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Research Article

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Sharp Constants and Optimizers for a Class of Caffarelli–Kohn–Nirenberg Inequalities

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Abstract: In this paper, we use a suitable transform of quasi-conformal mapping type to investigate the sharp constants and optimizers for the following Caffarelli–Kohn–Nirenberg inequalities for a large class of parameters $(r, p, q, s, \mu, \sigma)$ and $0 \le a \le 1$:

$$\left(\int |u|^r \frac{dx}{|x|^s} \right)^{\frac{1}{r}} \le C \left(\int |\nabla u|^p \frac{dx}{|x|^\mu} \right)^{\frac{a}{p}} \left(\int |u|^q \frac{dx}{|x|^\sigma} \right)^{\frac{1-a}{q}}.$$

We compute the best constants and the explicit forms of the extremal functions in numerous cases. When $0 < a < 1$, we can deduce the existence and symmetry of optimizers for a wide range of parameters. Moreover, in the particular cases $r = p\frac{q-1}{p-1}$ and $q = p\frac{r-1}{p-1}$, the forms of maximizers will also be provided in the spirit of Del Pino and Dolbeault [14, 15]. In the case $a = 1$, that is, the Caffarelli–Kohn–Nirenberg inequality without the interpolation term, we will provide the exact maximizers for all the range of $\mu \ge 0$. The Caffarelli–Kohn–Nirenberg inequalities with arbitrary norms on Euclidean spaces will also be considered in the spirit of Cordero-Erausquin, Nazaret and Villani [13]. Due to the absence of the classical Polyá–Szegö inequality in the weighted case, we establish a symmetrization inequality with power weights which is of independent interest.

Keywords: Caffarelli–Kohn–Nirenberg Inequality, Symmetry, Optimizers, Sharp Constants

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Dedicated to Richard Wheeden on the occasion of his retirement with appreciation and admiration

1 Introduction

Geometric and functional inequalities have a wide range of applications and play a crucial role in geometric analysis, partial differential equations and other branches of modern mathematics. In many situations, the validity of the inequality and some explicit bounds for its best constant are enough to run the process. However, there are numerous circumstances where we need to know the exact sharp constant and information on extremal functions.

Among those inequalities, the Caffarelli–Kohn–Nirenberg (CKN) inequality is one of the most important and interesting ones. It is worth noting that many well-known and important inequalities such as Gagliardo–Nirenberg inequalities, Sobolev inequalities, Hardy–Sobolev inequalities, Nash inequalities, etc. are just special cases of CKN inequalities.

The CKN inequalities were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in their celebrated work [6].

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Theorem A. *There exists a positive constant $C = C(N, r, p, q, \gamma, \alpha, \beta)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\||x|^\gamma u\|_r \le C\||x|^\alpha |\nabla u|\|_p^a \||x|^\beta u\|_q^{1-a}, \tag{1.1}$$

where

$$\begin{gathered} p, q \ge 1, \quad r > 0, \quad 0 \le a \le 1, \\ \frac{1}{p} + \frac{\alpha}{N}, \frac{1}{q} + \frac{\beta}{N}, \frac{1}{r} + \frac{\gamma}{N} > 0, \quad \gamma = a\sigma + (1-a)\beta, \\ \frac{1}{r} + \frac{\gamma}{N} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{N}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{N}\Big) \end{gathered}$$

and

$$\begin{gathered} 0 \le \alpha - \sigma \quad \text{if } a > 0, \\ \alpha - \sigma \le 1 \quad \text{if } a > 0 \text{ and } \frac{1}{p} + \frac{\alpha - 1}{N} = \frac{1}{r} + \frac{\gamma}{N}. \end{gathered}$$

If we perform, as in [37], the change of exponents

$$\alpha = -\frac{\mu}{p}, \quad \beta = -\frac{\theta}{q}, \quad \gamma = -\frac{s}{r},$$

then (1.1) can be written in the following equivalent form:

$$\Big(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\Big)^{\frac{1}{r}} \le C\Big(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\Big)^{\frac{a}{p}} \Big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\Big)^{\frac{1-a}{q}}, \tag{1.2}$$

where

$$a = \frac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r}.$$

In this paper, we will restrict our consideration to the case $1 < p < N$.

When $s = \mu = \theta = 0$ and $a = 1$, we recover the well-known Sobolev inequality

$$\Big(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\Big)^{\frac{1}{p^*}} \le S(N, p)\Big(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\Big)^{\frac{1}{p}},$$

where $p^* = \frac{Np}{N-p}$. When $p > 1$ the best constant $S(N, p)$ was found in the works of Aubin [3] and Talenti [34], using rather classical tools such as Schwartz rearrangement. The case $p = 2$ was explored more by Beckner in [4], due to its conformal invariance. For $p = 1$, it has been known that the Sobolev inequality is equivalent to the classical Euclidean isoperimetric inequality.

When $a = 1$, $\mu = 0$, $0 \le s \le p < N$ and $r = p^*(s) = \frac{N-s}{N-p}p$, the CKN inequality becomes the Hardy–Sobolev (HS) inequality:

$$\Big(\int_{\mathbb{R}^N} |u|^{p^*(s)} \frac{dx}{|x|^s}\Big)^{\frac{1}{p^*(s)}} \le \mathrm{HS}(N, p, s)\Big(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\Big)^{\frac{1}{p}}. \tag{1.3}$$

In this situation, Lieb in [26] applied the symmetrization argument to study (1.3) in the case $p = 2$ and gave the best constants and explicit maximizers. The study of the best constant $\mathrm{HS}(N, p, s)$ and extremal functions for inequalities (1.3) in the general range goes back to Ghoussoub and Yuan [25] and maybe even earlier (see the references in [25]). The maximizers for the HS inequality when $0 \le s < p < N$ are the functions

$$u_{c,\lambda}(x) = c(\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{N-p}{p-s}} \quad \text{for some } c \ne 0,\ \lambda > 0.$$

Actually, $u_{c,\lambda}$ (after rescaling) is the only positive radial solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{u^{p^*(s)-1}}{|x|^s} \quad \text{on } \mathbb{R}^N.$$

When $a = 1$ and $0 < \mu, s < N$, the CKN inequality does not contain the interpolation term. There are great efforts to investigate the sharp constants, existence/nonexistence and symmetry/symmetry breaking of maximizers in this situation, especially when $p = 2$. See [5, 9, 12, 21, 35], among others. For instance, Chou and Chu [12] considered the case $p = 2$ and $\frac{\mu}{2} \leq \frac{s}{r} \leq \frac{\mu}{2} + 1$, and provided the best constants and explicit optimizers. In [35], Wang and Willem studied the compactness of all maximizing sequences up to dilations in the spirit of Lions [28–31]. In [9], Catrina and Wang investigated the class of $p = 2$ and $\mu < 0$, and established the attainability/unattainability and symmetry breaking of extremal functions. In [7], Caldiroli and Musina studied the symmetry breaking of extremals for CKN inequalities in a non-Hilbertian setting. In a recent paper, [16], Dolbeault, Esteban and Loss studied the characterization of the optimal symmetry breaking region in HS inequalities with $p = 2$. As a consequence, maximizers and best constants are calculated in the symmetry region. Their result solves a longstanding conjecture on the optimal symmetry range.

In the case $0 < a < 1$, the CKN inequality includes the interpolation term. This situation is much harder to study. When there is no singular term, i.e., $s = \theta = \mu = 0$, the nonweighted CKN inequality, namely, the Gagliardo–Nirenberg inequality, has been studied at length by many authors. Especially, for very particular classes, the best constant and the maximizers for the Gagliardo–Nirenberg inequality are provided explicitly by Del Pino and Dolbeault in [14, 15]. Indeed, in the special class $r = p\frac{q-1}{p-1}$, Del Pino and Dolbeault proved that the maximizers for the Gagliardo–Nirenberg inequality have the form $A(1 + B|x - \overline{x}|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}$, while in the case $q = p\frac{r-1}{p-1}$, the maximizers are $A(1 - B|x - \overline{x}|^{\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}}$ for some $A \in \mathbb{R}$, $B > 0$ and $\overline{x} \in \mathbb{R}^N$. See also [1, 2], where Agueh gives a proof by studying a p-Laplacian type equation by transforming the unknown of the equation via some change of functions. We also cite [13] where Cordero-Erausquin, Nazaret, and Villani set up a beautiful link between optimal transportation and certain Sobolev inequalities and Gagliardo–Nirenberg inequalities.

However, as far as we know, there are only a few papers concerning the full weighted CKN inequalities (i.e., $0 < a < 1$ and at least one of s, μ, θ is nonzero). Compared with the special cases of Gagliardo–Nirenberg inequalities without the interpolation term (i.e., $a = 1$), dealing with such CKN inequalities is considerably more difficult. For instance, the Fourier analysis techniques cannot be applied in this setting. Moreover, the classical Schwarz rearrangement, which is based on an isoperimetric inequality, is unavailable due to the presence of singular terms (i.e., the weights $\frac{1}{|x|^s}$, $\frac{1}{|x|^\theta}$ and $\frac{1}{|x|^\mu}$). It is worth noting that symmetrization has been a very useful and efficient (and almost inevitable) method when dealing with the sharp geometric inequalities. Hence, in general, we are not able to reduce our problem on CKN inequalities to a radial setting. Actually, the problem of symmetry and symmetry breaking of optimizers for CKN inequalities has been investigated by many researchers, see, for instance, [7, 10, 11, 16].

Concerning inequality (1.2), for the special class

$$q = \frac{p(r-1)}{p-1}, \quad 1 < p < r, \quad N - \theta < \Big(1 + \frac{\mu}{p} - \frac{\theta}{p}\Big)\frac{(r-1)p}{r-p}, \quad s = \frac{\mu}{p} + 1 + \frac{p-1}{p}\theta,$$

Xia could guess and then verify in [36] that $(\lambda + |x|^{1+\frac{\mu}{p}-\frac{\theta}{p}})^{-\frac{p-1}{r-p}}$, $\lambda > 0$, are extremal functions. But he could not prove that these are all possible optimizers. Moreover, this case does not cover the interesting situations in [14, 15].

In [8], Catrina and Costa studied best constants and explicit optimizers for the CKN inequality when $p = q = r = 2$, $\mu = 2a$, $\theta = 2b$ and $s = a + b + 1$. They were able to show that when $a < b + 1$, $b \leq \frac{N-2}{2}$, or $a > b + 1$, $b \geq \frac{N-2}{2}$, the optimizers are of the form

$$D \exp\Big(\frac{t|x|^{b+1-a}}{b+1-a}\Big),$$

while if $a > b + 1$, $b \leq \frac{N-2}{2}$, or $a < b + 1$, $b \geq \frac{N-2}{2}$, the extremal functions are

$$D|x|^{2(b+1)-N} \exp\Big(\frac{t|x|^{b+1-a}}{b+1-a}\Big).$$

We note that the case $a = b + 1$ was treated in [10]. More precisely, in this case, the best constant is $\sqrt{\frac{2}{|N-s(b+1)|}}$, and is not achieved.

In [37], Zhong and Zou studied the existence of extremal functions for the CKN inequality under a wider region, and used it to set up the continuity and compactness of embeddings on weighted Sobolev spaces. However, there is no information about the maximizers provided there.

In a very recent paper, [17], Dolbeault, Muratori and Nazaret studied the CKN inequality in the regime $s = \theta > 0$, $p = 2$ and $r = 2(q-1) > 2$. In this case, they were able to show that for $s = \theta > 0$ small enough, the CKN inequality can be achieved by optimizers of the form $(1 + |x|^{2-s})^{-\frac{1}{q-2}}$, up to multiplications by a constant and scalings.

In [18, 19], when dealing with the sharp singular Trudinger–Moser inequalities, which can be considered as limiting Sobolev embeddings, where again the classical Schwarz rearrangement could not be used, the authors in collaboration with Dong proposed a new approach. Namely, we used suitable quasi-conformal mappings to convert those sharp singular weighted Trudinger–Moser inequalities to the nonweighted ones. (We will not discuss in detail weighted Trudinger–Moser inequalities here, but we refer the interested reader to, e.g., [32] for more references on weighted Trudinger–Moser inequalities for singular weights.) Moreover, in [19], we established the existence of the optimizers for weighted Trudinger–Moser inequalities for all functions which are not necessarily spherically symmetric by using this type of quasi-conformal mapping to reduce to the case of inequalities for spherically symmetric ones, and in [18], we treated successfully CKN inequalities in the special case $p = N$, $\mu = 0$, $0 \le s = \theta < N$, $1 \le q < r$ and $a = 1 - \frac{q}{r}$, using this new transform. Especially, for a 1-parameter family of inequalities, the best constants and the maximizers for the CKN inequality are calculated explicitly there.

Motivated by the results in [18, 19] and [1, 2, 13, 15, 17], in this paper, we will use convenient vector fields to investigate the CKN inequality in some special regions. Our main idea is that under our suitable transforms, CKN inequalities can be converted to simpler versions, namely, the Hardy–Sobolev inequalities and the Gagliardo–Nirenberg inequalities. Since the sharp constants and optimizers of those inequalities are easier to study, and are known in some particular classes, we can get the best constants and maximizers for CKN inequalities in the corresponding regions.

More precisely, we study the extremal functions for the CKN inequality involving the interpolation term (i.e., $0 < a < 1$). We will consider the following class:

$$\begin{cases} 1 < p < p+\mu < N, \quad \theta \le \dfrac{N\mu}{N-p} \le s < N, \\ 1 \le q < r < \dfrac{Np}{N-p}, \quad a = \dfrac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r}. \end{cases} \tag{C1}$$

We denote by $D^{p,q}_{\mu,\theta}(\mathbb{R}^N)$ the completion of the space of smooth compactly supported functions with the norm

$$\Big(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\Big)^{\frac{1}{q}},$$

and set

$$\mathrm{CKN}(N,\mu,\theta,s,p,q,r) = \sup_{u \in D^{p,q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\big)^{\frac{a}{p}} \big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}}. \tag{1.4}$$

Then we have the following result.

Theorem 1.1. *Assume that* (C1) *holds. Then* $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ *can be achieved. Moreover, all the extremal functions of* $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ *are radially symmetric.*

We will also give the explicit forms for all maximizers and the exact best constant for $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ in the following special cases.

Theorem 1.2. *Assume that* (C1) *holds with* $\theta = s = \frac{N\mu}{N-p}$. *If* $p < r = p\frac{q-1}{p-1} < \frac{Np}{N-p}$, *then, for* $\delta = Np - q(N-p)$, *we have*

$$\begin{aligned}&\mathrm{CKN}(N,\mu,\theta,s,p,q,r)\\&\quad = \Big(\frac{N-p}{N-p-\mu}\Big)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\Big(\frac{q-p}{p\sqrt{\pi}}\Big)^{a}\Big(\frac{pq}{N(q-p)}\Big)^{\frac{a}{p}}\Big(\frac{\delta}{pq}\Big)^{\frac{1}{r}}\Big(\frac{\Gamma(q\frac{p-1}{q-p})\Gamma(\frac{N}{2}+1)}{\Gamma(\frac{p-1}{p}\frac{\delta}{q-p})\Gamma(N\frac{p-1}{p}+1)}\Big)^{\frac{a}{N}},\end{aligned}$$

and all the maximizers have the form

$$V_0(x) = A(1+B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})^{-\frac{p-1}{q-p}} \quad \textit{for some } A\in\mathbb{R},\ B>0.$$

Theorem 1.3. *Assume that* (C1) *holds with* $\theta = s = \frac{N\mu}{N-p}$. *If* $1 < q = p\frac{r-1}{p-1} < p$, *then, for* $\delta = Np - r(N-p)$, *we have*

$$\begin{aligned}&\mathrm{CKN}(N,\mu,\theta,s,p,q,r)\\&\quad = \Big(\frac{N-p}{N-p-\mu}\Big)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\Big(\frac{p-r}{p\sqrt{\pi}}\Big)^{a}\Big(\frac{pr}{N(p-r)}\Big)^{\frac{a}{p}}\Big(\frac{pr}{\delta}\Big)^{\frac{1-a}{q}}\Big(\frac{\Gamma(\frac{p-1}{p}\frac{\delta}{p-r}+1)\Gamma(\frac{N}{2}+1)}{\Gamma(r\frac{p-1}{p-r}+1)\Gamma(N\frac{p-1}{p}+1)}\Big)^{\frac{a}{N}}.\end{aligned}$$

If $r > 2 - \frac{1}{p}$, *then all the maximizers have the form*

$$V_0(x) = A(1-B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}} \quad \textit{for some } A\in\mathbb{R},\ B>0.$$

We also provide the explicit optimizers for CKN inequalities in the following regime:

$$\begin{cases} p = 2 < 2+\mu < N, & 2 < r = 2(q-1) < \dfrac{2N}{N-2},\\[2mm] \mu+2 > s = \theta > \dfrac{N\mu}{N-2}, & a = \dfrac{(N-s)[q-2]}{[(N-s)2-(N-\mu-2)q](q-1)}.\end{cases} \tag{C2}$$

Again, we define

$$\mathrm{CKN}(N,\mu,s,q) = \sup_{u\in D^{2,q}_{\mu,s}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N}|u|^{2(q-1)}\frac{dx}{|x|^s}\big)^{\frac{1}{2(q-1)}}}{\big(\int_{\mathbb{R}^N}|\nabla u|^2\frac{dx}{|x|^\mu}\big)^{\frac{a}{2}}\big(\int_{\mathbb{R}^N}|u|^q\frac{dx}{|x|^s}\big)^{\frac{1-a}{q}}}. \tag{1.5}$$

We will prove the following theorem.

Theorem 1.4. *There exists* $s^* = s^*(N,q,\mu) \in (0, N-(q-1)(N-2-\mu))$ *such that* $\mathrm{CKN}(N,\mu,s,q)$ *is attained for all* $\frac{N\mu}{N-2} < s < s^*$ *by optimizers of the form*

$$V_0(x) = A(1+B|x|^{\mu+2-s})^{-\frac{1}{q-2}} \quad \textit{for some } A\in\mathbb{R},\ B>0.$$

2 Preliminaries and Some Important Lemmata

To carry through our argument, it is necessary to show that our quasi-conformal changes of variable can indeed be used to reduce CKN inequalities with more complicated weights to simpler ones and vice versa. This interchange is nicely done through the following lemmata which are of independent interests and can be found useful in other settings as well.

Lemma 2.1. *We have that* $|x\cdot\nabla u(x)| = |x||\nabla u(x)|$ *for a.e.* $x\in\mathbb{R}^N$ *if and only if* u *is radially symmetric, that is,* $u(x) = u(y)$ *when* $|x| = |y|$.

Proof. If u is radial, then we have

$$\frac{\partial u}{\partial x_j}(x) = u'(|x|)\frac{x_j}{|x|}.$$

Hence, $|\nabla u(x)| = |u'(|x|)|$. Also,

$$\left|\sum_{j=1}^{N} x_j \frac{\partial u}{\partial x_j}(x)\right| = |u'(|x|)| \left|\sum_{j=1}^{N} x_j \frac{x_j}{|x|}\right| = |u'(|x|)||x| = |x||\nabla u(x)|.$$

Now, assume that $|x \cdot \nabla u(x)| = |x||\nabla u(x)|$ for all x. This means that $\nabla u(x)$ has the same direction with x. That is, we can find a scalar function $g(x)$ such that $\nabla u(x) = g(x)x$. Now, let a and b be two points on the sphere with radius $r > 0$ (that is, $|a| = |b| = r$). We connect x and y by a piecewise smooth curve $r(t)$ on this sphere, i.e., $|r(t)| = r$, $r(0) = a$ and $r(1) = b$. Then we have $\nabla u(r(t)) = g(r(t))r(t)$. Note that since $|r(t)| = r$ for all t, we can get that $\nabla r(t) \cdot r(t) = 0$. Thus,

$$\nabla u(r(t)) \cdot \nabla r(t) = g(r(t))r(t) \cdot \nabla r(t) = 0.$$

So,

$$u(b) - u(a) = u(r(1)) - u(r(0)) = \int_0^1 \nabla u(r(t)) \cdot \nabla r(t)\, dt = 0.$$

This completes the proof of the lemma. □

Let $d > 0$. We define the vector-valued function $L_{N,d} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$L_{N,d}(x) = |x|^{d-1}x.$$

This is a quasi-conformal mapping type of transform, which was used earlier in [20] to establish weighted Poincaré and Sobolev type inequalities for powers of the Jacobian of a quasi-conformal mapping (these are not necessarily of Muckenhoupt A_p type weights), in particular, for appropriate power weights. The results of [20] have been subsequently extended in a greater generality to weighted Sobolev inequalities with a product of power weights by Gatto and Wheeden [22], and then further by Chanillo and Wheeden for weighted Poincaré inequalities [10]. We note that the best constants and maximizers of the inequalities were not of concern in the aforementioned works. The determinant of the Jacobian of this type of map was already calculated in the literature (see, e.g., [20, 23]). Since the calculation is quite elementary, we include below another elementary and simple way of calculation by using the characteristic polynomials of a matrix for the reader's convenience.

The Jacobian matrix of the function $L_{N,d}$ is

$$\mathbf{J}_{L_{N,d}} = |x|^{d-1}\mathbf{I}_N + \mathbf{A},$$

where

$$\mathbf{A} = \begin{pmatrix} (d-1)|x|^{d-3}x_1^2 & (d-1)|x|^{d-3}x_1x_2 & \dots & (d-1)|x|^{d-3}x_1x_N \\ (d-1)|x|^{d-3}x_2x_1 & (d-1)|x|^{d-3}x_2^2 & \dots & (d-1)|x|^{d-3}x_2x_N \\ \vdots & \vdots & \ddots & \vdots \\ (d-1)|x|^{d-3}x_Nx_1 & (d-1)|x|^{d-3}x_Nx_2 & \dots & (d-1)|x|^{d-3}x_N^2 \end{pmatrix}.$$

It is easy to check that

$$\operatorname{rank}(\mathbf{A}) = 1 \quad \text{and} \quad \operatorname{tr}(\mathbf{A}) = (d-1)|x|^{d-1}.$$

Hence, its characteristic polynomial is

$$\det(\lambda \mathbf{I}_N - \mathbf{A}) = \lambda^N - (d-1)|x|^{d-1}\lambda^{N-1}.$$

For $\lambda = -|x|^{d-1}$, we get $\det(\mathbf{J}_{L_{N,d}}) = d|x|^{N(d-1)}$.

We now define the mapping $D_{N,d,p}$, with $p > 1$, by

$$D_{N,d,p}u(x) := \left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(L_{N,d}(x)) = \left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(|x|^{d-1}x).$$

We also define

$$D_{N,d,p}^{-1}u = v \quad \text{if } u = D_{N,d,p}v.$$

Under the transform $D_{N,d,p}$, we have the following result that will play an important role in our paper. This is a powerful replacement of the classical Polyá–Szegö inequality in the weighted case with power weights.

Lemma 2.2. (1) *For a continuous function f, we have*

$$\int_{\mathbb{R}^N} \frac{f((\frac{1}{d})^{\frac{p-1}{p}} u(x))}{|x|^t}\,dx = d \int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{|x|^{N+td-Nd}}\,dx.$$

In particular, we obtain that $u \in L^s(dx/|x|^t)$ if and only if $D_{N,d,p}u \in L^s(dx/|x|^{N+td-Nd})$.

(2) *For $d > 1$, if $\nabla u \in L^p(dx/|x|^\mu)$, then $\nabla D_{N,d,p}u \in L^p(dx/|x|^{d(p+\mu-N)+N-p})$. Moreover,*

$$\int_{\mathbb{R}^N} \frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}\,dx \le \int_{\mathbb{R}^N} \frac{|\nabla u(x)|^p}{|x|^\mu}\,dx.$$

The equality occurs if and only if u is radially symmetric.

Proof. (1) We have

$$\int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{|x|^{N+td-Nd}}\,dx = \int_{\mathbb{R}^N} \frac{f((\frac{1}{d})^{\frac{p-1}{p}} u(|x|^{d-1}x))}{|x|^{N+td-Nd}}\,dx.$$

Using the change of variables $y_i = |x|^{d-1}x_i$, $i = 1, 2, \dots, N$, we have

$$dy = \det(\mathbf{J}_{L_{N,d}})\,dx = d|x|^{N(d-1)}\,dx \quad \text{and} \quad dx = \frac{1}{d}\frac{dy}{|y|^{N\frac{d-1}{d}}}.$$

Hence,

$$\int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{|x|^{N+td-Nd}}\,dx = \frac{1}{d}\int_{\mathbb{R}^N} \frac{f((\frac{1}{d})^{\frac{p-1}{p}} u(y))}{|y|^{N\frac{d-1}{d}}|y|^{\frac{N+td-Nd}{d}}}\,dy = \frac{1}{d}\int_{\mathbb{R}^N} \frac{f((\frac{1}{d})^{\frac{p-1}{p}} u(y))}{|y|^t}\,dy.$$

(2) Now we begin to consider the gradient of $D_{N,d,p}u$. After calculations, we have

$$\begin{pmatrix} \frac{\partial D_{N,d,p}u}{\partial x_1}(x) \\ \frac{\partial D_{N,d,p}u}{\partial x_2}(x) \\ \vdots \\ \frac{\partial D_{N,d,p}u}{\partial x_N}(x) \end{pmatrix} = \nabla D_{N,d,p}u(x) = \Big(\frac{1}{d}\Big)^{\frac{p-1}{p}} \nabla(u(|x|^{d-1}x)) = \Big(\frac{1}{d}\Big)^{\frac{p-1}{p}} \mathbf{J}^T_{L_{N,d}} \begin{pmatrix} \frac{\partial u}{\partial x_1}(|x|^{d-1}x) \\ \frac{\partial u}{\partial x_2}(|x|^{d-1}x) \\ \vdots \\ \frac{\partial u}{\partial x_N}(|x|^{d-1}x) \end{pmatrix}.$$

Hence, we have

$$\frac{\partial D_{N,d,p}u}{\partial x_i}(x) = \Big(\frac{1}{d}\Big)^{\frac{p-1}{p}} \Big(|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + A_i\Big)$$

for $i = 1, 2, \dots, N$, where

$$A_i := \sum_{j=1}^N (d-1)|x|^{d-3}x_i x_j \frac{\partial u}{\partial x_j}(|x|^{d-1}x).$$

Hence, we obtain

$$\begin{aligned} |\nabla D_{N,d,p}u(x)|^2 &= \sum_{i=1}^N \Big(\frac{\partial D_{N,d,p}u}{\partial x_i}(x)\Big)^2 \\ &= d^{-2\frac{p-1}{p}} \sum_{i=1}^N \Big(|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + A_i\Big)^2 \\ &= d^{-2\frac{p-1}{p}} \Big[\sum_{i=1}^N |x|^{2(d-1)}\Big(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\Big)^2 + \sum_{i=1}^N 2A_i|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + \sum_{i=1}^N A_i^2\Big] \\ &=: d^{-2\frac{p-1}{p}}(I_1 + I_2 + I_3). \end{aligned}$$

Direct computations show

$$I_1 = \sum_{i=1}^N |x|^{2(d-1)}\Big(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\Big)^2 = |x|^{2(d-1)}|\nabla u(|x|^{\frac{t}{N-t}}x)|^2.$$

By applying the Cauchy–Schwarz inequality to estimate the second term, we get

$$
\begin{aligned}
I_2 &= \sum_{i=1}^{N} 2A_i |x|^{d-1} \frac{\partial u}{\partial x_i}(|x|^{d-1}x) \\
&= \sum_{i=1}^{N} 2|x|^{d-1} \frac{\partial u}{\partial x_i}(|x|^{d-1}x) \sum_{j=1}^{N} (d-1)|x|^{d-3} x_i x_j \frac{\partial u}{\partial x_j}(|x|^{d-1}x) \\
&= 2(d-1)|x|^{2d-2} \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{x_i x_j}{|x|^2} \frac{\partial u}{\partial x_j}(|x|^{d-1}x) \frac{\partial u}{\partial x_i}(|x|^{d-1}x) \\
&= 2(d-1)|x|^{2d-2} \bigg(\sum_{i=1}^{N} \frac{x_i}{|x|} \frac{\partial u}{\partial x_i}(|x|^{d-1}x) \bigg)^2 \\
&\le 2(d-1)|x|^{2d-2} \bigg[\sum_{i=1}^{N} \Big(\frac{x_i}{|x|} \Big)^2 \bigg] \bigg[\sum_{i=1}^{N} \Big(\frac{\partial u}{\partial x_i}(|x|^{d-1}x) \Big)^2 \bigg] \\
&= 2(d-1)|x|^{2d-2} |\nabla u(|x|^{d-1}x)|^2.
\end{aligned}
$$

Similarly, for the last term, we have

$$
\begin{aligned}
I_3 = \sum_{i=1}^{N} A_i^2 &= \sum_{i=1}^{N} \bigg(\sum_{j=1}^{N} (d-1)|x|^{d-3} x_i x_j \frac{\partial u}{\partial x_j}(|x|^{d-1}x) \bigg)^2 \\
&\le (d-1)^2 |x|^{2d-6} \sum_{i=1}^{N} \bigg[\sum_{j=1}^{N} (x_i x_j)^2 \bigg] \bigg[\sum_{j=1}^{N} \Big(\frac{\partial u}{\partial x_j}(|x|^{\frac{t}{N-t}}x) \Big)^2 \bigg] \\
&= (d-1)^2 |x|^{2d-6} \sum_{i=1}^{N} |x|^2 x_i^2 |\nabla u(|x|^{d-1}x)|^2 \\
&= (d-1)^2 |x|^{2d-2} |\nabla u(|x|^{d-1}x)|^2.
\end{aligned}
$$

Combining them together, we have

$$
|\nabla D_{N,d,p} u(x)|^2 \le d^{-2\frac{p-1}{p}} d^2 |x|^{2d-2} |\nabla u(|x|^{d-1}x)|^2.
$$

This leads to

$$
|\nabla D_{N,d,p} u(x)| \le d^{\frac{1}{p}} |x|^{d-1} |\nabla u(|x|^{d-1}x)|.
$$

Using the change of variables again, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \frac{|\nabla u(y)|^p}{|y|^\mu}\, dy &= \int_{\mathbb{R}^N} \frac{|\nabla u(|x|^{d-1}x)|^p}{||x|^{d-1}x|^\mu} d|x|^{N(d-1)}\, dx \\
&\ge \frac{1}{d} \int_{\mathbb{R}^N} \frac{|\nabla D_{N,d,p} u(x)|^p}{|x|^{p(d-1)} ||x|^{d-1}x|^\mu} d|x|^{N(d-1)}\, dx \\
&= \int_{\mathbb{R}^N} \frac{|\nabla D_{N,d,p} u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}\, dx.
\end{aligned}
$$

Finally, by Lemma 2.1, it is easy to check that the equalities hold if and only if u is radial. □

3 The CKN Inequality when $0 < a < 1$ Under Condition (C1)

Theorems 1.1–1.3 will be proved via the following series of lemmata. Recall that the conditions on the parameters are given in (C1), and $\mathrm{CKN}(N, \mu, \theta, s, p, q, r)$ is defined in (1.4). We now set

$$
\mathrm{GN}(N, p, q, r, \mu, \theta, s) = \sup_{u \in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}\, dx \big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} |\nabla u|^p\, dx \big)^{\frac{a}{p}} \big(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\, dx \big)^{\frac{1-a}{q}}},
$$

where

$$d = \frac{N-p}{N-p-\mu}.$$

It is important to note here that since $\theta \le \frac{N\mu}{N-p} \le s < N$, we have

$$N + \theta d - Nd \le 0 \le N + sd - Nd < N.$$

Lemma 3.1. *The variational problem*

$$A(N,p,q,r,\mu,\theta,s) = \inf\Big\{ I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\,dx : u \in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}\,dx = 1 \Big\}$$

has a minimizer. Moreover, $A(N,p,q,r,\mu,\theta,s) > 0$.

Proof. By the classical Schwarz rearrangement, we can assume that there exists a sequence of radial functions (u_n) such that

$$I(u_n) \downarrow A(N,p,q,r,\mu,\theta,s) \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|u_n|^r}{|x|^{N+sd-Nd}}\,dx = 1.$$

We can assume, without loss of generality, that $u_n \rightharpoonup u$ in $u \in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)$. Since it is evident that $I(u) \le A(N,p,q,r,\mu,\theta,s)$, it is enough to show

$$\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}\,dx = 1.$$

But this is easy to observe since for $R > 0$ sufficiently large, we can write

$$\int_{\mathbb{R}^N} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}}\,dx = \int_{B_R} + \int_{B_R^c} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}}\,dx.$$

Then, by the radial lemma, we get

$$\int_{B_R^c} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}}\,dx \to 0.$$

Also, by the compactness of Sobolev embeddings, we can deduce

$$\int_{B_R} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}}\,dx \to 0.$$

As a consequence, $u \neq 0$ and $A(N,p,q,r,\mu,\theta,s) > 0$. Moreover, noting that $u_\lambda(x) = \lambda^{\frac{Nd-sd}{r}} u(\lambda x)$ for $\lambda > 0$, we have

$$\|\nabla u_\lambda\|_p = \lambda^{\frac{Nd-sd}{r} + \frac{p-N}{p}} \|\nabla u\|_p, \quad \|u_\lambda\|_k = \lambda^{\frac{Nd-sd}{r} - \frac{N}{k}} \|u\|_k$$

and

$$\int_{\mathbb{R}^N} \frac{|u_\lambda|^r}{|x|^{N+sd-Nd}}\,dx = 1.$$

Also,

$$\begin{aligned} I(u_\lambda) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_\lambda|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} \frac{|u_\lambda|^q}{|x|^{N+\theta d-Nd}}\,dx \\ &= \frac{1}{p}\lambda^{\frac{Nd-sd}{r}p+p-N}\|\nabla u\|_p^p + \frac{1}{q}\lambda^{q\frac{Nd-sd}{r}-N+N+\theta d-Nd}\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\,dx \\ &= \lambda^m A + \lambda^{-n} B, \end{aligned}$$

with

$$m = \frac{Nd - sd}{r}p + p - N, \quad n = Nd - \theta d - q\frac{Nd - sd}{r}, \quad A = \frac{1}{p}\|\nabla u\|_p^p, \quad B = \frac{1}{q}\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d - Nd}}\,dx.$$

Hence,

$$A(N, p, q, r, \mu, \theta, s) = \inf_{\lambda>0} I(u_\lambda) = I(u_{\lambda_0}),$$

where

$$\lambda_0 = \Big(\frac{nB}{mA}\Big)^{\frac{1}{m+n}}.$$

This means that

$$A(N, p, q, r, \mu, \theta, s) = \frac{m+n}{m}\Big(\frac{n}{m}\Big)^{-\frac{n}{m+n}} A^{\frac{n}{m+n}} B^{\frac{m}{m+n}}.$$ □

Lemma 3.2. GN(N, p, q, r) *can be achieved and*

$$\mathrm{GN}(N, p, q, r, \mu, \theta, s) = \left[\frac{\frac{m+n}{m}\big(\frac{n}{m}\big)^{-\frac{n}{m+n}}\big(\frac{1}{p}\big)^{\frac{n}{m+n}}\big(\frac{1}{q}\big)^{\frac{m}{m+n}}}{A(N, p, q, r, \mu, \theta, s)}\right]^{\frac{a/p}{n/(m+n)}}.$$

Proof. For any v with

$$\int_{\mathbb{R}^N} \frac{|v|^r}{|x|^{N+sd-Nd}}\,dx = 1,$$

we use the above process and get

$$A(N, p, q, r, \mu, \theta, s) \le \inf_{\lambda>0} I(v_\lambda) = \frac{m+n}{m}\Big(\frac{n}{m}\Big)^{-\frac{n}{m+n}}\Big(\frac{1}{p}\|\nabla v\|_p^p\Big)^{\frac{n}{m+n}}\Big(\frac{1}{q}\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\,dx\Big)^{\frac{m}{m+n}}.$$

Noting that $\frac{n/(m+n)}{m/(m+n)} = \frac{a/p}{(1-a)/q}$, we obtain

$$\frac{\big(\int_{\mathbb{R}^N} \frac{|v|^r}{|x|^{N+sd-Nd}}\,dx\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N}|\nabla v|^p\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N}\frac{|v|^q}{|x|^{N+\theta d-Nd}}\,dx\big)^{\frac{1-a}{q}}} \le \left[\frac{\frac{m+n}{m}\big(\frac{n}{m}\big)^{-\frac{n}{m+n}}\big(\frac{1}{p}\big)^{\frac{n}{m+n}}\big(\frac{1}{q}\big)^{\frac{m}{m+n}}}{A(N, p, q, r, \mu, \theta, s)}\right]^{\frac{a/p}{n/(m+n)}}.$$

Combining this with the previous lemma, we conclude that GN$(N, p, q, r, \mu, \theta, s)$ can be achieved and

$$\mathrm{GN}(N, p, q, r, \mu, \theta, s) = \left[\frac{\frac{m+n}{m}\big(\frac{n}{m}\big)^{-\frac{n}{m+n}}\big(\frac{1}{p}\big)^{\frac{n}{m+n}}\big(\frac{1}{q}\big)^{\frac{m}{m+n}}}{A(N, p, q, r, \mu, \theta, s)}\right]^{\frac{a/p}{n/(m+n)}}.$$

This completes the proof. □

Using Lemma 3.1 and Lemma 3.2, we will now show that CKN$(N, \mu, \theta, s, p, q, r)$ can be achieved.

Lemma 3.3. *Under condition* (C1), CKN$(N, \mu, \theta, s, p, q, r)$ *can be attained and*

$$\mathrm{CKN}(N, \mu, \theta, s, p, q, r) = \Big(\frac{N-p}{N-p-\mu}\Big)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\mathrm{GN}(N, p, q, r, \mu, \theta, s).$$

Proof. We begin by the observation that if $u \ge 0$ is a maximizer for GN$(N, p, q, r, \mu, \theta, s)$, then we can assume that u is radial. Indeed, this fact is just a consequence of the Schwarz rearrangement; see, for instance, [27]. Now, let us assume that $U_0 \ge 0$ is a radial maximizer of GN$(N, p, q, r, \mu, \theta, s)$. We set $V_0 = D_{N,d,p}^{-1}U_0$ with $d = \frac{N-p}{N-p-\mu}$. This means that $U_0 = D_{N,d,p}V_0$. We will show that V_0 is a maximizer of CKN$(N, \mu, \theta, s, p, q, r)$. Indeed, for any v, we need to show

$$\frac{\big(\int_{\mathbb{R}^N}|v|^r\frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N}|v|^q\frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}} \le \frac{\big(\int_{\mathbb{R}^N}|V_0|^r\frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N}|V_0|^q\frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}}.$$

By Lemma 2.2, by noting that when $d = \frac{N-p}{N-p-\mu}$, i.e., $d(p+\mu-N)+N-p=0$, we get

$$\int_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^r}{|x|^{N+sd-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^q}{|x|^{N+\theta d-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^\mu}\,dx \geq \int_{\mathbb{R}^N} |\nabla D_{N,d,p}v|^p\,dx,$$

and

$$\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu}\,dx = \int_{\mathbb{R}^N} |\nabla U_0|^p\,dx.$$

Hence,

$$\begin{aligned}
\frac{(\int_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s})^{\frac{1}{r}}}{(\int_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu})^{\frac{a}{p}} (\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta})^{\frac{1-a}{q}}} &\leq \frac{(d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^r}{|x|^{N+sd-Nd}}\,dx)^{\frac{1}{r}}}{(\int_{\mathbb{R}^N} |\nabla D_{N,d,p}v|^p\,dx)^{\frac{a}{p}} (d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^q}{|x|^{N+\theta d-Nd}}\,dx)^{\frac{1-a}{q}}} \\
&\leq d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{(\int_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}}\,dx)^{\frac{1}{r}}}{(\int_{\mathbb{R}^N} |\nabla U_0|^p\,dx)^{\frac{a}{p}} (\int_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}}\,dx)^{\frac{1-a}{q}}} \\
&= \frac{(\int |V_0|^r \frac{dx}{|x|^s})^{\frac{1}{r}}}{(\int |\nabla V_0|^p \frac{dx}{|x|^\mu})^{\frac{a}{p}} (\int |V_0|^q \frac{dx}{|x|^\theta})^{\frac{1-a}{q}}}.
\end{aligned}$$

We note that the last equality holds because U_0 and V_0 are radial. Hence, $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ is attained. Moreover, it is easy to see that

$$\mathrm{CKN}(N,\mu,\theta,s,p,q,r) = d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \mathrm{GN}(N,p,q,r,\mu,\theta,s). \qquad \square$$

Lemma 3.4. *Assume that* (C1) *holds. If V_0 is a maximizer of* $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$*, then V_0 is radially symmetric.*

Proof. let V_0 be a maximizer of $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$. We set $U_0 = D_{N,d,p}V_0$ where $d = \frac{N-p}{N-p-\mu}$. We will show that U_0 is a maximizer of $\mathrm{GN}(N,p,q,r,\mu,\theta,s)$. Indeed, for any radial function u (we can just choose radial functions because of symmetrization arguments), we define

$$v = D^{-1}_{N,d,p}u, \quad \text{i.e., } u = D_{N,d,p}v.$$

By Lemma 2.2, we get

$$\int_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^\mu}\,dx = \int_{\mathbb{R}^N} |\nabla u|^p\,dx,$$

and

$$\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}}\,dx,$$
$$\int_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu}\,dx \geq \int_{\mathbb{R}^N} |\nabla U_0|^p\,dx.$$

Hence,

$$\begin{aligned}
&d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\big(\int_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}}\,dx\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} |\nabla U_0|^p\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}}\,dx\big)^{\frac{1-a}{q}}}\\
&\geq \frac{\big(\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu}\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}}\\
&\geq \frac{\big(\int_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^\mu}\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}}\\
&= d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\big(\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}\,dx\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}\,dx\big)^{\frac{1-a}{q}}}.
\end{aligned}$$

Moreover, it is easy to see that

$$d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\big(\int_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}}\,dx\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} |\nabla U_0|^p\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}}\,dx\big)^{\frac{1-a}{q}}} = \frac{\big(\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu}\,dx\big)^{\frac{a}{p}}\big(\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta}\big)^{\frac{1-a}{q}}}.$$

Hence,

$$\int_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu}\,dx = \int_{\mathbb{R}^N} |\nabla U_0|^p\,dx.$$

So, V_0 is radial. □

Lemma 3.5. *Assume that* (C1) *holds with* $s = \theta = \frac{N\mu}{N-p}$. *If* $p < r = p\frac{q-1}{p-1} < \frac{Np}{N-p}$, *then, with* $\delta = Np - q(N-p)$, *we have*

$$\begin{aligned}
&\mathrm{CKN}(N, s, \mu, p, q, r)\\
&= \Big(\frac{N-p}{N-p-\mu}\Big)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \Big(\frac{q-p}{p\sqrt{\pi}}\Big)^a \Big(\frac{pq}{N(q-p)}\Big)^{\frac{a}{p}} \Big(\frac{\delta}{pq}\Big)^{\frac{1}{r}} \Bigg(\frac{\Gamma(q\frac{p-1}{q-p})\Gamma(\frac{N}{2}+1)}{\Gamma(\frac{p-1}{p}\frac{\delta}{q-p})\Gamma(N\frac{p-1}{p}+1)}\Bigg)^{\frac{a}{N}},
\end{aligned}$$

and all the maximizers have the form

$$V_0(x) = A(1 + B|x|^{\frac{1}{d}\frac{p}{p-1}})^{-\frac{p-1}{q-p}} \quad \textit{for some } A \in \mathbb{R},\ B > 0,$$

where $d = \frac{N-p}{N-p-\mu}$.

Proof. When $r = p\frac{q-1}{p-1}$ and $s = \theta = \frac{N\mu}{N-p}$, from [1, 2, 14, 15], we have that

$$\mathrm{GN}(N, p, q, r) = \Big(\frac{q-p}{p\sqrt{\pi}}\Big)^a \Big(\frac{pq}{N(q-p)}\Big)^{\frac{a}{p}} \Big(\frac{\delta}{pq}\Big)^{\frac{1}{r}} \Bigg(\frac{\Gamma(q\frac{p-1}{q-p})\Gamma(\frac{N}{2}+1)}{\Gamma(\frac{p-1}{p}\frac{\delta}{q-p})\Gamma(N\frac{p-1}{p}+1)}\Bigg)^{\frac{a}{N}},$$

and all the maximizers have the form

$$U_0(x) = A(1 + B|x-\overline{x}|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0,\ \overline{x} \in \mathbb{R}^N.$$

Now, let V_0 be a maximizer of $\mathrm{CKN}(N, s, \mu, p, q, r)$. By Lemma 3.4, $D_{N,d,p}V_0$ is a maximizer of $\mathrm{GN}(N, p, q, r)$. Hence,

$$D_{N,d,p}V_0(x) = A(1 + B|x - \overline{x}|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}.$$

This means that

$$V_0(x) = A'(1 + B||x|^{\frac{1}{d}-1}x - \overline{x}|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}.$$

Noting that V_0 is radial, we conclude that $\overline{x} = 0$. □

Lemma 3.6. *Assume that* (C1) *holds with* $s = \theta = \frac{N\mu}{N-p}$. *If* $1 < q = p\frac{r-1}{p-1} < p$, *then, with* $\delta = Np - r(N - p)$, *we have*

$$\mathrm{CKN}(N, s, \mu, p, q, r) = \Big(\frac{N-p}{N-p-\mu}\Big)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\Big(\frac{p-r}{p\sqrt{\pi}}\Big)^{a}\Big(\frac{pr}{N(p-r)}\Big)^{\frac{a}{p}}\Big(\frac{pr}{\delta}\Big)^{\frac{1-a}{q}}\Big(\frac{\Gamma(\frac{p-1}{p}\frac{\delta}{p-r}+1)\Gamma(\frac{N}{2}+1)}{\Gamma(r\frac{p-1}{p-r}+1)\Gamma(N\frac{p-1}{p}+1)}\Big)^{\frac{a}{N}}.$$

If $r > 2 - \frac{1}{p}$, *then all the maximizers of* $\mathrm{GN}(N, p, q, r)$ *have the form*

$$V_0(x) = A(1 - B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}} \quad \textit{for some } A \in \mathbb{R},\ B > 0.$$

Proof. When $q = p\frac{r-1}{p-1}$ and $s = \theta = \frac{N\mu}{N-p}$, from [1, 2, 14, 15], we have that

$$\mathrm{GN}(N, p, q, r) = \Big(\frac{p-r}{p\sqrt{\pi}}\Big)^{a}\Big(\frac{pr}{N(p-r)}\Big)^{\frac{a}{p}}\Big(\frac{pr}{\delta}\Big)^{\frac{1-a}{q}}\Big(\frac{\Gamma(\frac{p-1}{p}\frac{\delta}{p-r}+1)\Gamma(\frac{N}{2}+1)}{\Gamma(r\frac{p-1}{p-r}+1)\Gamma(N\frac{p-1}{p}+1)}\Big)^{\frac{a}{N}}.$$

Also, when $r > 2 - \frac{1}{p}$, all the maximizers of $\mathrm{GN}(N, p, q, r)$ have the form

$$U_0(x) = A(1 - B|x - \overline{x}|^{\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0,\ \overline{x} \in \mathbb{R}^N.$$

Now, let V_0 be a maximizer of $\mathrm{CKN}(N, s, \mu, p, q, r)$. By Lemma 3.4, $D_{N,d,p}V_0$ is a maximizer of $\mathrm{GN}(N, p, q, r)$. Hence,

$$D_{N,d,p}V_0(x) = A(1 - B|x - \overline{x}|^{\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}}.$$

This means that

$$V_0(x) = A'(1 - B||x|^{\frac{1}{d}-1}x - \overline{x}|^{\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}}.$$

Noting that V_0 is radially symmetric, we conclude that $\overline{x} = 0$. □

4 CKN Inequalities in the Region (C2)

In this section, we will be concerned with CKN inequalities in the class (C2). Recall that $\mathrm{CKN}(N, \mu, s, q)$ is given by (1.5). We also define

$$\mathrm{CKN}_1(N, \mu, s, q) = \sup_{u \in D^{2,q}_{0,s}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N} |u|^{2(q-1)} \frac{dx}{|x|^{N+sd-Nd}}\big)^{\frac{1}{2(q-1)}}}{\big(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\big)^{\frac{a}{2}}\big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^{N+sd-Nd}}\big)^{\frac{1-a}{q}}},$$

where $d = \frac{N-2}{N-2-\mu}$.

Proof of Theorem 1.4. For any $v \in D^{2,q}_{\mu,s}(\mathbb{R}^N)$, with $u = D_{N,d,2}v$, we have that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \frac{dx}{|x|^{\mu}} \geq \int_{\mathbb{R}^N} |\nabla u|^2\, dx,$$

$$\int_{\mathbb{R}^N} |v|^{2(q-1)} \frac{dx}{|x|^{s}} = d^q \int_{\mathbb{R}^N} \frac{|u|^{2(q-1)}}{|x|^{N+sd-Nd}}\, dx,$$

$$\int_{\mathbb{R}^N} |v|^{q} \frac{dx}{|x|^{s}} = d^{1+\frac{1}{2}q} \int_{\mathbb{R}^N} \frac{|u|^{q}}{|x|^{N+sd-Nd}}\, dx.$$

By a result in [17], we have that $U(x) = D_{N,d,2}V_0(x) = C(1 + D|x|^{2-N-sd+Nd})^{-\frac{1}{q-2}}$ for some $C \in \mathbb{R}$, where $D > 0$ is the maximizer for $\mathrm{CKN}_1(N, \mu, s, q)$ for $0 < N + sd - Nd < 2$ small enough. Hence, by Lemma 2.2, we have that

$$\begin{aligned}\frac{\left(\int_{\mathbb{R}^N}|v|^{2(q-1)}\frac{dx}{|x|^s}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N}|\nabla v|^2\frac{dx}{|x|^\mu}\right)^{\frac{a}{2}}\left(\int_{\mathbb{R}^N}|v|^q\frac{dx}{|x|^s}\right)^{\frac{1-a}{q}}} &\le \frac{d^{\frac{q}{2(q-1)}}}{d^{\frac{1+q/2}{(1-a)/q}}}\frac{\left(\int_{\mathbb{R}^N}|u|^{2(q-1)}\frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{\frac{a}{2}}\left(\int_{\mathbb{R}^N}|u|^q\frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1-a}{q}}}\\
&\le \frac{d^{\frac{q}{2(q-1)}}}{d^{\frac{1+q/2}{(1-a)/q}}}\frac{\left(\int_{\mathbb{R}^N}|U|^{2(q-1)}\frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N}|\nabla U|^2\,dx\right)^{\frac{a}{2}}\left(\int_{\mathbb{R}^N}|U|^q\frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1-a}{q}}}\\
&= \frac{\left(\int_{\mathbb{R}^N}|V_0|^{2(q-1)}\frac{dx}{|x|^s}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N}|\nabla V_0|^2\frac{dx}{|x|^\mu}\right)^{\frac{a}{2}}\left(\int_{\mathbb{R}^N}|V_0|^q\frac{dx}{|x|^s}\right)^{\frac{1-a}{q}}}.\end{aligned}$$

In other words, V_0 is the optimizer for $\mathrm{CKN}(N, \mu, s, q)$. □

5 The CKN Inequality Without the Interpolation Term: The Case $a = 1$

In this section, we will also consider CKN inequalities without the interpolation term for all $1 < p < N$, and we will be concerned with the following range:

$$\begin{cases} 1 < p < p + \mu < N, & \dfrac{\mu}{p} \le \dfrac{s}{r} < \dfrac{\mu}{p} + 1,\\ r = \dfrac{(N-s)p}{N-\mu-p}, & a = 1.\end{cases} \tag{C3}$$

Note that the condition $\frac{\mu}{p} \le \frac{s}{r} \le \frac{\mu}{p} + 1$ comes from the constraints of the CKN inequality. In this case, we define

$$D_\mu^{1,p}(\mathbb{R}^N; dx/|x|^s) = \left\{u \in L^r(dx/|x|^s) : \int\limits_{\mathbb{R}^N}|\nabla u|^p\frac{dx}{|x|^\mu} < \infty\right\}$$

and

$$\mathrm{CKN}(N, p, \mu, s) = \sup_{u\in D_\mu^{1,p}(\mathbb{R}^N;dx/|x|^s)}\frac{\left(\int_{\mathbb{R}^N}|u|^r\frac{dx}{|x|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N}|\nabla u|^p\frac{dx}{|x|^\mu}\right)^{\frac{1}{p}}}.$$

We will prove in this section the following result.

Theorem 5.1. *Assume that* (C3) *holds. Then* $\mathrm{CKN}(N, p, \mu, s)$ *is achieved with the extremals being of the following form:*

$$V_{c,\lambda}(x) = c(\lambda + |x|^{\frac{p+\mu-s}{p-1}})^{-\frac{N-p-\mu}{p+\mu-s}} \quad \text{for some } c \ne 0,\ \lambda > 0.$$

Theorem 5.1 was studied in [33] by solving the corresponding ODE. In this section, we will provide another proof using the transform $D_{N,d,p}$.

We note that $\frac{\mu}{p} \le \frac{s}{r} < \frac{\mu}{p} + 1$ means $\frac{N\mu}{N-p} \le s < p + \mu$.

We also define

$$\mathrm{HS}(N, p, \mu, s) = \sup_{D_0^{1,p}(\mathbb{R}^N;dx/|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}})}\frac{\left(\int_{\mathbb{R}^N}|u|^{p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right)}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{1}{p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right)}}}{\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{1}{p}}}.$$

Note that

$$p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right) = \frac{N-\frac{s(N-p)-N\mu}{N-p-\mu}}{N-p}p = \frac{(N-s)p}{N-\mu-p}$$

and

$$0 \le \frac{s(N-p)-N\mu}{N-p-\mu} < p.$$

Lemma 5.2. $\mathrm{CKN}(N,p,\mu,s)$ *can be attained.*

Proof. We will use the fact that $\mathrm{HS}(N,p,\mu,s)$ is attained by some radial functions U_0. Set $V_0 = D_{N,d,p}^{-1}U_0$ with $d = \frac{N-p}{N-p-\mu}$. This means that $U_0 = D_{N,d,p}V_0$. We will show that V_0 is a maximizer of $\mathrm{CKN}(N,p,\mu,s)$. Indeed, for any $v \in D_\mu^{1,p}(\mathbb{R}^N)$, we set $u = D_{N,d,p}v$. Then, by Lemma 2.2, we get

$$\int_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu} \ge \int_{\mathbb{R}^N} \frac{|\nabla D_{N,d,p}v|^p}{|x|^{d(p+\mu-N)+N-p}}\,dx = \int_{\mathbb{R}^N} |\nabla u|^p\,dx,$$

$$\int_{\mathbb{R}^N} |v|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s} = dd^{\frac{p-1}{p}\frac{(N-s)p}{N-\mu-p}} \int_{\mathbb{R}^N} |u|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{N+sd-Nd}} = d^{1+\frac{(N-s)(p-1)}{N-\mu-p}} \int_{\mathbb{R}^N} |u|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}},$$

$$\int_{\mathbb{R}^N} |\nabla V_0|^p \frac{dx}{|x|^\mu} = \int_{\mathbb{R}^N} |\nabla U_0|^p\,dx$$

and

$$\int_{\mathbb{R}^N} |V_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s} = d^{1+\frac{(N-s)(p-1)}{N-\mu-p}} \int_{\mathbb{R}^N} |U_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

Hence,

$$\begin{aligned}
\frac{\big(\int_{\mathbb{R}^N} |v|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s}\big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu}\big)^{\frac{1}{p}}} &\le d^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}} \frac{\Big(\int_{\mathbb{R}^N} |u|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\Big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\big)^{\frac{1}{p}}} \\
&\le d^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}} \frac{\Big(\int_{\mathbb{R}^N} |U_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\Big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N} |\nabla U_0|^p\,dx\big)^{\frac{1}{p}}} \\
&= \frac{\big(\int_{\mathbb{R}^N} |V_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s}\big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N} |\nabla V_0|^p \frac{dx}{|x|^\mu}\big)^{\frac{1}{p}}}.
\end{aligned}$$

In other words, V_0 is the maximizer for $\mathrm{CKN}(N,p,\mu,s)$. Moreover, we also deduce that

$$\mathrm{CKN}(N,p,\mu,s) = d^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}}\,\mathrm{HS}(N,p,\mu,s). \qquad \square$$

Lemma 5.3. *All the optimizers for* $\mathrm{CKN}(N,p,\mu,s)$ *are radially symmetric.*

Proof. Assume that V_0 is a maximizer for $\mathrm{CKN}(N,p,\mu,s)$ and $U_0 = D_{N,d,p}V_0$ where $d = \frac{N-p}{N-p-\mu}$. Again, by Lemma 2.2, we get

$$\int_{\mathbb{R}^N} |\nabla V_0|^p \frac{dx}{|x|^\mu} \ge \int_{\mathbb{R}^N} |\nabla U_0|^p\,dx$$

and

$$\int_{\mathbb{R}^N} |V_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s} = d^{1+\frac{(N-s)(p-1)}{N-\mu-p}} \int_{\mathbb{R}^N} |U_0|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

We will now prove that U_0 is a maximizer for $\mathrm{HS}(N,p,\mu,s)$. Indeed, for any radial function u (we can assume that u is radial by the Schwarz rearrangement argument), if we set $v = D_{N,d,p}^{-1}u$, that is, $u = D_{N,d,p}v$, then, by Lemma 2.2, we obtain

$$\int_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu} = \int_{\mathbb{R}^N} |\nabla u|^p\,dx$$

and

$$\int_{\mathbb{R}^N} |v|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^s} = d^{1+\frac{(N-s)(p-1)}{N-\mu-p}} \int_{\mathbb{R}^N} |u|^{\frac{(N-s)p}{N-\mu-p}} \frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

Hence,

$$
\begin{aligned}
\frac{\Big(\int_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\Big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla U_0|^p\,dx\big)^{\frac{1}{p}}}
&\geq \Big(\frac{1}{d}\Big)^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}}\frac{\big(\int_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\big)^{\frac{1}{p}}}\\
&\geq \Big(\frac{1}{d}\Big)^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}}\frac{\big(\int_{\mathbb{R}^N}|v|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\big)^{\frac{1}{p}}}\\
&= \frac{\Big(\int_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\Big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\big)^{\frac{1}{p}}}.
\end{aligned}
$$

Thus, U_0 is a maximizer for $\mathrm{HS}(N,p,\mu,s)$. Moreover, it is easy to see that the equality must occur in the first line, that is,

$$
\frac{\Big(\int_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\Big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla U_0|^p\,dx\big)^{\frac{1}{p}}}
= \Big(\frac{1}{d}\Big)^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}}\frac{\big(\int_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\big)^{\frac{N-\mu-p}{(N-s)p}}}{\big(\int_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\big)^{\frac{1}{p}}}.
$$

This means that

$$\int_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}=\int_{\mathbb{R}^N}|\nabla U_0|^p\,dx,$$

and thus V_0 is radial. □

Proof of Theorem 5.1. From Lemmata 5.2 and 5.3, we see that $\mathrm{CKN}(N,p,\mu,s)$ is attained,

$$\mathrm{CKN}(N,p,\mu,s)=d^{(1+\frac{(N-s)(p-1)}{N-\mu-p})\frac{N-\mu-p}{(N-s)p}}\mathrm{HS}(N,p,\mu,s),$$

and all maximizers for $\mathrm{CKN}(N,p,\mu,s)$ are radially symmetric. Furthermore, we can conclude that V_0 is a maximizer for $\mathrm{CKN}(N,p,mu,s)$ only if $U_0=D_{N,d,p}V_0$ is a maximizer for $\mathrm{HS}(N,p,\mu,s)$, where $d=\frac{N-p}{N-p-\mu}$. It is known (see, for instance, [24]) that $\mathrm{HS}(N,p,\mu,s)$ is attained with the maximizers being the functions

$$U_{c,\lambda}(x)=c\big(\lambda+|x|^{\frac{p-\frac{s(N-p)-N\mu}{N-p-\mu}}{p-1}}\big)^{-\frac{N-p}{p-\frac{s(N-p)-N\mu}{N-p-\mu}}}=c\big(\lambda+|x|^{\frac{(N-p)(p+\mu-s)}{(p-1)(N-p-\mu)}}\big)^{-\frac{N-p-\mu}{p+\mu-s}}\quad\text{for some } c\neq 0,\ \lambda>0.$$

Hence, $\mathrm{CKN}(N,p,\mu,s)$ could be achieved with the optimizers being the functions

$$V_{c,\lambda}(x)=D_{N,d,p}^{-1}U_{c,\lambda}(x)=c\big(\lambda+|x|^{\frac{p+\mu-s}{p-1}}\big)^{-\frac{N-p-\mu}{p+\mu-s}}\quad\text{for some } c\neq 0,\ \lambda>0.$$ □

Remark 5.4. If we have $\frac{s}{r}=\frac{\mu}{p}+1$ in condition (C2), then $s=p+\mu$ and $\frac{s(N-p)-N\mu}{N-p-\mu}=p$. So in this case, after applying the transform $D_{N,d,p}$, where $d=\frac{N-p}{N-p-\mu}$, the CKN inequality corresponds to the Hardy inequality. Hence, the best constant in this case is

$$\mathrm{CKN}(N,p,\mu,s)=\frac{p}{N-p-\mu},$$

and it is never achieved.

6 CKN Inequalities with Arbitrary Norm

In this section, we will investigate CKN inequalities under arbitrary norms in $\mathbb{R}^N$ in the spirit of Cordero-Erausquin, Nazaret and Villani [13]. More precisely, let $E=(\mathbb{R}^N,\|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^N$. Then its dual space is $E^*=(\mathbb{R}^N,\|\cdot\|_*)$, where for $X\in E^*$,

$$\|X\|_*=\sup_{Y\in E:\|Y\|\leq 1}X\cdot Y.$$

For simplicity, we will assume that $|\{\|x\|_* \le 1\}| = \omega_N$ and set $\kappa_N = |\{\|x\| \le 1\}|$. We will assume that for any $X \in \mathbb{R}^N$, there exists a unique $X^* \in \mathbb{R}^N$ such that $\|X^*\|_* = 1$ and

$$X \cdot X^* = \|X\| = \sup_{Y \in \mathbb{R}^N : \|Y\|_* \le 1} X \cdot Y.$$

It is clear that $\|\cdot\|$ is Lipschitz with Lipschitz constant 1, and thus, differentiable a.e. From the fact that $\|\lambda x\| = \lambda \|x\|$ for all $\lambda > 0$, we can see that the gradient of $\|\cdot\|$ at $x \in \mathbb{R}^N$ is the unique vector $\nabla(\|\cdot\|)(x) = x^*$. Recall that

$$\|x^*\|_* = 1, \quad x \cdot x^* = \|x\| = \sup_{\|y\|_* \le 1} x \cdot y.$$

Actually, first we will consider a more general situation. More precisely, we suppose that C is q-homogeneous, that is, there exists $q > 1$ such that

$$C(\lambda x) = \lambda^q C(x) \quad \text{for all } \lambda \ge 0 \text{ and all } x \in \mathbb{R}^N. \tag{6.1}$$

Then C^*, the Legendre transform of C defined by

$$C^*(x) = \sup_y \{\langle x, y\rangle - C(y)\},$$

is even, strictly convex function and is p-homogeneous with $p = \frac{q}{q-1}$.

We have that $\langle X, Y\rangle \le C^*(X) + C(Y)$ for all X, Y. Hence, $\langle X, Y\rangle \le \lambda^p C^*(X) + \lambda^{-q} C(Y)$ for all $\lambda > 0$, X, Y. Minimizing the right-hand side with respect to λ gives the Cauchy–Schwarz inequality

$$X \cdot Y \le [qC(Y)]^{\frac{1}{q}} [pC^*(X)]^{\frac{1}{p}}.$$

By Young's inequality, we have

$$X \cdot Y \le [qC(Y)]^{\frac{1}{q}} [pC^*(X)]^{\frac{1}{p}} \le C^*(x) + C(y).$$

Hence, we also have that

$$[pC^*(X)]^{\frac{1}{p}} = \sup_Y \frac{X \cdot Y}{[qC(Y)]^{\frac{1}{q}}}.$$

In other words,

$$C^*(X) = \sup_Y \frac{|X \cdot Y|^p}{p[qC(Y)]^{\frac{p}{q}}}.$$

We will assume that for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that

$$x \cdot x^* = qC(x) \quad \text{and} \quad C^*(x^*) = (q-1)C(x) = \frac{q}{p} C(x).$$

In other words, for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that the equality in the Cauchy–Schwarz inequality occurs.

Note that, from (6.1), we get that $C(\cdot)$ is differentiable a.e. We will assume that the gradient of $C(\cdot)$ at $x \in \mathbb{R}^N$ is the unique vector x^*. The example that we have in mind is $C(x) = \frac{1}{q}|x|^q$ and $C^*(x) = \frac{1}{p}|x|^p$, with $|\cdot|$ being the regular Euclidean norm on $\mathbb{R}^N$. Another example is the pair $C(x) = \frac{1}{q}\|x\|^q$ and $C^*(x) = \frac{1}{p}\|x\|_*^p$.

6.1 A Change of Variables

As in Lemma 2.1, we have the following result.

Lemma 6.1. *We have*

$$|x \cdot \nabla u(x)| = [qC(x)]^{\frac{1}{q}} [pC^*(\nabla u)]^{\frac{1}{p}} \quad \textit{for a.e. } x \in \mathbb{R}^N$$

if and only if u is C-radial, i.e., $u(x) = u(y)$ when $C(x) = C(y)$.

Proof. If u is C-radial, then recalling that $\nabla(C(\cdot))(x) = x^*$, we have

$$\frac{\partial u}{\partial x_j}(x) = u'(C(x))x_j^*.$$

Hence,

$$C^*(\nabla u) = C^*(u'(C(x))x^*) = |u'(C(x))|^p C^*(x^*) = |u'(C(x))|^p \frac{q}{p} C(x)$$

and

$$[qC(x)]^{\frac{1}{q}} [pC^*(\nabla u)]^{\frac{1}{p}} = [qC(x)]^{\frac{1}{q}} [|u'(C(x))|^p qC(x)]^{\frac{1}{p}} = |u'(C(x))| qC(x).$$

Also,

$$|x \cdot \nabla u(x)| = \left| \sum_{j=1}^{N} x_j \frac{\partial u}{\partial x_j}(x) \right| = |u'(\|x\|)| \left| \sum_{j=1}^{N} x_j x_j^* \right| = |u'(\|x\|)| qC(x).$$

Now, if for all $x \in \mathbb{R}^N$,

$$|x \cdot \nabla u(x)| = [qC(x)]^{\frac{1}{q}} [pC^*(\nabla u)]^{\frac{1}{p}},$$

then $\nabla u(x)$ has the same direction with x^*. That is, we can find a function $f(x)$ such that $\nabla u(x) = f(x)x^*$. Now let a and b be two points on the C-sphere with radius $r > 0$, that is, $C(a) = C(b) = r$. We connect a and b by a piecewise smooth curve $r(t)$ on the sphere, i.e., $C(r(t)) = r$ and $C(r(0)) = a$, $C(r(1)) = b$. Then we have

$$\nabla u(r(t)) = f(r(t))(r(t))^*.$$

Using that fact that $C(r(t)) = r$ for all t, we get $(r(t))^* \cdot \nabla r(t) = 0$. Hence,

$$\int_0^1 \nabla u(r(t)) \cdot \nabla r(t) dt = \int_0^1 f(r(t))(r(t))^* \cdot \nabla r(t)\, dt = 0.$$

In other words,

$$u(b) - u(a) = u(C(r(1))) - u(C(r(0))) = 0.$$

The proof is completed. □

Let $d > 0$. We define a vector-valued function $L_{N,d} \colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$L_{N,d}(x) = C(x)^d x.$$

The Jacobian matrix of this function $L_{N,d}$ is

$$\mathbf{J}_{L_{N,d}} = C(x)^d \mathbf{I}_N + \mathbf{A},$$

where

$$\mathbf{A} = \begin{pmatrix} dC(x)^{d-1}x_1x_1^* & dC(x)^{d-1}x_1x_2^* & \cdots & dC(x)^{d-1}x_1x_N^* \\ dC(x)^{d-1}x_2x_1^* & dC(x)^{d-1}x_2x_2^* & \cdots & dC(x)^{d-1}x_2x_N^* \\ \vdots & \vdots & \ddots & \vdots \\ dC(x)^{d-1}x_Nx_1^* & dC(x)^{d-1}x_Nx_2 & \cdots & dC(x)^{d-1}x_Nx_N^* \end{pmatrix}.$$

Then we get

$$\det(\mathbf{J}_{L_{N,d}}) = (-1)^N \det(-C(x)^d \mathbf{I}_N - \mathbf{A}) = (1 + dq)C(x)^{Nd}.$$

We now define the mapping $D_{N,d,p}$, with $p > 1$, by

$$D_{N,d,p}u(x) := \left(\frac{1}{1 + dq}\right)^{\frac{p-1}{p}} u(L_{N,d}(x)) = \left(\frac{1}{1 + dq}\right)^{\frac{p-1}{p}} u(C(x)^d x).$$

We also define

$$D_{N,d,p}^{-1}u = v \quad \text{if } u = D_{N,d,p}v.$$

Under the transform $D_{N,d,p}$, we also have the following result.

Lemma 6.2. (1) *For continuous function f, we have*

$$\int_{\mathbb{R}^N} \frac{f\big((\frac{1}{1+dq})^{\frac{p-1}{p}} u(x)\big)}{C(x)^t}\,dx = (1+dq)\int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-Nd}}\,dx.$$

In particular, we obtain that $u \in L^s(dx/C(x)^t)$ if and only if $D_{N,d,p}u \in L^s(dx/C(x)^{t(dq+1)-Nd})$.

(2) *For smooth functions u, we have*

$$\int_{\mathbb{R}^N} \frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-Nd}}\,dx \le \int_{\mathbb{R}^N} \frac{C^*(\nabla u(y))}{C(y)^{\mu}}\,dy.$$

The equality occurs if and only if u is C-radially symmetric.

Proof. (1) We have

$$\int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-Nd}}\,dx = \frac{1}{1+dq}\int_{\mathbb{R}^N} \frac{f\big((\frac{1}{1+dq})^{\frac{p-1}{p}} u(y)\big)}{C(y)^t}\,dy.$$

Using the change of variables $y_i = C(x)^d x_i$, $i = 1, 2, \dots, N$, we have

$$dy = \det(\mathbf{J}_{L_{N,d}})\,dx = (1+dq)C(x)^{Nd}\,dx \quad \text{and} \quad dx = \frac{1}{(1+dq)}\frac{dy}{C(y)^{\frac{Nd}{dq+1}}}.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-Nd}}\,dx &= \int_{\mathbb{R}^N} \frac{f\big((\frac{1}{1+dq})^{\frac{p-1}{p}} u(C(x)^d x)\big)}{C(x)^{t(dq+1)-Nd}}\,dx\\ &= \frac{1}{1+dq}\int_{\mathbb{R}^N} \frac{f\big((\frac{1}{1+dq})^{\frac{p-1}{p}} u(y)\big)}{C(y)^{\frac{t(dq+1)-Nd}{dq+1}}}\frac{dy}{C(y)^{\frac{Nd}{dq+1}}}\\ &= \frac{1}{1+dq}\int_{\mathbb{R}^N} \frac{f\big((\frac{1}{1+dq})^{\frac{p-1}{p}} u(y)\big)}{C(y)^t}\,dy.\end{aligned}$$

(2) Now we begin to consider the gradient of $D_{N,d,p}u$. After calculations, we have

$$\begin{pmatrix} \frac{\partial D_{N,d,p}u}{\partial x_1}(x)\\ \frac{\partial D_{N,d,p}u}{\partial x_2}(x)\\ \vdots\\ \frac{\partial D_{N,d,p}u}{\partial x_N}(x)\end{pmatrix} = \nabla D_{N,d,p}u(x) = \Big(\frac{1}{1+dq}\Big)^{\frac{p-1}{p}}\nabla(u(C(x)^d x)) = \Big(\frac{1}{1+dq}\Big)^{\frac{p-1}{p}}\mathbf{J}_{L_{N,d}}^T\begin{pmatrix} \frac{\partial u}{\partial x_1}(C(x)^d x)\\ \frac{\partial u}{\partial x_2}(C(x)^d x)\\ \vdots\\ \frac{\partial u}{\partial x_N}(C(x)^d x)\end{pmatrix}.$$

Hence, we have

$$\frac{\partial u(C(x)^d x)}{\partial x_i} = \Big(C(x)^d\frac{\partial u}{\partial x_i}(C(x)^d x) + A_i\Big)$$

for $i = 1, 2, \dots, N$, where

$$A_i := \sum_{j=1}^N dC(x)^{d-1}x_i^* x_j\frac{\partial u}{\partial x_j}(C(x)^d x)$$

and

$$C^*(X) = \sup\frac{|X\cdot Y|^p}{p[qC(Y)]^{\frac{p}{q}}}.$$

Hence, we obtain

$$\begin{aligned}C^*(\nabla D_{N,d,p}u(x)) &= C^*\Big(\Big(\frac{1}{1+dq}\Big)^{\frac{p-1}{p}}\nabla(u(C(x)^d x))\Big)\\ &= \Big(\frac{1}{1+dq}\Big)^{p-1} C^*(\nabla(u(C(x)^d x)))\\ &= \Big(\frac{1}{1+dq}\Big)^{p-1}\sup_y\Big\{\frac{(\nabla(u(C(x)^d x))\cdot y)^p}{p[qC(y)]^{\frac{p}{q}}}\Big\}\\ &= \Big(\frac{1}{1+dq}\Big)^{p-1}\sup_y\Big\{\frac{[\sum_{i=1}^N [C(x)^d \frac{\partial u}{\partial x_i}(C(x)^d x)y_i + A_i y_i]]^p}{p[qC(y)]^{\frac{p}{q}}}\Big\}.\end{aligned}$$

The first term is easy to compute. We have

$$\begin{aligned}I_1 &= \sum_{i=1}^N C(x)^d \frac{\partial u}{\partial x_i}(C(x)^d x)y_i\\ &= C(x)^d \nabla u(C(x)^d x)\cdot y\\ &\le C(x)^d [qC(y)]^{\frac{1}{q}}[pC^*(\nabla u(C(x)^d x))]^{\frac{1}{p}}.\end{aligned}$$

Applying the Cauchy–Schwarz inequality

$$X\cdot Y \le [qC(Y)]^{\frac{1}{q}}[pC^*(X)]^{\frac{1}{p}},$$

we can estimate the second term as follows:

$$\begin{aligned}I_2 &= \sum_{i=1}^N A_i y_i\\ &= \sum_{i=1}^N\sum_{j=1}^N dC(x)^{d-1}x_i^* x_j \frac{\partial u}{\partial x_j}(C(x)^d x)y_i\\ &= dC(x)^{d-1}\sum_{i=1}^N x_i^* y_i \sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}(C(x)^d x)\\ &\le dC(x)^{d-1}|x^*\cdot y||x\cdot \nabla u(C(x)^d x)|\\ &\le dC(x)^{d-1}[qC(y)]^{\frac{1}{q}}[pC^*(x^*)]^{\frac{1}{p}}[qC(x)]^{\frac{1}{q}}[pC^*(\nabla u(C(x)^d x))]^{\frac{1}{p}}\\ &\le dC(x)^{d-1}[qC(y)]^{\frac{1}{q}}[qC(x)]^{\frac{1}{p}}[qC(x)]^{\frac{1}{q}}[pC^*(\nabla u(C(x)^d x))]^{\frac{1}{p}}\\ &\le qdC(x)^d[qC(y)]^{\frac{1}{q}}[pC^*(\nabla u(C(x)^d x))]^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_y\Big\{\frac{[\sum_{i=1}^N [C(x)^d \frac{\partial u}{\partial x_i}(C(x)^d x)y_i + A_i y_i]]^p}{p[qC(y)]^{\frac{p}{q}}}\Big\} &\le \sup_y\Big\{\frac{[(1+qd)]^p C(x)^{pd}[qC(y)]^{\frac{p}{q}}pC^*(\nabla u(C(x)^d x))}{p[qC(y)]^{\frac{p}{q}}}\Big\}\\ &= [(1+qd)]^p C(x)^{pd}C^*(\nabla u(C(x)^d x)).\end{aligned}$$

In conclusion, we get

$$C^*(\nabla D_{N,d,p}u(x)) \le (1+qd)C(x)^{pd}C^*(\nabla u(C(x)^d x)).$$

Using the change of variables again, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{C^*(\nabla u(y))}{C(y)^\mu}\,dy &= \int_{\mathbb{R}^N}\frac{C^*(\nabla u(C(x)^d x))}{C(C(x)^d x)^\mu}(1+dq)C(x)^{Nd}\,dx\\ &\ge \int_{\mathbb{R}^N}\frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu}C(x)^{pd}}C(x)^{Nd}\,dx\\ &= \int_{\mathbb{R}^N}\frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-Nd}}\,dx.\end{aligned}$$

Finally, it is easy to check that the equalities hold if and only if the equality in the Cauchy–Schwarz inequality occurs. This means that u is C-radially symmetric. □

We note here that we will mainly apply the above change of variables with $C(x) = \frac{1}{q}\|x\|^q$ and $C^*(x) = \frac{1}{p}\|x\|_*^p$. In this case, for ease of reference, we will use the transform

$$T_{N,d,p}u(x) := \Big(\frac{1}{d}\Big)^{\frac{p-1}{p}} u(\|x\|^{d-1}x).$$

We also define

$$T_{N,d,p}^{-1}u = v \quad \text{if } u = T_{N,d,p}v.$$

The following lemma is a restatement of Lemma 6.2.

Lemma 6.3. (1) *For continuous function f, we have*

$$\int_{\mathbb{R}^N} \frac{f((\frac{1}{d})^{\frac{p-1}{p}}u(x))}{\|x\|^t}\,dx = d\int_{\mathbb{R}^N} \frac{f(T_{N,d,p}u(x))}{\|x\|^{N+td-Nd}}\,dx.$$

In particular, we obtain that $u \in L^s(dx/\|x\|^t)$ if and only if $T_{N,d,p}u \in L^s(dx/\|x\|^{N+td-Nd})$.

(2) *If $\nabla u \in L^p(dx/\|x\|^\mu)$, then $\nabla T_{N,d,p} \in L^p(dx/\|x\|^{d(p+\mu-N)+N-p})$. Moreover,*

$$\int_{\mathbb{R}^N} \frac{\|\nabla T_{N,d,p}u(x)\|_*^p}{\|x\|^{d(p+\mu-N)+N-p}}\,dx \le \int_{\mathbb{R}^N} \frac{\|\nabla u(x)\|_*^p}{\|x\|^\mu}\,dx.$$

The equality occurs if and only if u is $\|\cdot\|$-radial.

6.2 Maximizers for CKN Inequalities with Arbitrary Norms

Consider the following class:

$$\begin{cases} 1 < p < p+\mu < N, & \theta \le \dfrac{N\mu}{N-p} \le s < N, \\ 1 \le q < r < \dfrac{Np}{N-p}, & a = \dfrac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r}. \end{cases}$$

We denote by $D_{\mu,\theta}^{p,q}(\mathbb{R}^N)$ the completion of the space of smooth compactly supported functions with the norm

$$\Big(\int_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu}\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta}\Big)^{\frac{1}{q}},$$

and we set

$$\mathrm{CKN}(N,\mu,\theta,s,p,q,r) = \sup_{u\in D_{\mu,\theta}^{p,q}(\mathbb{R}^N)} \frac{(\int_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^s})^{\frac{1}{r}}}{(\int_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu})^{\frac{a}{p}} (\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta})^{\frac{1-a}{q}}},$$

$$\mathrm{GN}(N,p,q,r,\mu,\theta,s) = \sup_{u\in D_{0,N+\theta d-Nd}^{p,q}(\mathbb{R}^N)} \frac{(\int_{\mathbb{R}^N} \frac{|u|^r}{\|x\|^{N+sd-Nd}}\,dx)^{\frac{1}{r}}}{(\int_{\mathbb{R}^N} \|\nabla u\|_*^p\,dx)^{\frac{a}{p}} (\int_{\mathbb{R}^N} \frac{|u|^q}{\|x\|^{N+\theta d-Nd}}\,dx)^{\frac{1-a}{q}}}.$$

Then, similarly as in Section 3, we can prove the following theorem.

Theorem 6.4. *Assume that* (C3) *holds. Then* $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ *can be achieved. Moreover, all the extremal functions of* $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ *are $\|\cdot\|$-radially symmetric.*

The proof of Theorem 6.4 is similar to that of Theorem 1.1 and will be omitted.

Furthermore, we can provide the maximizers for $\mathrm{CKN}(N,\mu,\theta,s,p,q,r)$ in the following two classes.

Theorem 6.5. *Assume that* (C2) *holds with* $\theta = s = \frac{N\mu}{N-p}$. *If* $p < r = p\frac{q-1}{p-1} < \frac{Np}{N-p}$, *then* $\mathrm{CKN}(N, \mu, \theta, s, p, q, r)$ *is achieved by maximizers of the form*

$$V_0(x) = A(1 + B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})^{-\frac{p-1}{q-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0.$$

Theorem 6.6. *Assume that* (C2) *holds with* $\theta = s = \frac{N\mu}{N-p}$. *If* $1 < q = p\frac{r-1}{p-1} < p$, *then* $\mathrm{CKN}(N, \mu, \theta, s, p, q, r)$ *is achieved if* $r > 2 - \frac{1}{p}$, *by maximizers of the form*

$$V_0(x) = A(1 - B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0.$$

Proofs of Theorems 6.5–6.6. When $r = p\frac{q-1}{p-1}$ and $s = \theta = \frac{N\mu}{N-p}$, from [13], we have that $\mathrm{GN}(N, p, q, r)$ is achieved by maximizers of the form

$$U_0(x) = A(1 + B\|x - \overline{x}\|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0,\ \overline{x} \in \mathbb{R}^N.$$

Now, let V_0 be a maximizer of $\mathrm{CKN}(N, s, \mu, p, q, r)$. Then $T_{N,d,p}V_0$ is a maximizer of $\mathrm{GN}(N, p, q, r)$ with $d = \frac{N-p}{N-p-\mu}$. Hence, $\mathrm{CKN}(N, s, \mu, p, q, r)$ can be attained by

$$V_0(x) = T^{-1}_{N,d,p}A(1 + B\|x - \overline{x}\|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}.$$

This means that

$$V_0(x) = A'(1 + B\||x|^{\frac{1}{d}-1}x - \overline{x}\|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p}}.$$

Noting that V_0 is $\|\cdot\|$-radial, we conclude that $\overline{x} = 0$, that is,

$$V_0(x) = A'(1 + B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})^{-\frac{p-1}{q-p}}.$$

Similarly, when $\theta = s = \frac{N\mu}{N-p}$, if $q = p\frac{r-1}{p-1}$ and $r > 2 - \frac{1}{p}$, then $\mathrm{CKN}(N, \mu, \theta, s, p, q, r)$ is achieved by maximizers of the form

$$V_0(x) = A(1 - B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}})_+^{-\frac{p-1}{r-p}} \quad \text{for some } A \in \mathbb{R},\ B > 0.$$ □

7 Further Comments

Let $d > 1$. Then under the transform $T_{N,d,p}$, the CKN inequality with the triple (s, μ, θ) could be converted to the one with the triple $(N + sd - Nd, d(p + \mu - N) + N - p, N + \theta d - Nd)$. We should note that

$$a = \frac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r} = \frac{[(N - (N + \theta d - Nd))r - (N - (N + sd - Nd))q]p}{[(N - (N + \theta d - Nd))p - (N - (d(p + \mu - N) + N - p) - p)q]r}.$$

This fact may be used to simplify the study of symmetry/symmetry breaking phenomena. For instance, we could prove the following theorem.

Theorem 7.1. *Assume that* $d = \frac{N-p}{N-p-\mu} > 1$ *and* $0 < a = \frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r} \le 1$. *If*

$$\mathrm{CKN}_1 = \sup_{u \in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^{N+sd-Nd}}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} \|\nabla u\|_*^p\, dx\big)^{\frac{a}{p}} \big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^{N+\theta d-Nd}}\big)^{\frac{1-a}{q}}}$$

has a $\|\cdot\|$*-radially symmetric maximizer, then*

$$\mathrm{CKN}_2 = \sup_{u \in D^{p,q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\big(\int_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^s}\big)^{\frac{1}{r}}}{\big(\int_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu}\big)^{\frac{a}{p}} \big(\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta}\big)^{\frac{1-a}{q}}}$$

is attained by some $\|\cdot\|$*-radial optimizers.*

Proof. Assume that U_0 is a $\|\cdot\|$-radial maximizer of CKN_1. We set $V_0 = T_{N,d,p}^{-1} U_0$, which implies $U_0 = T_{N,d,p} V_0$. We note that V_0 is $\|\cdot\|$-radial. Then, for any v, we get

$$\int_{\mathbb{R}^N} |v|^r \frac{dx}{\|x\|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^r}{\|x\|^{N+sd-Nd}}\,dx,$$

$$\int_{\mathbb{R}^N} |v|^q \frac{dx}{\|x\|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^q}{\|x\|^{N+\theta d-Nd}}\,dx,$$

$$\int_{\mathbb{R}^N} \frac{\|\nabla v\|_*^p}{\|x\|^\mu}\,dx \geq \int_{\mathbb{R}^N} \|\nabla T_{N,d,p}v\|_*^p\,dx,$$

and

$$\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{\|x\|^s} = d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|U_0|^r}{\|x\|^{N+sd-Nd}}\,dx,$$

$$\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{\|x\|^\theta} = d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|U_0|^q}{\|x\|^{N+\theta d-Nd}}\,dx,$$

$$\int_{\mathbb{R}^N} \frac{\|\nabla V_0\|_*^p}{\|x\|^\mu}\,dx = \int_{\mathbb{R}^N} \|\nabla U_0\|_*^p\,dx.$$

Hence,

$$\begin{aligned}
\frac{\left(\int_{\mathbb{R}^N} |v|^r \frac{dx}{\|x\|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} \frac{\|\nabla v\|_*^p}{\|x\|^\mu}\,dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |v|^q \frac{dx}{\|x\|^\theta}\right)^{\frac{1-a}{q}}}
&\leq \frac{\left(d^{1+\frac{p-1}{p}r} \int_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^r}{\|x\|^{N+sd-Nd}}\,dx\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} \|\nabla T_{N,d,p}v\|_*^p\,dx\right)^{\frac{a}{p}} \left(d^{1+\frac{p-1}{p}q} \int_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^q}{\|x\|^{N+\theta d-Nd}}\,dx\right)^{\frac{1-a}{q}}} \\
&\leq d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int_{\mathbb{R}^N} \frac{|U_0|^r}{\|x\|^{N+sd-Nd}}\,dx\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} \|\nabla U_0\|_*^p\,dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} \frac{|U_0|^q}{\|x\|^{N+\theta d-Nd}}\,dx\right)^{\frac{1-a}{q}}} \\
&= \frac{\left(\int_{\mathbb{R}^N} |V_0|^r \frac{dx}{\|x\|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} \frac{\|\nabla V_0\|_*^p}{\|x\|^\mu}\,dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{\|x\|^\theta}\right)^{\frac{1-a}{q}}}.
\end{aligned}$$

We note that the last equality holds because U_0 and V_0 are $\|\cdot\|$-radial. Hence, V_0 is a $\|\cdot\|$-radial maximizer of CKN_2. □

As an application of Theorem 7.1, to study the symmetry problem of maximizers for the CKN inequality (with the assumption that $\frac{N-p}{N-p-\mu} > 1$), we can assume that $\mu = 0$.

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