Research Article

Genni Fragnelli, Dimitri Mugnai* and Nikolaos S. Papageorgiou

The Brezis–Oswald Result for Quasilinear Robin Problems

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Abstract: In this paper we consider a nonlinear elliptic problem driven by a nonhomogeneous differential operator with Robin boundary conditions. We produce conditions on the reaction term near 0^+ and near $+\infty$, which imply the existence and uniqueness of a positive solution. In the particular case of equations driven by the p-Laplacian, we show that these conditions are also necessary, extending in this way the semilinear Dirichlet work of Brezis and Oswald [5].

Keywords: Nonlinear Nonhomogeneous Differential Operator, p-Laplacian, Robin Boundary Conditions, Existence and Uniqueness of Positive Solutions, Principal Eigenvalue, Nonlinear Picone's Identity, Nonlinear Green's Identity, Nonlinear Maximum Principle

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In an interesting paper, Brezis and Oswald [5] examined the semilinear Dirichlet problem

$$\begin{cases} -\Delta u(z) = f(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0, \end{cases} \tag{1.1}$$

and, under very general conditions on the reaction term $f(z, x)$, established necessary and sufficient conditions for the existence of nontrivial solutions for problem (1.1).

The purpose of this work is to extend the Brezis–Oswald result to the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \\ u \geq 0. \end{cases} \tag{1.2}$$

In this problem the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$, involved in the definition of the differential operator, is continuous, strictly monotone (hence maximal monotone too) and satisfies certain other regularity and growth conditions, listed in hypotheses (Ha) below. These hypotheses provide a broad analytic framework, in which we

Genni Fragnelli: Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy, e-mail: genni.fragnelli@uniba.it. http://orcid.org/0000-0002-5436-7006
***Corresponding author: Dimitri Mugnai:** Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy, e-mail: dimitri.mugnai@unipg.it. http://orcid.org/0000-0001-8908-5220
Nikolaos S. Papageorgiou: Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece, e-mail: npapg@math.ntua.gr

can fit many nonlinear differential operators of interest, such as the p-Laplacian ($1 < p < \infty$). However, we stress that the differential operator in (1.2) need not be homogeneous and this is the source of difficulties in the analysis of problem (1.2). The reaction term $f(z, x)$ is a Carathéodory function (that is, the map $z \mapsto f(z, x)$ is measurable for all $x \in \mathbb{R}$, and the map $x \mapsto f(z, x)$ is continuous for a.a. $z \in \Omega$). On $f(z, \cdot)$ we impose only a unilateral growth restriction from above. In the boundary condition, $\partial/\partial n_a$ denotes the generalized directional derivative defined by

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \quad \text{for all } u \in W^{1,p}(\Omega),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This generalized directional derivative is dictated by the nonlinear Green's identity and was also used by Lieberman [15].

With the setting above, in this work we produce conditions which are sufficient for the existence of positive smooth solutions. In this special case of the p-Laplacian, we show that these conditions are also necessary, extending this way the work of Brezis and Oswald [5].

For references on the subject, see [3, 8, 10, 13, 25] for semilinear Dirichlet problems and [9, 12, 14] for Dirichlet problems driven by the p-Laplacian.

2 Auxiliary Results

Let $\Theta \in C^1(0, \infty)$ be a function such that

$$0 < \hat{c} \le \frac{\Theta'(t)t}{\Theta(t)} \le c_0 \quad \text{and} \quad c_1 t^{p-1} \le \Theta(t) \le c_2(1 + t^{p-1}) \tag{2.1}$$

for all $t > 0$ and some $c_1, c_2 > 0$.

(Ha) $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and, in addition, the following hold:

(i) $a_0 \in C^1(0, \infty)$, the map $t \mapsto a_0(t)t$ is strictly increasing,

$$\lim_{t \to 0^+} a_0(t)t = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1.$$

(ii) We have

$$|\nabla a(y)| \le c_3 \frac{\Theta(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\} \text{ and some } c_3 > 0.$$

(iii) We have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{\Theta(|y|)}{|y|}|\xi|^2 \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\} \text{ and all } \xi \in \mathbb{R}^N.$$

(iv) If $G_0(t) = \int_0^t s a_0(s)\, ds$, then there exist $1 < q \le p$, $\delta > 0$ and $\tilde{c} > 0$ such that the map $t \mapsto G_0(t^{\frac{1}{q}})$ is convex on $[0, +\infty)$ and

$$G_0(t) \le \tilde{c} t^q \quad \text{for all } t \in [0, \delta].$$

Remark 2.1. Hypotheses (Ha) (i)–(iii) are dictated by the nonlinear regularity theory of Lieberman [5] and by the nonlinear maximum principle of Pucci and Serrin [24]. Hypothesis (Ha) (iv) is used to prove the uniqueness of the positive solution.

Note that the primitive $G_0(t)$, $t \ge 0$, is strictly convex and strictly increasing. By setting $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$, we have that $G(\cdot)$ is convex and continuously differentiable. Moreover,

$$\nabla G(0) = 0 \quad \text{and} \quad \nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y)$$

for all $y \in \mathbb{R}^N \setminus \{0\}$. Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$ vanishing at $y = 0$. Finally, the convexity of $G(\cdot)$ implies that

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$ and is an easy consequence of hypotheses (Ha) above.

Lemma 2.2. *If hypotheses* (Ha) (i)–(iii) *hold, then*
(a) $y \mapsto a(y)$ *is continuous and strictly monotone, hence maximal monotone too,*
(b) $|a(y)| \le c_4(1 + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$ *and some* $c_4 > 0$,
(c) $(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

Using this lemma together with (2.1) and (2.2), we have the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.3. *If hypotheses* (Ha) (i)–(iii) *hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_5(1 + |y|^p)$$

for all $y \in \mathbb{R}^N$ *and some* $c_5 > 0$.

Examples. The following maps satisfy hypotheses (Ha):
(i) The map

$$a(y) = |y|^{p-2}y \quad \text{with } 1 < p < \infty,$$

which corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(ii) The map

$$a(y) = |y|^{p-2}y + |y|^{q-2}y \quad \text{with } 1 < q < p < \infty,$$

which corresponds to the (p, q)-differential operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics; see [4] and [6] for examples in quantum and plasma physics, respectively. Recently there have been some existence and multiplicity results for equations driven by such operators, see [7, 16, 20, 22, 23, 26].
(iii) The map

$$a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y \quad \text{with } 1 < p < \infty,$$

which corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{\frac{p-2}{2}}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(iv) The map

$$a(y) = |y|^{p-2}y\Big(1 + \frac{1}{1 + |y|^p}\Big) \quad \text{with } 1 < p < \infty.$$

The hypothesis on the boundary term $\beta(\cdot)$ is the following:
(Hβ) $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

We start by considering the following Robin eigenvalue problem for the negative p-Laplacian plus an indefinite potential $\xi(\cdot)$:

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $1 < p < \infty$ and

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ the outward unit normal on $\partial\Omega$. The Neumann version of this eigenvalue problem (that is, $\beta \equiv 0$), was investigated by Mugnai and Papageorgiou [18]. On the other hand, Papageorgiou and Rădulescu [21] studied the case $\xi \equiv 0$.

In addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Also, in the sequel, every Sobolev function restricted on $\partial\Omega$ is understood in the sense of traces.

Proposition 2.4. *If $\xi \in L^\infty(\Omega)$ and hypothesis* (Hβ) *holds, then problem* (2.3) *admits a smallest eigenvalue $\hat{\lambda}_1(\xi, \beta, p) \in \mathbb{R}$, which is simple and isolated, and there exists a corresponding L^p-normalized eigenfunction $\hat{u}_1$ (that is, $\|\hat{u}_1\|_p = 1$) which satisfies*

$$\hat{u}_1 \in C^{1,\eta}(\overline{\Omega}) \quad \text{for some } \eta \in (0,1), \qquad \hat{u}_1 \in \operatorname{int} C_+.$$

Moreover, every eigenfunction corresponding to an eigenvalue $\hat{\lambda} > \hat{\lambda}_1$ is nodal (that is, sign changing).

Proof. Let $\gamma \colon W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \, dz + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Here $\sigma(\cdot)$ denotes the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$. Let $M \subseteq W^{1,p}(\Omega)$ the C^1-Banach manifold defined by

$$M := \{u \in W^{1,p}(\Omega) : \|u\|_p = 1\},$$

and set

$$\hat{\lambda}_1(\xi, \beta, p) = \inf\{\gamma(u) : u \in M\}. \tag{2.4}$$

Evidently,

$$\hat{\lambda}_1(\xi, \beta, p) \geq -\|\xi\|_\infty,$$

see hypothesis (Hβ) and recall that $\xi \in L^\infty(\Omega)$.

Let $\{u_n\}_{n\geq 1} \subseteq M$ be a minimizing sequence for (2.4). So, $\gamma(u_n) \downarrow \hat{\lambda}_1(\xi, \beta, p)$. Evidently, $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded, and so we may assume that

$$u_n \rightharpoonup \hat{u}_1 \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to \hat{u}_1 \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{2.5}$$

Here we have used the Sobolev embedding theorem and the compactness of the trace map.

Because of (2.5), we have

$$\|D\hat{u}_1\|_p^p \leq \liminf_{n\to\infty} \|Du_n\|_p^p, \tag{2.6}$$

$$\int_\Omega \xi(z)|u_n|^p \, dz \to \int_\Omega \xi(z)|\hat{u}_1|^p \, dz, \quad \int_{\partial\Omega} \beta(z)|u_n|^p \, d\sigma \to \int_{\partial\Omega} \beta(z)|\hat{u}_1|^p \, d\sigma. \tag{2.7}$$

Using (2.6)–(2.7), we obtain

$$\gamma(\hat{u}_1) \leq \hat{\lambda}_1(\xi, \beta, p). \tag{2.8}$$

Note that $\|\hat{u}_1\|_p = 1$, see (2.5). Hence, $\hat{u}_1 \in M$, and so (2.8) becomes

$$\gamma(\hat{u}_1) = \hat{\lambda}_1(\xi, \beta, p).$$

The Lagrange multiplier rule (see, for example, [11, p. 701]), implies that $\hat{\lambda}_1(\xi, \beta, p)$ is an eigenvalue of problem (2.3), more precisely the smallest eigenvalue, with $\hat{u}_1 \in W^{1,p}(\Omega)$ a corresponding eigenfunction. From [23], we know that $\hat{u}_1 \in L^\infty(\Omega)$. Then the regularity result of Lieberman [15] implies that $\hat{u}_1 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Note that

$$\gamma(|u|) = \gamma(u) \quad \text{for all } u \in M.$$

So, we infer that $\hat{u}_1$ does not change sign and we may assume that $\hat{u}_1 \geq 0$. The nonlinear maximum principle (see, for example, [11, p. 738] and [24, pp. 111, 120]) implies that $\hat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$, hence $\hat{u}_1 \in \operatorname{int} C_+$.

Next we establish the simplicity of $\hat{\lambda}_1(\xi,\beta,p)$. To this end, let $\hat{v}_1 \in W^{1,p}(\Omega)$ be another eigenfunction corresponding to $\hat{\lambda}_1(\xi,\beta,p)$. As we did for $\hat{u}_1$, we can show that $\hat{v}_1 \in \operatorname{int} C_+$. Consider the function

$$R(\hat{u}_1,\hat{v}_1)(z) = |D\hat{u}_1(z)|^p - |D\hat{v}_1(z)|^{p-2}\Big(D\hat{v}_1(z),\ D\Big(\frac{\hat{u}_1^p}{\hat{v}_1^{p-1}}\Big)(z)\Big)_{\mathbb{R}^N}.$$

Invoking the nonlinear Picone's identity (see [2] and [17, p. 255]) and the nonlinear Green's identity (see [11, p. 211]), we have

$$\begin{aligned}
0 &\le \int_\Omega R(\hat{u}_1,\hat{v}_1)\,dz\\
&= \|D\hat{u}_1\|_p^p + \int_\Omega (\Delta_p\hat{v}_1)\frac{\hat{u}_1^p}{\hat{v}_1^{p-1}}\,dz + \int_{\partial\Omega}\beta(z)\hat{u}_1^p\,d\sigma\\
&= \|D\hat{u}_1\|_p^p + \int_\Omega (\xi(z)-\hat{\lambda}_1(\xi,\beta,p))\hat{u}_1^p\,dz + \int_{\partial\Omega}\beta(z)\hat{u}_1^p\,d\sigma \quad \text{(see (2.3))}\\
&= \gamma(\hat{u}_1) - \hat{\lambda}_1(\xi,\beta,p)\|\hat{u}_1\|_p^p\\
&= 0.
\end{aligned}$$

Recalling that $R(\hat{u}_1,\hat{v}_1)(z) \ge 0$, we get $R(\hat{u}_1,\hat{v}_1)(z) = 0$ for a.a. $z \in \Omega$, and so, by the nonlinear Picone's identity, $\hat{u}_1 = \mu\hat{v}_1$ for some $\mu > 0$.

Now, suppose that $\hat{\lambda} > \hat{\lambda}_1(\xi,\beta,p)$ is another eigenvalue of (2.3) with $u \in W^{1,p}(\Omega)$ a corresponding L^p-normalized eigenfunction. Assume that $\hat{u}$ has constant sign and to fix things suppose that $\hat{u}_1 \ge 0$. As before, the nonlinear regularity theory (see [15]) and the nonlinear maximum principle (see [11, 24]), imply that $\hat{u} \in \operatorname{int} C_+$. Using once again the nonlinear Picone's identity and the nonlinear Green's identity, we have

$$\begin{aligned}
0 &\le \int_\Omega R(\hat{u}_1,\hat{u})\,dz\\
&= \|D\hat{u}_1\|_p^p + \int_\Omega (\Delta_p\hat{u})\frac{\hat{u}_1^p}{\hat{u}^{p-1}}\,dz + \int_{\partial\Omega}\beta(z)\hat{u}_1^p\,d\sigma\\
&= \|D\hat{u}_1\|_p^p + \int_\Omega (\xi(z)-\hat{\lambda})\hat{u}_1^p\,dz + \int_{\partial\Omega}\beta(z)\hat{u}_1^p\,d\sigma\\
&= \gamma(\hat{u}_1) - \hat{\lambda}\|\hat{u}_1\|_p^p\\
&= \hat{\lambda}_1(\xi,\beta,p) - \hat{\lambda}\\
&< 0,
\end{aligned}$$

a contradiction. So, $\hat{u} \in C^1(\overline{\Omega})$ must be nodal.

Finally, we show that $\hat{\lambda}_1(\xi,\beta,p)$ is isolated in the spectrum of (2.3). Again we argue by contradiction. So, suppose that $\hat{\lambda}_1(\xi,\beta,p)$ is not isolated. Then we can find eigenvalues $\{\lambda_n\}_{n\ge1}$ such that $\lambda_n \downarrow \hat{\lambda}_1(\xi,\beta,p)$ as $n \to \infty$. Let $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a sequence of corresponding L^p-normalized eigenfunctions. Evidently, $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. Then, from [23], we know that there exists $M_1 > 0$ such that

$$\|u_n\|_\infty \le M_1 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, from [15], it follows that there exist $\eta \in (0,1)$ and $M_2 > 0$ such that

$$u_n \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\eta}(\overline{\Omega})} \le M_2 \quad \text{for all } n \in \mathbb{N}.$$

Because of the compact embedding of $C^{1,\eta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we may assume (at least for a subsequence) that

$$u_n \to \tilde{u}_1 \quad \text{in } C^1(\overline{\Omega}),\ \|\tilde{u}_1\|_p = 1. \tag{2.9}$$

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega). \tag{2.10}$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\, dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\, d\sigma = \lambda_n \int_\Omega |u_n|^{p-2}u_n h\, dz$$

for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$. Hence, passing to the limit as $n \to \infty$ and using (2.9), we find

$$\langle A(\tilde u_1), h\rangle + \int_\Omega \xi(z)|\tilde u_1|^{p-2}\tilde u_1 h\, dz + \int_{\partial\Omega} \beta(z)|\tilde u_1|^{p-2}\tilde u_1 h\, d\sigma = \hat\lambda_1(\xi, \beta, p) \int_\Omega |\tilde u_1|^{p-2}\tilde u_1 h\, dz$$

for all $h \in W^{1,p}(\Omega)$, that is,

$$\begin{cases} -\Delta_p \tilde u_1(z) + \xi(z)|\tilde u_1(z)|^{p-2}\tilde u_1(z) = \hat\lambda_1(\xi, \beta, p)|\tilde u_1(z)|^{p-2}\tilde u_1(z) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial \tilde u_1}{\partial n_p} + \beta(z)|\tilde u_1|^{p-2}\tilde u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

see [21]. This implies $\tilde u_1 = \pm\hat u_1$, see (2.9) and recall that $\hat\lambda_1(\xi, \beta, p)$ is simple. Then we can suppose that $\tilde u_1 \in \operatorname{int} C_+$, and from (2.9) we have that $u_n \in \operatorname{int} C_+$ for all $n \geq n_0$, which contradicts the fact established earlier, that every non principal eigenvalue has nodal eigenfunctions. This proves that $\hat\lambda_1(\xi, \beta, p)$ is isolated in the spectrum of (2.3). □

3 Positive Solutions

In this section, we establish the existence of positive solutions for problem (1.2) under general conditions on the reaction term $f(z, x)$. The precise hypotheses on the function $f(z, x)$ are as follows:

(Hf) The reaction $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:

(i) $f(\cdot, x) \in L^\infty(\Omega)$ for all $x \geq 0$, and there exists $c_6 > 0$ such that

$$f(z, x) \leq c_6(1 + x^{p-1}) \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0.$$

(ii) The function $x \mapsto f(z, x)/x^{q-1}$ is strictly decreasing on $(0, +\infty)$ for a.a. $z \in \Omega$, with $1 < q \leq p$ as in hypothesis (Ha) (iv).

(iii) If

$$\mu(z) := \lim_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} \quad \text{for a.a. } z \in \Omega, \qquad \hat\mu = \frac{p-1}{c_1}\mu, \qquad \hat\beta = \frac{p-1}{c_1}\beta,$$

then $\hat\lambda_1(-\hat\mu, \hat\beta, p) > 0$.

(iv) If

$$\mu_0(z) = \lim_{x\to0^+} \frac{f(z, x)}{x^{q-1}} \quad \text{for a.a. } z \in \Omega, \qquad \tilde\mu_0 = \frac{1}{\tilde c q}\mu_0, \qquad \tilde\beta = \frac{1}{\tilde c q}\beta,$$

then $\hat\lambda_1(-\tilde\mu_0, \tilde\beta_0, q) < 0$, where $\tilde c$ is the constant appearing in (Ha) (iv).

Remark 3.1. (1) Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume that

$$f(z, x) = f(z, 0) \quad \text{for a.a. } z \in \Omega \text{ and all } x \leq 0.$$

(2) Hypothesis (Hf) (i) is a unilateral growth condition on $f(z, \cdot)$. Clearly, both functions $\mu(\cdot)$ and $\mu_0(\cdot)$ in (Hf) (iii) and (iv), respectively, are measurable functions.

(3) Hypothesis (Hf) (iii) covers the case $\mu = -\infty$ in a set of positive measure.

(4) Hypothesis (Hf) (ii) implies that the function $x \mapsto f(z,x)/x^{p-1}$ is strictly decreasing on $(0,+\infty)$ for a.a. $z \in \Omega$. Then, using hypothesis (Hf) (i), we have

$$\frac{f(z,x)}{x^{p-1}} \le f(z,1) \le \|f(\cdot\,,1)\|_\infty = c_7 \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 1.$$

This implies $\mu(z) \le c_7$ for a.a. $z \in \Omega$ (see hypothesis (Hf) (iii)), hence

$$\hat{\lambda}_1(-\hat{\mu},\hat{\beta},p) \in (-\infty,+\infty].$$

On the other hand, again from hypotheses (Hf) (i) and (ii), we have

$$\frac{f(z,x)}{x^{q-1}} \ge f(z,1) \ge -\|f(\cdot\,,1)\|_\infty = -c_7 \quad \text{for a.a. } z \in \Omega \text{ and all } x \in (0,1].$$

Thus, $\mu_0(z) \ge -c_7$ for a.a. $z \in \Omega$ and

$$\hat{\lambda}_1(-\tilde{\mu}_0,\tilde{\beta},q) \in [-\infty,+\infty).$$

Of course, if $\mu,\mu_0 \in L^\infty(\Omega)$, then $\hat{\lambda}_1(-\hat{\mu},\hat{\beta},p)$ and $\hat{\lambda}_1(-\tilde{\mu}_0,\tilde{\beta},q) \in \mathbb{R}$ are principal eigenvalues of nonlinear eigenvalue problems similar to (2.3).

(5) If $f(z,x)=f(x)$ (autonomous case), then hypotheses (Hf) (iii) and (iv) are equivalent to writing

$$\mu < 0 < \mu_0.$$

Example. Consider the function

$$f(x) = \lambda(x^{q-1} - x^{r-1}) \quad \text{for all } x \ge 0$$

with $1 < q \le p < r$ and $\lambda > 0$. This function satisfies hypotheses (Hf) with $\mu_0 = \lambda > 0$ and $\mu = -\infty$. Note that $f(x)$ is the reaction term in the nonlinear logistic equation of sub-diffusive type, see [19].

We introduce the following Carathéodory function:

$$k(z,x) = \begin{cases} f(z,0) & \text{if } x \le 0,\\ f(z,x) + \dfrac{c_1}{p-1}x^{p-1} & \text{if } x > 0. \end{cases} \tag{3.1}$$

Let $\mathcal{K}(z,x) = \int_0^x k(z,s)\,ds$, and consider the C^1-functional $\varphi\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \int_\Omega G(Du)\,dz + \frac{c_1}{p(p-1)}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega \mathcal{K}(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 3.2. *If hypotheses* (Ha) (i)–(iii), (Hβ), (Hf) *hold, then the functional φ is coercive.*

Proof. We argue indirectly. So, suppose that φ is not coercive. Then, we can find $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| \to +\infty \qquad \text{and} \qquad \varphi(u_n) \le M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \in \mathbb{N}. \tag{3.2}$$

From (3.1) and hypothesis (Hf) (i), we have

$$\mathcal{K}(z,x) \le c_8(1+|x|^p) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_8 > 0. \tag{3.3}$$

Using Corollary 2.3 and (3.3), we have

$$\frac{c_1}{p(p-1)}(\|Du_n\|_p^p + \|u_n\|_p^p) + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u_n|^p\,d\sigma \le c_9(1+\|u_n\|_p^p) \tag{3.4}$$

for some $c_9 > 0$ and all $n \in \mathbb{N}$. From (3.2), (3.4) and hypothesis (Hβ), it follows that

$$\|u_n\|_p \to +\infty \quad \text{as } n \to \infty. \tag{3.5}$$

Let $y_n = u_n/\|u_n\|_p$, $n \in \mathbb{N}$. Then

$$\|y_n\|_p = 1 \quad \text{for all } n \in \mathbb{N}, \tag{3.6}$$

and from (3.4) we have

$$\frac{c_1}{p(p-1)}(\|Dy_n\|_p^p + 1) \le c_9\Big(\frac{1}{\|u_n\|_p^p} + 1\Big) \quad \text{for all } n \in \mathbb{N},$$

which implies that $\{y_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded by (3.5) and (3.6). So, we may assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega),\ \|y\|_p = 1. \tag{3.7}$$

From (3.2) and Corollary 2.3, we have

$$\begin{aligned} &\frac{c_1}{p(p-1)}(\|Dy_n\|_p^p + \|y_n\|_p^p) + \frac{1}{p}\int_{\partial\Omega} \beta(z)|y_n|^p \, d\sigma \\ &\le \frac{M_1}{\|u_n\|_p^p} + \int_\Omega \frac{\mathcal{K}(z,u_n)}{\|u_n\|_p^p}\,dz \\ &\le \frac{M_1}{\|u_n\|_p^p} + \int_{\{u_n>0\}} \Big(\frac{F(z,u_n)}{\|u_n\|_p^p} + \frac{c_1}{p(p-1)}y_n^p\Big)\,dz + \int_{\{u_n<0\}} \frac{f(z,0)u_n}{\|u_n\|_p^p}\,dz \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$ (see (3.1)), where $F(z,s) = \int_0^s f(z,t)\,dt$.

Hypothesis (Hf) (ii) implies that

$$\frac{f(z,x)}{x^{p-1}} \ge f(z,1) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in (0,1],$$

hence, by hypothesis (Hf) (i),

$$f(z,x) \ge -\|f(\cdot,1)\|_\infty x^{p-1} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in (0,1].$$

Therefore, $f(z,0) \ge 0$ for a.a. $z \in \Omega$, and so

$$\int_{\{u_n\le0\}} \frac{f(z,0)u_n}{\|u_n\|_p^p}\,dz \le 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$\frac{c_1}{p(p-1)}(\|Dy_n\|_p^p + \|y_n\|_p^p) \le \frac{M_1}{\|u_n\|_p^p} + \frac{c_1}{p(p-1)}\|y_n^+\|_p^p + \int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\,dz \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

If $\{u_n^+\}_{n\ge1} \subseteq L^p(\Omega)$ is bounded, then since $y_n^+ = u_n^+/\|u_n\|_p$ for all $n \in \mathbb{N}$, from (3.5) we infer that $y_n^+ \to 0$ in $L^p(\Omega)$, hence $y \le 0$. Hypothesis (Hf) (i) implies that

$$F(z,x) \le c_{10}(1 + x^p) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \ge 0 \text{ and some } c_{10} > 0, \tag{3.11}$$

hence

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\,dz \le \frac{c_{10}}{\|u_n\|_p^p} + c_{10}\|y_n^+\|_p^p \quad \text{for all } n \in \mathbb{N},$$

and by (3.5),

$$\limsup_{n\to\infty} \int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\,dz \le 0. \tag{3.12}$$

So, if in (3.10) we pass to the limit as $n \to \infty$ and use (3.7) and (3.12), then

$$\frac{c_1}{p(p-1)}(\|Dy\|_p^p + \|y\|_p^p) \le 0,$$

and so $y = 0$, in contradiction with (3.7).

If $\{u_n^+\}_{n\geq 1} \subseteq L^p(\Omega)$ is unbounded, then we may assume that $\|u_n^+\|_p \to +\infty$, and from (3.2) and (3.9) we have

$$\frac{c_1}{p(p-1)}(\|Dy_n^+\|_p^p + \|y_n^+\|_p^p) + \frac{1}{p}\int_{\partial\Omega} \beta(z)(y_n^+)^p \, d\sigma \leq \frac{M_1}{\|u_n^+\|_p^p} + \int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p} \, dz \quad \text{for all } n \in \mathbb{N}. \tag{3.13}$$

Note that

$$\int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p} \, dz = \int_{\{y^+=0\}} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p} \, dz + \int_{\{y>0\}\cap\{y_n>0\}} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p} \, dz \quad \text{for all } n \in \mathbb{N}. \tag{3.14}$$

From (3.7) we have that $y_n^+ \to y^+$ in $L^p(\Omega)$, hence

$$y_n^+(z) \to y^+(z) \quad \text{for a.a. } z \in \Omega,$$

at least for a subsequence. Then

$$u_n^+(z) \to +\infty \quad \text{for a.a. } z \in \{y^+ > 0\} = \{y > 0\} \tag{3.15}$$

and

$$\chi_{\{y>0\}\cap\{y_n>0\}} \to \chi_{\{y>0\}}(z) \quad \text{for a.a. } z \in \Omega. \tag{3.16}$$

Let $z \in \{\mu > -\infty\} \setminus E$ with $|E|_N = 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$) be such that (see hypothesis (Hf) (iii))

$$\frac{f(z, x)}{x^{p-1}} \to \mu(z) \quad \text{as } x \to +\infty.$$

Given $\epsilon > 0$, we can find $M_2 = M_2(z) > 0$ such that

$$f(z, x) \leq (\mu(z) + \epsilon)x^{p-1} \quad \text{for all } x \geq M_2,$$

hence

$$F(z, x) \leq \frac{1}{p}(\mu(z) + \epsilon)x^p \quad \text{for all } x \geq M_2,$$

so that

$$\limsup_{x\to+\infty} \frac{F(z, x)}{x^p} \leq \frac{1}{p}(\mu(z) + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \downarrow 0$ to conclude that

$$\limsup_{x\to+\infty} \frac{F(z, x)}{x^p} \leq \frac{1}{p}\mu(z).$$

Next suppose $z \in \{\mu = -\infty\} \setminus E$ (recall that $|E|_N = 0$). We have

$$\frac{f(z, x)}{x^{p-1}} \to -\infty \quad \text{as } x \to +\infty.$$

Then, given $S > 0$, we can find $M_3 = M_3(S) > 0$ such that

$$f(z, x) \leq -Sx^{p-1} \quad \text{for all } x \geq M_3,$$

thus

$$\frac{F(z, x)}{x^p} \leq -\frac{S}{p} \quad \text{for all } x \geq M_3,$$

and so

$$\limsup_{x\to+\infty} \frac{F(z, x)}{x^p} \leq -\frac{S}{p}.$$

Since $S > 0$ is arbitrary, we let $S \to +\infty$ to conclude that

$$\lim_{x\to+\infty} \frac{F(z, x)}{x^p} = -\infty = \mu(z).$$

From the discussion above, we infer that

$$\limsup_{x\to+\infty}\frac{F(z,x)}{x^p}\le\frac{1}{p}\mu(z)\quad\text{for a.a. } z\in\Omega. \tag{3.17}$$

Using (3.15), (3.16), (3.17) and Fatou's lemma, we have

$$\limsup_{n\to\infty}\int_{\{y>0\}\cap\{y_n>0\}}\frac{F(z,u_n^+)}{(u_n^+)^p}(y_n^+)^p\,dz\le\frac{1}{p}\int_{\{y^+\neq 0\}}\mu(y^+)^p\,dz. \tag{3.18}$$

Also, we have

$$\Bigg|\int_{\{y^+=0\}}\frac{F(z,u_n^+)}{\|u_n^+\|_p^p}\,dz\Bigg|\le c_{10}\int_{\{y^+=0\}}\Big(\frac{1}{\|u_n^+\|_p^p}+(y_n^+)^p\Big)\,dz\quad\text{for all } n\in\mathbb{N}$$

(see (3.11)), thus

$$\lim_{n\to\infty}\int_{\{y^+=0\}}\frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz=0. \tag{3.19}$$

We return to (3.14), pass to the limit as $n\to\infty$ and use (3.18) and (3.19). Then

$$\limsup_{n\to\infty}\int_\Omega\frac{F(z,u_n^+)}{\|u_n^+\|_p^p}\,dz\le\frac{1}{p}\int_{\{y^+\neq 0\}}\mu(y^+)^p\,dz. \tag{3.20}$$

In (3.13), if we pass to the limit as $n\to\infty$ and use (3.7) and (3.20), then we obtain

$$\frac{c_1}{p(p-1)}\Big[\|Dy^+\|_p^p+\int_{\partial\Omega}\hat\beta(z)(y^+)^p\,d\sigma\Big]\le\frac{1}{p}\int_{\{y^+\neq 0\}}\mu(y^+)^p\,dz.$$

Hence,

$$\|Dy^+\|_p^p+\int_{\partial\Omega}\hat\beta(z)(y^+)^p\,d\sigma\le\int_\Omega\hat\mu(y^+)^p\,dz. \tag{3.21}$$

If $y^+=0$, then from (3.7), (3.10) and (3.19), we have $\|Dy\|_p^p+\|y\|_p^p\le 0$, and thus $y=0$, in contradiction with (3.7). Therefore, $y^+\not\equiv 0$, and so from (3.21) we have (see the proof of Proposition 2.4) $\hat\lambda_1(-\hat\mu,\hat\beta,p)\le 0$, which contradicts hypothesis (Hf) (iii). This proves the coercivity of φ. ☐

Proposition 3.3. *If hypotheses* (Ha) (i)–(iii), (Hβ), (Hf) *hold, then the functional φ is sequentially weakly lower semicontinuous.*

Proof. Let $\hat\gamma(u)\colon W^{1,p}(\Omega)\to\mathbb{R}$ and $\hat\psi\colon W^{1,p}(\Omega)\to\mathbb{R}$ be the C^1-functionals defined by

$$\hat\gamma(u)=\int_\Omega G(Du)\,dz+\frac{c_1}{p(p-1)}\|u\|_p^p+\frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma$$

and

$$\hat\psi(u)=-\int_\Omega\mathcal{K}(z,u)\,dz\quad\text{for all } u\in W^{1,p}(\Omega).$$

The convexity of $G(\cdot)$, the Sobolev embedding Theorem and the compactness of the trace map, imply that $\hat\gamma$ is sequentially weakly lower semicontinuous. Since $\varphi=\hat\gamma+\hat\psi$, to prove the sequential weak lower semicontinuity of φ, we need to show that $\hat\psi$ is sequentially weakly lower semicontinuous.

To this end, let $S\in\mathbb{R}$ and consider the set

$$L_S:=\{u\in W^{1,p}(\Omega):\hat\psi(u)\le S\}.$$

We need to show that L_S is sequentially weakly closed. So, let $\{u_n\}_{n\ge1}\subseteq L_S$ and assume that

$$u_n\rightharpoonup u\quad\text{in } W^{1,p}(\Omega),$$

hence, by the Sobolev embedding theorem,

$$u_n\to u\quad\text{in } L^p(\Omega).$$

This implies that

$$u_n^+ \to u^+ \quad \text{and} \quad u_n^- \to u^- \quad \text{in } L^p(\Omega), \tag{3.22}$$

and, in particular,

$$u_n^+(z) \to u^+(z) \quad \text{for a.a. } z \in \Omega, \tag{3.23}$$

at least for a subsequence.

We have (see (3.1))

$$S \geq -\int_\Omega \mathcal{K}(z, u_n)\,dz = -\int_\Omega F(z, u_n^+)\,dz - \frac{c_1}{p(p-1)}\|u_n^+\|_p^p - \int_\Omega f(z,0)(-u_n^-)\,dz, \tag{3.24}$$

Note that, by (3.22),

$$\frac{c_1}{p(p-1)}\|u_n^+\|_p^p \to \frac{c_1}{p(p-1)}\|u^+\|_p^p \tag{3.25}$$

and, by (3.22) and hypothesis (Hf) (i),

$$\int_\Omega f(z,0)(-u_n^-)\,dz \to \int_\Omega f(z,0)(-u^-)\,dz. \tag{3.26}$$

Also, from (3.23) and Fatou's lemma,

$$\liminf_{n\to\infty}\Bigl(-\int_\Omega F(z, u_n^+)\,dz\Bigr) = -\limsup_{n\to\infty}\int_\Omega F(z, u_n^+)\,dz \geq -\int_\Omega F(z, u^+)\,dz. \tag{3.27}$$

So, if in (3.24) we pass to the limit as $n \to \infty$ and use (3.25)–(3.27), then

$$S \geq -\int_\Omega F(z, u^+)\,dz - \frac{c_1}{p(p-1)}\|u^+\|_p^p - \int_\Omega f(z,0)(-u^-)\,dz = -\int_\Omega \mathcal{K}(z, u)\,dz,$$

see also (3.1).

Hence, $u \in L_S$ and L_S is sequentially weakly closed. It follows that $\hat{\psi}$ is sequentially weakly lower semicontinuous, hence φ is sequentially weakly lower semicontinuous as well. ☐

Now, we are ready to produce a positive smooth solution for problem (1.2). So far we have not used hypothesis (Ha) (iv); in the next result we will use it to establish the nontriviality of the solution we produce.

Proposition 3.4. *If hypotheses* (Ha), (Hβ), (Hf) *hold, then problem* (1.2) *has a positive solution* $u_0 \in \operatorname{int} C_+$.

Proof. Propositions 3.2 and 3.3 permit the use of the Weierstrass–Tonelli theorem. Therefore, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi(u_0) = \inf\{\varphi(u) : u \in W^{1,p}(\Omega)\}. \tag{3.28}$$

Claim 1. $u_0 \geq 0$, $u_0 \neq 0$.

If $u_0^- \neq 0$, then, using (3.1) and recalling that $G \geq 0$ and $f(z,0) \geq 0$ for a.a. $z \in \Omega$, we have

$$\begin{aligned}\varphi(u_0^+) &= \int_\Omega G(Du_0^+)\,dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u_0^+)^p\,d\sigma - \int_\Omega F(z, u_0^+)\,dz\\ &< \int_\Omega G(Du_0)\,dz + \frac{c_1}{p(p-1)}\|u_0\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u_0|^p\,d\sigma - \int_\Omega F(z, u_0^+)\,dz - \int_\Omega f(z,0)(-u_0^-)\,dz\\ &= \varphi(u_0).\end{aligned}$$

But this contradicts (3.28). Thus, $u_0^- \equiv 0$, and so $u_0 \geq 0$.

Next we show that $u_0 \neq 0$. By hypothesis (Hf) (iv) we have $\hat{\lambda}_1(-\tilde{\mu}_0, \tilde{\beta}, q) < 0$. The variational characterization of $\hat{\lambda}_1(-\tilde{\mu}_0, \tilde{\beta}, q)$ (see the proof of Proposition 2.4) and the density of $W^{1,p}(\Omega)$ into $W^{1,q}(\Omega)$, imply that we can find $u \in W^{1,p}(\Omega)$ such that

$$\|u\|_q = 1 \quad \text{and} \quad \|Du\|_q^q + \int_{\partial\Omega}\tilde{\beta}(z)|u|^q\,d\sigma - \int_\Omega \tilde{\mu}_0(z)|u|^q\,dz < 0. \tag{3.29}$$

Clearly, we may assume that $u \geq 0$ (just replace, if necessary, u by $|u|$).

We can find $\{u_n\}_{n\geq 1} \subseteq C^1(\overline{\Omega})$ such that

$$u_n \to u \quad \text{in } W^{1,p}(\Omega), \tag{3.30}$$

see, for example, [11, p. 189]. Since we have that

$$u_n^+ \to u^+ = u \quad \text{in } W^{1,p}(\Omega),$$

we can always assume that $u_n \geq 0$ for all $n \in \mathbb{N}$. Let

$$\hat{u}_n := \min\{u, u_n\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \text{for all } n \in \mathbb{N}.$$

Then, by (3.30), $\hat{u}_n \to u$ in $W^{1,p}(\Omega)$ and, in particular,

$$\hat{u}_n(z) \to u(z) \quad \text{for a.a. } z \in \Omega,$$

at least for a subsequence. Recall that hypothesis (Hf) (ii) implies that

$$\tilde{\mu}_0(z) \geq \tilde{f}(z,1) \geq -\|\tilde{f}(\cdot,1)\|_\infty \quad \text{for a.a. } z \in \Omega,$$

where $\tilde{f} = f/(\tilde{c}q)$, see Remark 3.1 (4), so that

$$\mu_0(z)\hat{u}_n^q(z)\chi_{\{u_n\neq 0\}}(z) \geq -\|\tilde{f}(\cdot,1)\|_\infty u^q(z) \quad \text{for a.a. } z \in \Omega. \tag{3.31}$$

Note that

$$\|\tilde{f}(\cdot,1)\|_\infty u^q(\cdot) \in L^1(\Omega) \tag{3.32}$$

and

$$\tilde{\mu}_0(z)\hat{u}_n^q(z)\chi_{\{u_n\neq 0\}}(z) \to \tilde{\mu}_0(z)u^q(z) \quad \text{for a.a. } z \in \Omega. \tag{3.33}$$

Then (3.31)–(3.33) permit the use of Fatou's lemma, obtaining

$$\liminf_{n\to\infty} \int_{\{u_n\neq 0\}} \tilde{\mu}_0(z)\hat{u}_n^q\,dz \geq \int_{\{u\neq 0\}} \tilde{\mu}_0(z)u^q\,dz. \tag{3.34}$$

Since $\hat{u}_n \to u$ in $W^{1,p}(\Omega)$, using the trace theorem, we have

$$\|D\hat{u}_n\|_q^q + \int_{\partial\Omega} \tilde{\beta}(z)\hat{u}_n^q\,d\sigma \to \|D\hat{u}\|_q^q + \int_{\partial\Omega} \tilde{\beta}(z)u^q\,d\sigma. \tag{3.35}$$

From (3.29), (3.34) and (3.35), we see that we can find $n_0 \in \mathbb{N}$ such that

$$\|D\hat{u}_n\|_q^q + \int_{\partial\Omega} \tilde{\beta}(z)\hat{u}_n^q\,d\sigma - \int_{\{u_n\neq 0\}} \tilde{\mu}_0(z)\hat{u}_n^q\,dz < 0 \quad \text{for all } n \geq n_0.$$

This means that we can find $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\|Du\|_q^q + \int_{\partial\Omega} \tilde{\beta}(z)u^q\,d\sigma - \int_\Omega \tilde{\mu}_0(z)u^q\,dz < 0, \quad u \geq 0. \tag{3.36}$$

The q-homogeneity of the left-hand side in (3.36) implies that we can always assume that $\|u\|_q = 1$.

Let $\tilde{F}(z,x) = \int_0^x \tilde{f}(z,s)\,ds$. For every $(z,x) \in \Omega \times (0,\infty)$, by the chain rule, we have

$$\tilde{F}(z,x) = \int_0^1 \frac{d}{dt}\tilde{F}(z,tx)\,dt = \int_0^1 \tilde{f}(z,tx)x\,dt,$$

hence, by (Hf) (ii),

$$\frac{\tilde{F}(z,x)}{x^q} = \int_0^1 \frac{\tilde{f}(z,tx)}{x^{q-1}}\,dt \geq \int_0^1 \frac{\tilde{f}(z,x)}{x^{q-1}}t^{q-1}\,dt = \frac{1}{q}\frac{\tilde{f}(z,x)}{x^{q-1}},$$

and, by (Hf) (iv),

$$\liminf_{x\to 0^+} \frac{\tilde{F}(z,x)}{x^q} \geq \frac{1}{q}\tilde{\mu}_0(z) \quad \text{for a.a. } z \in \Omega. \tag{3.37}$$

Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\|u\|_q = 1$ be as in (3.36). We can find $\tau \in (0,1]$ so small that

$$\tau u(z) \in [0, \delta] \quad \text{for a.a. } z \in \Omega.$$

Here $\delta \in (0,1]$ is the one given in (Ha) (iv); evidently, we can always assume $\delta \le 1$.

Hence, by (Hf) (ii), we have

$$\begin{aligned} \frac{\tilde{F}(z, \tau u(z))}{\tau^q} &= \frac{1}{\tau^q} \int_0^{\tau u(z)} \tilde{f}(z,s)\,ds \\ &\ge -\frac{1}{\tau^q} \|\tilde{f}(\cdot, 1)\|_\infty \int_0^{\tau u(z)} s^{q-1}\,ds \\ &= -\frac{\|\tilde{f}(\cdot,1)\|_\infty}{q} u^q(z) \\ &\ge -\frac{\|\tilde{f}(\cdot,1)\|_\infty}{q} \|u\|_\infty^q. \end{aligned} \tag{3.38}$$

Then (3.38) permits the use of Fatou's lemma and thus, by (3.37), we have

$$\liminf_{\tau \to 0^+} \int_{\{u \neq 0\}} \frac{\tilde{F}(z, \tau u)}{\tau^q}\,dz \ge \frac{1}{q} \int_{\{u \neq 0\}} \tilde{\mu}_0(z) u^q\,dz.$$

Then, from (3.36) and for $\tau \in (0,1)$ small, we have

$$\tilde{c}\|D(\tau u)\|_q^q + \frac{1}{q} \int_{\partial\Omega} \beta(z)(\tau u)^q\,d\sigma - \tilde{c} \int_\Omega q\tilde{F}(z, \tau u)\,dz < 0,$$

recall that $\tilde{\beta} = \beta/(\tilde{c}q)$. Thus, from hypothesis (Ha) (iv) and by recalling that $\delta \in (0,1]$ and $q \le p$, we have

$$\int_\Omega G(D(\tau u))\,dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)(\tau u)^p\,d\sigma - \int_\Omega F(z, \tau u)\,dz < 0.$$

From (3.1) and the fact that $u \ge 0$, this implies $\varphi(\tau u) < 0$. Hence, from (3.28), $\varphi(u_0) < 0 = \varphi(0)$ and $u_0 \neq 0$ with $u_0 \ge 0$. This proves Claim 1.

Claim 2. $u_0 \in L^\infty(\Omega)$.

For every $k \in \mathbb{N}$, we introduce the following truncation of the reaction $f(z, \cdot)$:

$$f_k(z,x) = \begin{cases} f(z,0) & \text{if } x \le 0, \\ \max\{f(z,x), -kx^{p-1}\} & \text{if } x > 0 \end{cases}$$

(recall that $f(z,0) \ge 0$ for a.a. $z \in \Omega$). This is a Carathéodory function and $f_k(\cdot, x) \in L^\infty(\Omega)$ for all $x \in \mathbb{R}$ (see hypothesis (Hf) (i)). Also, we have

$$|f_k(z,x)| \le c_{11}(1 + |x|^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_{11} > 0. \tag{3.39}$$

We set

$$\mu_0^k(z) = \liminf_{x \to 0^+} \frac{f_k(z,x)}{x^{q-1}} \quad \text{and} \quad \mu^k(z) = \limsup_{x \to +\infty} \frac{f_k(z,x)}{x^{p-1}}.$$

Note that

$$f_k(z,x) \ge f(z,x) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

hence

$$\mu_0^k \ge \mu_0 \quad \text{and} \quad \hat{\lambda}_1(-\tilde{\mu}_0^k, \tilde{\beta}, q) \le \hat{\lambda}_1(-\tilde{\mu}_0, \tilde{\beta}, q) \quad \text{for all } k \in \mathbb{N}.$$

Here $\tilde{\mu}_0^k = \mu_0^k/(\tilde{c}q)$. Moreover, we have $\mu^k \downarrow \mu$ as $k \to +\infty$, thus

$$\hat{\lambda}_1(-\hat{\mu}^k, \hat{\beta}, p) \to \hat{\lambda}_1(-\hat{\mu}, \hat{\beta}, p) > 0 \quad \text{and} \quad \hat{\lambda}_1(-\mu^k, \hat{\beta}, p) > 0$$

for all k sufficiently large, where $\hat{\mu}^k = (p-1)\mu^k/c_1$.

Let $\varphi_k : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined as φ, with the primitive $F(z,x) = \int_0^x f(z,s)\,ds$ replaced by $F_k(z,x) = \int_0^x f_k(z,s)\,ds$. Reasoning as for φ, via the direct method, we can find $u_0^k \geq 0$, $u_0^k \neq 0$ such that

$$\varphi_k(u_0^k) = \inf\{\varphi_k(u) : u \in W^{1,p}(\Omega)\}.$$

Then $\varphi_k'(u_0^k) = 0$, thus, by (2.10) we have

$$\langle A(u_0^k), h\rangle + \int_{\partial\Omega} \beta(z)(u_0^k) h\,d\sigma = \int_{\Omega} f_k(z, u_0^k) h\,dz \quad \text{for all } h \in W^{1,p}(\Omega)$$

and (see [21])

$$\begin{cases} -\operatorname{div} a(Du_0^k(z)) = f_k(z, u_0^k(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0^k}{\partial n_a} + \beta(z)(u_0^k)^{p-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

Because of (3.39), from [23] we have $u_0^k \in L^\infty(\Omega)$. Then, from [15] we have that

$$u_0^k \in C^1(\overline{\Omega}) \quad \text{for all } k \in \mathbb{N}.$$

Let

$$y_k = \min\{u_0, u_0^k\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \text{for all } k \in \mathbb{N}.$$

Recalling that u_0^k is a minimizer of the functional φ_k, we get

$$\varphi_k(u_0^k) \leq \varphi_k(v) \quad \text{for all } v \in W^{1,p}(\Omega). \tag{3.40}$$

Taking $v := \max\{u_0, u_0^k\} \in W^{1,p}(\Omega)$ and using (3.40), we have

$$\begin{aligned} &\int_{\{u_0^k<u_0\}} G(Du_0^k)\,dz + \frac{1}{p}\int_{\{\gamma_0(u_0^k)<\gamma_0(u_0)\}} \beta(z)(u_0^k)^p\,d\sigma - \int_{\{u_0^k<u_0\}} F_k(z, u_0^k)\,dz \\ &\quad\leq \int_{\{u_0^k<u_0\}} G(Du_0)\,dz + \frac{1}{p}\int_{\{\gamma_0(u_0^k)<\gamma_0(u_0)\}} \beta(z)u_0^p\,d\sigma - \int_{\{u_0^k<u_0\}} F_k(z, u_0)\,dz, \end{aligned}$$

with $\gamma_0(\cdot)$ being the trace map. Thus,

$$\int_{\{u_0^k<u_0\}} [G(Du_0^k) - G(Du_0)]\,dz + \frac{1}{p}\int_{\{\gamma_0(u_0^k)<\gamma_0(u_0)\}} \beta(z)((u_0^k)^p - u_0^p)\,d\sigma \leq \int_{\{u_0^k<u_0\}} [F_k(z, u_0^k) - F_k(z, u_0)]\,dz. \tag{3.41}$$

Then, using (3.41) and recalling that $f_k \geq f$, we have

$$\begin{aligned} \varphi(y_k) - \varphi(u_0) &= \int_{\{u_0^k<u_0\}} [G(Du_0^k) - G(Du_0)]\,dz + \frac{1}{p}\int_{\{\gamma_0(u_0^k)<\gamma_0(u_0)\}} \beta(z)((u_0^k)^p - u_0^p)\,d\sigma - \int_{\{u_0^k<u_0\}} [F(z, u_0^k) - F(z, u_0)]\,dz \\ &\leq \int_{\{u_0^k<u_0\}} [F_k(z, u_0^k) - F_k(z, u_0) - F(z, u_0^k) + F(z, u_0)]\,dz \\ &= \int_{\{u_0^k<u_0\}} \int_{u_0^k}^{u_0} (-f_k(z,s) + f(z,s))\,ds\,dz \leq 0. \end{aligned}$$

Hence, $\varphi(y_k) \leq \varphi(u_0)$, and so, by (3.28),

$$\varphi(y_k) = \varphi(u_0) = \inf\{\varphi(u) : u \in W^{1,p}(\Omega)\}.$$

Since $y_k \in L^\infty(\Omega)$, we may assume that $u_0 \in L^\infty(\Omega)$. This proves Claim 2.

Conclusion. Let $h \in C^1(\overline{\Omega})$. We set

$$d(h) = \int_\Omega [\mathcal{K}(z, u_0 + h) - \mathcal{K}(z, u_0) - k(z, u_0)h]\, dz.$$

Then, using Fubini's theorem and Hölder's inequality, we have

$$|d(h)| \le \int_\Omega |h| \int_0^1 |k(z, u_0 + th) - k(z, u_0)|\, dt\, dz \le \|h\|_p \int_0^1 \|N_k(u_0 + th) - N_k(u_0)\|_{p'}\, dt,$$

where $N_k(v) = k(\cdot\,, v(\cdot))$ for all $v \in W^{1,p}(\Omega)$ and $1/p + 1/p' = 1$.

Note that $u_0 + th \in L^\infty(\Omega)$ (see Claim 2). By hypothesis (Hf) (ii), for a.a. $z \in \{u_0 + th > 0\}$, we have

$$\begin{aligned} f(z, (u_0 + th)(z)) &\ge \frac{f(z, \|u_0 + th\|_\infty)}{\|u_0 + th\|_\infty}(u_0 + th)(z) \\ &\ge -\|f(\cdot\,, \|u_0 + th\|_\infty)\|_\infty \frac{(u_0 + th)^{q-1}(z)}{\|u_0 + th\|_\infty^{q-1}} \\ &\ge -\|f(\cdot\,, \|u_0 + th\|_\infty)\|_\infty \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

On the other hand, from hypothesis (Hf) (i) we have

$$f(z, (u_0 + th)(z)) \le c_{12} \quad \text{for a.a. } z \in \{u_0 + th > 0\} \text{ and some } c_{12} > 0.$$

Therefore, from (3.1) and the Lebesgue dominated convergence theorem,

$$\int_0^1 \|N_k(u_0 + th) - N_k(u_0)\|_{p'}\, ds \to 0 \quad \text{as } h \to 0,$$

thus

$$\frac{1}{\|h\|}|d(h)| \to 0 \quad \text{as } h \to 0.$$

It follows that φ is Gateaux differentiable at u_0 in the direction of all $h \in C^1(\overline{\Omega})$. Let $\varphi'_G(u_0)$ denote this Gateaux derivative. We have

$$\langle \varphi'_G(u_0), h\rangle = 0 \quad \text{for all } h \in C^1(\overline{\Omega}).$$

Using the fact that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, one has

$$\langle \varphi'_G(u_0), h\rangle = 0 \quad \text{for all } h \in W^{1,p}(\Omega).$$

Hence, by (2.10) and (3.1),

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z) u_0^{p-1} h\, d\sigma = \int_\Omega f(z, u_0) h\, dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

This implies that (see [21])

$$\begin{cases} -\operatorname{div} a(Du_0(z)) = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_a} + \beta(z) u_0^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.42}$$

Thus, $u_0 \in C_+ \setminus \{0\}$ (see [15]). For a.a. $z \in \{u_0 > 0\}$, hypothesis (Hf) (ii) implies also that

$$\frac{f(z, u_0(z))}{u_0^{p-1}(z)} \ge \frac{f(z, \|u_0\|_\infty)}{\|u_0\|_\infty^{p-1}}.$$

Hence,

$$f(z, u_0(z)) \ge -\|f(\cdot\,, \|u_0\|_\infty)\|_\infty \frac{u_0^{p-1}(z)}{\|u_0\|_\infty^{p-1}}. \tag{3.43}$$

Recall that $f(z,0) \geq 0$ for a.a. $z \in \Omega$. Therefore, from (3.42) and (3.43), it follows that

$$\operatorname{div} a(Du_0(z)) \leq c_{13} u_0^{p-1}(z) \quad \text{for a.a. } z \in \Omega \text{ and some } c_{13} > 0.$$

Thus, $u_0 \in \operatorname{int} C_+$ (see [24, pp. 111, 120]). □

Next we show the uniqueness of this positive solution.

Theorem 3.5. *If hypotheses* (Ha), (Hβ), (Hf) *hold, then problem* (1.2) *admits a unique positive solution* $u_0 \in \operatorname{int} C_+$.

Proof. From Proposition 3.4, we already have a positive solution $u_0 \in \operatorname{int} C_+$. To show the uniqueness of this positive solution, we proceed as follows. Let $j \colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{\frac{1}{q}})\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}}\,d\sigma & \text{if } u \geq 0,\ u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$), and set

$$v = \big((1-t)u_1 + tu_2\big)^{\frac{1}{q}} \quad \text{with } t \in [0,1].$$

From [9, Lemma 1] we have

$$|Dv(z)| \leq \big[(1-t)|Du_1^{\frac{1}{q}}(z)|^q + t|Du_2^{\frac{1}{q}}(z)|^q\big]^{\frac{1}{q}} \quad \text{for a.a. } z \in \Omega.$$

In this way, since $G_0(\cdot)$ is increasing, we get

$$\begin{aligned} G_0(|Dv(z)|) &\leq G_0\Big(\big((1-t)|Du_1^{\frac{1}{q}}(z)|^q + t|Du_2^{\frac{1}{q}}(z)|^q\big)^{\frac{1}{q}}\Big) \\ &\leq (1-t)G_0(|Du_1^{\frac{1}{q}}(z)|) + tG_0(|Du_2^{\frac{1}{q}}(z)|), \end{aligned}$$

see hypothesis (Ha) (iv). Hence,

$$G(Dv(z)) \leq (1-t)G(Du_1^{\frac{1}{q}}(z)) + tG(Du_2^{\frac{1}{q}}(z)) \quad \text{for a.a. } z \in \Omega,$$

and it follows that the function $u \mapsto \int_\Omega G(Du^{\frac{1}{q}})\,dz$ is convex on $\operatorname{dom} j$.

Since $p \geq q$, using hypothesis (Hβ), we see that

$$u \to \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}}\,d\sigma$$

is convex on $\operatorname{dom} j$ too. So, it follows that $j(\cdot)$ is convex. Also, by Fatou's lemma, it is lower semicontinuous.

Suppose that $\hat{u} \in W^{1,p}(\Omega)$ is another positive solution of problem (1.2). As we did for u_0 (see the proof of Proposition 3.4), we can show that $\hat{u} \in \operatorname{int} C_+$. Then, for $h \in C^1(\overline{\Omega})$ and $|t|$ small, we have

$$u_0^q + th,\ \hat{u}^q + th \in \operatorname{dom} j.$$

Then we can easily check that $j(\cdot)$ is Gateaux differentiable at both u_0^q and $\hat{u}^q$ in the direction h. Moreover, the chain rule and the nonlinear Green's identity imply that

$$j'(u_0^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du_0)}{u_0^{q-1}} h\,dz, \quad j'(\hat{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\hat{u})}{\hat{u}^{q-1}} h\,dz \quad \text{for all } h \in C^1(\overline{\Omega}).$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence,

$$\begin{aligned} 0 &\leq \langle j'(u_0^q) - j'(\hat{u}^q),\ u_0^q - \hat{u}^q \rangle \\ &= \frac{1}{q}\int_\Omega \Big(\frac{-\operatorname{div} a(Du_0)}{u_0^{q-1}} - \frac{-\operatorname{div} a(D\hat{u})}{\hat{u}^{q-1}}\Big)(u_0^q - \hat{u}^q)\,dz \\ &= \frac{1}{q}\int_\Omega \Big(\frac{f(z,u_0)}{u_0^{q-1}} - \frac{f(z,\hat{u})}{\hat{u}^{q-1}}\Big)(u_0^q - \hat{u}^q)\,dz. \end{aligned}$$

This implies $u_0 = \hat{u}$ (see hypothesis (Hf) (ii)), and thus $u_0 \in \operatorname{int} C_+$ is the unique positive solution of (1.2). □

Remark 3.6. In the case where the differential operator is the p-Laplacian, the proof of the uniqueness can be based on the nonlinear Picone's identity (see the proof of Proposition 2.4), thus extending the result of Abdellaoui and Peral (see [1, Corollary 4.4]). We present this proof. Note that if the differential operator is the p-Laplacian, then $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ and $q = p$ in hypothesis (Ha) (iv). Thus, suppose that $u_0, \hat{u}$ are two positive solutions of problem (1.2). From Proposition 3.4, we know that $u_0, \hat{u} \in \operatorname{int} C_+$. We consider the function $R(u_0, \hat{u})(\cdot)$ introduced in the proof of Proposition 2.4. Recall that

$$0 \le R(u_0, \hat{u})(z) \quad \text{for a.a. } z \in \Omega. \tag{3.44}$$

By the nonlinear Green's identity (see [11, p. 211]), we have

$$\begin{aligned}\int_\Omega \frac{f(z,u_0)}{u_0^{p-1}}(u_0^p - \hat{u}^p)\,dz &= \int_\Omega (-\Delta_p u_0)\Big(u_0 - \frac{\hat{u}^p}{u_0^{p-1}}\Big)\,dz \\ &= \int_\Omega |Du_0|^{p-2}\Big(Du_0, D\Big(u_0 - \frac{\hat{u}^p}{u_0^{p-1}}\Big)\Big)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}\Big(u_0 - \frac{\hat{u}^p}{u_0^{p-1}}\Big)\,d\sigma \\ &= \|Du_0\|_p^p - \|D\hat{u}\|_p^p + \int_\Omega R(\hat{u}, u_0)\,dz + \int_{\partial\Omega} \beta(z)(u_0^p - \hat{u}^p)\,d\sigma.\end{aligned} \tag{3.45}$$

Interchanging the roles of u_0 and $\hat{u}$, we also have

$$\int_\Omega \frac{f(z,\hat{u})}{\hat{u}^{p-1}}(\hat{u}^p - u_0^p)\,dz = \|D\hat{u}\|_p^p - \|Du_0\|_p^p + \int_\Omega R(u_0, \hat{u})\,dz - \int_\Omega \beta(z)(u_0^p - \hat{u}^p)\,d\sigma. \tag{3.46}$$

Adding (3.45) and (3.46), and using (3.44), we obtain

$$\int_\Omega \Big(\frac{f(z,u_0)}{u_0^{p-1}} - \frac{f(z,\hat{u})}{\hat{u}^{p-1}}\Big)(u_0^p - \hat{u}^p)\,dz = \int_\Omega [R(\hat{u}, u_0) + R(u_0, \hat{u})]\,dz \ge 0. \tag{3.47}$$

As we already remarked at the beginning of this section, hypothesis (Hf) (ii) implies that the function $x \to f(z,x)/x^{p-1}$ is strictly decreasing on $(0, +\infty)$ for a.a. $z \in \Omega$. Therefore, from (3.47) we infer that $u_0 = \hat{u}$, and this proves the uniqueness of the positive solution for the particular case of problem (1.2) in which the differential operator is the p-Laplacian.

4 The p-Laplacian Case

In this section we continue with the setting introduced in the last Remark. Namely, we deal with the particular case of problem (1.2) in which the differential operator is the p-Laplacian. So, now the problem under consideration is

$$\begin{cases} -\Delta_p u(z) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \\ u \ge 0. \end{cases} \tag{4.1}$$

Problem (4.1) is a particular case of problem (1.2) with

$$a(y) = |y|^{p-2}y \quad \text{for all } y \in \mathbb{R}^N.$$

Therefore, $c_1 = p - 1$, and so we have

$$\hat{\mu} = \mu \quad \text{and} \quad \hat{\beta} = \beta.$$

Moreover, $q = p$ and $\tilde{c} = 1/p$ (see hypothesis (Ha) (iv), and hypotheses (Hf) (ii) and (iv)). Then

$$\tilde{\mu}_0 = \mu_0 \quad \text{and} \quad \tilde{\beta} = \beta.$$

In this case we show that hypotheses (Hf) (iii) and (iv) are also necessary for the existence of a unique positive solution for problem (4.1).

Theorem 4.1. *If hypothesis* (Hβ) *holds and* $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory function which satisfies hypotheses* (Hf) (i) *and* (ii), *then problem* (4.1) *admits a unique positive solution* $u_0\in \operatorname{int}C_+$ *if and only if*

$$\hat\lambda_1(-\mu_0,\beta,p)<0<\hat\lambda_1(-\mu,\beta,p).$$

Proof. “ $\Leftarrow$ ” This was established in Theorem 3.5 for a broader class of problems.

“ $\Rightarrow$ ” From the proof of Proposition 2.4, we have

$$\hat\lambda_1(-\mu_0,\beta,p)\le\frac{\gamma(u_0)-\int_\Omega \mu_0(z)u_0^p\,dz}{\|u_0\|_p^p}=\frac{\int_\Omega f(z,u_0)u_0\,dz-\int_\Omega \mu_0(z)u_0^p\,dz}{\|u_0\|_p^p}. \tag{4.2}$$

Denoting $m_0=\min_{\overline\Omega}u_0>0$ (recall that $u_0\in\operatorname{int}C_+$), then, from hypothesis (Hf) (ii) (since $q=p$), we have

$$\frac{f(z,u_0(z))}{u_0^{p-1}(z)}\le\frac{f(z,m_0)}{m_0^{p-1}}<\mu_0(z)\quad\text{for a.a. } z\in\Omega.$$

Using this in (4.2), we infer that

$$\hat\lambda_1(-\mu_0,\beta,p)<\frac{\int_\Omega \mu_0(z)u_0^p(z)\,dz-\int_\Omega \mu_0(z)u_0^p(z)\,dz}{\|u_0\|_p^p}=0.$$

Now, set

$$\xi(z)=-\frac{f(z,\|u_0\|_\infty+1)}{(\|u_0\|_\infty+1)^{p-1}};$$

clearly $\xi\in L^\infty(\Omega)$ (see hypothesis (Hf) (i)).

Consider the eigenvalue problem (2.3) with this particular $\xi(\cdot)$ as potential function. From Proposition 2.4, we know that this eigenvalue problem has a principal eigenfunction $\hat u_1\in\operatorname{int}C_+$. We choose $\tau>0$ so big that

$$u_0<\tau\hat u_1=\tilde u_1\in\operatorname{int}C_+. \tag{4.3}$$

We use (3.45) first with $f(z,x)$ (replacing $\hat u$ with $\tilde u_1$), and then with $(\hat\lambda_1(\xi,\beta,p)-\xi(z))x^{p-1}x\ge0$ and $\tilde u_1$ as solution to the related problem (2.3). We have

$$\int_\Omega\frac{f(z,u_0)}{u_0^{p-1}}(u_0^p-\tilde u_1^p)\,dz=\int_\Omega R(\tilde u_1,u_0)\,dz+\|Du_0\|_p^p-\|D\tilde u_1\|_p^p+\int_{\partial\Omega}\beta(z)(u_0^p-\tilde u_1^p)\,d\sigma, \tag{4.4}$$

$$\int_\Omega(\hat\lambda_1(\xi,\beta,p)-\xi(z))(\tilde u_1^p-u_0^p)\,dz=\int_\Omega R(u_0,\tilde u_1)\,dz+\|D\tilde u_1\|_p^p-\|Du_0\|_p^p+\int_{\partial\Omega}\beta(z)(\tilde u_1^p-u_0^p)\,d\sigma. \tag{4.5}$$

Adding (4.4) and (4.5) we obtain

$$\int_\Omega\Big[\frac{f(z,u_0)}{u_0^{p-1}}+\xi(z)-\hat\lambda_1(\xi,\beta,p)\Big](u_0^p-\tilde u_1^p)\,dz=\int_\Omega[R(\tilde u_1,u_0)+R(u_0,\tilde u_1)]\,dz\ge0. \tag{4.6}$$

Note that (see hypothesis (Hf) (ii) and recall that $q=p$)

$$\frac{f(z,u_0)}{u_0^{p-1}}>\frac{f(z,\|u_0\|_\infty+1)}{(\|u_0\|_\infty+1)^{p-1}}=-\xi(z)\quad\text{for a.a. } z\in\Omega.$$

Hence,

$$\frac{f(z,u_0)}{u_0^{p-1}}+\xi(z)>0\quad\text{for a.a. } z\in\Omega. \tag{4.7}$$

From (4.3) we have

$$(u_0^p-\tilde u_1^p)(z)<0\quad\text{for all } z\in\overline\Omega. \tag{4.8}$$

From (4.6) we have

$$\int_\Omega\Big[\frac{f(z,u_0)}{u_0^{p-1}}+\xi(z)\Big](u_0^p-\tilde u_1^p)\,dz\ge\hat\lambda_1(\xi,\beta,p)\int_\Omega(u_0^p-\tilde u_1^p)\,dz.$$

Thus, by (4.7) and (4.8), we get

$$0 > \hat{\lambda}_1(\xi, \beta, p) \int_\Omega (u_0^p - \tilde{u}_1^p)\, dz,$$

which, by (4.8), implies that

$$\hat{\lambda}_1(\xi, \beta, p) > 0. \tag{4.9}$$

But, by (Hf) (ii), note that

$$\xi(z) \le -\mu(z) \quad \text{for a.a. } z \in \Omega,$$

hence

$$\hat{\lambda}_1(\xi, \beta, p) \le \hat{\lambda}_1(-\mu, \beta, p),$$

and, by (4.9),

$$\hat{\lambda}_1(-\mu, \beta, p) > 0. \qquad \square$$

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