}\mathcal{F}_{A_\sigma^m,i}(a_i) g_i(a_i)}{(2\alpha_i - 1)(1+\gamma_i)\int_\Omega K e^u\,dx}, \tag{3.1}$$

where $\mathcal{F}_{A_\sigma^m,i}$ *is defined in* (1.2) *and* (1.3) *and* g_i *is defined in* (A.7) *of Lemma A.4.*

(ii) *If* $\sigma < m$, *for* $i \geq \sigma + 1$ *we have*

$$\Big\langle \nabla J(u), \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \Big\rangle = 16\pi \frac{\tau_i}{\alpha_i} + O\Big(|\alpha_i - 1|^2 + \frac{\ln \lambda_i}{\lambda_i^2}\Big).$$

Proof. In each proof, for simplicity, we will write $\mathcal{F}_i$ and $P\delta_i$ instead of $\mathcal{F}_{A_\sigma^m,i}$ and $P\delta_{a_i,\lambda_i}$, respectively. Before giving the proof, we remark that $I_n(\gamma)$ satisfies

$$n(1+\gamma)I_{n+1}(\gamma) = ((n-1)\gamma + (n-2))I_n(\gamma) \quad \text{for all } n \geq 2. \tag{3.2}$$

Now, for $u := \sum \alpha_j \widetilde{P\delta}_j$, we have

$$\Big\langle \nabla J(u), \lambda_i \frac{\partial \widetilde{P\delta_i}}{\partial \lambda_i} \Big\rangle = \sum \alpha_j \Big\langle \widetilde{P\delta}_j, \lambda_i \frac{\partial \widetilde{P\delta_i}}{\partial \lambda_i} \Big\rangle - \frac{\varrho_*}{\int K e^u} \int_\Omega K e^u \lambda_i \frac{\partial \widetilde{P\delta_i}}{\partial \lambda_i}.$$

The first term is given by Lemma A.3. We focus on the second one. By Lemma A.4, we have

$$\int_{B_i} K e^u \lambda_i \frac{\partial \widetilde{P\delta_i}}{\partial \lambda_i} = \int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(1 + \alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2 + O(\|\overline{\psi}_i\|_\infty^3)\Big)$$
$$\times \Big(1 + O\Big(\frac{1}{\lambda_i^2}\Big)\Big)\Big(\lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \lambda_i \frac{\partial \psi_i}{\partial \lambda_i}\Big). \tag{3.3}$$

We will divide this integral into six ones. For the first one observe that, using Lemma A.1, we get

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \int_{B_i} \frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}} + O(\lambda_i^{4\alpha_i-4})$$
$$= \frac{2\pi\lambda_i^{4\alpha_i-2}}{\alpha_i(1+\gamma_i)}\mathcal{F}_i(a_i)g_i(a_i) + \frac{1}{2}\int_{B_i} \frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}} D^2(\mathcal{F}_i g_i)(a_i)(x-a_i, x-a_i)$$
$$+ O(\lambda_i^{4\alpha_i-4}) + O\Big(\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i+4}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}}\Big)$$
$$= \frac{2\pi\lambda_i^{4\alpha_i-2}}{\alpha_i(1+\gamma_i)}(\mathcal{F}_i g_i)(a_i) + \frac{2\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Delta(\mathcal{F}_i g_i)(a_i)(I_3(\gamma_i) + O(|\alpha - 1|)) + O\Big(\lambda_i^{4\alpha_i-4} + \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{4/(1+\gamma_i)}}\Big).$$

Concerning the second one, since

$$\lambda_i \partial \psi_i / \partial \lambda_i = \lambda_i \partial \overline{\psi}_i / \partial \lambda_i + O(1/\lambda_i^2) = O(1/\lambda_i^{1/(1+\gamma_i)})$$

and using the oddness of $\lambda_i \partial\overline{\psi}_i/\partial\lambda_i$ by expanding $\mathcal{F}_i g_i$ around a_i, we obtain

$$\begin{aligned}
&\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\lambda_i \frac{\partial\psi_i}{\partial\lambda_i}\\
&\quad= \int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\nabla(\mathcal{F}_i g_i)(a_i)(x-a_i)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\lambda_i \frac{\partial\overline{\psi}_i}{\partial\lambda_i}\\
&\qquad+ O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^2}\Big) + O\Big(\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i+3}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\frac{\lambda_i^2|x-a_i|^{2\gamma_i+3}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^2}\Big)\\
&\quad= \underbrace{\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\nabla(\mathcal{F}_i g_i)(a_i)(x-a_i)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\lambda_i \frac{\partial\overline{\psi}_i}{\partial\lambda_i}}_{E_1} + O\Big(\lambda_i^{4\alpha_i-2}\Big(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i^{4/(1+\gamma_i)}}\Big)\Big).
\end{aligned} \tag{3.4}$$

Note that $g_i = 1 + O(\sum|\alpha_j - 1|)$ and $\nabla g_i = O(\sum|\alpha_j - 1|)$, hence the integral E_1 in (3.4) is equal to

$$\begin{aligned}
E_1 &= O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\sum|\alpha_j-1|\Big) + 4\frac{1+\gamma_i}{\gamma_i}\frac{1}{\mathcal{F}_i(a_i)}\int_{B_i}\frac{\lambda_i^{4\alpha_i+2}|x-a_i|^{4\gamma_i+2}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+2}}\Big(\sum_{k=1,2}\Big(\frac{\partial\mathcal{F}_i}{\partial x_k}(a_i)\Big)^2(x-a_i)_k^2\Big)\\
&= \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big(\frac{1+\gamma_i}{\gamma_i}\frac{4\pi}{\mathcal{F}_i(a_i)}|\nabla\mathcal{F}_i(a_i)|^2\Big(I_3(\gamma_i) - I_4(\gamma_i) + O(|\alpha_i-1|)\Big) + O\Big(\sum|\alpha_j-1|\Big)\Big).
\end{aligned}$$

Thus we derive that

$$\begin{aligned}
\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\lambda_i \frac{\partial\psi_i}{\partial\lambda_i} &= 4\pi\frac{1+\gamma_i}{\gamma_i}\frac{1}{\mathcal{F}_i(a_i)}|\nabla\mathcal{F}_i(a_i)|^2\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}(I_3(\gamma_i)-I_4(\gamma_i))\\
&\quad+ O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\sum|\alpha_j-1|\Big) + O\Big(\lambda_i^{4\alpha_i-2}\Big(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i^{4/(1+\gamma_i)}}\Big)\Big).
\end{aligned}$$

For the third integral in (3.3) we have

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\overline{\psi}_i\lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i} = \int_{B_i}\frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}}\overline{\psi}_i + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2+1/(1+\gamma_i)}}\Big).$$

Arguing as in the previous estimate, we get

$$\begin{aligned}
&\int_{B_i}\frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}}\overline{\psi}_i\\
&\quad= \int_{B_i}\frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}}\nabla(\mathcal{F}_i g_i)(a_i)(x-a_i)\overline{\psi}_i + O\Big(\|\overline{\psi}_i\|_\infty\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i+3}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}}\Big)\\
&\quad= E_3 + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{4/(1+\gamma_i)}}\Big).
\end{aligned}$$

Arguing as in the proof of estimate E_1, we obtain

$$\begin{aligned}
E_3 &= -8\frac{1+\gamma_i}{\gamma_i}\frac{1}{\mathcal{F}_i(a_i)}\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+2}}\nabla(\mathcal{F}_i g_i)(a_i)(x-a_i)\nabla(\mathcal{F}_i)(a_i)(x-a_i)\\
&= -8\pi\frac{1+\gamma_i}{\gamma_i}\frac{1}{\mathcal{F}_i(a_i)}|\nabla\mathcal{F}_i(a_i)|^2\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}I_4(\gamma_i) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\sum|\alpha_j-1|\Big).
\end{aligned}$$

For the fourth integral in (3.3) we have

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \overline{\psi}_i \lambda_i \frac{\partial \psi_i}{\partial \lambda_i} = \int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \overline{\psi}_i \lambda_i \frac{\partial \overline{\psi}_i}{\partial \lambda_i} + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2+1/(1+\gamma_i)}}\Big).$$

As in the proof of estimate E_1, we derive

$$\begin{aligned}&\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \overline{\psi}_i \lambda_i \frac{\partial \overline{\psi}_i}{\partial \lambda_i}\\ &\qquad = -8\pi \frac{(1+\gamma_i)^2}{\gamma_i^2} \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big(\frac{(\mathcal{F}_i g_i)(a_i)}{\mathcal{F}_i(a_i)^2}|\nabla \mathcal{F}_i(a_i)|^2(I_4(\gamma_i)-I_5(\gamma_i)) + O(1+|\alpha_i-1|)\Big).\end{aligned}$$

For the fifth integral in (3.3), as before, we derive that

$$\begin{aligned}&\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \overline{\psi}_i^2 \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} = \int_{B_i} \frac{4\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}} \overline{\psi}_i^2 + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2+2/(1+\gamma_i)}}\Big)\\ &\qquad = 16\pi \frac{(1+\gamma_i)^2}{\gamma_i^2} \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big(\frac{(\mathcal{F}_i g_i)(a_i)}{\mathcal{F}_i(a_i)^2}|\nabla \mathcal{F}_i(a_i)|^2 I_5(\gamma_i) + O\Big(\frac{1}{\lambda_i^{2/(1+\gamma_i)}} + \frac{1}{\lambda_i^2} + |\alpha_i - 1|\Big)\Big).\end{aligned}$$

Finally, concerning the sixth one, we have

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \overline{\psi}_i^2 \lambda_i \frac{\partial \psi_i}{\partial \lambda_i} = O\Big(\|\overline{\psi}_i^2 \lambda_i \frac{\partial \psi_i}{\partial \lambda_i}\|_\infty \lambda_i^{4\alpha_i-2}\Big) = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{3/(1+\gamma_i)}}\Big),$$

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \|\overline{\psi}_i^3\|_\infty = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{3/(1+\gamma_i)}}\Big).$$

Summing up the above estimates and using the fact that $(2\alpha_i - 1)\alpha_i^{-2} = 1 + O((\alpha_i - 1)^2)$ as well as Lemma A.3 concludes the proof of the claimed estimate. □

Recall that for $u \in V(\mathbf{q}_\sigma, m, \varepsilon)$ we have $|\nabla J(u)| < c\varepsilon$, hence we derive the following result.

Corollary 3.2. *For all i we have $|\tau_i| = o(1)$ as $\varepsilon \to 0$.*

Summing the estimates proved in Proposition 3.1 for $i = 1, \dots, m$, we derive the following corollary.

Corollary 3.3. *Let $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. Then we have*

$$\begin{aligned}&\Big\langle \nabla J(u), \sum_{i=1}^m \alpha_i \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle\\ &\qquad = 8\pi(1+\gamma_*) I_2(\gamma_*) \sum_{i:\gamma_i=\gamma_*} \frac{\Delta \ln(\mathcal{F}_{A_\sigma^m,i})(a_i)}{\lambda_i^{2/(1+\gamma_*)}} + \sum O(|\alpha_i - 1|^2 + |\tau_i|^2) + o\Big(\frac{1}{\lambda^{2/(1+\gamma_*)}}\Big),\end{aligned}$$

where $\gamma_* = \max\{\gamma_i : i = 1, \dots, \sigma\}$.

Corollary 3.4. *Let $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. Assume that $(\alpha_k - 1)\ln \lambda_k$ is small for each k. Then we get*

$$\begin{aligned}&\Big\langle \nabla J(u), \sum_{i=1}^m \alpha_i \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle\\ &= \frac{8\pi(1+\gamma_*)I_2(\gamma_*)}{(\lambda_1^2 \mathcal{F}_{A_\sigma^m,1}(q_1))^{1/(1+\gamma_*)}} \sum_{i:\gamma_i=\gamma_*} (\mathcal{F}_{A_\sigma^m,i}(q_i))^{1/(1+\gamma_*)} \Delta \ln(\mathcal{F}_{A_\sigma^m,i})(q_i) + \sum O(|\alpha_i - 1|^2 + |\tau_i|^2) + o\Big(\frac{1}{\lambda^{2/(1+\gamma_*)}}\Big).\end{aligned}$$

Proof. By Corollary 3.2 we have $\tau_i = o_\varepsilon(1)$ for each i. Since we assumed that $|\alpha_k - 1|\ln \lambda_k$ is small for each k, we get that $\lambda_i^2 \mathcal{F}_{A_\sigma^m,i}(q_i) = \lambda_j^2 \mathcal{F}_{A_\sigma^m,j}(q_j)(1+o(1))$ for each i, j. Therefore, we derive that

$$\lambda_i^{\frac{2}{1+\gamma_*}} = (\lambda_i^2 \mathcal{F}_{A_\sigma^m,i}(q_i))^{\frac{1}{1+\gamma_*}} \mathcal{F}_{A_\sigma^m,i}(q_i)^{\frac{-1}{1+\gamma_*}} = (\lambda_1^2 \mathcal{F}_{A_\sigma^m,1}(q_1))^{\frac{1}{1+\gamma_*}} \mathcal{F}_{A_\sigma^m,i}(q_i)^{\frac{-1}{1+\gamma_*}}(1+o(1)).$$

The result follows by using Corollary 3.3. □

Proposition 3.5. *Let $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. Then we have the following results:*
(i) *For $i \le \sigma$ we have*

$$\begin{aligned}\langle \nabla J_{\varrho_*}(u), \widetilde{P\delta}_{a_i,\lambda_i}\rangle &= 32\pi(1+\gamma_i)(\alpha_i - 1)\ln\lambda_i + O\Big(\sum|\alpha_k - 1| + |\tau_k| + \frac{1}{\lambda_k^{2/(1+\gamma_*)}}\Big)\\ &\quad + 16\pi(1+\gamma_i)\ln\lambda_i\Big(2\tau_i + (1+\gamma_i)\frac{I_2(\gamma_i)}{\lambda_i^{2/(1+\gamma_i)}}\Big(-\Delta(\ln\mathcal{F}_{A_\sigma^m,i})(a_i) - \frac{|\nabla\mathcal{F}_{A_\sigma^m,i}(a_i)|^2}{\mathcal{F}_{A_\sigma^m,i}(a_i)^2}\frac{(2+\gamma_i)}{3\gamma_i}\Big)\Big).\end{aligned}$$

(ii) *For $i \ge \sigma + 1$ we have*

$$\langle \nabla J_{\varrho_*}(u), P\delta_{a_i,\lambda_i}\rangle = 32\pi((\alpha_i - 1) + \tau_i)\ln\lambda_i + O\Big(\sum|\alpha_k - 1| + |\tau_k| + \ln(\lambda_k)/\lambda_k^2\Big).$$

Proof. For simplicity, we will write $\mathcal{F}_i$ and $\widetilde{P\delta}_i$ instead of $\mathcal{F}_{A_\sigma^m,i}$ and $\widetilde{P\delta}_{a_i,\lambda_i}$, respectively. It is easy to see that

$$\langle \nabla J_{\varrho_*}(u), \widetilde{P\delta}_i\rangle = \sum \alpha_j\langle \widetilde{P\delta}_j, \widetilde{P\delta}_i\rangle - \frac{\varrho_*}{\int Ke^u}\int_\Omega Ke^u\widetilde{P\delta}_i.$$

The first term is given by Lemma A.3. We focus on the second one. We note that the integral on $\Omega \setminus (\bigcup B_k)$ is bounded. It remains to estimate it on $\bigcup B_k$. On B_i, by Lemmas A.1 and A.4, we have

$$\begin{aligned}\int_{B_i} Ke^u\widetilde{P\delta}_i &= \int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(1 + \alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2 + O(\|\overline{\psi}_i\|_\infty^3)\Big)\Big(1 + O\Big(\frac{1}{\lambda_i^2}\Big)\Big)\\ &\quad \times\Big(4\ln\lambda_i - 2\ln(1+\lambda_i^2|x-a_i|^{2+2\gamma_i}) - 8\pi(1+\gamma_i)H(a_i,x) + \overline{\psi}_i + O\Big(\frac{1}{\lambda_i^2}\Big)\Big).\end{aligned}$$

Using the oddness of $\overline{\psi}_i$, we easily see that

$$\begin{aligned}&\ln\lambda_i\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\|\overline{\psi}_i\|_\infty^3 = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{3/(1+\gamma_i)}}\ln\lambda_i\Big),\\ &\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(1 + \alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2\Big)\overline{\psi}_i = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big),\\ &\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(\alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2\Big)H(a_i,x) = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big),\\ &\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(\alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2\Big)\ln(1+\lambda_i^2|x-a_i|^{2+2\gamma_i}) = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big).\end{aligned}$$

Now, using (A.11) and (A.12), we derive that

$$\begin{aligned}&4\ln\lambda_i\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(\alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2\Big)\\ &\quad = 4\ln\lambda_i\Big(-\frac{1+\gamma_i}{\gamma_i}\frac{|\nabla\mathcal{F}_i(a_i)|^2}{\mathcal{F}_i(a_i)}\frac{2\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}I_3(\gamma_i) + O\Big(\frac{1}{\lambda^2}\Big) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{4/(1+\gamma_i)}}\Big)\Big)\\ &\qquad + 2\ln\lambda_i\Big(4\frac{(1+\gamma_i)^2}{\gamma_i^2}\frac{(\mathcal{F}_i g_i)(a_i)}{\mathcal{F}_i(a_i)^2}\frac{1}{2}|\nabla\mathcal{F}_i(a_i)|^2\frac{2\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}I_4(\gamma_i) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\sum|\alpha_k - 1|\Big)\Big).\end{aligned}$$

Concerning the last integral, by (A.10), we have

$$\begin{aligned}&4\ln\lambda_i\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\\ &\quad = 4\ln\lambda_i\Big(\frac{\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)\pi}{(2\alpha_i-1)(1+\gamma_i)} + \frac{1}{4}\Delta(\mathcal{F}_i g_i)(a_i)\frac{2\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}(I_2(\gamma_i) + O(|\alpha_i - 1|)) + O(1)\Big),\end{aligned}$$

$$
\begin{aligned}
2\int_{B_i} & \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} \ln(1+\lambda_i^2|x-a_i|^{2+2\gamma_i}) \\
&= 2\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)\int_{\tilde{B}_i} \frac{|y|^{2\gamma_i}}{(1+|y|^{2+2\gamma_i})^{2\alpha_i}} \ln(1+|y|^{2+2\gamma_i}) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big) \\
&= \frac{2\pi\alpha_i\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)}{(2\alpha_i-1)^2(1+\gamma_i)} + O\Big(\ln\lambda + \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big),
\end{aligned}
$$

$$
\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} H(a_i,x) = \frac{\pi\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)}{(2\alpha_i-1)(1+\gamma_i)} H(a_i,a_i) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big).
$$

Finally, we need to estimate the integral on the ball B_j for $j \neq i$. We have

$$
\begin{aligned}
\int_{B_j} Ke^u \widetilde{P\delta}_i &= \int_{B_j} \frac{\lambda_j^{4\alpha_j}|x-a_j|^{2\gamma_j}\mathcal{F}_j g_j}{(1+\lambda_j^2|x-a_j|^{2+2\gamma_j})^{2\alpha_j}} (1+\alpha_j\overline{\psi}_j + O(\|\overline{\psi}_j\|_\infty^2))\Big(1+O\Big(\frac{1}{\lambda_j^2}\Big)\Big)\Big(8\pi(1+\gamma_i)G(a_i,x) + O\Big(\frac{1}{\lambda_i^2}\Big)\Big) \\
&= \frac{8\pi^2(1+\gamma_i)}{(2\alpha_j-1)(1+\gamma_j)}\lambda_j^{4\alpha_j-2}(\mathcal{F}_j g_j)(a_j)G(a_i,a_j) + O\Big(\frac{\lambda_j^{4\alpha_j-2}}{\lambda_j^{2/(1+\gamma_j)}}\Big).
\end{aligned}
$$

Now, using the fact that

$$
\frac{\varrho_*\lambda_k^{4\alpha_k-2}}{\int Ke^u}(\mathcal{F}_k g_k)(a_k) = 8(1+\gamma_k)^2(1-\tau_k)(2\alpha_k-1) \quad \text{and} \quad g_k(a_k) = 1 + O\Big(\sum|\alpha_j - 1|\Big), \tag{3.5}
$$

we infer by the previous estimates that

$$
\begin{aligned}
\langle \nabla J(u), \widetilde{P\delta}_i\rangle &= 32\pi(1+\gamma_i)(\alpha_i-1)\ln\lambda_i + O\Big(\sum|\alpha_k-1| + |\tau_k| + \frac{1}{\lambda_k^{2/(1+\gamma_*)}}\Big) \\
&\quad + 16\pi(1+\gamma_i)\ln\lambda_i\Big(2\tau_i - \frac{\Delta\mathcal{F}_i(a_i)}{\mathcal{F}_i(a_i)}\frac{(1+\gamma_i)I_2(\gamma_i)}{\lambda_i^{2/(1+\gamma_i)}} - \frac{4}{3}\frac{|\nabla\mathcal{F}_i(a_i)|^2}{\mathcal{F}_i(a_i)^2}\frac{\gamma_i+2}{\lambda_i^{2/(1+\gamma_i)}}\frac{(1+\gamma_i)^2}{\gamma_i^2}I_3(\gamma_i)\Big).
\end{aligned}
$$

Note that $\Delta(\ln\mathcal{F}_i) = \Delta\mathcal{F}_i/\mathcal{F}_i - |\nabla\mathcal{F}_i|^2/\mathcal{F}_i^2$ and $2(1+\gamma_i)I_3(\gamma_i) = \gamma_i I_2(\gamma_i)$. □

Corollary 3.6. *We have*

$$
\Big\langle \nabla J_{\varrho_*}(u), \frac{\widetilde{P\delta}_{a_i,\lambda_i}}{\ln\lambda_i} - 2\alpha_i\lambda_i\frac{\partial\widetilde{P\delta}_{a_i,\lambda_i}}{\partial\lambda_i}\Big\rangle = 32\pi(1+\gamma_i)(\alpha_i-1) + O\Big(\frac{1}{\ln\lambda_i}\Big(\sum|\alpha_k-1| + |\tau_k| + \frac{1}{\lambda_k^{2/(1+\gamma_*)}}\Big)\Big).
$$

The following proposition is extracted from [1], by taking account of $a_i = q_i \in \Sigma$ for $i \leq \sigma$.

Proposition 3.7. *For $\sigma < m$ and $i \geq \sigma + 1$ there holds*

$$
\Big\langle \nabla J_{\varrho_*}(u), \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big\rangle = -\frac{8\pi}{\lambda_i}\frac{\nabla\mathcal{F}_{A_\sigma^m,i}(a_i)}{\mathcal{F}_{A_\sigma^m,i}(a_i)} + O\Big(\frac{1}{\lambda^2} + \frac{1}{\lambda^{4/(1+\gamma_*)}} + \sum|\alpha_k-1|^2 + |\tau_k|^2\Big).
$$

4 A Finite-Dimensional Reduction

The goal of this section is to split the variables of the concentration into finite-dimensional ones and an infinite-dimensional part and to obtain a good control on the infinite-dimensional part in terms of the finite-dimensional variables. Moreover, the dynamic of the infinite-dimensional part under the gradient flow is very well understood.

We start with a uniform expansion of the Euler–Lagrange functional around a sum of bubbles.

Proposition 4.1. *For $\bar{u} = \sum_{i=1}^{m} \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} + w = u + w \in V(\mathbf{q}_\sigma, m, \varepsilon)$ with $w \in E^m_{a,\Lambda}$ we have that*

$$J_{\varrho_*}(\overline{u}) = J_{\varrho_*}(u) - f(w) + \tfrac{1}{2}Q(w) + o(\|w\|^2),$$

where

$$f(w) = -\varrho \frac{\int K e^u w}{\int K e^u},$$

$$Q(w) = \|w\|^2 - \varrho \frac{\int K e^u w^2}{\int K e^u} + \varrho\Big(\frac{\int K e^u w}{\int K e^u}\Big)^2.$$

Proof. The proof follows from a Taylor expansion combined with the uniform estimates in Lemma A.7. In particular,

$$\int_\Omega K e^u |e^w - 1 - w - w^2/2| \le c\|w\|^3 \sum \lambda_k^{4\alpha_k - 2},$$

as desired. □

Now we prove that the second variation $\partial^2 J_{\varrho_*}(\overline{u})$ is a non-degenerate quadratic form on the space $E^m_{a,\Lambda}$.

Proposition 4.2. *Let $u = \sum_{i=1}^{m} \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. Then there exists a constant C_0 such that*

$$Q(w) = Q_0(w) + o(\|w\|^2)$$

for all $w \in E^m_{a,\Lambda}$, where

$$Q_0(w) := \|w\|^2 - \sum_{i=1}^{m} \int_\Omega |x - a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} w^2.$$

Moreover, Q_0 is a non-degenerate quadratic form on $E^m_{a,\Lambda}$ whose morse index is given by

$$\sum_{i=1}^{\sigma} 2(1 + [\gamma_i]),$$

where $[\gamma_i]$ is the integer part of γ_i.

Proof. We first notice that it follows from Lemma A.8 that the last term in the definition of $Q(w)$ is $o(\|w\|^2)$. Now, we will focus on the second one.

It follows from Lemma A.1, that e^u is bounded outside of the union of the B_i's. Hence we have

$$\int_{\Omega \setminus \bigcup B_i} K e^u w^2 = O(\|w\|^2).$$

Moreover, we have, using Lemmas A.4 and A.7, that

$$\begin{aligned}\int_{B_i} K e^u w^2 &= \int_{B_i} \frac{\lambda_i^{4\alpha_i} |x - a_i|^{2\gamma_i}}{(1 + \lambda_i^2 |x - a_i|^{2+2\gamma_i})^{2\alpha_i}} \mathcal{F}_i g_i (1 + O(\|\overline{\psi}_i\|_\infty))\Big(1 + O\Big(\frac{1}{\lambda_i^2}\Big)\Big) w^2 \\ &= \frac{\lambda_i^{4\alpha_i - 2}}{8(1+\gamma_i)^2} \int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} \mathcal{F}_i g_i w^2 + o(\|w\|^2 \lambda_i^{4\alpha_i - 2}).\end{aligned}$$

To estimate the previous integral by expanding $\mathcal{F}_i g_i$ around a_i, we obtain

$$\int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} \mathcal{F}_i g_i w^2 = \mathcal{F}_i(a_i) g_i(a_i) \int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} w^2 + O\Big(\frac{\|w\|^2}{\lambda_i^{1/(1+\gamma_i)}}\Big).$$

Using (3.5) and Corollary 3.2, we deduce that

$$\frac{\varrho_*}{\int_\Omega K e^u} \int_\Omega K e^u w^2 = \sum_{i=1}^{m} \int_\Omega |x - a_i|^{2\gamma_i} e^{\delta_i} w^2 + o(\|w\|^2),$$

therefore the first claim follows.

The non-degeneracy of this quadratic form and its Morse index are given in [26, Lemma 3.3]. □

As a consequence of Proposition 4.2 and Lemma A.8, we have the following proposition.

Proposition 4.3. *Let* $u := \sum_{i=1}^{m} \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. *Then the map*

$$w \in E^m_{a,\Lambda} \mapsto J(u+w)$$

has a unique critical point $\overline{w}$. *The Morse index of this critical point is* $\sum_{i=1}^{\sigma} 2(1+[\gamma_i])$. *Furthermore, there exists a constant C such that*

$$\|\overline{w}\| \le C\Big(\sum_{i=\sigma+1}^{m} \frac{|\nabla \mathcal{F}_{A^m_\sigma,i}(a_i)|}{\lambda_i} + \sum_{j=1}^{m} |\alpha_j - 1| + \frac{1}{\lambda^{2/(1+\gamma_*)}}\Big).$$

Note that since $\overline{w}$ is a non-degenerate critical point for J_{ϱ_*} in the space $E^m_{a,\Lambda}$ for all fixed variables (α, a, Λ), by using the Morse lemma we derive the existence of a change of variables $w - \overline{w} \to (V^+, V^-)$ such that

$$J_{\varrho_*}\Big(\sum \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} + w\Big) = J_{\varrho_*}\Big(\sum \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} + \overline{w}\Big) + \|V^+\|^2 - \|V^-\|^2.$$

Hence, to decrease the functional J_{ϱ_*} we need to define

$$\dot{V^+} = -V^+ \quad \text{and} \quad \dot{V^-} = V^-.$$

This vector field will bring the variable V^+ to zero.

Our aim now is to construct a vector field on the remaining variables α_i, a_i and λ_i.

5 Characterization of the Critical Points at Infinity

Critical points at infinity are non-compact orbits of the gradient flow (or any other pseudo-gradient) along which the functional remains bounded. They are besides the critical points responsible for the difference of topology between the level sets of the Euler–Lagrange functional. The goal of this section is to identify the ends of these orbits, to perform a Morse-type reduction around them and to associate to them an index which enables us to compute their contribution in the difference of topology.

Inspired by the ideas introduced by Bahri in his study of Yamabe- and scalar curvature-type equations (see [3, 4]), we look for a *normal form* of the gradient around such points. The way to find such a normal form is to construct a pseudo-gradient for the functional in the neighborhood at infinity whose dynamic is easier to understand. As a starting point in our search for such an appropriate pseudo-gradient, we perform an accurate expansion of the functional J_{ϱ_*} in $V(\mathbf{q}_\sigma, m, \varepsilon)$.

Proposition 5.1. *Let* $u = \sum_{i=1}^{m} \alpha_i \widetilde{P\delta}_{(a_i,\lambda_i)} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. *Then we have*

$$\begin{aligned} J_{\varrho_*}(u) = &-\varrho_*\Big(1 + \ln\Big(\frac{\varrho_*}{8}\Big)\Big) + 16\pi \sum (1+\gamma_i)\ln(1+\gamma_i) - 8\pi \mathcal{F}^m_{\mathbf{q}_\sigma}(a_{\sigma+1}, \dots, a_m) \\ &+ 16\pi \sum (1+\gamma_i)(\alpha_i - 1)^2 \ln \lambda_i - 4\pi \sum (1+\gamma_i)\tilde{\tau}_i^2 \\ &- 4\pi I_2(\gamma_*)(1+\gamma_*)^2 \sum_{i=1}^{l} \frac{1}{\lambda_i^{2/(1+\gamma_*)}} \Delta \ln(\mathcal{F}_{A^m_\sigma,i})(a_i) + O\Big(\sum |\alpha_i - 1|^2 + |\tilde{\tau}_i|^3\Big) + o\Big(\frac{1}{\lambda^{2/(1+\gamma_*)}}\Big), \end{aligned}$$

where

$$\tilde{\tau}_i := 1 - \frac{\varrho_*}{8\pi(1+\gamma_i)} \frac{\Gamma_i}{\sum_{k=1}^{m} \Gamma_k} \quad \textit{with } \Gamma_k := \frac{1}{(2\alpha_k - 1)(1+\gamma_k)} \lambda_k^{4\alpha_k - 2} (\mathcal{F}_{A^m_\sigma,k} g_k)(a_k).$$

Proof. First, let $\Gamma := \sum \Gamma_k$ with

$$\Gamma_k := \frac{1}{(2\alpha_k - 1)(1+\gamma_k)} \lambda_k^{4\alpha_k - 2} (\mathcal{F}_k g_k)(a_k).$$

Using Lemma A.2, we have

$$\begin{aligned} \|u\|^2 = &\ 16\pi \sum \alpha_i^2 (1+\gamma_i)(2\ln(\lambda_i) - 1 - 4\pi(1+\gamma_i) H(a_i, a_i)) \\ &+ \sum \alpha_i^2 \|\psi_i\|^2 + 64\pi^2 \sum \alpha_i \alpha_j (1+\gamma_i)(1+\gamma_j) G(a_i, a_j) + O\Big(\frac{1}{\lambda^{\min(2, 4/(1+\gamma_*))}}\Big). \end{aligned}$$

Secondly, using Lemma A.5, we have

$$\ln\Big(\int_\Omega Ke^u\Big) = \ln(\pi\Gamma) + \frac{1}{\Gamma}\sum_{i=1}^{\sigma}\Big(\frac{1}{2}I_2(\gamma_i)\Delta\mathcal{F}_i(a_i) + \frac{2}{3}\frac{1-\gamma_i^2}{\gamma_i^2}I_3(\gamma_i)\frac{|\nabla\mathcal{F}_i(a_i)|^2}{\mathcal{F}_i(a_i)}\Big)\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}} + O\Big(\frac{1}{\Gamma} + \sum\Big(\frac{1}{\lambda^{3/(1+\gamma_*)}} + \|w\|^2 + |f(w)| + \sum|\alpha_i - 1|^2\Big)\Big).$$

Observe that

$$\begin{aligned}\varrho_*\ln(\pi\Gamma) &= \varrho_*\ln(\pi) + 8\pi\sum(1+\gamma_i)\ln(\Gamma)\\ &= \varrho_*\ln(\pi) + 8\pi\sum(1+\gamma_i)\ln\Big(\frac{\Gamma}{\frac{\varrho_*}{8\pi(1+\gamma_i)}\Gamma_i}\frac{\varrho_*}{8\pi(1+\gamma_i)}\Gamma_i\Big)\\ &= \varrho_*\ln(\pi) + 8\pi\sum(1+\gamma_i)\Big(\ln\Big(\frac{\varrho_*}{8\pi(1+\gamma_i)}\Gamma_i\Big) - \ln\Big(\frac{\varrho_*}{8\pi(1+\gamma_i)}\frac{\Gamma_i}{\Gamma}\Big)\Big)\\ &= \varrho_*\ln(\pi) + 8\pi\sum(1+\gamma_i)\{\ln(\varrho_*) - \ln(8\pi(1+\gamma_i)) - \ln(2\alpha_i - 1) - \ln(1+\gamma_i)\\ &\qquad + (4\alpha_i - 2)\ln\lambda_i + \ln((\mathcal{F}_i g_i)(a_i)) - \ln(1-\tilde\tau_i)\}.\end{aligned}$$

We remark that

$$\begin{aligned}&\frac{\varrho_*\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)}{\Gamma} = 8\pi(1+\gamma_i)^2 + O(|\alpha_i - 1| + |\tilde\tau_i|),\\ &\ln(1-\tilde\tau_i) = -\tilde\tau_i - \frac{\tilde\tau_i^2}{2} + O(\tilde\tau_i^3), \quad \sum_{i=1}^{m}(1+\gamma_i)\tilde\tau_i = 0,\\ &\ln(2\alpha_i - 1) = \ln(1 + 2(\alpha_i - 1)) = 2(\alpha_i - 1) + O(|\alpha_i - 1|^2),\\ &\alpha_i^2 = 1 + 2(\alpha_i - 1) + (\alpha_i - 1)^2,\\ &g_i(a_i) = 1 + O\Big(\sum|\alpha_k - 1|\Big).\end{aligned}$$

Combining the previous estimates, we obtain the result. □

From this proposition, it is easy to see that the function J_{ϱ_*} will decrease if we bring the variables α_i to 1 and if we increase the parameters $\tilde\tau_k$. In fact, we have the following proposition which defines a vector field W along which the weights α_i will be brought to 1, the $\tilde\tau_k$ to zero, if $\sigma < m$, the concentration points to a critical point of the functional $\mathcal{F}^m_{\mathbf{q}_\sigma}$, while the concentration speed will be kept under control everywhere but in one specific region, where the speed will increase dramatically and leads to a "critical point at infinity."

Proposition 5.2. *Let $\varrho_* \in \mathcal{G}$ and assume that the function K satisfies condition $(\mathrm{ND}_{m,\sigma})$. Then there exist a pseudo-gradient W defined in $V(\mathbf{q}_\sigma, m, \varepsilon)$ and a constant C independent of $u = \sum_{i=1}^m \alpha_i\widetilde{P\delta}_{a_i,\lambda_i}$ such that the following properties hold:*

(i) *We have*

$$\langle -\nabla J_{\varrho_*}(u), W\rangle \geq C\sum_{i=1}^{m}\Big(|\alpha_i - 1| + |\tau_i| + \frac{1}{\lambda_i^{2/(1+\gamma_*)}}\Big) + \underbrace{C\sum_{i=\sigma+1}^{m}\frac{|\nabla\mathcal{F}_{A^m_\sigma,i}(a_i)|}{\lambda_i}}_{\text{if } \sigma<m}.$$

(ii) *We have*

$$\Big\langle -\nabla J_{\varrho_*}(u+\overline{w}), W + \frac{\partial\overline{w}(W)}{\partial(\alpha,\lambda,a)}\Big\rangle \geq C\sum_{i=1}^{m}\Big(|\alpha_i - 1| + |\tau_i| + \frac{1}{\lambda_i^{2/(1+\gamma_*)}}\Big) + \underbrace{C\sum_{i=\sigma+1}^{m}\frac{|\nabla\mathcal{F}_{A^m_\sigma,i}(a_i)|}{\lambda_i}}_{\text{if } \sigma<m}.$$

(iii) *$|W|$ is bounded and the only region where the variables λ_i increase along the flow lines of W is*
   (a) *a small neighborhood of some critical point $P := (P_1, \dots, P_m)$ of $\mathcal{F}^m_0$ with $P \in \mathcal{K}^-_{m,0}$ if $\sigma = 0$;*
   (b) *the region where $(a_{\sigma-1}, \dots, a_m)$ is very close to a critical point $P := (p_{\sigma+1}, \dots, p_m)$ of $\mathcal{F}^m_{q_\sigma}$ with $P \in \mathcal{K}^-_{m,\sigma}$ if $0 < \sigma < m$;*
   (c) *a small neighborhood of an m-tuple $Q := (q_1, \dots, q_m)$ of zeros of the function K such that $\mathcal{L}(Q) < 0$ if $\sigma = m$.*

Proof. We first observe that in the case $\sigma = 0$ the statement is proved in [1, Proposition 5.2]. So we will focus on the case $0 < \sigma \le m$.

We start the proof by defining special vector fields which will be used to construct W. To this end we set $W_\alpha := \sum W_{\alpha_i}$, where

$$W_{\alpha_i} := \frac{(1-\alpha_i)}{|\alpha_i - 1|}(1 - \xi_1(\lambda_i^{2/(1+\gamma_*)}|\alpha_i - 1|))\Big(\frac{\widetilde{P\delta_i}}{\ln(\lambda_i)} - 2\alpha_i\lambda_i\frac{\partial\widetilde{P\delta_i}}{\partial\lambda_i}\Big),$$

where ξ_1 is a positive cut-off function defined by $\xi_1(t) = 1$ if $|t| \le 1$ and $\xi_1(t) = 0$ if $|t| \ge 2$. Using Propositions 3.7 and 3.1, we derive that

$$\begin{aligned}\langle -\nabla J_{\varrho_*}(u), W_{\alpha_i}\rangle = (1 - \xi_1(\lambda_i^{2/(1+\gamma_*)}|\alpha_i - 1|))\Big(&32\pi(1+\gamma_i)|\alpha_i - 1| \\ &+ O\Big(\frac{1}{\ln\lambda_i}\sum(|\tau_k| + |\alpha_k - 1| + \frac{1}{\lambda_k^{2/(1+\gamma_*)}})\Big)\Big).\end{aligned} \tag{5.1}$$

Furthermore, we define

$$W_\lambda^1 := -4\sum_{i=1}^m \alpha_i\lambda_i\frac{\partial\widetilde{P\delta_i}}{\partial\lambda_i} \quad \text{and} \quad W_\lambda^2 := -\sum_{i=1}^m \frac{\tau_i}{|\tau_i|}(1 - \xi_1(\lambda_i^{2/(1+\gamma_*)}|\tau_i|))\alpha_i\lambda_i\frac{\partial\widetilde{P\delta_i}}{\partial\lambda_i}.$$

Using Proposition 3.1, we obtain

$$\langle -\nabla J_{\varrho_*}(u), W_\lambda^1\rangle = O\Big(\sum\frac{1}{\lambda_i^{2/(1+\gamma_*)}} + |\alpha_i - 1|^2 + |\tau_i|^2\Big), \tag{5.2}$$

$$\langle -\nabla J_{\varrho_*}(u), W_\lambda^2\rangle = (1 - \xi_1(\lambda_i^{2/(1+\gamma_*)}|\tau_i|))\Big(16\pi(1+\gamma_i)|\tau_i| + O\Big(\frac{1}{\lambda^{2/(1+\gamma_*)}} + |\alpha_i - 1|^2\Big)\Big). \tag{5.3}$$

In the case $\sigma < m$, to move the points, we define

$$W_a := \sum_{i=\sigma+1}^m \nabla\mathcal{F}_{A_\sigma^m, i}(a_i)\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}.$$

Using Proposition 3.7, we obtain

$$\langle -\nabla J_{\varrho_*}(u), W_a\rangle \ge c\sum_{i=\sigma+1}^m\Big(\frac{|\nabla\mathcal{F}_i(a_i)|^2}{\lambda_i} + \sum O\Big(\frac{1}{\lambda_k^2} + \frac{1}{\lambda_k^{4/(1+\gamma_*)}} + |\tau_k|^2 + |\alpha_k - 1|^2\Big)\Big).$$

We divide the set $V(\mathbf{q}_\sigma, m, \varepsilon)$ into two subsets and on each one we define a pseudo-gradient. The global vector field will be a convex combination of them. Let us define

$$\begin{aligned}\mathcal{V}_1(\mathbf{q}_\sigma, m, \varepsilon) &:= \{u \in V(\mathbf{q}_\sigma, m, \varepsilon) : \text{there exists an } i \text{ such that } |\alpha_i - 1| + |\tau_i| \ge 2M/\lambda_i^{2/(1+\gamma_*)}\}, \\ \mathcal{V}_2(\mathbf{q}_\sigma, m, \varepsilon) &:= \{u \in V(\mathbf{q}_\sigma, m, \varepsilon) : \text{for all } i \text{ there holds } |\alpha_i - 1| + |\tau_i| \le 3M/\lambda_i^{2/(1+\gamma_*)}\},\end{aligned}$$

where M is a large positive constant.

In $\mathcal{V}_1(\mathbf{q}_\sigma, m, \varepsilon)$, we define

$$W^1 := W_\alpha + W_\lambda^1 + W_\lambda^2.$$

Note that, denoting $A := \{i : \lambda_i^{2/(1+\gamma_*)}|\alpha_i - 1| \ge 2\}$ and $D := \{i : \lambda_i^{2/(1+\gamma_*)}|\tau_i| \ge 2\}$ and using (5.1)–(5.3), we obtain that in the lower bound of $\langle -\nabla J_{\varrho_*}(u), W^1\rangle$ the terms $\sum_{i\in A}|\alpha_i - 1|$ and $\sum_{i\in D}|\tau_i|$ appear. Moreover, since $u \in \mathcal{V}_1(\mathbf{q}_\sigma, m, \varepsilon)$, there exists at least an index i_0 such that $|\alpha_{i_0} - 1|$ or $|\tau_{i_0}|$ is larger that $M/\lambda_{i_0}^{2/(1+\gamma_*)}$. Thus, in the lower bound of $\langle -\nabla J_{\varrho_*}(u), W^1\rangle$ all variables $|\alpha_i - 1|$, $|\tau_i|$ and $\lambda_i^{2/(1+\gamma_*)}$ appear and we obtain

$$\langle -\nabla J_{\varrho_*}(u), W^1\rangle \ge c\sum\frac{1}{\lambda_i^{2/(1+\gamma_*)}} + |\alpha_i - 1| + |\tau_i|$$

by choosing M large enough.

Note that along the flow lines generated by W^1 the variables λ_i are decreasing functions.

In $\mathcal{V}_2(\mathbf{q}_\sigma, m, \varepsilon)$, for $\sigma < m$, we observe that the tuple $a := (a_{\sigma+1}, \dots, a_m)$ is close to a critical point z of $\mathcal{F}^m_{\mathbf{q}_\sigma}$. Let $A^m_\sigma := (q_1, \dots, q_\sigma, z)$. We have

$$\sum_{i:\gamma_i=\gamma_*} \mathcal{F}_i(q_i)^{1/(1+\gamma_*)} \Delta \ln(\mathcal{F}_i)(q_i) = \mathcal{L}(A^m_\sigma) + o(1).$$

In this set, we define

$$W^2 := -\mathrm{sign}(\mathcal{L}(A^m_\sigma)) \sum_{i=1}^m \alpha_i \lambda_i \frac{\partial \widetilde{P\delta}_i}{\partial \lambda_i},$$

and using Proposition 3.4, we get

$$\begin{aligned} \langle -\nabla J_{\varrho_*}(u), W^2_\lambda \rangle &\geq c \frac{1}{\lambda_1^{2/(1+\gamma_*)}} + \sum O(|\alpha_i - 1|^2 + |\tau_i|^2) \\ &\geq c \sum \frac{1}{\lambda_i^{2/(1+\gamma_*)}} + |\alpha_i - 1| + |\tau_i| \end{aligned}$$

since $u \in \mathcal{V}_2(\mathbf{q}_\sigma, m, \varepsilon)$.

Now, in the case $\sigma < m$, we define the pseudo-gradient W by a convex combination of $W^1 + W_a$ and $W^2_\lambda + W_a$. In the other case, that is, $\sigma = m$, the term W_a does not appear. We notice that $|W|$ is bounded and the only region where the concentration speeds λ_i increase along the flow lines of W is the subset $\mathcal{V}_2^-(\mathbf{q}_\sigma, m, \varepsilon) \subset \mathcal{V}_2(\mathbf{q}_\sigma, m, \varepsilon)$ where $\mathcal{L}(A^m_\sigma) < 0$. Hence Claim (i) follows.

Concerning Claim (ii), we remark that $\partial \overline{w}/\partial s$ does not necessary lie in $E^m_{a,\Lambda}$. Then we write

$$\frac{\partial \overline{w}}{\partial s} = \sum_{i=1}^m \Big(t_i \frac{\widetilde{P\delta}_i}{\sqrt{\ln \lambda_i}} + \mu_i \lambda_i \frac{\partial \widetilde{P\delta}_i}{\partial \lambda_i} \Big) + \sum_{i=\sigma+1}^m \sum_{j=1,2} \nu_{i,j} \frac{1}{\lambda_i} \Big(\alpha_i \frac{\partial P\delta_i}{\partial (a_i)_j} + \sum_{k=1}^\sigma \alpha_k \frac{\partial \psi_k}{\partial (a_i)_j} \Big) + \underline{w}, \tag{5.4}$$

where $\underline{w} \in E^m_{a,\Lambda}$. Hence, for $\overline{u} := u + \overline{w} := \sum \alpha_i \widetilde{P\delta}_i + \overline{w}$ we have

$$\begin{aligned} \Big\langle \nabla J_{\varrho_*}(\overline{u}), \frac{\partial \overline{w}}{\partial s} \Big\rangle = &\sum_{i=1}^m \Big(t_i \Big\langle \nabla J_{\varrho_*}(\overline{u}), \frac{\widetilde{P\delta}_i}{\sqrt{\ln \lambda_i}} \Big\rangle + \mu_i \Big\langle \nabla J_{\varrho_*}(\overline{u}), \lambda_i \frac{\partial \widetilde{P\delta}_i}{\partial \lambda_i} \Big\rangle \Big) \\ &+ \sum_{i=\sigma+1}^m \sum_{j=1,2} \nu_{i,j} \Big\langle \nabla J_{\varrho_*}(\overline{u}), \frac{\alpha_i}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} + \sum_{k=1}^\sigma \frac{\alpha_k}{\lambda_i} \frac{\partial \psi_k}{\partial (a_i)_j} \Big\rangle \end{aligned}$$

since $\overline{w}$ minimizes $J_{\varrho_*}(\sum \alpha_i \widetilde{P\delta}_i + w)$ in $E^m_{a,\Lambda}$.

It remains to estimate the variables t_i, μ_i and $\nu_{i,j}$. For this purpose, we multiply (5.4) by $\widetilde{P\delta}_i/\sqrt{\ln \lambda_i}$, $\lambda_i(\partial \widetilde{P\delta}_i/\partial \lambda_i)$ and $(1/\lambda_i)(\partial P\delta_i/\partial (a_i)_j)$, respectively (the norm of each term is of the order $c + o(1)$, c is a positive constant, and the scalar product of two of them is of order $o(1)$). We get the following system:

$$\begin{cases} c_i t_i + o\big(\sum |t_j| + |\mu_j| + |\nu_{k,j}|\big) = \big\langle \partial \overline{w}/\partial s, \widetilde{P\delta}_i/\sqrt{\ln \lambda_i} \big\rangle, & i = 1, \dots, m, \\ c'_i \mu_i + o\big(\sum |t_j| + |\mu_j| + |\nu_{k,j}|\big) = \langle \partial \overline{w}/\partial s, \lambda_i \partial \widetilde{P\delta}_i/\partial \lambda_i \rangle, & i = 1, \dots, m, \\ c''_{i,j} \nu_{i,j} + o\big(\sum |t_j| + |\mu_j| + |\nu_{k,j}|\big) = \langle \partial \overline{w}/\partial s, \lambda^{-1} \partial P\delta_{a_i,\lambda_i}/\partial (a_i)_j \rangle, & \sigma + 1 \leq i \leq m,\ j = 1, 2, \end{cases}$$

where c_i, c'_i and c''_i are fixed positive constants.

Recall that $\overline{w}$ is in $E^m_{a,\Lambda}$. Then we deduce that

$$\frac{\partial}{\partial s} \langle \overline{w}, \widetilde{P\delta}_i \rangle = 0,$$

which implies

$$\Big\langle \frac{\partial \overline{w}}{\partial s}, \widetilde{P\delta}_i \Big\rangle = 0.$$

Furthermore, in the same way we get (recall that $\dot{\lambda}_i = O(\lambda_i)$ and $\dot{a}_{i,j} = O(\lambda_i^{-1})$)

$$\Big\langle \frac{\partial \overline{w}}{\partial s}, \frac{\partial \widetilde{P\delta}_i}{\partial \lambda_i}\Big\rangle = -\Big\langle \overline{w}, \frac{\partial^2 \widetilde{P\delta}_i}{\partial \lambda_i \partial s}\Big\rangle = O(\lambda^{-1}\|\overline{w}\|).$$

Indeed,

$$\begin{cases} \dfrac{\partial^2 P\delta_i}{\partial\lambda_i\partial s} = \dfrac{\partial^2 P\delta_i}{\partial\lambda_i^2}\dot{\lambda}_i + \dfrac{\partial^2 P\delta_i}{\partial\lambda_i\partial a_i}\dot{a}_i & \text{if } i \geq \sigma+1,\\ \dfrac{\partial^2 \widetilde{P\delta}_i}{\partial\lambda_i\partial s} = \dfrac{\partial^2 \widetilde{P\delta}_i}{\partial\lambda_i^2}\dot{\lambda}_i + \displaystyle\sum_{k=\sigma+1}^{m} \dfrac{\partial^2 \widetilde{P\delta}_i}{\partial\lambda_i\partial a_k}\dot{a}_k & \text{if } i \leq \sigma. \end{cases}$$

Moreover, for $i \geq \sigma + 1$ we have

$$\Big\langle \frac{\partial \overline{w}}{\partial s}, \frac{\partial P\delta_i}{\partial a_{i,j}}\Big\rangle = -\Big\langle \overline{w}, \frac{\partial^2 P\delta_i}{\partial a_{i,j}\partial s}\Big\rangle = -\Big\langle \overline{w}, \frac{\partial^2 P\delta_i}{\partial a_{i,j}\partial\lambda_i}\Big\rangle\dot{\lambda}_i - \sum_{k=1,2}\Big\langle \overline{w}, \frac{\partial^2 P\delta_i}{\partial a_{i,j}\partial a_{i,k}}\Big\rangle \dot{a}_{i,k} = O(\lambda_i\|\overline{w}\|).$$

Hence we get

$$t_i, \mu_i, \nu_{i,j} = O(\|\overline{w}\|).$$

This implies that the term $\langle \nabla J_{\varrho_*}(u), (\partial\overline{w})/(\partial s)\rangle$ is small with respect to the lower bound in Claim (i) (by using the estimate of $\|\overline{w}\|$ given in Proposition 4.3).

Now, we will focus on

$$\langle \nabla J_{\varrho_*}(\overline{u}), W\rangle = \langle u, W\rangle + \langle \overline{w}, W\rangle - \frac{\varrho_*}{\int Ke^{\overline{u}}}\int Ke^{\overline{u}}W.$$

For the second term on the right-hand side, since $\overline{w}$ is in $E^m_{a,\Lambda}$, we get that it is equal to $o(\|\overline{w}\|)$. Indeed, we have

$$\begin{aligned} \langle \overline{w}, \widetilde{P\delta}_i\rangle = \Big\langle \overline{w}, \frac{\partial\widetilde{P\delta}_i}{\partial\lambda_i}\Big\rangle &= 0 && \text{for all } i,\\ \Big\langle \overline{w}, \frac{\partial P\delta_i}{\partial a_i}\Big\rangle = -\sum_{k=1}^{\sigma}\frac{\alpha_k}{\alpha_i}\Big\langle \overline{w}, \frac{\partial\psi_k}{\partial a_i}\Big\rangle &= o(\|\overline{w}\|) && \text{for all } i \geq \sigma+1. \end{aligned}$$

Now, we write

$$\int Ke^{\overline{u}} = \int Ke^{u} + \int Ke^{u}\overline{w} + \int Ke^{u}(e^{\overline{w}} - 1 - \overline{w}),$$

and therefore, from Lemmas A.7 and A.8 we derive that

$$\int Ke^{\overline{u}} = \int Ke^{u} + o\Big(\sum \lambda_k^{4\alpha_k-2}\|\overline{w}\|\Big).$$

Finally, observe that

$$\int Ke^{\overline{u}}W = \int Ke^{u}W + \int Ke^{u}\overline{w}W + \int Ke^{u}(e^{\overline{w}} - 1 - \overline{w})W.$$

Note that W is a linear combination of the functions $\widetilde{P\delta}_{a_i,\lambda_i}$, $\partial\widetilde{P\delta}_{a_i,\lambda_i}/\partial\lambda_i$ and, if $\sigma < m$, also $\partial\widetilde{P\delta}_{a_i,\lambda_i}/\partial(a_i)_j$. Hence, the second claim follows from these estimates and Lemma 5.3 given below. □

Lemma 5.3. *Let $u := \sum \alpha_i\widetilde{P\delta}_{a_i,\lambda_i}$. Then there holds*

$$\int Ke^{u}\overline{w}\frac{\widetilde{P\delta}_{a_i,\lambda_i}}{\ln\lambda_i} = o\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}\Big), \tag{5.5}$$

$$\int Ke^{u}\overline{w}\lambda_i\frac{\partial\widetilde{P\delta}_{a_i,\lambda_i}}{\partial\lambda_i} = o\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}\Big), \tag{5.6}$$

$$\int Ke^{u}\overline{w}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial(a_i)_j} = o\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}\Big) \quad \textit{if } \sigma < m. \tag{5.7}$$

Proof. Claim (5.7) follows from [1], since for $i \geq \sigma + 1$ the concentration point $a_i \notin \Sigma$. Concerning claim (5.5), by using Lemma A.4 and the fact that $\widetilde{P\delta_i}$ is bounded outside of B_i, we have

$$\begin{aligned}\int Ke^u \overline{w}\widetilde{P\delta_i} &= \frac{\lambda_i^{4\alpha_i-2}}{8(1+\gamma_i)^2}\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\mathcal{F}_i g_i \xi_i\Big(1+O\Big(\frac{1}{\lambda_i^{1/(1+\gamma_i)}}\Big)\Big)\overline{w}\widetilde{P\delta_i} + O\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}\Big)\\ &= \frac{\lambda_i^{4\alpha_i-2}}{8(1+\gamma_i)^2}(\mathcal{F}_i g_i)(a_i)4\ln(\lambda_i)\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\overline{w} + O\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}(1+|\alpha_i-1|\ln\lambda_i)\Big)\\ &= O\Big(\|\overline{w}\|\sum\lambda_k^{4\alpha_k-2}(1+|\alpha_i-1|\ln\lambda_i)\Big),\end{aligned}$$

by using the fact that $\langle\widetilde{P\delta_i},\overline{w}\rangle = 0$. Hence Claim (5.5) follows. Claim (5.6) follows as in the proof of (5.5) by using the fact that $\langle\widetilde{P\delta_i},\overline{w}\rangle = 0$ and $\langle\partial\widetilde{P\delta_i}/\partial\lambda_i,\overline{w}\rangle = 0$. For more details, one can see the proof of the analogous estimate in [1]. □

As a consequence of Propositions 5.2 and 5.1, we are able to identify the critical points at infinity of J_{ϱ_*} and to compute their Morse indices.

Corollary 5.4. *Let $\mathbf{q}_\sigma := (q_{i_1},\dots,q_{i_\sigma}) \subset \Sigma$. Furthermore, denote $\varrho_{\sigma,m} := 8\pi(m-\sigma) + \sum_{j=1}^{\sigma} 8\pi(1+\gamma_j)$. Then, for $0 < \sigma < m$, the critical points at infinity in $V(\mathbf{q}_\sigma, m, \varepsilon)$ are in a one-to-one correspondence with the critical points $\mathbf{a} := (a_{\sigma+1},\dots,a_m)$ of $\mathcal{F}^m_{\mathbf{q}_\sigma}$ such that $\mathbf{A}^m_\sigma := (a_1,\dots,a_\sigma,\dots,a_m) \in \mathcal{K}^-_{\sigma,m}$.*

The morse index of such a critical point at infinity, which will be denoted by $(\mathbf{A}^m_\sigma)_\infty$, is given by

$$\operatorname{ind}((\mathbf{A}^m_\sigma)_\infty) := 3m - 1 - 2\sigma - \operatorname{ind}(\mathcal{F}^m_{\mathbf{q}_\sigma},\mathbf{a}) + 2\sum_{i=1}^{\sigma}(1+[\gamma_i]).$$

If $\sigma = m$, there are two cases: either $\mathcal{L}(\mathbf{A}^\sigma_\sigma) < 0$, and then $(\mathbf{A}^\sigma_\sigma)_\infty$ is the only critical point at infinity in $V(\mathbf{q}_\sigma,\sigma,\varepsilon)$ with morse index

$$\operatorname{ind}((\mathbf{A}^\sigma_\sigma)_\infty) := \sigma - 1 + 2\sum_{i=1}^{\sigma}(1+[\gamma_i]),$$

or $\mathcal{L}(\mathbf{A}^\sigma_\sigma) > 0$, and then there is no critical point at infinity in $V(\mathbf{q}_\sigma,\sigma,\varepsilon)$.

Furthermore, for $0 < \sigma \leq m$ the critical value of this critical point at infinity is

$$C_\infty((\mathbf{A}^m_\sigma)_\infty) := -\varrho_{\sigma,m}\Big(1+\ln\Big(\frac{\varrho_{\sigma,m}}{8}\Big)\Big) + 16\pi\sum(1+\gamma_i)\ln(1+\gamma_i) - 8\pi\mathcal{F}^m_{\mathbf{q}_\sigma}(a_{\sigma+1},\dots,a_m).$$

If $\sigma = 0$, the critical points at infinity in $V(0,m,\varepsilon) := V(m,\varepsilon)$ are in a one-to-one correspondence with the critical points $A^m_0 := (a_1,\dots,a_m)$ of $\mathcal{F}^m_0$ such that $A^m_0 \in \mathcal{K}^-_{0,m}$. The Morse index of such a critical point at infinity is given by

$$\operatorname{ind}((\mathbf{A}^m_0)_\infty) := 3m - 1 - \operatorname{ind}(\mathcal{F}^m_0, A^m_0),$$

and its critical level is

$$C_\infty((A^m_0))_\infty = -8\pi m(1+\ln(m\pi)) - 8\pi\sum_{i=1}^{m}\ln(K(a_i)) + 32\pi^2\sum_{i=1}^{m}\Big(H(a_i,a_i) - \underbrace{\sum_{j\neq i}G(a_i,a_j)}_{\text{if } m\geq 2}\Big).$$

6 Proofs of the Main Results

Proof of Theorem 1.1. We argue by contradiction, assuming that J_{ϱ_*} has no critical points. Hence from Corollary 5.4 and [9, Theorem 7.2] we have the following:

(i) There exists $L_1 > 0$ large such that all the critical points at infinity are contained between the subsets $J^{L_1}_{\varrho_*}$ and $J^{-L_1}_{\varrho_*}$.

(ii) $J_{\varrho_*}^{L_1}$ retracts by deformation onto

$$\bigcup W_u((A_\sigma^m)_\infty) \cup J_{\varrho_*}^{-L_1},$$

where the $(A_\sigma^m)_\infty$'s are the critical points at infinity and $W_u((A_\sigma^m)_\infty)$ denotes the unstable manifold at the critical point at infinity $(A_\sigma^m)_\infty$. Hence

$$1 = \chi(J_{\varrho_*}^{L_1}) = \sum_{i=1}^{l} \sum_{\mathcal{L}(A_{\sigma_i}^{m_i})<0} (-1)^{\operatorname{ind}(A_{\sigma_i}^{m_i})} + \chi(J_{\varrho_*}^{-L_1}). \tag{6.1}$$

Furthermore, for ε small enough, we have $\chi(J_{\varrho_*}^{-L}) = \chi(J_{\varrho_*-\varepsilon}^{-L})$ for L large.

Now, it follows from the a-priori estimate of [11, 16], that all the solutions u_ε of $J_{\varrho_*-\varepsilon}$ are bounded in $C^{2,\alpha}(\overline{\Omega})$ and $-L_2 \le J_{\varrho_*-\varepsilon}(u_\varepsilon) \le L_2$ for some large L_2. Taking L larger than $\max(L_1, L_2)$, we have that

$$\deg_{\varrho_*-\varepsilon} = \chi(J_{\varrho_*-\varepsilon}^{L}, J_{\varrho_*-\varepsilon}^{-L}), \tag{6.2}$$

where $\deg_{\varrho_*-\varepsilon}$ denotes the Leray–Schauder degree of

$$\begin{cases} -\Delta u = (\varrho_* - \varepsilon)\dfrac{Ke^u}{\int_\Omega Ke^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from the result of Chen and Lin [26] that

$$\deg_{\varrho_*-\varepsilon} = 1 + \sum_{\substack{(\mathbf{q}_\sigma, m): \\ \sigma\le m,\ 8\pi\le \varrho(\mathbf{q}_\sigma, m)<\varrho_*}} (-1)^\sigma \binom{g+N+m-\sigma-2}{m-\sigma}. \tag{6.3}$$

Combining (6.1)–(6.3) and the fact that

$$1 = \chi(J_{\varrho_*}^{L_2}) = \chi(J_{\varrho_*}^{L_2}, J_{\varrho_*}^{-L_2}) + \chi(J_{\varrho_*}^{-L_2}),$$

we get a contradiction with the assumption of the theorem. □

Proof of Corollary 1.5. Observe that in the case $\varrho_* = 8\pi(1 + \gamma_{\min})$ the sum on the right-hand side of Theorem 1.1 will be on $\sigma = 0$ and $1 \le m \le [\gamma_{\min}] + 1$. Moreover, the critical point at infinity will be in a one-to-one correspondence with the q_i such that $\gamma(q_i) = \gamma_{\min}$ and $\mathcal{L}(q_i) < 0$. These critical points have an even index. Therefore the condition in the theorem will be

$$\#\{i : \gamma(q_i) = \gamma_{\min},\ \mathcal{L}(q_i) < 0\} = \sum_{i:\, \gamma(q_i)=\gamma_{\min},\ \mathcal{L}(q_i)<0} 1 \ne \binom{g+N+[\gamma_{\min}]}{1+[\gamma_{\min}]},$$

where the sum is zero if the set $\{i : \gamma(q_i) = \gamma_{\min},\ \mathcal{L}(q_i) < 0\}$ is empty. Now it is easy to see that this condition is always satisfied under the assumption of the corollary. Hence Theorem 1.1 implies the existence of at least one solution. □

Proof. of Corollary 1.6 Observe that in the case $\varrho_* = 8\pi(2 + \gamma_1 + \gamma_2)$ we have $A_2^2 = (q_1, q_2)$. On the left-hand side we have 0 if $\mathcal{L}(A_2^2) > 0$ (that is, there is no critical point at infinity) or -1 if $\mathcal{L}(A_2^2) < 0$ (in this case, $(A_2^2)_\infty$ is the only critical point at infinity). Moreover, the sum on the right-hand side of Theorem 1.1 will be on (q_1, m) with $m = 1, \dots, [\gamma_2] + 2$ (in this case we have $\sigma = 1$), (q_2, m) with $m = 1, \dots, [\gamma_1] + 2$ (in this case we have $\sigma = 1$) and over $m = 1, \dots, [\gamma_1 + \gamma_2] + 2$ (in this case $\sigma = 0$). Thus, the right-hand side will be

$$1 + \sum_{m=1}^{[\gamma_1+g_2]+2} \binom{g+N+m-2}{m} - \sum_{m=1}^{[\gamma_2]+2} \binom{g+N+m-3}{m-1} - \sum_{m=1}^{[\gamma_1]+2} \binom{g+N+m-3}{m-1},$$

which is equal to

$$\binom{g+N+[\gamma_1+\gamma_2]+1}{[\gamma_1+\gamma_2]+2} - \binom{g+N+[\gamma_2]}{[\gamma_2]+1} - \binom{g+N+[\gamma_1]}{[\gamma_1]+1}.$$

Under the assumption of the corollary, the assumption of Theorem 1.1 is satisfied. □

A Appendix

In this appendix, the concentration points are assumed to be in a compact set of Ω and the concentration speeds are of the same order and large enough. Furthermore, for the sake of simplicity, $O(1/\lambda^\alpha)$ designs some quantities like $O(\sum 1/\lambda_i^\alpha)$. Moreover, we observe that for $i \geq \sigma + 1$ we have $\gamma_i = 0$, $\psi_i = 0$ and $\overline{\psi_i} = 0$, hence $\widetilde{P\delta}_{a_i,\lambda_i} = P\delta_{a_i,\lambda_i}$.

Lemma A.1. *We have the following results:*

(i) *On Ω there holds*

$$\widetilde{P\delta}_{a_i,\lambda_i} = \ln\Big(\frac{\lambda_i^4}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^2}\Big) - 8\pi(1+\gamma_i)H(x,a_i) + \overline{\psi}_i + O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \leq i \leq m,$$

$$\lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} = \frac{4}{1+\lambda_i^2|x-a_i|^{2+2\gamma_i}} + O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \leq i \leq m,$$

$$\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} = \frac{4\lambda_i(x-a_i)}{1+\lambda_i^2|x-a_i|^2} - \frac{8\pi}{\lambda_i}\frac{\partial H(a_i,x)}{\partial a_i} + O\Big(\frac{1}{\lambda^3}\Big), \quad \sigma+1 \leq i \leq m.$$

(ii) *Let $0 < \eta < d(a_i, \partial\Omega)$ be a fixed small constant. On $\Omega \setminus B(a_i,\eta)$ there holds*

$$\widetilde{P\delta}_{a_i,\lambda_i} = 8\pi(1+\gamma_i)G(a_i,x) + O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \leq i \leq m,$$

$$\lambda\frac{\partial \widetilde{P\delta}_{a,\lambda}}{\partial \lambda} = O\Big(\frac{1}{\lambda^2}\Big) \quad \text{and} \quad \lambda\frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \leq i \leq m,$$

$$\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} = \frac{8\pi}{\lambda_i}\frac{\partial G(a_i,x)}{\partial a_i} + O\Big(\frac{1}{\lambda^3}\Big), \quad \sigma+1 \leq i \leq m.$$

(iii) *For $i \leq \sigma$ we have*

$$\psi_i = \overline{\psi}_i + O\Big(\frac{1}{\lambda_i^2}\Big), \quad \lambda_i\frac{\partial \psi_i}{\partial \lambda_i} = \lambda_i\frac{\partial \overline{\psi}_i}{\partial \lambda_i} + O\Big(\frac{1}{\lambda_i^2}\Big), \quad \frac{1}{\lambda_i}\frac{\partial \psi_k}{\partial a_i} = \frac{1}{\lambda_i}\frac{\partial \overline{\psi}_k}{\partial a_i} + O\Big(\frac{1}{\lambda^2}\Big) \quad \text{on } \Omega$$

and

$$\|\psi_i\| + |\psi_i|_\infty \leq \frac{c}{\lambda_i^{1/(1+\gamma_i)}}, \quad \lambda_i\Big|\frac{\partial \psi_i}{\partial \lambda_i}\Big|_\infty \leq \frac{c}{\lambda_i^{1/(1+\gamma_i)}}.$$

Proof. The first and the third assertions follow from the maximum principle. However, the second one follows from the first one. □

Lemma A.2. *For $1 \leq i \leq m$ (for $i \geq \sigma + 1$ we take $\gamma_i = 0$ and $\psi_i = 0$) we have*

$$\int_\Omega |\nabla \widetilde{P\delta}_{a_i,\lambda_i}|^2\,dx = 32\pi(1+\gamma_i)\ln\lambda_i - 16\pi(1+\gamma_i) - 64\pi^2(1+\gamma_i)^2 H(a_i,a_i) + \|\psi_i\|^2 + O\Big(\frac{1}{\lambda^{\min(2,\frac{4}{1+\gamma})}}\Big).$$

For $i \leq \sigma$ we have

$$\|\psi_i\|^2 = \frac{16\pi}{3}\frac{(1+\gamma_i)^3(2+\gamma_i)}{\gamma_i^2}I_3(\gamma_i)\frac{|\nabla\mathcal{F}_i(a_i)|^2}{\mathcal{F}_i(a_i)^2}\frac{1}{\lambda_i^{2/(1+\gamma_i)}} + O\Big(\frac{1}{\lambda_i^{2+1/(1+\gamma_i)}}\Big),$$

where $I_3(\gamma)$ is defined in (3.1). *For $i \neq j$ and $|a_i - a_j| \geq c$ let $\gamma := \min(\gamma_i, \gamma_j)$. There holds*

$$\langle \widetilde{P\delta}_{a_i,\lambda_i}, \widetilde{P\delta}_{a_j,\lambda_j}\rangle = 64\pi^2(1+\gamma_i)(1+\gamma_j)G(a_i,a_j) + O\Big(\frac{1}{\lambda^{\min(2,\frac{4}{1+\gamma})}}\Big).$$

Proof. For $i \geq \sigma + 1$ the result follows from [1, Lemma A3]. For $i \leq \sigma$ we have

$$\|\widetilde{P\delta}_{a_i,\lambda_i}\|^2 = \|P\delta_{a_i,\lambda_i}\|^2 + \|\psi_i\|^2 + 2\langle P\delta_{a_i,\lambda_i}, \psi_i\rangle.$$

From Lemma A.1 (iii) and using the oddness of $\overline{\psi_i}$, we get

$$\langle P\delta_{a_i,\lambda_i}, \psi_i\rangle = \int_\Omega |x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}}\Big(\overline{\psi_i} + O\Big(\frac{1}{\lambda_i^2}\Big)\Big) = \int_{\Omega\setminus B(a_i,\eta)} |x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}}\overline{\psi_i}\,dx + O\Big(\frac{1}{\lambda_i^2}\Big) = O\Big(\frac{1}{\lambda_i^2}\Big).$$

From Lemma A.1, we have

$$\begin{aligned}\|P\delta_{a,\lambda}\|^2 &= \int_\Omega |x-a|^{2\gamma} e^{\delta_{a,\lambda}}\Big(4\ln\lambda - 2\ln(1+\lambda^2|x-a|^{2+2\gamma}) - 8\pi(1+\gamma)H(a,x) + O\Big(\frac{1}{\lambda^2}\Big)\Big)\\ &= \int_{\mathbb{R}^2} |x-a|^{2\gamma} e^{\delta_{a,\lambda}}(4\ln\lambda - 2\ln(1+\lambda^2|x-a|^{2+2\gamma})) + O\Big(\frac{1}{\lambda^2}\Big)\\ &\quad - \int_{\mathbb{R}^2\setminus\Omega} |x-a|^{2\gamma} e^{\delta_{a,\lambda}} \ln\Big(\frac{\lambda^4}{(1+\lambda^2|x-a|^{2+2\gamma})^2}\Big) - 8\pi(1+\gamma)\int_\Omega |x-a|^{2\gamma}e^{\delta_{a,\lambda}}H(a,x)\\ &= 32\pi(1+\gamma)\ln\lambda - 16\pi(1+\gamma) - 64\pi^2(1+\gamma)^2 H(a,a) + O\Big(\frac{1}{\lambda^{\min(2,\frac{4}{1+\gamma})}}\Big).\end{aligned} \tag{A.1}$$

Indeed, the first integral gives $32\pi(1+\gamma)\ln(\lambda) - 16\pi(1+\gamma)$. The second one is $O(\frac{1}{\lambda^2})$ (since $d(a,\partial\Omega) \ge c > 0$). Concerning the last one, letting $B := B(a,d)$, we have

$$\int_{\Omega\setminus B(a,d)} |x-a|^{2\gamma} e^{\delta_{a,\lambda}} H(a,x) = O\Big(\frac{1}{\lambda^2}\Big), \tag{A.2}$$

$$\int_B |x-a|^{2\gamma} e^{\delta_{a,\lambda}} H(a,x) = \int_B |x-a|^{2\gamma} e^{\delta_{a,\lambda}}\Big(H(a,a) + \frac{1}{2}D^2H(a,a)(x-a,x-a) + O(|x-a|^4)\Big) \tag{A.3}$$

(in equation (A.3) the terms ∇H and D^3H do not appear since their contribution is zero). The integral on the right-hand side of (A.3) will be divided into three integrals. The first one is equal to $8\pi(1+\gamma)H(a,a) + O(\frac{1}{\lambda^2})$. The last one is $O(1/\lambda^{\min(2,4/(1+\gamma))})$ (since $\gamma \ne 1$). Concerning the second one, we have

$$\begin{aligned}\int_{B(a,d)} |x-a|^{2\gamma} e^{\delta_{a,\lambda}} D^2H(a,a)(x-a,x-a) &= \sum_{i,j}\frac{\partial^2 H(a,a)}{\partial x_i\partial x_j}\int_{B(a,d)} |x-a|^{2\gamma}\frac{8\lambda^2(x-a)_i(x-a)_j}{(1+\lambda^2|x-a|^2)^2}\\ &= \sum_i \frac{\partial^2 H(a,a)}{\partial x_i^2}\int_{B(a,d)}\frac{8\lambda^2|x-a|^{2\gamma}}{(1+\lambda^2|x-a|^2)^2}(x-a)_i^2.\end{aligned}$$

We notice that the integrals for $i = 1, 2$ have the same value. Thus, it follows that the integral vanishes since $\Delta H(a,\cdot)_{|x=a} = 0$. Hence the first claim is proved.

Concerning the third estimate, assuming that $\gamma_j \le \gamma_i$, we write

$$\langle \widetilde{P\delta}_{a_i,\lambda_i}, \widetilde{P\delta}_{a_j,\lambda_j}\rangle = \langle \widetilde{P\delta}_{a_i,\lambda_i}, P\delta_{a_j,\lambda_j}\rangle + \langle \widetilde{P\delta}_{a_i,\lambda_i}, \psi_j\rangle.$$

Observe that

$$\langle \widetilde{P\delta}_{a_i,\lambda_i}, \psi_j\rangle = \int_{B_i} -\Delta\widetilde{P\delta}_{a_i,\lambda_i} O\Big(\frac{1}{\lambda_j^2}\Big) + \int_{\Omega\setminus B_i} O\Big(\frac{1}{\lambda_i^2}\Big)\psi_j = O\Big(\frac{1}{\lambda^2}\Big).$$

For the other term, using Lemma A.1, we get

$$\langle \widetilde{P\delta}_{a_i,\lambda_i}, P\delta_{a_j,\lambda_j}\rangle = \int_{B_j} |x-a_j|^{2\gamma_j} e^{\delta_{a_j,\lambda_j}}\Big(8\pi(1+\gamma_i)G(a_i,x) + O\Big(\frac{1}{\lambda_i^2}\Big)\Big) + \int_{\Omega\setminus B_j} O\Big(\frac{1}{\lambda_j^2}\Big)\widetilde{P\delta}_{a_i,\lambda_i}.$$

Finally, arguing as in the proof of the last integral in (A.1), we derive the result.

Concerning the second claim, we have

$$\begin{aligned}\|\psi_i\|^2 &= \int_\Omega |x-q_i|^{2\gamma_i} e^{\delta_{q_i,\lambda_i}}\Big(\overline{\psi_i} + \frac{1}{\mathcal{F}^m_{\mathbf{q}_\sigma;i}(q_i)}\nabla\mathcal{F}^m_{\mathbf{q}_\sigma;i}(q_i)(x-q_i)\Big)\Big(\overline{\psi_i} + O\Big(\frac{1}{\lambda_i^2}\Big)\Big)\\ &= \int_{B_i}\cdots + O\Big(\frac{1}{\lambda_i^{2+1/(1+\gamma_i)}}\Big) := A_1 + A_2 + O\Big(\frac{1}{\lambda_i^{2+1/(1+\gamma_i)}}\Big).\end{aligned} \tag{A.4}$$

For the first integral A_1 we have that

$$\int_\Omega |x-q_i|^{2\gamma_i} e^{\delta_{q_i,\lambda_i}} \overline{\psi}_i^2 = 32\frac{(1+\gamma_i)^4}{\gamma_i^2 \mathcal{F}_i(q_i)^2} \int_{B_i} \frac{\lambda_i^2 |x-q_i|^{2\gamma_i}}{(1+\lambda_i^2|x-q_i|^{2+2\gamma_i})^4} \{\nabla \mathcal{F}_i(q_i)(x-q_i)\}^2$$
$$= 16\frac{(1+\gamma_i)^4}{\gamma_i^2 \mathcal{F}_i(q_i)^2} |\nabla \mathcal{F}_i(q_i)|^2 \int_{B_i} \frac{\lambda_i^2 |x-q_i|^{2\gamma_i+2}}{(1+\lambda_i^2|x-q_i|^{2+2\gamma_i})^4}$$
$$= 32\pi\frac{(1+\gamma_i)^4}{\gamma_i^2 \mathcal{F}_i(q_i)^2} |\nabla \mathcal{F}_i(q_i)|^2 \frac{1}{\lambda_i^{2/(1+\gamma_i)}} I_4(\gamma_i) + O\Big(\frac{1}{\lambda_i^6}\Big), \tag{A.5}$$

where $I_4(\gamma)$ is defined in (3.1). Concerning the second integral A_2, we have

$$A_2 = -16\frac{(1+\gamma_i)^3}{\gamma_i \mathcal{F}_i(q_i)^2} \int_{B_i} \frac{\lambda_i^2 |x-q_i|^{2\gamma_i}}{(1+\lambda_i^2|x-q_i|^{2+2\gamma_i})^3} \{\nabla \mathcal{F}_i(q_i)(x-q_i)\}^2$$
$$= -8\frac{(1+\gamma_i)^3}{\gamma_i \mathcal{F}_i(q_i)^2} |\nabla \mathcal{F}_i(q_i)|^2 \int_{B_i} \frac{\lambda_i^2 |x-q_i|^{2\gamma_i+2}}{(1+\lambda_i^2|x-q_i|^{2+2\gamma_i})^3}$$
$$= -8\frac{(1+\gamma_i)^3}{\gamma_i \mathcal{F}_i(q_i)^2} |\nabla \mathcal{F}_i(q_i)|^2 \frac{2\pi}{\lambda_i^{2/(1+\gamma_i)}} I_3(\gamma_i) + O\Big(\frac{1}{\lambda_i^4}\Big). \tag{A.6}$$

Now, (A.4)–(A.6) imply the result. □

Lemma A.3. *There holds*

$$\Big\langle \widetilde{P\delta}_{a_i,\lambda_i}, \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle = 16\pi(1+\gamma_i) + \Big\langle \psi_i, \lambda_i\frac{\partial \psi_i}{\partial \lambda_i}\Big\rangle + O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \le i \le m,$$
$$\Big\langle P\delta_{a_i,\lambda_i}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big\rangle = -64\pi^2 \frac{1}{\lambda_i}\frac{\partial H}{\partial a}(a_i,a_i) + O\Big(\frac{\ln\lambda}{\lambda^3}\Big), \quad \sigma+1 \le i \le m,$$

where $\partial H/\partial a$ denotes the derivative with respect to the first variable. Furthermore, there holds

$$\Big\langle \widetilde{P\delta}_{a_j,\lambda_j}, \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle = O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \le i \ne j \le m,$$
$$\Big\langle \widetilde{P\delta}_{a_j,\lambda_j}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big\rangle = 64\pi^2(1+\gamma_j)\frac{1}{\lambda_i}\frac{\partial G(a_i,a_j)}{\partial a_i} + O\Big(\frac{1}{\lambda^2}\Big), \quad 1 \le j \le m,\ \sigma+1 \le i \ne j.$$

Proof. It is easy to see that

$$\Big\langle \widetilde{P\delta}_{a_i,\lambda_i}, \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle = \Big\langle P\delta_{a_i,\lambda_i}, \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle + \Big\langle P\delta_{a_i,\lambda_i}, \lambda_i \frac{\partial \psi_i}{\partial \lambda_i}\Big\rangle + \Big\langle \psi_i, \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle + \Big\langle \psi_i, \lambda_i \frac{\partial \psi_i}{\partial \lambda_i}\Big\rangle.$$

Since $\overline{\psi}_i$ is an odd function, we get that the second and the third terms on the right-hand side are $O(1/\lambda^2)$. Concerning the first one, using Lemma A.1, we get

$$\Big\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\Big\rangle = \int_\Omega \frac{8(1+\gamma)^2\lambda^2|x-a|^{2\gamma}}{(1+\lambda^2|x-a|^{2+2\gamma})^2}\Big(\frac{4}{1+\lambda^2|x-a|^{2+2\gamma}} + O\Big(\frac{1}{\lambda^2}\Big)\Big) = 16\pi(1+\gamma) + O\Big(\frac{1}{\lambda^2}\Big).$$

The second estimate is extracted from [1, Lemma A.2] since, for $i \ge \sigma+1$, we have $\gamma_i = 0$. Concerning the third and the fourth, let $d > 0$ such that $B_i := B(a_i, d) \subset \Omega$ and $B_i \cap B_j = \emptyset$. We have

$$\Big\langle \widetilde{P\delta}_{a_j,\lambda_j}, \lambda_i \frac{\partial \widetilde{P\delta}_{a_i,\lambda_i}}{\partial \lambda_i}\Big\rangle = \int_{B_j} -\Delta \widetilde{P\delta}_{a_j,\lambda_j} O\Big(\frac{1}{\lambda_i^2}\Big) + \int_{\Omega\setminus B_j} -\Delta \widetilde{P\delta}_{a_j,\lambda_j} O(1) = O\Big(\frac{1}{\lambda^2}\Big),$$
$$\Big\langle \widetilde{P\delta}_{a_j,\lambda_j}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big\rangle = \int_\Omega -\Delta \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} \widetilde{P\delta}_{a_j,\lambda_j}$$
$$= \int_{B_i} \frac{32\lambda_i^3(x-a_i)}{(1+\lambda_i^2|x-a_i|^2)^3} 8\pi(1+\gamma_j) G(a_j,x)\,dx + O\Big(\frac{1}{\lambda^2}\Big).$$

Now, we expand $G(a_j,x)$ around a_i, as above and the result follows. □

Using Lemma A.1, we derive the following result.

Lemma A.4. *Let* $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$. *On* $B(a_i, \eta)$ *there holds*

$$Ke^u = \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_{A_\sigma^m,i}g_i e^{\alpha_i\overline{\psi}_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big(1+O\Big(\frac{1}{\lambda^2}\Big)\Big)$$
$$= \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_{A_\sigma^m,i}g_i e^{\alpha_i\overline{\psi}_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} + O\Big(\frac{1}{\lambda_i^2}Ke^u\Big),$$

where $A_\sigma^m := (a_1, \dots, a_m)$, $\mathcal{F}_{A_\sigma^m,i}$ *is defined by* (1.2), (1.3) *and*

$$g_i(x) := \exp\Big(-8\pi(\alpha_i-1)(1+\gamma_i)H(a_i,x) + 8\pi\sum_{j\neq i}(\alpha_j-1)(1+\gamma_j)G(a_j,x)\Big). \tag{A.7}$$

We introduce the following function:

$$\xi(x) = \frac{1}{(1+\lambda^2|x-a|^{2+2\gamma})^{2(\alpha-1)}}.$$

We remark that

$$|\xi(x)-1| = \Big|\int_0^1 \frac{4(1-\alpha)(1+\gamma)\lambda^2 t^{1+2\gamma}|x-a|^{2+2\gamma}}{(1+\lambda^2 t^{2+2\gamma}|x-a|^{2+2\gamma})^{2\alpha-1}}\,dt\Big| \le c|\alpha-1|\sqrt{\lambda|x-a|^{1+\gamma}}. \tag{A.8}$$

Lemma A.5. *Let* $u := \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i\lambda_i} + w \in V(\mathbf{q}_\sigma, m, \varepsilon)$ *and let* $\gamma_* = \max\{\gamma_i : i \le \sigma\}$. *Then we have*

$$\int_\Omega Ke^u = \sum_{i=1}^m \frac{\pi\lambda_i^{4\alpha_i-2}(\mathcal{F}_{A_\sigma^m,i}g_i)(a_i)}{(2\alpha_i-1)(1+\gamma_i)} + \sum_{i=1}^\sigma \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{\frac{2}{1+\gamma_i}}}\Big(\frac{\pi}{2}I_2(\gamma_i)\Delta\mathcal{F}_{A_\sigma^m,i}(a_i) + \frac{2\pi}{3}\frac{1-\gamma_i^2}{\gamma_i^2}I_3(\gamma_i)\frac{|\nabla\mathcal{F}_{A_\sigma^m,i}(a_i)|^2}{\mathcal{F}_{A_\sigma^m,i}(a_i)}\Big)$$
$$+ O\Big(1+\sum\lambda_i^{4\alpha_i-2}\Big(\min\Big(\frac{1}{\lambda^{3/(1+\gamma_*)}}, \frac{\ln\lambda}{\lambda^2}\Big) + \|w\|^2 + |f(w)| + \sum|\alpha_i-1|^2\Big)\Big).$$

Proof. Let $\overline{u} := u - w = \sum\alpha_i\widetilde{P\delta}_i$. We have

$$\int_\Omega Ke^u = \int_\Omega Ke^{\overline{u}} + \int_\Omega Ke^{\overline{u}}w + \int_\Omega Ke^{\overline{u}}(e^w-1-w).$$

Concerning the last integral, from Lemma A.7 we obtain

$$\int_\Omega Ke^{\overline{u}}|e^w-1-w| \le c\sum\lambda_k^{4\alpha_k-2}\|w\|^2.$$

The estimate of the second integral is given in Lemma A.8. It remains to estimate the first one. Using Lemma A.1, we derive that e^u is bounded outside of the union of the B_i's. Hence the integral in this set is bounded. Now, using Lemma A.4, we have

$$\int_{B_i} Ke^{\overline{u}} = \int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i(x)g_i(x)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}(1+\alpha_i\overline{\psi}_i + \frac{1}{2}\alpha_i^2\overline{\psi}_i^2 + 0(|\overline{\psi}_i|^3)) + O\Big(\frac{1}{\lambda_i^2}\int Ke^{\overline{u}}\Big). \tag{A.9}$$

For $i \ge \sigma+1$ we have $\gamma_i = 0$ and $\overline{\psi}_i = 0$, and therefore this integral is estimated in the estimate of I_3 in [1, Lemma A.8] and we have

$$\int_{B_i} Ke^{\overline{u}} = \int_{B_i}\frac{\lambda_i^{4\alpha_i}\mathcal{F}_i(x)g_i(x)}{(1+\lambda_i^2|x-a_i|^2)^{2\alpha_i}}\Big(1+O\Big(\frac{1}{\lambda_i^2}\Big)\Big)$$
$$= \lambda_i^{4\alpha_i-2}\Big(\frac{\pi(\mathcal{F}_ig_i)(a_i)}{(2\alpha_i-1)} + O\Big(\frac{\ln\lambda_i}{\lambda_i^2} + \frac{|\alpha_i-1|}{\lambda_i^{3/2}}\Big)\Big).$$

To estimate (A.9) for $i \le \sigma$ four integrals have to be computed. For the first one, observe that

$$\int_{B_i} \frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i(x)g_i(x)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} = (\mathcal{F}_i g_i)(a_i)\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}} + O\Big(\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i+4}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\Big)$$
$$+\frac{1}{2}\int_{B_i} D^2(\mathcal{F}_i g_i)(a_i)(x-a_i,x-a_i)\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}$$
$$= \frac{\lambda_i^{4\alpha_i-2}(\mathcal{F}_i g_i)(a_i)\pi}{(2\alpha_i-1)(1+\gamma_i)} + \frac{1}{4}\Delta(\mathcal{F}_i g_i)(a_i)\frac{2\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}(I_2(\gamma_i)+O(|\alpha_i-1|)) + O(1). \quad \text{(A.10)}$$

Concerning the second one, using the facts that $g_i = 1 + O(\sum|\alpha_k - 1|)$ and $\nabla g_i = O(\sum|\alpha_k - 1|)$ and the oddness of $\overline{\psi}_i$, we derive

$$\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i g_i}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\overline{\psi}_i$$
$$= \int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\nabla(\mathcal{F}_i g_i)(a_i)(x-a_i)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\overline{\psi}_i + O\Big(\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i+3}}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}|\overline{\psi}_i|\Big)$$
$$= -2\frac{1+\gamma_i}{\gamma_i}\frac{1}{\mathcal{F}_i(a_i)}\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\{\nabla\mathcal{F}_i(a_i)(x-a_i)\}^2}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}} + O\Big(\sum|\alpha_j-1|\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}} + \frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{4/(1+\gamma_i)}}\Big). \quad \text{(A.11)}$$

The previous integral can be estimated as

$$\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\{\nabla\mathcal{F}_i(a_i)(x-a_i)\}^2}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i+1}} = |\nabla\mathcal{F}_i(a_i)|^2\frac{\pi\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}(I_3(\gamma_i)+O(|\alpha_i-1|)+O\Big(\frac{1}{\lambda^2}\Big).$$

For the third one of (A.9), expanding $\mathcal{F}_i g_i$ around a_i and using the definition of $\overline{\psi}_i$ (see (2.1)), we obtain

$$\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i(x)g_i(x)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\overline{\psi}_i^2$$
$$= 4\pi\frac{(1+\gamma_i)^2}{\gamma_i^2}\frac{(\mathcal{F}_i g_i)(a_i)}{\mathcal{F}_i(a_i)^2}|\nabla\mathcal{F}_i(a_i)|^2\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}(I_4(\gamma_i)+O(|\alpha_i-1|)) + O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{4/(1+\gamma_i)}}+\frac{1}{\lambda^4}\Big). \quad \text{(A.12)}$$

Finally, the fourth integral of (A.9) can be estimated as

$$\int_{B_i}\frac{\lambda_i^{4\alpha_i}|x-a_i|^{2\gamma_i}\mathcal{F}_i(x)g_i(x)}{(1+\lambda_i^2|x-a_i|^{2+2\gamma_i})^{2\alpha_i}}\|\overline{\psi}_i\|_\infty^3 = O\Big(\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{3/(1+\gamma_i)}}\Big).$$

Now, we sum the previous estimates and use the recurrence formula of $I_n(\gamma)$ (see (3.2)). □

Corollary A.6. *Let $u = \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} + w \in V(\mathbf{q}_\sigma, m, \varepsilon)$ with $w \in E^m_{a,\Lambda}$. Then we have*

$$8\pi\sum_{i=1}^m(1+\gamma_i)\tau_i = \frac{\varrho_*}{\int_\Omega Ke^u}\Big\{\sum_{i=1}^\sigma\frac{\lambda_i^{4\alpha_i-2}}{\lambda_i^{2/(1+\gamma_i)}}\Big(\frac{\pi}{2}I_2(\gamma_i)\Delta\mathcal{F}_{A^m_\sigma,i}(a_i) + \frac{2\pi}{3}\frac{1-\gamma_i^2}{\gamma_i^2}I_3(\gamma_i)\frac{|\nabla\mathcal{F}_{A^m_\sigma,i}(a_i)|^2}{\mathcal{F}_{A^m_\sigma,i}(a_i)}\Big)$$
$$+ O\Big(1+\sum\lambda_i^{4\alpha_i-2}\Big(\frac{1}{\lambda^{3/(1+\gamma_*)}} + \|w\|^2 + |f(w)| + \sum|\alpha_i-1|^2\Big)\Big)\Big\}.$$

Proof. Recall that

$$\tau_i := 1 - \frac{\varrho_*}{8\pi(1+\gamma_i)}\frac{\pi\Gamma_i}{\int_\Omega Ke^u}, \quad \text{where } \Gamma_k := \frac{1}{(2\alpha_k-1)(1+\gamma_k)}\lambda_k^{4\alpha_k-2}(\mathcal{F}_{A^m_\sigma,k}g_k)(a_k).$$

Thus we get that

$$8\pi \sum (1+\gamma_i)\tau_i = \frac{\varrho_*}{\int_\Omega Ke^u}\Big(\int_\Omega Ke^u - \sum \pi\Gamma_i\Big).$$

Using Lemma A.5, we derive the claimed estimate. □

Lemma A.7. *Let $u = \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$ and $w \in E^m_{a,\Lambda}$. Then there holds*

$$\begin{aligned}
&\Big(\int_{B_i} |x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} |w|^q\Big)^{1/q} \le C_0\sqrt{q}\|w\|,\\
&\int_{B_i} |x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}} (1+\xi_i)|w| \le c\|w\|,\\
&\int_{B_i} |x-a_i|^{2\gamma_i+2} e^{\delta_{a_i,\lambda_i}} |w| \le c\frac{\|w\|}{\lambda^{2/(1+\gamma_i)}},\\
&\int_{B_i} e^{\delta_{a_i,\lambda_i}} |x-a_i|^2|w| \le c\frac{\ln\lambda}{\lambda^2}\|w\| && \text{for } i\ge\sigma+1,\\
&\int_\Omega Ke^u\Big|e^w - \sum_{k=0}^p \frac{w^k}{k!}\Big| \le c\|w\|^{p+1}\sum \lambda_k^{4\alpha_k-2} && \text{for all } p\in\mathbb{N},\\
&\int_{B_i} |\xi_i-1||x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}}|w|^\beta \le c|\alpha_i-1|\|w\|^\beta && \text{for } \beta = 1,2.
\end{aligned}$$

Proof. The fourth estimate is extracted from [1, Lemma A.5]. Now, arguing as in the proof of [1, Lemma A.4], we prove the first estimate. The other ones follow from the first by using the Hölder inequality. □

Lemma A.8. *Let $u = \sum_{i=1}^m \alpha_i \widetilde{P\delta}_{a_i,\lambda_i} \in V(\mathbf{q}_\sigma, m, \varepsilon)$ and $w \in E^m_{a,\Lambda}$. Then there holds*

$$\int_\Omega Ke^u w = O\Big(\|w\|\Big(1+\sum_{i=\sigma+1}^m \lambda_i^{4\alpha_i-2}\frac{|\nabla \mathcal{F}_{A^m_\sigma,i}(a_i)|}{\lambda_i} + \Big(\sum|\alpha_j-1| + \frac{1}{\lambda^{2/(1+\gamma_*)}}\Big)\sum_{i=1}^m \lambda_i^{4\alpha_i-2}\Big)\Big).$$

Proof. We recall that we have

$$\langle \widetilde{P\delta}_{a_i,\lambda_i}, w\rangle = 0 = \int_\Omega |x-a_i|^{2\gamma_i} e^{\delta_{a_i,\lambda_i}}\Big(1+\overline{\psi}_i + \frac{1}{\mathcal{F}_i(a_i)}\nabla\mathcal{F}_i(a_i)(x-a_i)\Big)w \tag{A.13}$$

Now, using Lemma A.4, we get

$$\begin{aligned}
\int_\Omega Ke^u w &= \sum_{i=1}^m \frac{\lambda_i^{4\alpha_i-2}}{8(1+\gamma_i)^2}\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\xi_i\mathcal{F}_i g_i(1+\alpha_i\overline{\psi}_i + O(|\psi_i|^2_\infty))\Big(1+O\Big(\frac{1}{\lambda_i^2}\Big)\Big)w + O(\|w\|)\\
&= \sum_{i=1}^m \frac{\lambda_i^{4\alpha_i-2}}{8(1+\gamma_i)^2}\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\xi_i\mathcal{F}_i g_i(1+\alpha_i\overline{\psi}_i)w + \|w\|O\Big(1+\lambda_i^{4\alpha_i-2}\Big(|\psi_i|^2_\infty + \frac{1}{\lambda_i^2}\Big)\Big).
\end{aligned}$$

Using (A.8) and the fact that $g_i = 1 + O(\sum|\alpha_j-1|)$, we get

$$\begin{aligned}
&\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\xi_i\mathcal{F}_i g_i(1+\alpha_i\overline{\psi}_i)w\\
&\quad= \int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}\mathcal{F}_i(1+\alpha_i\overline{\psi}_i)w + O\Big(\sum|\alpha_j-1|\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}|w|\Big)\\
&\qquad + O\Big(|\alpha_i-1|\int_{B_i}|x-a_i|^{2\gamma_i}e^{\delta_i}|w|\sqrt{\lambda_i|x-a_i|^{1+\gamma_i}}\Big).
\end{aligned}$$

To estimate the first integral on the right-hand side, we expand the function $\mathcal{F}_i$ around a_i. For $i \geq \sigma + 1$, as in [1], we get

$$\int_{B_i} e^{\delta_i} \mathcal{F}_i w = O\Big(\|w\|\Big(\frac{|\nabla \mathcal{F}_i(a_i)|}{\lambda_i} + \frac{|\ln \lambda_i|^{3/2}}{\lambda_i^2}\Big)\Big).$$

For $i \leq \sigma$, that is, $\gamma_i > 0$, we obtain

$$\begin{aligned}
&\int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_i} \mathcal{F}_i (1 + \alpha_i \overline{\psi}_i) w \\
&\quad = \mathcal{F}_i(a_i) \int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_i} (1 + \alpha_i \overline{\psi}_i) w + \int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_i} (1 + \alpha_i \overline{\psi}_i) \nabla \mathcal{F}_i(a_i)(x - a_i) w \\
&\qquad + O\Big(\int_{B_i} |x - a_i|^{2+2\gamma_i} e^{\delta_i} |w|\Big) \\
&\quad = \mathcal{F}_i(a_i) \int_{B_i} |x - a_i|^{2\gamma_i} e^{\delta_i} \Big(1 + \alpha_i \overline{\psi}_i + \frac{1}{\mathcal{F}_i(a_i)} \nabla \mathcal{F}_i(a_i)(x - a_i)\Big) w \\
&\qquad + O\Big(\int_{B_i} |x - a_i|^{2+2\gamma_i} e^{\delta_i} |w| + |\overline{\psi}_i|_\infty \int_{B_i} |x - a_i|^{1+2\gamma_i} e^{\delta_i} |w|\Big) \\
&\quad = O\Big(\frac{1}{\lambda_i^2} \|w\| + \frac{1}{\lambda_i^{2/(1+\gamma_i)}} \|w\|\Big)
\end{aligned}$$

by using (A.13) and Lemma A.7. □

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