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Research Article

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Two-Phase Free Boundary Problems: From Existence to Smoothness

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Abstract: We describe the theory we developed in recent times concerning two-phase free boundary problems governed by elliptic operators with forcing terms. Our results range from existence of viscosity solutions to smoothness of both solutions and free boundaries. We also discuss some open questions, possible object of future investigation.

Keywords: Two-Phase Free Boundary Problem, Free Boundary Regularity, Perron Solutions, Elliptic Systems

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Dedicated to Ireneo for his 70th birthday with all our friendship

1 Introduction

In the last few years, significant progress has been achieved in the analysis of two-phase free boundary problems governed by elliptic equations with forcing terms. In this survey paper we give an account of our main contributions and to avoid technicalities we refer to the following model problem:

$$\begin{cases} \Delta u = f_+ & \text{in } B_1^+(u), \\ \Delta u = f_- & \text{in } B_1^-(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial B_1^+. \end{cases} \tag{1.1}$$

Here B_1 is the unit ball in $\mathbb{R}^n$, centered at the origin, $f_\pm \in C(B_1) \cap L^\infty(B_1)$ and

$$B_1^+(u) := \{x \in B_1 : u(x) > 0\}, \quad B_1^-(u) := \{x \in B_1 : u(x) \le 0\}^\circ.$$

Moreover, u^+ and u^- denote the positive and negative part of u, respectively. $F(u)$ is the so-called free boundary of u.

This type of problem arises in a number of applied contexts, such as the Prandtl–Bachelor model in fluid-dynamics (see, e.g., [2, 15]), the eigenvalue problem in magnetohydrodynamics [21], or in flame propagation models [24]. We will comment on more general problems in the final section of this paper.

The theory for problem (1.1) can be developed according to a well-established paradigm:

(a) Existence and optimal regularity of solutions, e.g., viscosity or variational solutions, or solutions obtained as a limit of singular perturbations.

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(b) Weak regularity properties of the free boundary, such as finite perimeter and density properties for the positivity set.
(c) Strong regularity properties of the free boundary. For instance Lipschitz or "flat" free boundaries are C^1 or better.
(d) Higher regularity: Schauder estimates and analyticity for both solution and free boundary.

In the homogeneous case, i.e. $f_\pm = 0$, in his pioneer work [3, 5] Caffarelli obtained strong regularity properties of the free boundary. Subsequently, in [4] he showed existence of Lipschitz viscosity solutions which enjoy weak regularity properties of the free boundary. The Lipschitz regularity of viscosity solutions to homogeneous problems relies on a monotonicity formula by Alt, Caffarelli and Friedman [1]. For the inhomogeneous case, Lipschitz regularity was obtained by Caffarelli, Jerison and Kenig in [6]. For further results on homogeneous free boundary problem see for example [17–20, 28, 29].

The paper is organized as follows. In Sections 2 and 3 we provide a brief description of the results and the main ideas introduced in the papers [8, 9, 11, 12], concerning points (a), (b) and (c) above. We focus mostly on the strong regularity results, as our novel approach differs from the above mentioned work of Caffarelli. Section 4 focuses on point (d), i.e. higher regularity issues with an account of our recent results in [13]. Our higher regularity results for two-phase problems are new even in the homogeneous case, with the exception of [16], where a free boundary problem for the harmonic measure is considered. Interestingly, the proof of higher regularity presents somewhat unexpected features, proper of genuine two-phase problems. In Section 5 we indicate possible generalization of our results and emphasize some open questions.

2 Existence of Lipschitz Viscosity Solutions and Weak Regularity Properties of the Free Boundary

In this section we describe our existence result. In [11], we use Perron's method to construct Lipschitz viscosity solutions to free boundary problems with forcing terms (for a given boundary data), thus extending the results of [4] to the inhomogeneous case. Our results hold for operators $\mathcal{L} = \operatorname{div}(A(x)\nabla)$ with Hölder coefficients and general free boundary conditions $|\nabla u^+| = G(|\nabla u^-|)$, with G Lipschitz continuous, strictly increasing and $G(0) > 0$. In this section however we consider the model problem (1.1).

We start by recalling the definition of viscosity solution.

We say that a point $x_0 \in F(u)$ is regular from the right (resp. left) if there is a ball $B \subset B_1^+(u)$ (resp. $B_1^-(u)$), such that $B \cap F(u) = \{x_0\}$.

Definition 2.1. We say that $u \in C(B_1)$ is a viscosity solution of free boundary problem (1.1) if the following holds:
(i) $\Delta u = f_+$ in $B_1^+(u)$ and $\Delta u = f_-$ in $B_1^-(u)$.
(ii) u satisfies the free boundary condition in the following sense:
(1) If $x_0 \in F(u)$ is regular from the right with tangent ball B then

$$u^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B, \text{ with } \alpha \geq 0,$$
$$u^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \beta \geq 0$$

with equality along every non-tangential domain, and $\alpha^2 - \beta^2 \leq 1$.
(2) If $x_0 \in F(u)$ is regular from the left with tangent ball B, then

$$u^-(x) \geq \beta\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B, \text{ with } \beta \geq 0,$$
$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \alpha \geq 0$$

with equality along every non-tangential domain, and $\alpha^2 - \beta^2 \geq 1$.
Here $\nu = \nu(x_0)$ denotes the unit normal to ∂B at x_0, pointing towards $B_1^+(u)$.

The notion of viscosity solution can be also given in terms of test functions. Given $u, \varphi \in C(B_1)$, we say that φ touches u by below (resp. above) at $x_0 \in B_1$, if $u(x_0) = \varphi(x_0)$ and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

Then $u \in C(B_1)$ is a viscosity solution to (1.1) if (i) holds and (ii) is replaced by (ii′):

(ii′) Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ ($B = B_\delta(x_0)$, δ small) with $F(v) \in C^2$. If v touches u by below (resp. above) at x_0, then

$$|\nabla v^+(x_0)|^2 - |\nabla v^-(x_0)|^2 \leq 1 \quad (\text{resp. } \geq 1).$$

Our solution is constructed as the infimum over a class of admissible supersolutions $\mathcal{F}$ which we define below.

Definition 2.2. A function $w \in C(\overline{B}_1)$ is in $\mathcal{F}$ if the following holds:

(i) w is a solution to

$$\begin{cases} \Delta w \leq f_+ & \text{in } B_1^+(w), \\ \Delta w \leq f_- \chi_{\{w<0\}} & \text{in } B_1^-(w). \end{cases}$$

(ii) If $x_0 \in F(u)$ is regular from the left, then, near x_0,

$$\begin{aligned} w^+ &\leq \alpha\langle x - x_0, \nu(x_0)\rangle^+ + o(|x - x_0|), \quad \alpha \geq 0, \\ w^- &\geq \beta\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|), \quad \beta \geq 0, \end{aligned}$$

with $\alpha^2 - \beta^2 < 1$.

(iii) If $x_0 \in F(w)$ is not regular from the left, then, near x_0,

$$w(x) = o(|x - x_0|).$$

We also need to introduce a minorant subsolution. We say that a *locally Lipschitz* function $\underline{u}$, defined in B_1, is a *minorant* if the following holds:

(i) $\underline{u}$ is a weak solution to

$$\begin{cases} \Delta \underline{u} \geq f_+ & \text{in } B_1^+(\underline{u}), \\ \Delta \underline{u} \geq f_- \chi_{\{\underline{u}<0\}} & \text{in } B^-(\underline{u}). \end{cases}$$

(ii) Every $x_0 \in F(u)$ is regular from the right and, near x_0,

$$\begin{aligned} \underline{u}^- &\leq \beta\langle x - x_0, \nu(x_0)\rangle^+ + o(|x - x_0|), \\ \underline{u}^+ &\geq \alpha\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|), \end{aligned}$$

with $\alpha^2 - \beta^2 > 1$.

We are now ready to present our main result. Consider the problem

$$\begin{cases} \Delta u = f_+ & \text{in } B_1^+(u), \\ \Delta u = f_- \chi_{\{u<0\}} & \text{in } B_1^-(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial B_1^+. \end{cases} \tag{2.1}$$

Theorem 2.3. *Let φ be a continuous function on ∂B_1 and $\underline{u}$ a minorant of our free boundary problem, with boundary data φ. Then*

$$u = \inf\{w : w \in \mathcal{F},\ w \geq \underline{u} \text{ in } \overline{B_1}\}$$

is a locally Lipschitz viscosity solution to (2.1) such that $u = \varphi$ on ∂B_1, as long as the set on the right is non-empty. The free boundary $F(u)$ has finite $(n-1)$-dimensional Hausdorff measure and there exist universal positive constants c, C, r_0 such that for every $r < r_0$ and every $x_0 \in F(u)$,

$$cr^{n-1} \leq \mathcal{H}^{n-1}(F(u) \cap B_r(x_0)) \leq Cr^{n-1}.$$

Moreover, if $F^*(u)$ denotes the reduced part of $F(u)$, then

$$\mathcal{H}^{n-1}(F(u) \setminus F^*(u)) = 0.$$

The proof follows the main guidelines of [4]. The presence of a distributed source requires to face some new technical delicate points. For instance, the classical harmonic replacement technique does not work in this context and one has to resort to suitable auxiliary obstacle problems. For the details we refer to [11].

The following theorem is a consequence of the results in the next sections.

Theorem 2.4. *$F(u)$ is a $C^{1,\bar{\gamma}}$ surface in a neighborhood of $\mathcal{H}^{n-1}$ a.e. point $x_0 \in F(u)$. Moreover, if $f_\pm \in C^{k,\gamma}$ (resp. C^∞, analytic) then $F(u)$ is C^{k+2,γ^*} (resp. C^∞, analytic) in a neighborhood of $\mathcal{H}^{n-1}$ a.e. point $x_0 \in F(u)$, for a universal γ^* depending on $n, \gamma, \|f_\pm\|_{k,\gamma}$.*

3 Strong Regularity Results

In this section we describe our strong regularity results for the free boundary. Precisely, we explain the strategy to show that *flat* or *Lipschitz* free boundaries are $C^{1,\gamma}$. Our strategy differs from the one developed in [3, 5] for the homogeneous case. For the details of the proofs we refer to [9].

A way to express the flatness of the free boundary is to assume that $F(u)$ (or the zero set of u^+) is trapped between two parallel hyperplanes at δ-distance from each other, for a small δ (δ-flatness). While this looks like a somewhat strong assumption, it is indeed a natural one since it is satisfied for example by rescaling a solution around a point of the free boundary where there is a normal in some weak sense (*regular point*), for instance in the measure theoretical one (see Theorem 2.4).

Our main theorems read as follows [9]. We assume that $f_\pm$ are continuous with $\|f_\pm\|_{L^\infty(B_1)} \le L$ and let u be a Lipschitz viscosity solution to (1.1) in B_1, with $\mathrm{Lip}(u) \le L$. Universal constants depend only on n, L.

Theorem 3.1. *There exists a universal constant $\bar{\delta} > 0$ such that, if $0 \le \delta \le \bar{\delta}$ and*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\} \quad (\delta\text{-flatness}), \tag{3.1}$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.

Theorem 3.2. *If $F(u)$ is a Lipschitz graph in B_1, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.*

Theorem 3.2 follows from Theorem 3.1. Here is an outline of the proof. First we show that we can find $\sigma > 0$, small, depending on u such that $F(u)$ is a $C^{1,\gamma}$ graph in B_σ. Indeed, there exists a blow-up sequence $u_k = u(r_k x)/r_k$ which converges as $r_k \to 0$, up to rotations, to a two-plane solution

$$U_\beta(x_n) = \alpha x_n^+ - \beta x_n^-, \quad \beta \ge 0, \quad \alpha^2 - \beta^2 = 1.$$

This follows from a Weiss-type monotonicity formula and a dimension reduction argument. The conclusion now follows from the flatness Theorem 3.1.

Next we use a compactness argument to show that σ depends only on the Lipschitz constant L of $F(u)$. For this we need to show that $F(u)$ is $\bar{\delta}$-flat in B_r for some $r \ge \sigma$ depending on L. If by contradiction no such σ exists, then we can find a sequence of solutions u_k and of $\sigma_k \to 0$ such that $F(u_k)$ is not $\bar{\delta}$-flat in any B_r with $r \ge \sigma_k$. Then the u_k converge uniformly (up to a subsequence) to a solution u_* and we reach a contradiction since $F(u_*)$ is $C^{1,\gamma}$ in a neighborhood of 0 by the first part of the proof.

3.1 Non-degenerate Versus Degenerate

The proof of Theorem 3.1 is based on an iterative procedure that "squeezes" our solution around an optimal limiting configuration $U_\beta(x \cdot \nu)$ at a geometric rate in dyadically decreasing balls. Here ν is a unit vector, which plays the role of the normal vector at the origin (say $0 \in F(u)$). This plan of flatness improvement works nicely in the one-phase case ($\beta = 0$) or as long as the two phases u^+, u^- are, say, comparable (*non-degenerate case*). The difficulties arise when the negative phase becomes very small but at the same time not negligible (*degenerate case*). In this case the flatness assumption in Theorem 3.1 gives a control of the positive phase only, through the closeness to a *one-plane solution* $U_0(x \cdot \nu) = (x \cdot \nu)^+$.

First of all, the flatness condition (3.1) implies that u is close to U_β for some $\beta \geq 0$. Indeed we prove that, for small universal parameters η and $\bar\rho$,

$$\|u - U_\beta\|_{L^\infty(B_{\bar\rho})} \leq \eta\bar\rho. \tag{3.2}$$

A closer look to (3.2) reveals that, when α and β are comparable, a nice control on the location of $F(u)$ is available. However, when $\beta \ll \alpha$, only a one-side control of $F(u)$ is possible. This dichotomy is made evident if we translate the “vertical” closeness between the graphs of u and U_β, given by (3.2), into “horizontal” closeness. By rescaling we may take $\bar\rho = 1$ in (3.2). Then, setting $\eta^{1/3} = \varepsilon$, we get the following lemma.

Lemma 3.3. *If $\beta \geq \varepsilon$, then*

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_{3/4}.$$

If $\beta < \varepsilon$, then

$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_{3/4}.$$

Thus, the dichotomy *non-degenerate versus degenerate* translates quantitatively into the two cases:

$$\beta \geq \varepsilon : \text{non-degenerate}, \qquad \beta < \varepsilon : \text{degenerate}.$$

3.2 Improvement of Flatness. Non-degenerate Case

In this case, the basic step in the improvement of flatness reads as follows. Assume that for some $\varepsilon > 0$, small, we have

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1, \tag{3.3}$$

with $0 < \beta \leq L$, $\alpha^2 - \beta^2 = 1$, and say $0 \in F(u)$. One would like to get in a smaller ball an improvement of (3.3). After a rescaling we may assume that f is small compared to β, in particular,

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^3 \leq \varepsilon^2 \min\{\alpha, \beta\}.$$

Lemma 3.4. *If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then*

$$U_{\beta'}\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \leq u(x) \leq U_{\beta'}\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \text{in } B_r, \tag{3.4}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde C\varepsilon$, and $|\beta - \beta'| \leq \tilde C\beta\varepsilon$ for a universal constant $\tilde C$.

To prove Theorem 3.1 we rescale considering a blow-up sequence

$$u_k(x) = \frac{u(\bar r^k x)}{\bar r^k}, \quad x \in B_1$$

for suitable $\bar r$, and iterate Lemma 3.4 to get, at the k-th step,

$$U_{\beta_k}(x \cdot \nu_k - \bar r^k \varepsilon_k) \leq u(x) \leq U_{\beta_k}(x \cdot \nu_k + \bar r^k \varepsilon_k) \quad \text{in } B_{\bar r^k},$$

with $\varepsilon_k = 2^{-k}\varepsilon$, $|\nu_k| = 1$, $|\nu_k - \nu_{k-1}| \leq \tilde C\varepsilon_{k-1}$, and

$$|\beta_k - \beta_{k-1}| \leq \tilde C\beta_{k-1}\varepsilon_{k-1}, \quad \varepsilon_k \leq \beta_k \leq L.$$

Note that at each step we have the correct inductive hypotheses.

This implies that $F(u)$ is $C^{1,\gamma}$ at the origin. Repeating the procedure for points in a neighborhood of $x = 0$, since all estimates are universal, we conclude that there exist a unit vector $\nu_\infty = \lim \nu_k$ and $C > 0$, $\gamma \in (0, 1]$, both universal, such that, in the coordinate system $e_1, \dots, e_{n-1}, \nu_\infty, \nu_\infty \perp e_j, e_j \cdot e_k = \delta_{jk}$, $F(u)$ is a $C^{1,\gamma}$ graph, say $x_n = f(x')$, with $f(0') = 0$ and

$$\big|f(x') - \nu_\infty \cdot x'\big| \leq C|x'|^{1+\gamma}$$

in a neighborhood of $x = 0$.

The main question is: *Where is the information allowing one to realize the step from* (3.3) *to* (3.4) *hidden*? Here a *linearized problem* comes into play.

3.3 The Linearized Problem

The flatness condition (3.3) suggests the renormalization

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha\varepsilon}, & x \in B_1^+(u) \cup F(u), \\ \dfrac{u(x) - \beta x_n}{\beta\varepsilon}, & x \in B_1^-(u) \end{cases}$$

or

$$u(x) = \begin{cases} \alpha x_n + \varepsilon\alpha\tilde{u}_\varepsilon(x), & x \in B_1^+(u) \cup F(u), \\ \beta x_n + \varepsilon\beta\tilde{u}_\varepsilon(x), & x \in B_1^-(u). \end{cases} \tag{3.5}$$

In (3.5), u appears as a first-order perturbation of $U_\beta(x_n)$. The idea is that the key information we are looking for is stored precisely in the "coefficient" $\tilde{u}_\varepsilon$. To extract it, we look at what happens to $\tilde{u}_\varepsilon$, asymptotically as $\varepsilon \to 0$. Note that, as $\varepsilon \to 0$, $B_1^+(u) \to \{x_n > 0\}$, $B_1^-(u) \to \{x_n < 0\}$ and $F(u)$ goes to $\{x_n = 0\}$, all in Hausdorff distance. We have

$$\Delta\tilde{u}_\varepsilon = \frac{f_+}{\alpha\varepsilon} \sim \varepsilon \quad \text{in } B_1^+(u) \cup B_1^-(u).$$

On $F(u)$,

$$|\nabla u^+| = \alpha|e_n + \varepsilon\nabla\tilde{u}_\varepsilon^+(x)| \sim \alpha(1 + \varepsilon(\tilde{u}_\varepsilon^+)_{x_n} + \varepsilon^2|\nabla\tilde{u}_\varepsilon^+|^2),$$
$$|\nabla u^-| = \beta|e_n + \varepsilon\nabla\tilde{u}_\varepsilon^-(x)| \sim \beta(1 + \varepsilon(\tilde{u}_\varepsilon^-)_{x_n} + \varepsilon^2|\nabla\tilde{u}_\varepsilon^-|^2)$$

and

$$0 = |\nabla u^+|^2 - |\nabla u^-|^2 - 1 \sim 2\varepsilon[\alpha^2(\tilde{u}_\varepsilon^+)_{x_n} - \beta^2(\tilde{u}_\varepsilon^-)_{x_n}] + O(\varepsilon^2).$$

Dividing by ε and letting $\varepsilon \to 0$, we get for "the limit" $\tilde{u}$ of u_ε, the following problem:

$$\Delta\tilde{u} = 0 \quad \text{in } B_1^+ \cup B_1^- \tag{3.6}$$

and the transmission condition (linearization of the free boundary condition)

$$\alpha^2\tilde{u}_{x_n}^+ - \beta^2\tilde{u}_{x_n}^- = 0 \quad \text{on } B_1 \cap \{x_n = 0\}. \tag{3.7}$$

The crucial information we were mentioning before is contained in the following regularity result.

Theorem 3.5. *Let $\tilde{u}$ be a viscosity solution to the transmission problem* (3.6)–(3.7) *in B_1 such that $\|\tilde{u}\|_\infty \le 1$. Then $\tilde{u} \in C^\infty(\bar{B}_1^\pm)$ and in particular, there exists a universal constant $\bar{C}$ such that*

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \le \bar{C}r^2 \quad \textit{in } B_r \tag{3.8}$$

for all $r \le \frac{1}{2}$ and with $\alpha^2\tilde{p} - \beta^2\tilde{q} = 0$.

The question is now to transfer estimate (3.8) to $\tilde{u}_\varepsilon$ and then read it in terms of flatness for u through the formulas (3.5). The right way to proceed is to argue by contradiction.

Fix $r \le r_0$, to be chosen suitably. Assume that for a sequence $\varepsilon_k \to 0$ there is a sequence u_k of solutions of our free boundary problem in B_1, with right-hand side f_k such that $\|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^2\beta_k$ and

$$U_{\beta_k}(x_n - \varepsilon_k) \le u_k(x) \le U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1 \; 0 \in F(u_k),$$

with $0 \le \beta_k \le L$, $\alpha_k^2 - \beta_k^2 = 1$, but the conclusion of Lemma 3.1 does not hold for every $k \ge 1$.

Construct the corresponding sequence of renormalized functions

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k), \\ \dfrac{u_k(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(u_k). \end{cases} \tag{3.9}$$

At this point we need a compactness property to show that u_k converges uniformly (up to a subsequence) to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$. Also $\alpha_k \to \alpha$, $\beta_k \to \beta$, with $\alpha^2 - \beta^2 = 1$. The compactness is provided by the Harnack inequality stated in Theorem 3.6 and its corollary, as we shall see later.

It turns out that the limit function $\tilde{u}$ satisfies the linearized problem (3.6)–(3.7) in the viscosity sense. Hence, from (3.8), having $\tilde{u}(0) = 0$,

$$|\tilde{u}(x) - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \le Cr^2, \quad x \in B_r, \tag{3.10}$$

for all $r \le \frac{1}{4}$ (say), with

$$\alpha^2\tilde{p} - \beta^2\tilde{q} = 0, \quad |\nu'| = |\nabla_{x'}\tilde{u}(0)| \le C.$$

Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ in $B_{1/2}$, inequality (3.10) transfers to $\tilde{u}_k$:

$$|\tilde{u}_k(x) - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \le C'r^2, \quad x \in B_r.$$

Set

$$\beta_k' = \beta_k(1 + \varepsilon_k\tilde{q}), \quad \nu_k = \frac{1}{\sqrt{1 + \varepsilon_k^2|\nu'|^2}}(e_n + \varepsilon_k(\nu', 0)).$$

Then,

$$\alpha_k' = \sqrt{1 + \beta_k'^2} = \alpha_k(1 + \varepsilon_k\tilde{p}) + O(\varepsilon_k^2), \quad \nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2\tau, \quad |\tau| \le C,$$

where to obtain the first equality we used that $\alpha^2\tilde{p} - \beta^2\tilde{q} = 0$ and hence

$$\frac{\beta_k^2}{\alpha_k^2}\tilde{q} = \tilde{p} + o(1).$$

With these choices we can now show that (for k large and $r \le r_0$)

$$U_{\beta_k'}\Big(x \cdot \nu_k - \varepsilon_k\frac{r}{2}\Big) \le u_k(x) \le U_{\beta_k'}\Big(x \cdot \nu_k + \varepsilon_k\frac{r}{2}\Big) \quad \text{in } B_r$$

leading to a contradiction.

We are left to prove the compactness claim. The Harnack inequality takes the following form.

Theorem 3.6. *There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_1$ and u satisfies the condition*

$$U_\beta(x_n + a_0) \le u(x) \le U_\beta(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_1 \tag{3.11}$$

with

$$\|f\|_{L^\infty(B_2)} \le \varepsilon^2\beta, \quad 0 < \beta \le L$$

and

$$0 < b_0 - a_0 \le \varepsilon r$$

for some $0 < \varepsilon \le \tilde{\varepsilon}$, then

$$U_\beta(x_n + a_1) \le u(x) \le U_\beta(x_n + b_1) \quad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \le a_1 \le b_1 \le b_0 \quad \text{and} \quad b_1 - a_1 \le (1 - c)\varepsilon r$$

and $0 < c < 1$ universal.

If u satisfies (3.11) with, say $r = 1$, then we can apply the Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \le u(x) \le U_\beta(x_n + b_m) \quad \text{in } B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \le (1 - c)^m\varepsilon$$

for all m's such that

$$(1 - c)^m 20^m \varepsilon \le \tilde{\varepsilon}.$$

This implies that for all such m's, the oscillation of the renormalized functions $\tilde{u}_k$ in $B_r(x_0)$, $r = 20^{-m}$, is less than $(1 - c)^m = 20^{-\gamma m} = r^\gamma$. Thus, the following corollary holds.

Corollary 3.7. *Let $\tilde{u}_k$ be as defined in formula* (3.9). *Then*

$$\left|\tilde{u}_k(x) - \tilde{u}_k(x_0)\right| \le C|x - x_0|^\gamma$$

for all $x \in B_1(x_0)$ such that $|x - x_0| \ge \varepsilon_k/\bar{\varepsilon}$.

Note now that

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in B_1$$

and $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. These facts together with Ascoli–Arzelà theorem give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$.

Thus the improvement of flatness proof in the non-degenerate case can be concluded.

3.4 Improvement of Flatness. Degenerate Case

In this case, the negative part of u is negligible and the positive part is close to a one-plane solution, i.e. for some $\varepsilon > 0$ small, we have

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1.$$

This time the key lemma is the following.

Lemma 3.8. *Assume that $\|f\|_{L^\infty(B_1)} \le \varepsilon^4$ and*

$$\|u^-\|_{L^\infty(B_1)} \le \varepsilon^2. \tag{3.12}$$

There exists a universal r_1 such that if $0 < r \le r_1$ and $0 < \varepsilon \le \varepsilon_1$ for some ε_1 depending on r, then

$$U_0\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \le u^+(x) \le U_0\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \text{in } B_r,$$

with $|\nu_1| = 1$ and $|\nu_1 - e_n| \le C\varepsilon$ for a universal constant C.

The proof follows the pattern of the non-degenerate case.

Fix $r \le r_1$, to be chosen suitably. By contradiction assume that, for some sequences $\varepsilon_k \to 0$ and u_k, solutions of our free boundary problem in B_1 with right-hand side f_k, we have $\|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^4$ and

$$\|u_k^-\|_{L^\infty(B_1)} \le \varepsilon_k^2,$$
$$U_0(x_n - \varepsilon_k) \le u_k(x) \le U_0(x_n + \varepsilon_k) \quad \text{in } B_1, \ 0 \in F(u_k),$$

but the conclusion of the lemma does not hold.

Then one proves, via a one-phase version of the Harnack inequality in Theorem 3.6, that the sequence of normalized functions

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$

converges to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$. The limit function $\tilde{u}$ is a viscosity solution of the linearized problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_{x_n} = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

The regularity of $\tilde{u}$ is not a problem and the contradiction argument proceeds as before with obvious changes.

Lemma 3.8 provides the first step in the flatness improvement. Notice that this improvement is obtained through the closeness of the positive phase to a *one-plane solution*, as long as inequality (3.12) holds. This inequality expresses in another quantitative way the degeneracy of the negative phase and should be kept valid at each step of the iteration of Lemma 3.8. However, it could happen that this is not the case and in some step of the iteration, say at the level ε_k of flatness, the norm $\|u^-\|_{L^\infty(B_1)}$ becomes of order ε_k^2. When this occurs, a suitable rescaling restores a non-degenerate situation. This gives rise in the final iteration to a new dichotomy. The situation is precisely described in the following lemma.

Lemma 3.9. *Let u be a solution in B_1 satisfying*

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1$$

with

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4,$$

and for $\tilde{C}$ universal,

$$\|u^-\|_{L^\infty(B_2)} \le \tilde{C}\varepsilon^2, \quad \|u^-\|_{L^\infty(B_1)} > \varepsilon^2.$$

There exists (universal) ε_1 such that, if $0 < \varepsilon \le \varepsilon_1$, the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2} u(\varepsilon^{1/2} x)$$

satisfies, in $B_{2/3}$,

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \le u_\varepsilon(x) \le U_{\beta'}(x_n + C'\varepsilon^{1/2})$$

with $\beta' \sim \varepsilon^2$ and C' depending on $\tilde{C}$.

Let us see how the dichotomy arises. To prove our theorem in the degenerate case, choose $\bar{r}$ small (e.g., $\le \frac{1}{16}$) and assume $\beta < \varepsilon$. From

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1,$$

since

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \bar{\eta} = \varepsilon^3,$$

we infer

$$\|u^-\|_{L^\infty(B_1)} \le \beta + \varepsilon^3 \le 2\varepsilon.$$

Set $\varepsilon' = \sqrt{2\varepsilon}$. Then

$$U_0(x_n - \varepsilon') \le u^+(x) \le U_0(x_n + \varepsilon') \quad \text{in } B_1$$

and

$$\|f\|_{L^\infty(B_1)} \le (\varepsilon')^4, \quad \|u^-\|_{L^\infty(B_1)} \le (\varepsilon')^2.$$

From Lemma 3.8, we get

$$U_0\Big(x \cdot \nu_1 - \bar{r}\frac{\varepsilon'}{2}\Big) \le u^+(x) \le U_0\Big(x \cdot \nu_1 + \bar{r}\frac{\varepsilon'}{2}\Big) \quad \text{in } B_{\bar{r}}$$

with $|\nu_1| = 1$ and $|\nu_1 - e_n| \le C\varepsilon'$ for a universal constant C.

We now rescale considering the blow-up sequence for $k = 1, 2, \dots,$

$$u_k(x) = \frac{u(\bar{r}^k x)}{\bar{r}^k}, \quad x \in B_1$$

and set $\varepsilon_k = 2^{-k}\varepsilon'$,

$$f_k(x) = \bar{r}^k f(\bar{r}^k x), \quad x \in B_1.$$

Note that

$$\|f_k\|_{L^\infty(B_1)} \le \bar{r}^k (\varepsilon')^4 \le \frac{1}{16}(\varepsilon')^4 = \varepsilon_k^4.$$

We can iterate Lemma 3.8 and obtain

$$U_0(x \cdot \nu_k - \varepsilon_k) \le u_k^+(x) \le U_0(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1$$

with $|\nu_k - \nu_{k-1}| \le C\varepsilon_{k-1}$, *as long as*

$$\|u_k^-\|_{L^\infty(B_1)} \le \varepsilon_k^2.$$

Let $k^* > 1$ be the first integer for which this fails:

$$\|u_{k^*}^-\|_{L^\infty(B_1)} > \varepsilon_{k^*}^2$$

and

$$\|u^-_{k^*-1}\|_{L^\infty(B_1)} \le \varepsilon^2_{k^*-1}.$$

We also have

$$U_0(x \cdot \nu_{k^*-1} - \varepsilon_{k^*-1}) \le u^+_{k^*-1}(x) \le U_0(x \cdot \nu_{k^*-1} + \varepsilon_{k^*-1}) \quad \text{in } B_1.$$

By a usual comparison argument we can write

$$u^+_{k^*-1}(x) \le C|x_n - \varepsilon_{k^*-1}|\varepsilon^2_{k^*-1} \quad \text{in } B_{19/20}$$

for C universal. Rescaling, we have

$$\|u^-_{k^*}\|_{L^\infty(B_1)} \le C_1\varepsilon^2_{k^*}$$

where C_1 is universal (C_1 depends on $\bar{r}$). Then u_{k^*} satisfies the assumptions of Lemma 3.9 and therefore the rescaling

$$v(x) = \varepsilon^{-1/2}_{k^*} u_{k^*}(\varepsilon^{1/2}_{k^*} x)$$

satisfies, in $B_{2/3}$,

$$U_{\beta'}(x \cdot \nu_{k^*} - C'\varepsilon^{1/2}_{k^*}) \le v(x) \le U_{\beta'}(x \cdot \nu_{k^*} + C'\varepsilon^{1/2}_{k^*})$$

with $\beta' \sim \varepsilon^2_{k^*}$. Set $\hat{\varepsilon} = C'\varepsilon^{1/2}_{k^*}$. Then v is a solution of our free boundary problem in $B_{2/3}$ with right-hand side

$$g(x) = \varepsilon^{1/2}_{k^*} f_{k^*}(\varepsilon^{1/2}_{k^*} x)$$

and the flatness assumption

$$U_{\beta'}(x \cdot \nu_{k^*} - \hat{\varepsilon}) \le v(x) \le U_{\beta'}(x \cdot \nu_{k^*} + \hat{\varepsilon}).$$

Since $\beta' \sim \varepsilon^2_{k^*}$, we have

$$\|g\|_{L^\infty(B_1)} \le \varepsilon^{1/2}_{k^*}\varepsilon^4_{k^*} \le \hat{\varepsilon}^2\beta'$$

as long as ε is small enough. Under these restrictions, v satisfies the assumptions of the non-degenerate case and we can proceed accordingly.

4 Higher Regularity

4.1 Smoothness of Flat Free Boundaries

Assume now that $f_\pm \in C^\infty(B_1)$ or (real) analytic and, still, that the free boundary is flat. Thanks to the results of Section 3 we know that $F(u)$ is $C^{1,\gamma}$. As a consequence u is a classical solution, i.e. the free boundary condition is satisfied in a point-wise sense. The question is if also u and $F(u)$ are C^∞ or (real) analytic, respectively. It is well known that if u is at least C^2 up to the free boundary from both sides, through a zero-order hodograph transformation and a suitable reflection map, as in [23, Theorem 3.2] it follows that u and $F(u)$ are C^∞ or analytic. Thus the main point is to show that flat free boundaries are at least C^2. Our main theorem gives indeed C^{2,γ^*} regularity of flat free boundaries for a universal $\gamma^* \le \gamma$, provided $f_\pm \in C^{0,\gamma}(B_1)$. Precisely:

Theorem 4.1. *Let u be a (Lipschitz) viscosity solution to* (1.1) *in B_1. There exists a universal constant $\bar{\eta} > 0$ such that, if*

$$\{x_n \le -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \eta\} \quad \textit{for } 0 \le \eta \le \bar{\eta},$$

then $F(u)$ is C^{2,γ^*} in $B_{1/2}$ for a small γ^* universal, with the C^{2,γ^*} norm bounded by a universal constant.

Having proved C^{2,γ^*} regularity, we can also prove intermediate Schauder estimates:

Theorem 4.2. *Let k be a nonnegative integer. Assume that $f_\pm \in C^{k,\gamma}(B_1)$. Then $F(u) \cap B_{1/2}$ is C^{k+2,γ^*}. If $f_\pm$ are C^∞ or real analytic in B_1, then $F(u) \cap B_{1/2}$ is C^∞ or real analytic, respectively.*

Indeed, Theorem 4.2 follows by a direct application of [27, Theorem 6.8.2], after transforming problem (1.1) into an elliptic system with coercive boundary conditions. This can be done as in [23]. We recall the main computations. For σ small, the partial hodograph map

$$y' = x', \quad y_n = u^+(x)$$

is 1-1 from $\overline{B_1^+}(u) \cap B_\sigma(0)$ onto a neighborhood of the origin $U \subset \{y_n \geq 0\}$, and flattens $F(u)$ into a set $\Sigma \subset \{y_n = 0\}$. The inverse mapping is the partial Legendre transformation

$$x' = y', \quad x_n = \psi(y),$$

where ψ satisfies $y_n = u^+(y', \psi(y))$, $y \in U$. The free boundary is the graph of $x_n = \psi(y', 0)$. Differentiating, we get

$$dy_n = (\nabla' u^+ + \partial_{x_n} u^+ \nabla' \psi) \cdot dy' + \partial_{x_n} u^+ \partial_{y_n} \psi dy_n$$

from which

$$\partial_{x_n} u^+(y, \psi(y)) = \frac{1}{\partial_{y_n} \psi(y)}, \quad \nabla' u^+(y, \psi(y)) = -\frac{\nabla' \psi(y)}{\partial_{y_n} \psi(y)}$$

in U. Moreover, $\Delta u^+ = f_+$ transforms into

$$\mathcal{F}_1(\psi) :\equiv -\frac{\partial_{y_n y_n} \psi}{(\partial_{y_n} \psi)^3} + \sum_{j=1}^{n-1} \left(-\partial_{y_j} \frac{\partial_{y_j} \psi}{\partial_{y_n} \psi} + \frac{\partial_{y_j} \psi}{\partial_{y_n} \psi} \partial_{y_n} \frac{\partial_{y_j} \psi}{\partial_{y_n} \psi} \right) = f_+(y', \psi(y))$$

in U.

Concerning the negative part, let C be a constant larger than $\partial_{y_n} \psi$ in U. Introduce the reflection map

$$x' = y', \quad x_n = \psi(y) - C y_n,$$

which is 1-1 from a neighborhood of the origin $U_1 \subseteq U$ onto $\overline{B_1^-}(u) \cap B_\sigma(0)$ (choosing σ smaller, if necessary). Define, in U_1,

$$\varphi(y) = u^-(y', \psi(y) - C y_n).$$

Differentiating, we get

$$\nabla' \varphi \cdot dy' + \partial_{y_n} \varphi dy_n = (\nabla' u^- + \partial_{x_n} u^- \nabla' \psi) \cdot dy' + \partial_{x_n} u^- (\partial_{y_n} \psi - C) dy_n$$

from which

$$\partial_{x_n} u^- = \frac{\partial_{y_n} \varphi}{\partial_{y_n} \psi - C}, \quad \nabla' u^- = \nabla' \varphi - \frac{\partial_{y_n} \varphi}{\partial_{y_n} \psi - C} \nabla' \psi.$$

The equation $\Delta u^- = f_-$ in $\overline{B_1^-}(u) \cap B_\sigma(0)$ transforms into the equation

$$\begin{aligned} \mathcal{F}_2(\varphi, \psi) \equiv \frac{1}{\partial_{y_n} \psi - C} \partial_{y_n} \left(\frac{\partial_{y_n} \varphi}{\partial_{y_n} \psi - C} \right) + \sum_{j=1}^{n-1} \partial_{y_j} \left(\partial_{y_j} \varphi - \frac{\partial_{y_n} \varphi}{\partial_{y_n} \psi - C} \partial_{y_j} \psi \right) \\ - \sum_{j=1}^{n-1} \frac{\partial_{y_j} \psi}{\partial_{y_n} \psi - C} \partial_{y_n} \left(\partial_{y_j} \varphi - \frac{\partial_{y_n} \varphi}{\partial_{y_n} \psi - C} \partial_{y_j} \psi \right) = f_-(y', \psi(y) - C y_n) \end{aligned}$$

in U_1.

Thus, in U_1 we have the following nonlinear system:

$$\begin{cases} \mathcal{F}_1(\psi) = f_+(y', \psi(y)), \\ \mathcal{F}_2(\varphi, \psi) = f_-(y', \psi(y) - C y_n). \end{cases}$$

The free boundary conditions

$$u^+ = u^- \quad \text{and} \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 1 \quad \text{on } F(u)$$

become

$$\begin{cases} \varphi(y',0)=0, \\ \dfrac{1+|\nabla'\psi(y',0)|^2}{(\partial_{y_n}\psi(y',0))^2}-\dfrac{(\partial_{y_n}\varphi(y',0))^2}{(\partial_{y_n}\psi(y',0)-C)^2}-\Big\|\nabla'\varphi(y',0)-\dfrac{\partial_n\varphi(y',0)}{\partial_{y_n}-C}\nabla'\psi(y',0)\Big\|^2_{\mathbb{R}^{n-1}}=1. \end{cases}$$

That is, after a simple computation,

$$\begin{cases} \varphi(y',0)=0, \\ (1+|\nabla'\psi(y',0)|^2)\Big(\dfrac{1}{(\partial_{y_n}\psi(y',0))^2}-\dfrac{(\partial_{y_n}\varphi(y',0))^2}{(\partial_{y_n}\psi(y',0)-C)^2}\Big)=1. \end{cases}$$

Linearization at $y=0$ gives (setting $A=C-\partial_{y_n}\psi(0)$)

$$\begin{cases} \mathcal{L}_1(\psi)=|\nabla u^+(0)|^2\partial_{y_ny_n}\psi+\sum_{k=1}^{n-1}\partial_{y_ky_k}\psi=0, \\ \mathcal{L}_2(\psi,\varphi)=\dfrac{1}{A^2}\partial_{y_ny_n}\varphi+\sum_{k=1}^{n-1}\partial_{y_ky_k}\varphi-|\nabla u^-(0)|\Big(\dfrac{1}{A^2}\partial_{y_ny_n}\psi+\sum_{k=1}^{n-1}\partial_{y_ky_k}\psi\Big)=0, \\ \mathcal{B}_1(\varphi)=\varphi=0, \\ \mathcal{B}_2(\psi,\varphi)=\Big(|\nabla u^+(0)|^3+\dfrac{1}{A}|\nabla u^-(0)|^2\Big)\partial_{y_n}\psi-\dfrac{1}{A}|\nabla u^-(0)|\partial_{y_n}\varphi=0. \end{cases}$$

This system is elliptic with coercive boundary conditions under the natural choices of weights $s_1=s_2=0$, $t_1=t_2=2$ for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and $r_1=-2$, $r_2=-1$ for $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Indeed

$$\operatorname{order}\mathcal{L}_j=s_j+t_j=2\quad(j=1,2)$$

and

$$\operatorname{order}\mathcal{B}_1=t_1+r_1=0,\quad \operatorname{order}\mathcal{B}_2=t_2+r_2=1.$$

4.2 From $C^{1,\gamma}$ to C^{2,γ^*}. Outline and Strategy

The overall strategy for the proof of Theorem 4.1 is based again on a compactness argument leading to a limiting linearized problem in which the information for an improvement of flatness is stored. However, to reach the $C^{2,\gamma}$ regularity requires a much more involved process because of the possible degeneracy of the negative part. Indeed this causes a delicate interplay between the two phases, as we shall try to explain below. We give here an idea of the complexity of the proof by outlining the overall strategy. Ultimately the main source of difficulties is due to the presence of a forcing term of general sign in the negative phase. Indeed, if $f_-\geq 0$, the Hopf maximum principle would imply non-degeneracy (also) on the negative side, making the two phases of comparable size and considerably simplifying the final iteration procedure. It is worth noting that, even in this easier scenario (and in particular in the homogeneous case), if one wants to attain uniform estimates with universal constants, then one must employ the more involved methods developed in [12] for the degenerate case.

The first thing to do is to reinforce the notion of flatness, tailoring it for the attainment of $C^{2,\gamma}$ regularity. This can be done by introducing a suitable class of functions that we call *two-phase* and *one-phase* polynomials. In principle second-order polynomials should be enough but it turns out that we need a small third-order perturbation.

Let $\omega\in\mathbb{R}^n$, with $|\omega|=1$, and let S_ω be an orthonormal basis containing ω. Let $M\in S^{n\times n}$ satisfy $M\omega=0$. Define

$$P_{M,\omega}(x)=x\cdot\omega-\frac{1}{2}x^TMx.$$

Set

$$V^{\alpha,\beta}_{M,\omega,a,b}(x)=\alpha(1+a\cdot x)P^+_{M,\omega}(x)-\beta(1+b\cdot x)P^-_{M,\omega}(x),\quad \alpha>0,\ \beta\geq 0,\ a,b\in\mathbb{R}^n.$$

These are our two-phase polynomials, one-phase if $\beta = 0$. In the particular case when $M = 0$, $a = b = 0$ and $\omega = e_n$, we obtain the two-plane function

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-.$$

The unit vector ω establishes the "direction of flatness".

We shall need to work with a subclass, strictly related to problem (1.1), at least at the origin. We denote by $\mathcal{V}_{f_\pm}$ the class of functions of the form $V^{\alpha,\beta}_{M,\omega,a,b}$ for which

$$\begin{aligned} &2\alpha a \cdot \omega - \alpha trM = f_+(0) \\ &2\beta b \cdot \omega - \beta trM = f_-(0) \quad \text{if } \beta \neq 0, \\ &\alpha^2 - \beta^2 = 1, \quad \text{if } \beta \neq 0, \end{aligned}$$

and

$$\alpha^2 a \cdot \omega^\perp = \beta^2 b \cdot \omega^\perp \quad \text{for all } \omega^\perp \in S_\omega.$$

The role of the last condition is to make $V^{\alpha,\beta}_{M,\omega,a,b}$ an "almost" viscosity subsolution.

When $\beta = 0$, then there is no dependence on b and $a \cdot \omega^\perp = 0$. Thus, we drop the dependence on β, b and f_- in our notation above and we indicate the dependence on $a_\omega := a \cdot \omega$.

We introduce the following definitions.

Definition 4.3. Let $V = V^{\alpha,\beta}_{M,\omega,a,b}$. We say that u is (V, ε, δ) flat in B_1 if

$$V(x - \varepsilon\omega) \leq u(x) \leq V(x + \varepsilon\omega) \quad \text{in } B_1$$

and

$$|a|, |b'|, \|M\| \leq \delta\varepsilon^{1/2}, \quad |b_n| \leq \delta^2, \quad |b_n|\|M\| \leq \delta^2\varepsilon.$$

Given $V = V^{\alpha,\beta}_{M,\omega,a,b}$, set

$$V_r(x) = \frac{V(rx)}{r}$$

and notice that

$$V_r = V^{\alpha,\beta}_{rM,\omega,ra,rb}.$$

Definition 4.4. Let $V = V^{\alpha,\beta}_{M,\omega,a,b}$. We say that u is (V, ε, δ) flat in B_r if the rescaling

$$u_r(x) := \frac{u(rx)}{r}$$

is $(V_r, \frac{\varepsilon}{r}, \delta)$ flat in B_1.

Notice that if u is (V, ε, δ) flat in B_r then

$$V(x - \varepsilon\omega) \leq u(x) \leq V(x + \varepsilon\omega) \quad \text{in } B_r.$$

The parameter ε measures the level of polynomial approximation and δ is a flatness parameter (also controlling the $C^{0,\gamma}$ norms of f_+ and f_-).

To obtain uniform point-wise C^{2,γ^*} regularity both for the solution and the free boundary in $B_{1/2}$ we have to show that u is $(V_k, \lambda_k^{2+\gamma^*}, \delta)$ flat in B_{λ_k} for $\lambda_k = \eta^k$ and all $k \geq 0$, for some δ, η small and a sequence of V_k converging to a final profile V_0.

The starting point in the proof of Theorem 4.1 is to show that the flatness condition (3.1) allows us to normalize our solution so that a rescaling $u_{\bar{r}}$ of u is close to a one- or two-phase polynomial. This kind of dichotomy parallels in a sense what happens in the *flatness to* $C^{1,\gamma}$ case but at a quadratic order of approximation. Set

$$u_r(x) := \frac{u(rx)}{r}, \quad f_{\pm r}(x) = rf_\pm(rx), \quad x \in B_1.$$

Lemma 4.5. *There exist universal constants $\bar\varepsilon$, $\bar\delta$, $\bar\lambda$ such that if u satisfies (3.1) with $\bar\eta = \bar\eta(\bar\varepsilon)$ then either of these flatness conditions holds with $\bar r = \bar r(\bar\varepsilon)$.*

(i) *Non-degenerate case: $u_{\bar r}$ is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ flat in B_1, with $V = V^{\alpha,\beta}_{0,e_n,a,b} \in \mathcal{V}_{f_\pm}$,*

$$a' = b' = 0, \quad \beta \geq \frac{1}{2}\bar\delta^{1/2}\bar\lambda^{2+\gamma},$$

and

$$|f_{+\bar r}(x) - f_{+\bar r}(0)| \leq \bar\delta|x|^\gamma, \quad |f_{-\bar r}(x) - f_{-\bar r}(0)| \leq \beta\bar\delta|x|^\gamma.$$

(ii) *Degenerate case: $u^+_{\bar r}$ is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ flat in B_1, for $V = V^1_{0,e_n,a_n} \in \mathcal{V}_{f_+}$,*

$$\Big|u^-_{\bar r} + \frac{1}{2}f_{-\bar r}(0)x_n^2\Big| \leq \bar\delta^{1/2}\bar\lambda^{2+\gamma} \quad \text{in } B_1^-(u_{\bar r})$$

and

$$\|f_{-\bar r}\|_\infty \leq \bar\delta, \quad |f_{\pm\bar r}(x) - f_{\pm\bar r}(0)| \leq \bar\delta|x|^\gamma.$$

We describe the dichotomy as follows.

Case 1. It corresponds to a non-degenerate configuration, in which the two phases have comparable size and $u_{\bar r}$ is trapped between two translations of a genuine two-phase polynomials, with a positive slope β (not too small).

Case 2. It corresponds to a degenerate configuration, where the negative phase that has either zero slope or a small one (but not negligible) with respect to $u^+_{\bar r}$ and $u^+_{\bar r}$ is trapped between two translations of a one-phase polynomial. Note that this situation cannot occur if $f_- \geq 0$ unless u^- is identically zero.

Next we examine how the initial flatness corresponding to cases 1 and 2 above improves successively at a smaller scale. We construct the following two "subroutines", to be implemented in the course of the final iteration towards C^{2,γ^*} regularity.

The first subroutine provides a *two-phase* $C^{2,\gamma}$ flatness improvement: if u is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ flat in B_λ then u is $(\bar V, (\eta\lambda)^{2+\gamma}, \bar\delta)$ flat in $B_{\lambda\eta}$, with $\bar V$ close to V. This result applies to the non-degenerate case.

Proposition 4.6 (Two-Phase Flatness Improvement). *There exist $\bar\eta$, $\bar\delta$, $\bar\lambda$ universal such that if for $\beta > 0$, u is $(V, \lambda^{2+\gamma}, \bar\delta)$ flat in B_λ, $\lambda \leq \bar\lambda$ with $V = V^{\alpha,\beta}_{M,e_n,a,b} \in V_{f_\pm}$,*

$$|f_+(x) - f_+(0)| \leq \bar\delta|x|^\gamma, \quad |f_-(x) - f_-(0)| \leq \beta\bar\delta|x|^\gamma$$

and

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 1 \quad \text{on } F(u) \cap B_{2/3\lambda},$$

then u is $(\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta)$ in $B_{\bar\eta\lambda}$ with $\bar V = V^{\bar\alpha,\bar\beta}_{\bar M,\bar\nu,\bar a,\bar b} \in V_{f_\pm}$ and $|\beta - \bar\beta| \leq C\lambda^{1+\gamma}$ for C universal.

The second subroutine provides a *one-phase* flatness improvement that will be used when we will deal with the degenerate case, that is when the flatness of the free boundary only guarantees closeness of the positive part u^+ to a quadratic profile. More precisely if u^+ is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ flat in B_λ and $|\nabla u^+|$ is close to α on $F(u)$, then u^+ enjoys a $C^{2,\gamma}$ flatness improvement, with $\bar V$ close to V.

Proposition 4.7 (One-Phase Flatness Improvement). *There exist $\bar\eta$, $\bar\delta$, $\bar\lambda$ such that if for $\beta = 0$, u^+ is $(V, \lambda^{2+\gamma}, \bar\delta)$ flat in B_λ, $\lambda \leq \bar\lambda$ with $V = V^\alpha_{M,e_n,a_n} \in V_{f_+}$,*

$$|f_+(x) - f_+(0)| \leq \bar\delta|x|^\gamma$$

and

$$\big||\nabla u^+| - \alpha\big| \leq \bar\delta^{1/2}\lambda^{1+\gamma} \quad \text{on } F(u) \cap B_{2/3\lambda},$$

in the viscosity sense, then u^+ is $(\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta)$ in $B_{\bar\eta\lambda}$ with $\bar V = V^\alpha_{\bar M,\bar\nu,\bar a_{\bar\nu}} \in V_{f_+}$.

The achievement of the improvements above relies on a higher order refinement of the Harnack inequalities in Section 3. This gives the necessary compactness to pass to the limit in a sequence of renormalizations of u of the type (e.g., in the genuine two-phase case)

$$\tilde{v}^{\varepsilon}(x) = \begin{cases} \dfrac{v(x) - \alpha(1 + a \cdot x)P_{M,e_n}}{\alpha\varepsilon}, & x \in B_1^+(u) \cup F(u), \\ \dfrac{v(x) - \beta(1 + b \cdot x)P_{M,e_n}}{\beta\varepsilon}, & x \in B_1^-(u),\ \beta > 0, \\ 0, & x \in B_1^-(u),\ \beta = 0, \end{cases}$$

and to obtain a limiting transmission or Neumann problem, which turns out to be the same as in Section 3. From the regularity of the solution of this problem we get the information to improve the two-phase or one-phase approximation for u or u^+, respectively, and hence their flatness.

4.3 The New Dichotomy

Now we can start iterating. As we have seen, according to Case 1 above, after a suitable rescaling, we face a first dichotomy "degenerate versus non-degenerate".

In the latter case the two-phase subroutine of Proposition 4.6 can be applied indefinitely to reach pointwise C^{2,γ^*} regularity for some universal γ^*.

When u falls into the degenerate case, a *new kind of dichotomy appears*. First of all, to run the *one-phase* subroutine in Proposition 4.7 we need to make sure that the closeness of u^- to a purely quadratic profile makes u^+ to be a (viscosity) solution of a one-phase free boundary problem with $|\nabla u_\nu^+|$ close to an appropriate α on $F(u)$. At this point two alternatives occur at a smaller scale:

(D1) either u^- is closer to a purely quadratic profile at a proper $C^{2,\gamma}$ rate and u^+ enjoys a $C^{2,\gamma}$ flatness improvement;

(D2) or u^- is closer (at a $C^{2,\gamma}$ rate) to a one-phase polynomial profile with a small non-zero slope but u^+ only enjoys an "intermediate" C^2 flatness improvement.

To give a precise statement it is convenient to introduce a new class $\mathcal{Q}_{f_-}$ of functions, defined as

$$Q_{p,q,\omega,M} = \Big(x \cdot \omega - \frac{1}{2}x^T Mx\Big)(p + q \cdot x) - \frac{1}{2}(f_-(0) + p \operatorname{tr} M)(x \cdot \omega)^2,$$

with $p \in \mathbb{R}$, $q \in \mathbb{R}^n$, $M \in S^{n\times n}$, such that

$$q \cdot \omega = 0, \quad M\omega = 0, \quad \|M\| \le 1.$$

In the degenerate case, we use these functions to approximate u^- in a $C^{2,\gamma}$ fashion at a smaller and smaller scale. We have:

Proposition 4.8. *There exist universal constants $\bar\lambda, \bar\delta, \bar\eta$ such that if u^+ is $(V, \lambda^{2+\gamma}, \bar\delta)$ flat in B_λ, $\lambda \le \bar\lambda$ with $V = V^1_{M,e_n,a_n} \in \mathcal{V}_{f_+}$,*

$$|f_\pm(x) - f_\pm(0)| \le \bar\delta|x|^\gamma, \quad \|f_-\|_\infty \le \bar\delta$$

and

$$|u^- - Q_{0,0,e_n,0}| \le \bar\delta^{1/2}\lambda^{2+\gamma} \quad \text{in } B_\lambda^-(u),$$

then either one of the following holds:

(D1) *There exists $\bar V = V^1_{\bar M,\bar{\mathbf{e}},\bar a_{\bar{\mathbf{e}}}} \in \mathcal{V}_{f_+}$ such that u^+ is $(\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta)$ flat in $B_{\bar\eta\lambda}$, and*

$$|u^- - Q_{0,0,\bar{\mathbf{e}},0}| \le \bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma} \quad \text{in } B_{\bar\eta\lambda}^-(u).$$

(D2) *There exists $V^* = V^{\alpha^*}_{M^*,\mathbf{e}^*,a^*_{\mathbf{e}^*}} \in \mathcal{V}_{f_+}$ such that u^+ is $(V^*, \bar\eta^2\lambda^{2+\gamma}, \bar\delta)$ flat in $B_{\bar\eta\lambda}$, and*

$$|u^- - Q_{p^*,q^*,\mathbf{e}^*,M^*}| \le \bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma} \quad \text{in } B_{\bar\eta\lambda}^-(u),$$

for $(\alpha^*)^2 - (p^*)^2 = 1$ and $p^* < 0$, $|p^*| \sim (\bar\delta^{1/2}\lambda^{1+\gamma})$, $|q^*| = O(\bar\delta^{1/2}\lambda^\gamma)$.

If (D1) occurs indefinitely, we are done. If it does not, we prove that the intermediate improvement in (D2) is kept for a while, at smaller and smaller scale. The final and crucial step is to prove that, at a given universally small enough scale, the $C^{2,\gamma}$ one-phase approximation of u^-, together with the intermediate C^2 flatness improvement of u^+, is good enough to recover a full C^{2,γ^*} two-phase improvement of u with a universal $\gamma^* < \gamma$.

More precisely, at the beginning u^+ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat while u^- is $C^{2,\gamma}$ close to a pure quadratic profile. This closeness improves at a $C^{2,\gamma}$ rate until (possibly) the slope of the approximating polynomial Q is no longer zero, say at scale λ. However, to obtain the desired full flatness of u, we need to reach a scale $\rho = \lambda r$ for $r \sim \lambda^{1+1/\gamma}$ (see the proposition below). It is necessary to exploit also the information that the flatness of u^+ is in fact improving at a C^2 rate for a little while, hence allowing us to continue the iteration on the negative side and to obtain that u^- is $C^{2,\gamma}$ close to a non-degenerate configuration at an even smaller scale. We have seen that in the case of the $C^{1,\gamma}$ estimates this issue is not present. The key result is the following:

Proposition 4.9. *There exist $\bar{\lambda}, \bar{\delta}, \gamma^*$ universal such that if u^+ is $(V, r^2\lambda^{2+\gamma}, \bar{\delta})$ flat in $B_{r\lambda}$, $\lambda \le \bar{\lambda}$ with $V = V^{\alpha}_{M,e_n,a_n} \in \mathcal{V}_{f_+}$, for r such that $\bar{\delta}^{1/2} r^\gamma \in [2\bar{\eta}^\gamma \lambda^{1+\gamma}, 2\lambda^{1+\gamma})$, and*

$$|u^- - Q_{p,q,e_n,M}| \le \bar{\delta}^{1/2}(r\lambda)^{2+\gamma} \quad \text{in } B^-_{r\lambda}(u),$$

for $\alpha^2 - p^2 = 1$ and $p < 0$, $|p| \sim \bar{\delta}^{1/2}\lambda^{1+\gamma}$, $|q| = O(\bar{\delta}^{1/2}\lambda^\gamma)$, then u is $(\bar{V}, (r\lambda)^{2+\gamma^*}, \bar{\delta})$ flat in $B_{r\lambda}$ with $\bar{V} = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm}$, $\beta = |p|$.

From this point on we can go back to the two-phase subroutine to reach pointwise C^{2,γ^*} regularity.

5 Generalization and Further Developments

The results in Sections 3 and 4 extend without much effort to more general linear uniformly elliptic equations with $C^{0,\gamma}$ (C^∞, analytic) coefficients and to more general free boundary jump conditions

$$|\nabla u^+| = G(|\nabla u^-|, \nu, x),$$

where G is C^2 (C^∞, analytic) with respect to all its arguments. For these operators, the theory of viscosity solutions to inhomogeneous free boundary problems has reached a considerable level of completeness.

For problems governed by fully nonlinear operators, we proved in [10] that for a fairly general class of operators, Lipschitz viscosity solutions with Lipschitz or flat (in the sense of (3.1)) free boundaries are indeed classical ($C^{1,\gamma}$). The questions of Lipschitz continuity of solutions and higher regularity of the free boundary remain open problems.

Strong regularity of the free boundary for homogeneous problems governed by the p-Laplace operator has been developed by Lewis and Nystrom in [25, 26]. Nothing is known in presence of distributed sources.

Also of great importance, we believe, is to have information on the Hausdorff measure or dimension of the *singular* (*nonflat*) points of the free boundary. For instance, in three and four dimensions, the free boundary for local energy minimizer in the variational problem

$$\int_\Omega \{|\nabla u|^2 + \chi_{\{u>0\}}\} \to \min$$

is a smooth surface (see [7, 22]). In dimension $n = 7$, De Silva and Jerison in [14] provided an example of a minimizer with singular free boundary. Thus the conjecture is that energy minimizing free boundaries should be smooth for $n < 7$.

Nothing is known in the nonhomogeneous case.

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