Research Article

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Multiplicity of Radial Solutions of Quasilinear Problems with Minimum and Maximum

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Abstract: We show the existence and multiplicity of radial solutions for the problems with minimum and maximum involving mean curvature operators in the Minkowski space:

$$\begin{cases} \operatorname{div}(\phi_N(\nabla v)) = F(v)(|x|) & \text{for a.e. } R_1 < |x| < R_2,\ x \in \mathbb{R}^N,\ N \geq 2, \\ \min\{v(x) \mid R_1 \leq |x| \leq R_2\} = A, & \max\{v(x) \mid R_1 \leq |x| \leq R_2\} = B, \end{cases}$$

where $\phi_N(z) = z/\sqrt{1-|z|^2}$, $z \in \mathbb{R}^N$, $R_1, R_2, A, B \in \mathbb{R}$ are constants satisfying $1 < R_1 < R_2 - 1$ and $A < B$; $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, and $F : C^1[R_1, R_2] \to L^1[R_1, R_2]$ is an unbounded operator. By using the Leray–Schauder degree theory and the Borsuk theorem, we prove that the problem has at least two different radial solutions.

Keywords: Mean Curvature Operators, Radial Solutions, Multiplicity, Minkowski Space, Leray–Schauder Degree, Borsuk Theorem

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1 Introduction

In this paper we shall consider the quasilinear differential equation

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = F(v)(|x|) \quad \text{for a.e. } R_1 < |x| < R_2,\ x \in \mathbb{R}^N, \tag{1.1}$$

with the nonlinear boundary conditions

$$\min\{v(x) \mid R_1 \leq |x| \leq R_2\} = A, \quad \max\{v(x) \mid R_1 \leq |x| \leq R_2\} = B, \tag{1.2}$$

where $N \geq 2$, $F : C^1[R_1, R_2] \to L^1[R_1, R_2]$ is an unbounded operator, and $R_1, R_2, A, B \in \mathbb{R}$ are constants satisfying
(H0) $1 < R_1 < R_2 - 1$, $A < B$.

The differential operator we consider is known as the mean curvature operator in the Minkowski space

$$\mathbb{L}^{N+1} := \{(x, t) \mid x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$$

endowed with the Lorentzian metric

$$\sum_{j=1}^{N} (dx_j)^2 - (dt)^2,$$

where (x, t) are the canonical coordinates in $\mathbb{R}^{N+1}$, see [1, 3, 7, 11, 20].

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It is well known (see [1, 20]) that the study of spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

$$\mathcal{M}v = H(x, v) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where

$$\mathcal{M}v = \nabla \cdot \Big(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \Big),$$

Ω is a bounded domain in $\mathbb{R}^N$, and the nonlinearity $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

Bartnik and Simon [2] showed that if H is bounded, then problem (1.3) has at least one solution $v \in C^1(\Omega) \cap W^{2,2}(\Omega)$. Also, when Ω is a ball or an annulus in $\mathbb{R}^N$ and the nonlinearity H has a radial structure (no boundedness assumptions), Bereanu, Jebelean and Mawhin [4] proved that (1.3) has at least one classical radial solution. This can be seen as a *universal* existence result for (1.3) in radial solution. Very recently, Bereanu, Jebelean and Mawhin [6] proved the existence of positive radial solutions with $H(x, v) = f(|x|, v)$, and Bereanu, Jebelean and Torres [5] obtained the multiplicity of positive radial solutions with $H(x, v) = \lambda[\mu(|x|)v^q]$ ($q > 1$).

In his celebrated paper [9], Brykalov studied the second-order functional problem

$$\begin{cases} u'' = h(t, u, u'), & t \in (a, b), \\ \min\{u(t) \mid t \in [a, b]\} = \alpha, & \max\{u(t) \mid t \in [a, b]\} = \beta, \end{cases} \tag{1.4}$$

where α, β are constants. He proved the following theorem.

Theorem 1.1. *Let h satisfy the Carathéodory conditions on $[a, b] \times \mathbb{R}^2$ and*

$$|h(t, x, y)| \le M \quad \textit{for a.e. } t \in [a, b],\ (x, y) \in \mathbb{R}^2$$

with some positive constant M. Let $\beta - \alpha > \frac{M}{8}(b - a)^2$. Then problem (1.4) *has at least two different solutions.*

Theorem 1.1 has been extended to the case that h is unbounded by Staněk [17], he considered the following problem:

$$\begin{cases} u'' = (Gu)(t), & t \in (a, b), \\ \min\{u(t) \mid t \in [a, b]\} = \alpha, & \max\{u(t) \mid t \in [a, b]\} = \beta, \end{cases} \tag{1.5}$$

where $G : C^1[a, b] \to L^1[a, b]$, and α, β are constants. Staněk also proved the following theorem.

Theorem 1.2. *Let $G : C^1[a, b] \to L^1[a, b]$ be such that there exists a continuous nondecreasing function $f : [0, \infty) \to [0, \infty)$ satisfying*

$$|G(u)(t)| \le f(|u'(t)|), \quad t \in [a, b],\ u \in C^1[a, b].$$

and

$$\int_0^\infty \frac{dt}{f(t)} \ge b - a, \quad \int_0^\infty \frac{t}{f(t)} dt = \infty.$$

Let α, β satisfy

$$\beta - \alpha > \frac{b - a}{2} P^{-1}\Big(\frac{b - a}{2}\Big), \quad \textit{where } P(u) = \int_0^u \frac{dt}{f(t)}.$$

Then problem (1.5) *has at least two different solutions.*

Moreover, Theorems 1.1 and 1.2 have been extended to the p-Laplacian case by Staněk [15]. Of course the natural question is what would happen if we replace the p-Laplacian operator with the mean curvature operator $\mathcal{M}$? The purpose of this paper is to use the topological degree theory and the Borsuk theorem to investigate the existence and multiplicity radial solutions for (1.1), (1.2), which is a problem with minimum and maximum involving mean curvature operators in the Minkowski space.

Setting $r = |x|$ and $v(x) = u(r)$, we reduce (1.1), (1.2) to the problem

$$(r^{N-1}\phi_1(u'))' = r^{N-1}(Fu)(r) \quad \text{for a.e. } r \in (R_1, R_2), \tag{1.6}$$

$$\min\{u(r) \mid r \in [R_1, R_2]\} = A, \quad \max\{u(r) \mid r \in [R_1, R_2]\} = B, \tag{1.7}$$

where $N \geq 2$ and

$$\phi_1(s) = \frac{s}{\sqrt{1-s^2}}, \quad s \in (-1, 1).$$

Obviously $\phi_1 : (-1, 1) \to \mathbb{R}$ is an odd increasing homeomorphism with $\phi_1(0) = 0$, whose inverse is also an odd increasing homeomorphism with the form $\phi_1^{-1}(s) = s/\sqrt{1+s^2}$ and $\phi_1^{-1}(0) = 0$.

A *solution of problem* (1.6), (1.7) is a function $u \in C^1[R_1, R_2]$ that satisfies (1.6) and (1.7) for a.e. $r \in (R_1, R_2)$, as well as $\max_{r\in[R_1,R_2]}|u'(r)| < 1$ and $r^{N-1}\phi_1(u') \in AC[R_1, R_2]$.

Throughout this paper we shall assume that the operator F satisfies the following assumptions:

(H1) There exists a continuous nondecreasing function $g : [0, \infty) \to [0, \infty)$ such that

$$|r^{N-1}F(u)(r)| \leq g(|u'(r)|), \quad r \in [R_1, R_2],\ u \in C^1[R_1, R_2].$$

(H2) $\displaystyle\int_0^\infty \frac{ds}{g(\phi_1^{-1}(s))} \geq R_2 - R_1.$

The remainder of this paper is arranged as follows. In Section 2, we give some notations and the a priori estimate for the possible solutions of problem (1.6), (1.7). Section 3 is devoted to prove the existence and multiplicity solution of problem (1.6), (1.7), and also gives an application to illustrate our main results.

For other results concerning the linear or quasilinear problems in annuli we refer the reader to [4, 13, 14, 21, 22].

2 Preliminaries

Let $L^1(R_1, R_2)$ be the Banach space of all Lebesgue integrable functions on $[R_1, R_2]$ with the norm

$$\|u\|_{L^1} = \int_{R_1}^{R_2} |u(r)|dr.$$

Let $X := C[R_1, R_2]$ be the Banach space of continuous functions on $[R_1, R_2]$ endowed with the uniform norm

$$\|u\|_0 = \max\{u(r) \mid r \in [R_1, R_2]\}.$$

Let $Y := C^1[R_1, R_2]$ be the Banach space of continuously differentiable functions on $[R_1, R_2]$ equipped with the uniform norm

$$\|u\|_{C^1} = \|u\|_0 + \|u'\|_0.$$

The corresponding open ball of center in 0 and radius ρ is denoted by B_ρ.

Definition 2.1. A functional $\tau : X \to \mathbb{R}$ is increasing if

$$x(r) < y(r) \quad \text{for all } r \in [R_1, R_2] \quad \Rightarrow \quad \tau(x) < \tau(y)$$

for $x, y \in X$. For every $\tau : X \to \mathbb{R}$, $\mathrm{Im}(\tau)$ denotes the range of τ. Denote

$$\mathcal{A} = \{\tau \mid \tau : X \to \mathbb{R} \text{ is continuous and increasing}\}, \quad \mathcal{A}_0 = \{\tau \mid \tau \in \mathcal{A},\ \tau(0) = 0\}.$$

Remark 2.2. If we take

$$\tau(u) = \min\{u(r) \mid r \in [R_1, R_2]\}, \quad \zeta(u) = \max\{u(r) \mid r \in [R_1, R_2]\},$$

then the functionals τ, ζ belong to $\mathcal{A}_0$. It is easy to observe that the boundary condition (1.7) can reduce to

$$\tau(u) = A, \quad \max\{u(r) \mid r \in [R_1, R_2]\} - \min\{u(r) \mid r \in [R_1, R_2]\} = B - A. \tag{2.1}$$

So, in the remaining part of the paper, we only need to deal with problem (1.6), (2.1).

Lemma 2.3. *Let $\tau \in \mathcal{A}$, $h \in \mathrm{Im}(\tau)$. Then there exists a unique $d \in \mathbb{R}$ such that $\tau(d) = h$.*

Proof. Suppose that there exist $x \in X$ and $c_1, c_2 \in \mathbb{R}$ satisfying $\tau(x) = h$ and $c_1 < x(t) < c_2$. By the definition of $\mathcal{A}$, we get $\tau(c_1) < \tau(x) < \tau(c_2)$. Define the increasing function $p \in C([c_1, c_2])$ by

$$p(c) = \tau(c).$$

Obviously, $p(c_1) < h < p(c_2)$. Since p is continuous and strictly increasing, the equation $p(c) = h$ has a unique solution $c = d$. Of course, there exists a unique $d \in \mathbb{R}$ such that $\tau(d) = h$. □

Lemma 2.4. *Let $\tau \in \mathcal{A}$ and $e \in [0, 1]$. For each $u \in X$, let u satisfy the equality $\tau(u) - e\tau(-u) = 0$. Then there exists a $\xi \in [R_1, R_2]$ such that $u(\xi) = 0$.*

Proof. Suppose on the contrary that $u(\xi) \neq 0$. If $u(r) < 0$ on $[R_1, R_2]$, then $\tau(u) < \tau(0) = 0$, $\tau(-u) > \tau(0) = 0$. Moreover, $\tau(u) - e\tau(-u) < 0$, this is a contradiction. Similarly, for $u(r) > 0$, we also can get a contradiction. □

In the proof of Lemma 2.7 we will apply the well-known Bihari lemma (see, e.g., [8, 15, 17]). We state this lemma in the following form.

Lemma 2.5. *Let $q : [0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function, $Q : [0, +\infty) \to [0, +\infty)$ defined by $Q(u) = \int_0^u \frac{dt}{q(t)}$ and let $a \in [b, c] \subset \mathbb{R}$. Let Q^{-1} denote the inverse function to Q. If $y \in X$ satisfies the inequality*

$$|y(r)| \le \left| \int_a^r q(|y(s)|) ds \right| \quad \text{for } r \in (b, c),$$

then

$$|y(r)| \le Q^{-1}(a - r) \quad \text{for } r \in [b, a] \quad \text{provided } \lim_{y \to \infty} Q(y) > a - b,$$
$$|y(r)| \le Q^{-1}(r - b) \quad \text{for } r \in [a, c] \quad \text{provided } \lim_{y \to \infty} Q(y) > c - a.$$

As in [9, 10, 16, 17], we define the function $\varphi : X \to \mathbb{R}$ by the formula

$$\varphi(u) = \max \left\{ \int_k^m u(s) ds \mid k, m \in [R_1, R_2],\ k \le m \right\}. \tag{2.2}$$

Lemma 2.6 ([9, 18, 19]). *For every $u \in Y$, the functional φ is continuous and*

$$\max\{u(r) \mid r \in [R_1, R_2]\} - \min\{u(r) \mid r \in [R_1, R_2]\} = \max\{\varphi(u'), \varphi(-u')\}.$$

Lemma 2.7. *Assume that* (H0)–(H2) *are satisfied. Let u be a solution of* (1.6) *on $[R_1, R_2]$. Then*

$$\min\{\varphi(u'), \varphi(-u')\} \le \frac{R_2 - R_1}{2} \phi_1^{-1}\left(Q^{-1}\left(\frac{R_2 - R_1}{2}\right)\right), \tag{2.3}$$

where Q^{-1} denotes the inverse function to

$$Q(u) = \int_0^u \frac{ds}{g(\phi_1^{-1}(s))}.$$

Proof. Set

$$C_+ = \{r \mid u'(r) > 0,\ r \in (R_1, R_2)\}, \quad C_- = \{r \mid u'(r) < 0,\ r \in (R_1, R_2)\}.$$

Let $\mu(C_+)$ and $\mu(C_-)$ denote the Lebesgue measure of C_+ and C_-, respectively.

If $C_+ = \emptyset$ (resp. $C_- = \emptyset$), then $\varphi(u') = 0$ (resp. $\varphi(-u') = 0$), and (2.3) is obviously satisfied. Suppose $C_+ \neq \emptyset$ and $C_- \neq \emptyset$. Since $u' \in X$, we have that C_+, C_- are open subsets of $[R_1, R_2]$ and therefore C_+ (resp. C_-) is a union of at most countably many disjoint open intervals (a_i, b_i), $i \in I_+ \subset \mathbb{N}$ (resp. $(c_i, d_j) \in I_- \subset \mathbb{N}$) without common elements, i.e.,

$$C_+ = \bigcup_{i \in I_+} (a_i, b_i) \quad \text{resp.} \quad C_- = \bigcup_{i \in I_-} (c_i, d_i).$$

Of course, $u'(a_i) \neq 0$ or $u'(b_i) \neq 0$ for any $i \in I_+$ (resp. $u'(c_j) \neq 0$ or $u'(d_j) \neq 0$ for any $j \in I_-$) imply $a_i = R_1$ or $b_i = R_2$ (resp. $c_i = R_1$ or $d_i = R_2$). Furthermore $C_+ \neq (R_1, R_2)$, since in the opposite case $C_- = \emptyset$, a contradiction. Similarly, $C_- \neq (R_1, R_2)$.

From the inequality $\mu(C_+) + \mu(C_-) \leq R_2 - R_1$, it follows that

$$\min\{\mu(C_+), \mu(C_-)\} \leq \frac{R_2 - R_1}{2}. \tag{2.4}$$

For our next consideration, recall that the interval $[0, R_2 - R_1)$ is contained in the domains of Q^{-1} by assumption (H2).

We now show the inequality

$$\varphi(u') \leq \mu(C_+) \sup\{\phi_1^{-1}(Q^{-1}(b_i - a_i)) \mid i \in I_+\}. \tag{2.5}$$

Fix $i \in I_+$, let $u'(\varepsilon) = 0$, $\varepsilon \in \{a_i, b_i\}$. Integrating (1.6) from ε to r, we will easily get the following equality with $\phi_1(0) = 0$:

$$r^{N-1}\phi_1(u'(r)) = \int_\varepsilon^r s^{N-1}(Fu)(s)ds, \quad r \in [a_i, b_i].$$

Since $u'(r) \geq 0$ in $r \in [a_i, b_i]$, it follows that ϕ_1 is an increasing homeomorphism. Using (H0) and (H1), we have

$$\begin{aligned} 0 \leq \phi_1(u'(r)) &\leq r^{1-N}\left|\int_\varepsilon^r s^{N-1}|(Fu)(s)|ds\right| \\ &\leq r^{1-N}\left|\int_\varepsilon^r g(u'(s))ds\right| \\ &\leq \left|\int_\varepsilon^r g(\phi_1^{-1}(\phi_1(u'(s))))ds\right|. \end{aligned}$$

From Lemma 2.5 with $a = \varepsilon$, $b = a_i$, $c = b_i$, $y(s) = \phi_1(u'(s))$ and $q(y) = g(\phi_1^{-1}(y))$, it is easy to check that

$$\phi_1(u'(r)) \leq Q^{-1}(|\varepsilon - r|), \quad r \in [a_i, b_i].$$

Note that $0 \leq u'(r) \leq \phi_1^{-1}(Q^{-1}(b_i - a_i))$ for $r \in [a_i, b_i]$, $i \in I_+$. We can see that

$$\int_{a_i}^{b_i} u'(s)ds \leq (b_i - a_i)\phi_1^{-1}(Q^{-1}(b_i - a_i)).$$

Furthermore,

$$\begin{aligned} \varphi(u') &\leq \int_{C_+} u'(s)ds = \sum_{i \in I_+} \int_{a_i}^{b_i} u'(s)ds \\ &\leq \sup\{\phi_1^{-1}(Q^{-1}(b_i - a_i)) \mid i \in I_+\} \sum_{i \in I_+} (b_i - a_i) \\ &\leq \mu(C_+) \sup\{\phi_1^{-1}(Q^{-1}(b_i - a_i)) \mid i \in I_+\}. \end{aligned}$$

Consequently, (2.5) is fulfilled.

Next, we will prove that

$$\varphi(-u') \le \mu(C_-)\sup\{\phi_1^{-1}(Q^{-1}(d_j - c_j)) \mid j \in I_-\}. \tag{2.6}$$

Fix $j \in I_-$, let $u'(\eta) = 0$, $\eta \in \{c_j, d_j\}$. Integrating (1.6) from η to r with $\phi_1(0) = 0$, we get

$$r^{N-1}\phi_1(u'(r)) = \int_\eta^r s^{N-1}(Fu)(s)ds, \quad r \in [c_j, d_j].$$

Since $u'(r) \le 0$ in $[c_j, d_j]$ and using the fact that ϕ_1 is an odd increasing homeomorphism and (H0)–(H1), we obtain

$$\begin{aligned} 0 \le -\phi_1(u'(r)) &\le r^{1-N}\Big|\int_\eta^r s^{N-1}|(Fu)(s)|ds\Big| \\ &\le r^{1-N}\Big|\int_\eta^r g(-u'(s))ds\Big| \\ &\le \Big|\int_\eta^r g(-u'(s))ds\Big|. \end{aligned}$$

Hence,

$$\phi_1(|u'(r)|) = -\phi_1(u'(r)) \le \Big|\int_\eta^r g(\phi_1^{-1}(\phi_1(|u'(s)|)))ds\Big|.$$

If we take $a = \eta$, $b = c_j$, $c = d_j$, $y(s) = \phi_1(|u'(s)|)$ and $q(y) = g(\phi_1^{-1}(y))$, then by Lemma 2.5 we deduce that

$$\phi_1(|u'(r)|) \le Q^{-1}(|r - \eta|), \quad r \in [c_j, d_j].$$

Hence we obtain $0 \le -u'(r) \le \phi_1^{-1}(Q^{-1}(d_j - c_j))$ for $r \in [c_j, d_j]$, $j \in I_-$. As a consequence,

$$-\int_{c_j}^{d_j} u'(s)ds \le (d_j - c_j)\phi_1^{-1}(Q^{-1}(d_j - c_j)).$$

Moreover,

$$\begin{aligned} \varphi(-u') &\le -\int_{C_-} u'(s)ds = -\sum_{j\in I_-}\int_{c_j}^{d_j} u'(s)ds \\ &\le \sup\{\phi_1^{-1}(Q^{-1}(d_j - c_j)) \mid j \in I_-\}\sum_{i\in I_+}(d_i - c_j) \\ &\le \mu(C_-)\sup\{\phi_1^{-1}(Q^{-1}(d_j - c_j)) \mid j \in I_-\}. \end{aligned}$$

Thus, (2.6) is fulfilled.

From (2.4), (2.5) and (2.6), we can conclude inequality (2.3). □

Now, let us consider the functional differential equation

$$(r^{N-1}\phi_1(u'(r)))' = \lambda r^{N-1}(Fu)(r), \quad \lambda \in [0, 1], \tag{2.7}$$

depending on the parameter λ.

The next lemma gives a priori bounds for solutions of problem (2.7), (1.7) (with $A = 0$ and $B > 0$).

Lemma 2.8. *Suppose that u is a solution of (2.7) for any $\lambda \in [0, 1]$ and satisfies boundary condition (1.7) with $A = 0$. Then the following two conclusions are satisfied:*

$$\begin{aligned} \|u\|_0 &\le B, \\ \|u\|_{C^1} &\le B + 1, \end{aligned} \tag{2.8}$$

Proof. According to $\tau(u) = A = 0$ and Lemma 2.4, there exists a $\xi \in [R_1, R_2]$ such that $u(\xi) = 0$. It is easy to observe that

$$\min\{u(r) \mid r \in [R_1, R_2]\} \le 0, \quad \max\{u(r) \mid r \in [R_1, R_2]\} \ge 0,$$

and consequently the boundary condition

$$\max\{u(r) \mid r \in [R_1, R_2]\} - \min\{u(r) \mid r \in [R_1, R_2]\} = B$$

implies (2.8). Using the fact that $\phi_1 : (-1, 1) \to \mathbb{R}$ is an increasing homeomorphism and (2.8), we deduce that

$$\|u\|_{C^1} = \|u\|_0 + \|u'\|_0 < B + 1. \qquad \square$$

The following lemma plays a very crucial role in proving our main results.

If Ω is an open bounded subset in Banach space X and $T : \overline{\Omega} \to X$ is compact, with $0 \notin (I - T)(\partial\Omega)$, then $d_{\mathrm{LS}}(I - T, \Omega, 0)$ will denote the Leray–Schauder degree of T with respect to Ω and 0. For the definition and properties of the Leray–Schauder degree we refer the reader to, e.g., [12].

Lemma 2.9. *Let B be a positive constant, $\tau \in \mathcal{A}$ and φ be defined in* (2.2). *Set*

$$\Omega = \{(u, \mu, \nu) \mid (u, \mu, \nu) \in Y \times \mathbb{R}^2,\ \|u\|_{C^1} < \rho,\ \|u'\|_0 < 1,\ |\mu| < \rho,\ |\nu| < R_2^{N-1}\phi_1(1)\}.$$

Define $\Gamma_i : \overline{\Omega} \to Y \times \mathbb{R}^2$ for $i = 1, 2$ by

$$\Gamma_1(u, \mu, \nu) = \Big(\mu + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)\,ds,\ \mu + \tau(u),\ \nu + \varphi(u') - B\Big),$$

$$\Gamma_2(u, \mu, \nu) = \Big(\mu + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)\,ds,\ \mu + \tau(u),\ \nu + \varphi(-u') - B\Big).$$

Then

$$d_{\mathrm{LS}}(I - \Gamma_i, \Omega, 0) \ne 0, \quad i = 1, 2,$$

where $\rho = B + 1$ and $\rho < R_2 - R_1$, and I denotes the identity operator on $Y \times \mathbb{R}^2$.

Proof. Observe that Ω is a bounded open subset of the Banach space $Y \times \mathbb{R}^2$ with usual norm and symmetric with respect to $\theta \in \Omega$. Define $G_i : [0, 1] \times \Omega \to Y \times \mathbb{R}^2$ for $i = 1, 2$ by

$$\begin{aligned} G_1(\lambda, u, \mu, \nu) = \Big(&\mu + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)\,ds - (1 - \lambda)\int_{R_1}^{r} \phi_1^{-1}(-s^{1-N}\nu)\,ds, \\ &\mu + \tau(u) - (1 - \lambda)\tau(-u),\ \nu + \varphi(u') - \varphi((\lambda - 1)u') - \lambda B\Big), \\ G_2(\lambda, u, \mu, \nu) = \Big(&\mu + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)\,ds - (1 - \lambda)\int_{R_1}^{r} \phi_1^{-1}(-s^{1-N}\nu)\,ds, \\ &\mu + \tau(u) - (1 - \lambda)\tau(-u),\ \nu + \varphi(-u') - \varphi((1 - \lambda)u') - \lambda B\Big). \end{aligned}$$

For every $(u, \mu, \nu) \in \overline{\Omega}$, it is easy to check that

$$G_i(1, u, \mu, \nu) = \Gamma_i(u, \mu, \nu), \quad i = 1, 2.$$

By the Borsuk theorem [12, Theorem 8.3], to prove $d_{\mathrm{LS}}(I - \Gamma_i, \Omega, 0) \ne 0$, it is sufficient to show that the following hypotheses hold:

(1) $G_i(0, \cdot, \cdot, \cdot)$ is an odd operator on $\overline{\Omega}$, that is,

$$G_i(0, -u, -\mu, -\nu) = -G_i(0, u, \mu, \nu), \quad i = 1, 2,\ (u, \mu, \nu) \in \overline{\Omega};$$

(2) G_i is a completely continuous operator;
(3) $G_i(\lambda, u, \mu, \nu) \neq (u, \mu, \nu)$ for $(\lambda, u, \mu, \nu) \in [0, 1] \times \partial\Omega$.
At a first step, we take $(u, \mu, \nu) \in \overline{\Omega}$, for $i = 1$,

$$\begin{aligned} G_1(0, -u, -\mu, -\nu) &= \Big(-\mu + \int_{R_1}^{r} \phi_1^{-1}(-s^{1-N}\nu)ds - \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)ds, -\mu + \tau(-u) - \tau(u), -\nu + \varphi(-u') - \varphi(u')\Big) \\ &= -\Big(\mu + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu)ds - \int_{R_1}^{r} \phi_1^{-1}(-s^{1-N}\nu)ds, \mu + \tau(u) - \tau(-u), \nu + \varphi(u') - \varphi(-u')\Big) \\ &= -G_1(0, u, \mu, \nu). \end{aligned}$$

A similar argument shows that $G_2(0, -u, -\mu, -\nu) = -G_2(0, u, \mu, \nu)$. So (1) is satisfied.

Moreover, to prove (2), we need to show that G_i is continuous and that G_i takes bounded sets of $[0, 1] \times \Omega$ into relatively compact sets of $Y \times \mathbb{R}^2$.

Let $(\lambda_n, u_n, \mu_n, \nu_n)$ be a sequence in $[0, 1] \times \overline{\Omega}$. For each $n \in \mathbb{Z}^+$ and $r \in [R_1, R_2]$, $|\lambda_n| \le 1$, $\|u_n\|_0 < \rho$, $|\mu_n| \le \rho$ and $|\nu_n| < R_2^{N-1}\phi_1(1)$, at the same time, we have that $\{\tau(u_n)\}$, $\{\tau(-u_n)\}$, $\{\varphi(u_n)\}$ and $\{\varphi(-u_n)\}$ are bounded. From the Bolzano–Weierstrass theorem and the Arzelà–Ascoli theorem, it is not difficult to see that $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{u_n\}$ have convergent subsequences. Without loss of generality, we also denote the subsequences by $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{u_n\}$. Let

$$\lim_{n\to\infty} \lambda_n = \lambda_0, \quad \lim_{n\to\infty} u_n = u_0, \quad \lim_{n\to\infty} \mu_n = \mu_0, \quad \lim_{n\to\infty} \nu_n = \nu_0.$$

Hence,

$$\lim_{n\to\infty} G(\lambda_n, u_n, \mu_n, \nu_n) = G(\lambda_0, u_0, \mu_0, \nu_0).$$

Further, by the continuity of ϕ_1^{-1}, τ and φ, we obtain that G_1 and G_2 are continuous. So G_1 and G_2 are completely continuous.

To prove (3), suppose on the contrary that $G_i(\lambda_0, u_0, \mu_0, \nu_0) = (u_0, \mu_0, \nu_0)$ for some $(\lambda_0, u_0, \mu_0, \nu_0) \in [0, 1] \times \partial\Omega$. Observe that

$$\mu_0 + \int_{R_1}^{r} \phi_1^{-1}(s^{1-N}\nu_0)ds - (1 - \lambda_0) \int_{R_1}^{r} \phi_1^{-1}(-s^{1-N}\nu_0)ds = u_0(r), \tag{2.9}$$

$$\tau(u_0) - (1 - \lambda_0)\tau(-u_0) = 0, \tag{2.10}$$

$$\varphi(u_0') - \varphi((\lambda_0 - 1)u_0') = \lambda_0 B. \tag{2.11}$$

By Lemma 2.4 (take $u = u_0$, $e = 1 - \lambda_0$) and (2.10) we infer that, there exists $\delta \in [R_1, R_2]$ such that $u_0(\delta) = 0$. Together with (2.9), it is easy to see that

$$\mu_0 = -\Big(\int_{R_1}^{\delta} \phi_1^{-1}(s^{1-N}\nu_0)ds - (1 - \lambda_0) \int_{R_1}^{\delta} \phi_1^{-1}(-s^{1-N}\nu_0)ds\Big), \tag{2.12}$$

From (2.9), by simple computation, we get

$$u_0(r) = \int_{\delta}^{r} \phi_1^{-1}(s^{1-N}\nu_0)ds - (1 - \lambda_0) \int_{\delta}^{r} \phi_1^{-1}(-s^{1-N}\nu_0)ds. \tag{2.13}$$

Next, we will divide the proof into three cases.

Case 1. Let $\nu_0 = 0$. It follows from (2.12), (2.13) that $\mu_0 = 0$, $u_0 = 0$, so

$$(0, 0, 0) = (u_0, \mu_0, \nu_0) \in \partial\Omega,$$

which is a contradiction.

Case 2. If $\nu_0 > 0$, then

$$\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0) > 0.$$

By simple computation, from the definition of φ in (2.2), we deduce that

$$\varphi(u_0') - \varphi((\lambda_0 - 1)u_0') = (\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1).$$

Together with (2.11), we get

$$(\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1) = \lambda_0 B \tag{2.14}$$

and

$$\phi_1^{-1}(r^{1-N}\nu_0) \le \frac{\lambda_0 B}{R_2 - R_1},$$

since

$$-(1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0) \ge 0.$$

Thus,

$$\nu_0 \le r^{N-1}\phi_1\Big(\frac{\lambda_0 B}{R_2 - R_1}\Big) \le r^{N-1}\phi_1\Big(\frac{\lambda_0 \rho}{R_2 - R_1}\Big) \le r^{N-1}\phi_1\Big(\frac{\rho}{R_2 - R_1}\Big) < R_2^{N-1}\phi_1(1).$$

Furthermore, by (2.12)–(2.14), for every $r \in [R_1, R_2]$, we conclude that

$$\begin{aligned}
|u_0(r)| &= \Big|\int_\delta^r \phi_1^{-1}(s^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-s^{1-N}\nu_0)\,ds\Big| = \int_\delta^r \frac{\lambda_0 B}{R_2 - R_1}\,ds \le \frac{\lambda_0 B}{R_2 - R_1}|r-\delta| \le B,\\
|u_0'(r)| &= \phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0) = \frac{\lambda_0 B}{R_2 - R_1} \le \frac{\rho}{R_2 - R_1} < 1,\\
|\mu_0| &= |u_0(0)| < \|u\|_0 < \rho, \quad \|u\|_{C^1} < B + 1 = \rho.
\end{aligned}$$

Consequently, $(u_0, \alpha_0, \beta_0) \notin \partial\Omega$, a contradiction.

Case 3. If $\nu_0 < 0$, taking into account the inequality

$$\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0) < 0$$

and the definition of φ in (2.2), we have

$$\begin{aligned}
\varphi(u_0') - \varphi((\lambda_0 - 1)u_0') &= 0 - (\lambda_0 - 1)(\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1)\\
&= (1-\lambda_0)(\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1).
\end{aligned}$$

Combining this with (2.11), we deduce that

$$(1-\lambda_0)(\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1) = \lambda_0 B. \tag{2.15}$$

If $\lambda_0 = 0$, then (2.15) implies

$$\phi_1^{-1}(r^{1-N}\nu_0) - (1-0)\phi_1^{-1}(-r^{1-N}\nu_0) = 0,$$

which contradicts $\phi_1^{-1}(r^{1-N}\nu_0) - (1-0)\phi_1^{-1}(-r^{1-N}\nu_0) < 0$.

If $\lambda_0 = 1$, then $\lambda_0 B = 0$, i.e. $B = 0$, which is impossible.

If $\lambda_0 \in (0, 1)$, then

$$(1-\lambda_0)(\phi_1^{-1}(r^{1-N}\nu_0) - (1-\lambda_0)\phi_1^{-1}(-r^{1-N}\nu_0))(R_2 - R_1) < 0, \quad \text{also } \lambda_0 B > 0.$$

This is a contradiction. The proof is completed. □

3 Existence and Multiplicity Results

Lemma 3.1. *Let* (H0)–(H2) *hold and let Q be defined by Lemma 2.7. Then for $A = 0$ and every $B \in R$ satisfying*

$$\frac{R_2 - R_1}{2}\phi_1^{-1}\Big(Q^{-1}\Big(\frac{R_2 - R_1}{2}\Big)\Big) < B < R_2 - R_1 - 1, \tag{3.1}$$

problem (1.1), (1.2) *has at least two different radial solutions.*

Proof. Fix $B \in \mathbb{R}$ and let (3.1) hold. For $A = 0$, we consider the boundary conditions

$$\tau(u) = 0, \quad \varphi(u') = B - A = B, \tag{3.2}$$

and

$$\tau(u) = 0, \quad \varphi(-u') = B - A = B, \tag{3.3}$$

where $\varphi : X \to \mathbb{R}$ is defined by (2.2). Suppose u is a solution of (1.6). Then from Lemma 2.6,

$$\max\{u(r) \mid r \in [R_1, R_2]\} - \min\{u(r) \mid r \in [R_1, R_2]\} = \max\{\varphi(u'),\ \varphi(-u')\}.$$

Note that, if problem (1.6), (3.2) has a solution u_1, then Lemma 2.7 and (3.2) imply that $\varphi(-u_1') < B$ and

$$\max\{u_1(r) \mid r \in [R_1, R_2]\} - \min\{u_1(r) \mid r \in [R_1, R_2]\} = B.$$

So, u_1 is a solution of problem (1.6), (1.7), i.e., u_1 is a radial solution of problem (1.1), (1.2).

Similarly, if problem (1.6), (3.3) has a solution u_2, then $\varphi(u_2') < B$ and

$$\max\{u_2(r) \mid r \in [R_1, R_2]\} - \min\{u_2(r) \mid r \in [R_1, R_2]\} = B.$$

Therefore, u_2 is a radial solution of problem (1.1), (1.2).

Moreover, since $\varphi(u_1') = B$ and $\varphi(u_2') < B$, we have $u_1 \neq u_2$. Next, we need to prove that both problem (1.6), (3.2) and problem (1.6), (3.3) have solutions.

Since $\rho = B + 1$, according to (3.1), it is obvious that $\rho < R_2 - R_1$. Set

$$\Omega = \{(u, \mu, \nu) \mid (u, \mu, \nu) \in Y \times \mathbb{R}^2,\ \|u\|_{C^1} < \rho,\ \|u'\|_0 < 1,\ |\mu| < \rho,\ |\nu| < R_2^{N-1}\phi_1(1)\}.$$

Define $\Phi_1 : [0, 1] \times \overline{\Omega} \to Y \times \mathbb{R}^2$ by

$$\Phi_1(\lambda, u, \mu, \nu) = \Big(\mu + \int_{R_1}^{r} \phi_1^{-1}\Big(s^{1-N}\Big(\nu + \lambda \int_{R_1}^{s} \sigma^{N-1}F(u)(\sigma)d\sigma\Big)\Big)ds, \mu + \tau(u), \nu + \varphi(u') - B\Big).$$

It is easy to check that

$$\Phi_1(0, u, \mu, \nu) = \Gamma_1(u, \mu, \nu), \quad (u, \mu, \nu) \in \overline{\Omega}.$$

Now, we come to study the parameter equation

$$\Phi_1(\lambda, u, \mu, \nu) = (u, \mu, \nu), \quad \lambda \in [0, 1]. \tag{3.4}$$

Observe that u is a solution of problem (1.6), (3.2) if and only if $(u(r), u(R_1), R_1^{N-1}\phi_1(u'(R_1)))$ is a solution of (3.4), when $\lambda = 1$.

Next, we prove that (3.4) has a solution when $\lambda = 1$. By Lemma 2.9, we have $d_{\mathrm{LS}}(I - \Gamma_i, \Omega, 0) \neq 0$. It is sufficient to check the following two hypotheses:

(1) $\Phi_1(\lambda, u, \mu, \nu)$ is a completely continuous operator;
(2) $\Phi_1(\lambda, u, \mu, \nu) \neq (u, \mu, \nu)$ for every $(\lambda, u, \mu, \nu) \in [0, 1] \times \partial\Omega$.

According to the continuity of ϕ_1^{-1}, F, τ and φ, it is clear that $\Phi_1(\lambda, u, \mu, \nu)$ is continuous.

Let $\{(\lambda_n, u_n, \mu_n, \nu_n)\}_{n=1}^{\infty} \subset [0,1] \times \overline{\Omega}$ be a sequence. Set

$$(v_n, \alpha_n, \beta_n) = \Phi_1(\lambda_n, u_n, \mu_n, \nu_n) \quad \text{for } n \in \mathbb{N}.$$

It is easy to see that

$$v_n = \mu_n + \int_{R_1}^{r} \phi_1^{-1}\Big(s^{1-N}\Big(\nu_n + \lambda_n \int_{R_1}^{s} \sigma^{N-1} F(u_n)(\sigma)\,d\sigma\Big)\Big)ds, \quad \alpha_n = \mu_n + \tau(u_n), \quad \beta_n = \nu_n + \varphi(u_n') - B.$$

Then, using the fact that $\phi_1^{-1} : \mathbb{R} \to (-1,1)$ is an odd increasing homeomorphism, $0 \le \lambda_n \le 1$, $\|u_n\|_{C^1} < \rho$, $\|u_n'\|_0 < 1$, $|\mu_n| < \rho$, $|\nu_n| < R_2^{N-1}\phi_1(1)$ and (H1), we obtain

$$|v_n| \le \rho + \int_{R_1}^{r} \Big|\phi_1^{-1}\Big(s^{1-N}\Big(\nu_n + \lambda_n \int_{R_1}^{s} \sigma^{N-1} F(u_n)(\sigma)\,d\sigma\Big)\Big)\Big| ds < \rho + R_2, \tag{3.5}$$

$$|v_n'| = \phi_1^{-1}\Big(r^{1-N}\Big(\nu_n + \lambda_n \int_{R_1}^{r} s^{N-1} F(u_n)(s)\,ds\Big)\Big) < 1, \tag{3.6}$$

and

$$\begin{aligned} \big|\phi_1(v_n'(r_1)) - \phi_1(v_n'(r_2))\big| &\le |\nu_n|\big|r_1^{1-N} - r_2^{1-N}\big| + \lambda_n \int_{r_1}^{r_2} |s^{N-1} F(v_n)(s)|\,ds \\ &\le 2R_2^{1-N}|\nu_n| + \int_{r_1}^{r_2} g(v_n')\,ds \\ &< 2\phi_1(1) + g(1)(R_2 - R_1) \end{aligned} \tag{3.7}$$

for $n \in \mathbb{N}$, $r_1, r_2 \in [R_1, R_2]$.

Notice that ϕ_1 is increasing. Combining (3.5), (3.6) and (3.7) with the Arzelà–Ascoli theorem, we deduce that there exists a sequence ω_n such that v_{ω_n} is convergent in Y. By $\tau(u_n) \le \max\{\tau(R_2), \tau(-R_2)\}$ and $0 \le \varphi(u_n') \le \rho$, it is easy to check that α_n and β_n are bounded. Without loss of generality, we can assume that $\{\alpha_{\omega_n}\}$ and $\{\beta_{\omega_n}\}$ are convergent. Therefore $\{(u_n, \mu_n, \nu_n)\}$ is convergent in $Y \times \mathbb{R}^2$, which implies that $\Phi_1(\lambda, u, \mu, \nu)$ is completely continuous.

To show property (2), we suppose, on the contrary, that $\Phi_1(\lambda_0, u_0, \mu_0, \nu_0) = (u_0, \mu_0, \nu_0)$ for some $(\lambda_0, u_0, \mu_0, \nu_0) \in [0,1] \times \partial\Omega$. Moreover,

$$\mu_0 + \int_{R_1}^{r} \phi_1^{-1}\Big(s^{1-N}\Big(\nu_0 + \lambda_0 \int_{R_1}^{s} \sigma^{N-1} F(u_0)(\sigma)\,d\sigma\Big)\Big)ds = u_0(r), \quad r \in [R_1, R_2], \tag{3.8}$$

and

$$\tau(u_0) = 0, \quad \varphi(u_0') = B.$$

Taking into account (3.8), we get

$$(r^{N-1}\phi_1(u_0'(r)))' = \lambda_0 r^{N-1}(Fu_0)(r) \quad \text{for a.e. } r \in [R_1, R_2].$$

Consequently, u_0 is a solution of problem (3.4), (1.2). By Lemma 2.8,

$$\|u'\|_0 < 1, \quad \|u\|_0 \le B, \quad \|u\|_{C^1} < B + 1 = \rho.$$

In particular, $\mu_0 = u_0(R_1)$ and $R_1^{N-1}\phi_1(u_0'(R_1)) = \nu_0$, so

$$|\mu_0| < \|u_0\|_0 < \rho, \quad |\nu_0| < R_2^{N-1}\phi(1),$$

which contradicts $(u_0, \alpha_0, \beta_0) \in \partial\Omega$.

Analogously, we consider the operator $\Phi_2 : [0,1] \times \overline{\Omega} \to Y \times \mathbb{R}^2$ given by

$$\Phi_2(\lambda, u, \mu, \nu) = \Big(\mu + \int_{R_1}^{r} \phi_1^{-1}\Big(s^{1-N}\Big(\nu + \lambda \int_{R_1}^{s} \sigma^{N-1} F(u)(\sigma)d\sigma\Big)\Big)ds, \mu + \tau(u), \nu + \varphi(-u') - B\Big).$$

It is not difficult to obtain a solution of problem (1.6), (3.3). □

Theorem 3.2. *Let* (H0)–(H2) *hold and let Q be defined by Lemma 2.7. Then for $A, B \in \mathbb{R}$ such that $A \in \operatorname{Im}(\tau)$ and*

$$\frac{R_2 - R_1}{2}\phi_1^{-1}\Big(Q^{-1}\Big(\frac{R_2 - R_1}{2}\Big)\Big) < B - A < R_2 - R_1 - 1,$$

problem (1.1), (1.2) *has at least two different radial solutions.*

Proof. Assume that $A \in \operatorname{Im}(\tau)$. By Lemma 2.6, there exists a unique $d \in \mathbb{R}$ such that $\tau(d) = A$. Let $\tilde{\tau} : X \to \mathbb{R}$ be given by $\tilde{\tau}(u) = \tau(u + d) - \tau(d)$. Then $\tilde{\tau}(u) = 0$.

Define the continuous operator $\tilde{F} : Y \to L^1[0, T]$ by

$$(\tilde{F}u)(r) = (Fz)(r), \quad z(r) = u(r) + A.$$

Observe that, by (H1),

$$|r^{N-1}(\tilde{F}u)(r)| \le g(|(u(r) + A)'|) = g(|u'(r)|) \quad \text{for } u \in Y.$$

Indeed, by Lemma 3.1,

$$\begin{cases} (r^{N-1}\phi(u'(r)))' = r^{N-1}(\tilde{F}u)(r), & r \in (R_1, R_2), \\ \tilde{\tau}(u) = 0, \quad \max\{u(r) \mid r \in [R_1, R_2]\} - \min\{u(t) \mid r \in [R_1, R_2]\} = B - A \end{cases} \tag{3.9}$$

has at least two different solutions, u_1, u_2. It is easy to verify that $u(r)$ is a solution of (3.9) if and only if $u(r) + A$ is a solution of (1.6). As a consequence

$$u_i(r) = \tilde{u}_i(r) + A, \quad i = 1, 2$$

are two different solutions of problem (1.6), (2.1). Then $u_i(r)$ are two different radial solutions of problem (1.1), (1.2). □

Finally, we give an example to illustrate our main result.

Example 3.3. Consider the quasilinear problem

$$\nabla \cdot \Big(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\Big) = F_1(v)(|x|) + F_2(v)(|x|)h(v'(|x|)), \quad 5 < |x| < 9, \ x \in \mathbb{R}^N \tag{3.10}$$

submitted to the nonlinear boundary conditions

$$\min\{v(x) \mid 5 \le |x| \le 9\} = A, \quad \max\{v(x) \mid 5 \le |x| \le 9\} = B, \tag{3.11}$$

where $A, B \in \mathbb{R}$ are constants satisfying $A < B$, and $R_1 = 5$, $R_2 = 9$.

From above, we only need to study the equivalent problem

$$\begin{cases} \Big(r^{N-1}\dfrac{u'}{\sqrt{1 - u'^2}}\Big)' = r^{N-1}(F_1(u)(r) + F_2(u)(r)h(v'(r))) & \text{for a.e. } r \in (R_1, R_2), \\ \min\{u(r) \mid r \in [R_1, R_2]\} = A, \quad \max\{u(r) \mid r \in [R_1, R_2]\} = B, \end{cases}$$

Let $F_i : Y \to L^1[0, \pi]$ for $i = 1, 2$ be continuous operators such that $|r^{N-1}(F_i u)(r)| \le 1$ for any $u \in Y$ and $h \in X$, $|h(s)| \le s^2$ for $s \in \mathbb{R}$.

Notice that $\phi_1(s) = s/\sqrt{1 - s^2}$ and $\phi_1 : (-1, 1) \to \mathbb{R}$ is an increasing homeomorphism, $\phi_1(0) = 0$. Then $\phi_1^{-1} : \mathbb{R} \to (-1, 1)$ is given by $\phi_1^{-1}(s) = s/\sqrt{1 + s^2}$.

If we take $g(s) = 1 + s^2$ for $s \in [0, \infty)$. It is easy to observe that

$$\left|r^{N-1}((F_1u)(r) + (F_2u)(r)h(u'(r)))\right| \le g(|u'(r)|), \quad u \in Y.$$

Furthermore,

$$\int_0^\infty \frac{ds}{g(\phi_1^{-1}(s))} = \int_0^\infty \frac{1+s^2}{1+2s^2}\,ds = \frac{1}{2}(s + \arctan \sqrt{2}s)\Big|_{s=0}^{s=\infty} = \infty \ge R_2 - R_1.$$

Consequence, (H0)–(H2) are fulfilled. In addition,

$$Q(u) = \int_0^u \frac{ds}{g(\phi_1^{-1}(s))} = \int_0^u \frac{1+s^2}{1+2s^2}\,ds = \frac{1}{2}(u + \arctan \sqrt{2}u).$$

Since

$$Q'(u) = \frac{1}{2}\left(1 + \frac{\sqrt{2}}{1+2u^2}\right) > 0 \quad \text{for } u \in [0,\infty)$$

and Q is strictly monotone increasing, Q^{-1} certainly exists. By simple computation, we obtain

$$\frac{R_2 - R_1}{2}\phi_1^{-1}\left(Q^{-1}\left(\frac{R_2 - R_1}{2}\right)\right) = 2\phi_1^{-1}(Q^{-1}(2)) < \frac{5}{2} < R_2 - R_1 - 1 = 3.$$

Note that $\chi(u) = \min\{u(r) \mid r \in [5,9]\}$, $\psi(u) = \max\{u(r) \mid r \in [5,9]\}$ and $\chi, \psi \in \mathcal{A}$. By Theorem 3.2, it is easy to verify that

$$\frac{R_2 - R_1}{2}\phi_1^{-1}\left(Q^{-1}\left(\frac{R_2 - R_1}{2}\right)\right) < \frac{5}{2} \le B - A \le 3$$

for any $A, B \in \mathbb{R}$. Then problem (3.10), (3.11) has at least two different radial solutions.

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