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Research Article

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Existence and Asymptotic Behavior for the Ground State of Quasilinear Elliptic Equations

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Abstract: In this paper, we are concerned with the existence and asymptotic behavior of minimizers of a minimization problem related to some quasilinear elliptic equations. Firstly, we prove that there exist minimizers when the exponent q is the critical one $q^* = 2 + \frac{4}{N}$. Then, we prove that all minimizers are compact as q tends to the critical case q^* when $a < a_{q^*}$ is fixed. Moreover, we find that all the minimizers must blow up as the exponent q tends to the critical case q^* for any fixed $a > a_{q^*}$.

Keywords: Quasilinear Elliptic Equations, Minimization Problem, Asymptotic Behavior

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1 Introduction

In this paper, we consider the minimization problem

$$d_a(q) = \inf_{u \in M} E_q^a(u), \tag{1.1}$$

where

$$M = \Big\{ \int_{\mathbb{R}^N} |u|^2 \, dx = 1,\ u \in X \Big\}$$

and

$$E_q^a(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u^2|^2 \, dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u|^{q+2} \, dx. \tag{1.2}$$

Here, we assume that $0 < q \le q^* = 2 + \frac{4}{N}$, $a \in \mathbb{R}$ is a constant, the potential $V(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^+)$. The space X is defined by

$$X = \Big\{ u : \int_{\mathbb{R}^N} |\nabla u^2|^2 \, dx < \infty,\ u \in H \Big\}$$

with

$$H = \Big\{ u : \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx < \infty \Big\}.$$

Any minimizers of (1.1) solve the following quasilinear elliptic equation:

$$-\Delta u - \Delta(u^2)u + V(x)u = \mu u + a|u|^q u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

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which is the Euler–Lagrange equation to problem (1.1), where μ denotes the Lagrange multiplier under the constraint $\|u\|_{L^2}^2 = 1$. Solutions of problem (1.3) also correspond to the standing wave solutions of the following quasilinear Schrödinger equation:

$$i\partial_t\varphi = -\Delta\varphi - \Delta(\varphi^2)\varphi + W(x)\varphi - a|\varphi|^q\varphi, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $\varphi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ and $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential. Equation (1.4) arises in several physical phenomena such as the theory of plasma physics, exciton in one-dimensional lattices and dissipative quantum mechanics, see for examples [4, 16, 17, 20] and the references therein for more backgrounds. It is obvious that $e^{-i\mu t}u(x)$ solves (1.4) if and only if $u(x)$ is the solution of equation (1.3).

Equation (1.3) is usually called a semilinear elliptic equation if we ignore the term $-\Delta(u^2)u$. The constrained minimization problem associated to a semilinear elliptic equation has been widely studied, see, e.g., [2, 10–13]. Several authors [2, 10, 11, 13] considered the following minimization problem in dimension two:

$$I_a(q) = \inf_{u\in H,\, \int_{\mathbb{R}^2}|u|^2dx=1} J_q^a(u), \tag{1.5}$$

where

$$J_q^a(u) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)|u|^2dx - \frac{a}{q+2}\int_{\mathbb{R}^2}|u|^{q+2}dx$$

and $0 < q \le q^* = 2$, $a \in \mathbb{R}$ is a constant. By using some rescaling arguments, it was shown that there exists a constant a^*, such that (1.5) has at least one minimizer if and only if $a < a^*$. Moreover, the concentration and symmetry breaking of minimizers of (1.5) were discussed in [10, 11, 13] when $q = 2$ and a tends to a^* from left (denoted by $a \nearrow a^*$). Recently, these methods were also used to deal with the concentration behaviors of minimizers of many different kinds of minimization problems, see [12, 26, 27]. The concentration properties of minimizers of (1.5) as $q \nearrow 2$ for any fixed $a \ge a^*$ were studied in [12, 26]. Zeng and Zhang [27] proved the uniqueness and concentration of minimizers of a class of Kirchhoff equations.

Several works considered the existence of solutions for equation (1.3), see [6, 7, 20–22] for the subcritical case, and [8, 9, 23, 28, 29] for the critical case. For different types of potentials, the existence of positive solutions of problem (1.3) on the manifold $M = \{\int_{\mathbb{R}^N}|u|^{q+2} = c,\ u \in X\}$ and the Nahari manifold when $2 \le q < \frac{2(N+2)}{N-2}$ was established in [20, 22] by using a constrained minimization argument. In [6, 21], by changing of variables, problem (1.3) was transformed into a semilinear elliptic equation, then the existence of positive solutions was obtained by the mountain pass theorem in Orlicz space or Hilbert space framework. It is worth mentioning that several authors [5, 7, 14] investigated the following constrained minimization problem associated to the quasilinear elliptic equation (1.3) with $V(x) =$ constant:

$$m(c) = \inf\{E(u) : |u|_{L^2}^2 = c\}, \tag{1.6}$$

where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u^2|^2dx - \frac{1}{q+2}\int_{\mathbb{R}^N}|u|^{q+2}dx. \tag{1.7}$$

They mainly obtained that, for any $c > 0$,

$$m(c) = \begin{cases} -\infty & \text{if } q > q^* = 2 + \frac{4}{N}, \\ 0 & \text{if } q = q^* = 2 + \frac{4}{N}, \end{cases}$$

and (1.6) possesses no minimizer. On the other hand, when $q \in (0, 2 + \frac{4}{N})$, it holds true that $m(c) \in (-\infty, 0]$. Especially, if the energy is strictly less than zero, namely,

$$m(c) \in (-\infty, 0), \tag{1.8}$$

they proved that (1.6) possesses at least one minimizer by using Lions' concentration-compactness principle. In general, condition (1.8) can be verified for any $c > 0$ if $q \in (0, \frac{4}{N})$. But for the case of $q \in [\frac{4}{N}, 2 + \frac{4}{N})$, by setting

$$c(q, N) := \inf\{c > 0 : m(c) < 0\},$$

it was proved in [14] that $c(q, N) > 0$ and (1.6) is achieved if and only if $c \in [c(q, N), +\infty)$. Based on the above results, Jeanjean, Luo and Wang [15] recently discovered that there exists $\hat{c} \in (0, c(q, N))$, such that the functional (1.7) admits a local minimum on the manifold $\{u \in X : |u|_{L^2}^2 = c\}$ for all $c \in (\hat{c}, c(q, N))$ and $q \in (\frac{4}{N}, 2 + \frac{4}{N})$. Furthermore, a mountain pass type critical point of (1.7) was also obtained therein for all $c \in (\hat{c}, \infty)$, which is different from the minimum solution.

We note that, by taking the scaling $u_c(x) = u(c^{1/N}x)$,

$$E(u) = c^{1-\frac{2}{N}}\Big\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_c|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u_c^2|^2\,dx - \frac{c^{\frac{2}{N}}}{q+2}\int_{\mathbb{R}^N} |u_c|^{q+2}\,dx\Big\}.$$

It is easy to see that problem (1.6) can be equivalently transformed to problem (1.1) with $V(x) \equiv$ constant (without loss of generality, we assume $V(x) \equiv 0$) by setting $a = c^{2/N}$. Then we obtain the following minimization problem:

$$\widetilde{d}_a(q) = \inf_{u\in M} \widetilde{E}_q^a(u), \tag{1.9}$$

where $\widetilde{E}_q^a(\cdot)$ is given by

$$\widetilde{E}_q^a(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx - \frac{a}{q+2}\int_{\mathbb{R}^N} |u|^{q+2}\,dx.$$

From the above mentioned results on the minimization of problem (1.6), we see that (1.9) can be achieved only if $q < q^* = 2 + \frac{4}{N}$. The exponent q^* seems to be the critical exponent for the existence of minimizers of (1.9). A natural question one would ask is whether problem (1.1) admits minimizers if $V(x) \not\equiv$ constant? By taking the scaling $u^\sigma(x) = \sigma^{N/2}u(\sigma x)$, it is easy to see that $E_q^a(u^\sigma) \to -\infty$ as $\sigma \to +\infty$ if $q > q^*$ and $V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. This implies that there are no minimizers of problem (1.1) when $q > q^*$. However, when $q = q^*$, the result is quite different. Indeed, for a class of non-constant potentials, we will prove that there exists a threshold (with respect to the parameter a) independent of $V(x)$ for the existence of minimizers of (1.1), see our Theorem 1.2 below for details. Moreover, stimulated by [12], we are further interested in studying the limit behavior of minimizers of (1.1) as $q \nearrow q^*$.

Before stating our main results, we first recall the following sharp Gagliardo–Nirenberg inequality [1]:

$$\int_{\mathbb{R}^N} |u|^{\frac{q+2}{2}}\,dx \le \frac{1}{\Upsilon_q}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{\frac{(q+2)\theta_q}{4}} |u|_{L^1}^{\frac{(q+2)(1-\theta_q)}{2}} \quad \text{for all } u \in \mathcal{D}^{2,1}(\mathbb{R}^N), \tag{1.10}$$

where

$$1 < \frac{q+2}{2} < \frac{2N}{N-2}, \quad \theta_q = \frac{2qN}{(q+2)(N+2)}, \quad \Upsilon_q > 0$$

and

$$\mathcal{D}^{2,1}(\mathbb{R}^N) := \{u : \nabla u \in L^2(\mathbb{R}^N),\ u \in L^1(\mathbb{R}^N)\}.$$

As proved in [1], the optimal constant is $\Upsilon_q = \lambda_q a_q$ with

$$\lambda_q = (1-\theta_q)\Big(\frac{\theta_q}{1-\theta_q}\Big)^{\frac{qN}{2(N+2)}} \quad \text{and} \quad a_q = |v_q|_{L^1}^{\frac{q}{N+2}}. \tag{1.11}$$

Here, $v_q \ge 0$ optimizes (1.10) (that is, (1.10) is an identity if $u = v_q$) and is the unique non-negative radially symmetric solution of the following equation [25]:

$$-\Delta v_q + 1 = v_q^{\frac{q}{2}}, \quad x \in \mathbb{R}^N. \tag{1.12}$$

Remark 1.1. Strictly speaking, it has been proved in [25, Theorem 1.3 (iii)] that v_q has a compact support in $\mathbb{R}^N$ and exactly satisfies a Dirichlet–Neumann free boundary problem. Namely, there exists one $R > 0$ such that v_q is the unique positive solution of

$$\begin{cases} -\Delta u + 1 = u^{\frac{q}{2}}, \\ u > 0 \text{ in } B_R, \quad u = \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial B_R. \end{cases} \tag{1.13}$$

In what follows, if we say that u is a non-negative solution of an equation of the form (1.12), then we exactly mean that u is a solution of the free boundary problem (1.13).

From equation (1.12) and the classical Pohozaev identity, one can prove that

$$\int_{\mathbb{R}^N} |v_q|^{\frac{q+2}{2}} dx = \frac{1}{1-\theta_q} \int_{\mathbb{R}^N} |v_q| dx, \qquad \int_{\mathbb{R}^N} |\nabla v_q|^2 dx = \frac{\theta_q}{1-\theta_q} \int_{\mathbb{R}^N} |v_q| dx. \tag{1.14}$$

Using the above notations, we first obtain the following result which addresses the existence of minimizers of problem (1.1) for the critical case of $q = q^*$.

Theorem 1.2. *Let $q = q^*$ and let a_{q^*} be given by* (1.11). *Assume that $V(x)$ satisfies*

$$V(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^+), \quad \inf_{x\in\mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} V(x) = \infty. \tag{1.15}$$

Then,
(i) *$d_a(q^*)$ has at least one minimizer if $0 < a \le a_{q^*}$;*
(ii) *there is no minimizer of $d_a(q^*)$ if $a > a_{q^*}$.*

Theorem 1.2 is mainly stimulated by [2, Theorem 2.1] and [10, Theorem 1], where the semilinear minimization problem (1.5) was studied. The argument in these two references for studying the non-critical case, namely $a \ne a_{q^*}$, is useful for solving our problem. However, when a equals the threshold (i.e., $a = a^*$ in their problem), it was proved in [2, 10] that there is no minimizer of problem (1.5). This is quite different from our case since there exists at least one minimizer of (1.1) when $a = a_{q^*}$. This difference is mainly caused by the presence of the extra term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$ in (1.2), which makes the argument in [2, 10] unavailable for studying our problem. To deal with the critical case, we will introduce in Section 2 a suitable auxiliary functional and obtain the boundedness of the minimizing sequence by contradiction. Then, the existence of minimizers follows directly from the compactness Lemma 2.1.

We remark that if $V(x)$ satisfies condition (1.15), one can easily apply the Gagliardo–Nirenberg inequality (1.10) and Lemma 2.1 to prove that (1.1) possesses minimizers for any fixed $0 < q < q^*$. In what follows, we investigate the limit behavior of minimizers of (1.1) as $q \nearrow q^*$. Firstly, if $a < a_{q^*}$ is fixed, our result shows that the minimizers of (1.1) are compact in the space X as $q \nearrow q^*$. More precisely, we have:

Theorem 1.3. *Assume $V(x)$ satisfies* (1.15) *and let $u_q \in M$ be a non-negative minimizer of problem* (1.1) *with $0 < a < a_{q^*}$ and $0 < q < q^* = 2 + \frac{4}{N}$. Then*

$$\lim_{q\nearrow q^*} d_a(q) = d_a(q^*).$$

Moreover, there exists $u_0 \in M$ such that $\lim_{q\nearrow q^*} u_q = u_0$ in X, where u_0 is a non-negative minimizer of $d_a(q^*)$. Here, the sequence $\lim_{q\nearrow q^*} u_q = u_0$ in X means that

$$u_q \to u_0 \text{ in } H \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \to \int_{\mathbb{R}^N} |\nabla u_0^2|^2 dx \quad \text{as } q \nearrow q^*.$$

On the contrary, if $a > a_{q^*}$, the result is quite different and blow-up will happen in minimizers as $q \nearrow q^*$. Actually, our following theorem states that all minimizers of (1.1) must concentrate and blow up at one minimal point of the potentials.

Theorem 1.4. *Assume $V(x)$ satisfies* (1.15) *and $a > a_{q^*}$. Let $\bar{u}_q$ be a non-negative minimizer of* (1.1) *with $0 < q < q^*$. For any sequence of $\{\bar{u}_q\}$, by passing to a subsequence if necessary, there exist $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ and $y_0 \in \mathbb{R}^N$ such that*

$$\lim_{q \nearrow q^*} \varepsilon_q^N \bar{u}_q^2(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - y_0|) \quad \text{strongly in } \mathcal{D}^{2,1}(\mathbb{R}^N),$$

where v_{q^*} is the unique non-negative radially symmetric solution of (1.12) *and*

$$\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N}\right)^{\frac{1}{N+2}}, \quad \varepsilon_q = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{-\frac{2}{N(q^*-q)}} \to 0^+ \quad \text{as } q \nearrow q^*. \tag{1.16}$$

Moreover, the sequence $\{y_{\varepsilon_q}\}$ satisfies

$$\operatorname{dist}(\varepsilon_q y_{\varepsilon_q}, A) \to 0 \quad \text{as } q \nearrow q^*,$$

where $A := \{x : V(x) = 0\}$.

Throughout the paper, $|u|_{L^p}$ denotes the L^p-norm of a function u, and C, c_0, c_1 denote some constants.

This paper is organized as follows. In Section 2, we shall prove Theorem 1.2 by some rescaling arguments, especially, we prove that $d_{a^*}(q^*)$ possesses minimizers by introducing an auxiliary minimization problem. Section 3 is devoted to the proof of Theorem 1.3 on the compactness in space X for minimizers of $d_a(q^*)$ as $q \nearrow q^*$. In Section 4, we first establish optimal energy estimates for $d_a(q)$ as $q \nearrow q^*$ for any fixed $a > a_{q^*}$, upon which we then complete the proof of Theorem 1.4 on the concentration behavior of non-negative minimizers as $q \nearrow q^*$.

2 The Existence of Minimizers: Proof of Theorem 1.2

The main purpose of this section is to establish Theorem 1.2. We first introduce the following lemma, which was essentially proved in [24, Theorem XIII.67] and [3, Theorem 2.1].

Lemma 2.1. *Assume $V(x)$ satisfies* (1.15). *Then the embedding from H into $L^p(\mathbb{R}^N)$ is compact for all*

$$2 \le p < 2^* = \begin{cases} +\infty & \text{if } N = 1, 2, \\ \frac{2N}{N-2} & \text{if } N \ge 3. \end{cases}$$

Taking $q = q^*$ in (1.12), we get $\theta_{q^*} = \frac{N}{N+1}$, $\lambda_{q^*} = \frac{N}{N+1}$ and $a_{q^*} = |v_{q^*}|_{L^1}^{2/N}$. Moreover, (1.14) becomes

$$\int_{\mathbb{R}^N} |v_{q*}|^{\frac{q^*+2}{2}} dx = (N+1) \int_{\mathbb{R}^N} |v_{q*}| dx, \quad \int_{\mathbb{R}^N} |\nabla v_{q*}|^2 dx = N \int_{\mathbb{R}^N} |v_{q*}| dx, \tag{2.1}$$

and the Gagliardo–Nirenberg inequality (1.10) can be simply given as

$$\int_{\mathbb{R}^N} |u|^{\frac{q^*+2}{2}} dx \le \frac{N+1}{N a_{q*}} \int_{\mathbb{R}^N} |\nabla u|^2 dx \cdot |u|_{L^1}^{\frac{2}{N}}. \tag{2.2}$$

Inspired by the argument used in [2, 10], we first prove the following lemma which addresses Theorem 1.2 for the case of $a \neq a_{q^*}$.

Lemma 2.2. *Let $V(x)$ satisfy* (1.15) *and $q = q^*$. Then*
(i) *$d_a(q^*)$ has at least one minimizer if $0 < a < a_{q^*}$;*
(ii) *there is no minimizer of $d_a(q^*)$ if $a > a_{q^*}$.*

Proof. (i) If $a < a_{q^*}$, for any $u \in M$, it follows from (2.2) that

$$\int_{\mathbb{R}^N} |u|^{q^*+2} dx \le \frac{N+1}{N a_{q^*}} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx \left(\int_{\mathbb{R}^N} u^2 dx\right)^{\frac{2}{N}} \le \frac{N+1}{N a_{q^*}} \int_{\mathbb{R}^N} |\nabla u^2|^2 dx.$$

Thus,

$$\begin{aligned}E^a_{q^*}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u^2|^2dx - \frac{a}{q+2}\int_{\mathbb{R}^N}|u|^{q^*+2}dx\\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)dx + \frac{1}{4}\Big(1-\frac{a}{a_{q^*}}\Big)\int_{\mathbb{R}^N}|\nabla u^2|^2dx.\end{aligned} \tag{2.3}$$

Hence, if $\{u_n\}$ is a minimizing sequence of $d_a(q^*)$ with $a < a_{q^*}$, it is easy to deduce from the above formulas that there exists $C > 0$ independent of n such that

$$\sup_n \int_{\mathbb{R}^N}|\nabla u_n^2|^2dx \leq C < \infty, \quad \sup_n \|u_n\|_H \leq C < \infty.$$

It then follows from Lemma 2.1 that there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in M$ such that

$$u_n \overset{n}{\to} u \text{ in } L^p(\mathbb{R}^N) \quad \text{for all } 2 \leq p < 2^* \tag{2.4}$$

and

$$\int_{\mathbb{R}^N}|\nabla u^2|^2dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n^2|^2dx \leq C < \infty.$$

The latter inequality indicates that u_n^2 is bounded in $L^{2^*}(\mathbb{R}^N)$. Hence, we can deduce from (2.4) that

$$u_n \overset{n}{\to} u \text{ in } L^p(\mathbb{R}^N) \quad \text{for all } 2 \leq p < 2 \times 2^*.$$

Therefore,

$$d_a(q^*) = \liminf_{n\to\infty} E^a_{q^*}(u_n) \geq E^a_{q^*}(u) \geq d_a(q^*).$$

This indicates that

$$u_n \overset{n}{\to} u \text{ in } H \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n^2|^2dx = \int_{\mathbb{R}^N}|\nabla u^2|^2dx.$$

It means that u is a minimizer of $d_a(q^*)$ for $a < a_{q^*}$.

(ii) Let

$$u_\tau = \frac{\tau^{\frac{N}{2}}}{|v_{q^*}|^{\frac{1}{2}}_{L^1}}\sqrt{v_{q^*}(\tau|x-x_0|)} \quad \text{with some } x_0 \in \mathbb{R}^N.$$

Using (2.1), we have

$$\int_{\mathbb{R}^N} u_\tau^2dx = \frac{\tau^N}{|v_{q^*}|_{L^1}}\int_{\mathbb{R}^N}|v_{q^*}(\tau x)|dx = \frac{1}{|v_{q^*}|_{L^1}}\int_{\mathbb{R}^N}|v_{q^*}|dx = 1,$$

$$\int_{\mathbb{R}^N}|\nabla u_\tau^2|^2dx = \frac{\tau^{N+2}}{|v_{q^*}|^2_{L^1}}\int_{\mathbb{R}^N}|\nabla v_{q^*}|^2dx = \frac{N\tau^{N+2}}{|v_{q^*}|_{L^1}}, \tag{2.5}$$

$$\int_{\mathbb{R}^N} u_\tau^{q^*+2}dx = \frac{\tau^{N+2}}{|v_{q^*}|^{2+2/N}_{L^1}}\int_{\mathbb{R}^N} v_{q^*}^{\frac{2(N+1)}{N}}dx = \frac{(N+1)\tau^{N+2}}{|v_{q^*}|^{1+2/N}_{L^1}}, \tag{2.6}$$

$$\int_{\mathbb{R}^N}|\nabla u_\tau|^2dx = \frac{\tau^2}{|v_{q^*}|_{L^1}}\int_{\mathbb{R}^N}|\nabla\sqrt{v_{q^*}}|^2dx = c_0\tau^2, \tag{2.7}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)u_\tau^2dx &= \frac{\tau^N}{|v_{q^*}|_{L^1}}\int_{\mathbb{R}^N}V(x)|v_{q^*}(\tau x)|dx\\ &= \frac{1}{|v_{q^*}|_{L^1}}\int_{\mathbb{R}^N}V\Big(\frac{x}{\tau}+x_0\Big)|v_{q^*}|dx \to V(x_0) \quad \text{as } \tau \to +\infty.\end{aligned} \tag{2.8}$$

If $a > a_{q^*}$, from (2.5) and (2.6) we have

$$\frac{1}{4}\int_{\mathbb{R}^N} |\nabla u_\tau^2|^2 dx - \frac{a}{q^*+2}\int_{\mathbb{R}^N} u_\tau^{q^*+2} dx = \frac{N\tau^{N+2}}{4|v_{q^*}|_{L^1}}\Big(1 - \frac{a}{a_{q^*}}\Big).$$

This together with (2.7) and (2.8) gives that

$$d_a(q^*) \le V(x_0) + o(1) + \frac{N\tau^{N+2}}{4|v_{q^*}|_{L^1}}\Big(1 - \frac{a}{a_{q^*}}\Big) + c_0\tau^2 \to -\infty \quad \text{as } \tau \to +\infty.$$

Therefore, we deduce that $d_a(q^*) = -\infty$ and $d_a(q^*)$ possesses no minimizer if $a > a_{q^*}$. □

In view of the above lemma, to complete the proof of Theorem 1.2, it remains to deal with the case of $a = a_{q^*}$. In the following lemma, we first prove that there exist minimizers of $d_{a_{q^*}}(q^*)$ when $N \le 3$.

Lemma 2.3. *Assume that $V(x)$ satisfies (1.15), then $d_a(q^*)$ has at least one minimizer if $a = a_{q^*}$ and $N \le 3$.*

Proof. Assume $\{u_n\}$ is a minimizing sequence of $d_{a_{q^*}}(q^*)$. Using an argument similar to (2.3), we easily see that

$$\sup_n \|u_n\|_H \le C < +\infty.$$

Note that $q^* = 2 + \frac{4}{N} < 2^* - 2$ in view of $N \le 3$, we thus deduce from the above inequality that

$$\sup_n \int_{\mathbb{R}^N} |u_n|^{q^*+2} dx \le C < +\infty.$$

This further indicates that

$$\sup_n \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx \le C < +\infty.$$

Then similar to the proof of Lemma 2.2 (i), we see that there exists a minimizer of $d_a(q^*)$ and the proof is complete. □

When $N \ge 4$, the argument of Lemma 2.3 cannot be used to obtain the existence of minimizers of $d_{a_{q^*}}(q^*)$ since we have $q^* > 2^* - 2$. To deal with this case, we introduce the following auxiliary minimization problem:

$$m(c) = \inf\Big\{F(u) : \int_{\mathbb{R}^N} |u| dx = c,\ u \in \mathcal{D}^{2,1}(\mathbb{R}^N)\Big\}, \tag{2.9}$$

where

$$F(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{Na_{q^*}}{N+1}\int_{\mathbb{R}^N} |u|^{2+\frac{2}{N}} dx.$$

Lemma 2.4. (i) $m(c) = 0$ *if* $0 < c \le 1$; $m(c) = -\infty$ *if* $c > 1$.

(ii) *Problem (2.9) possesses a minimizer if and only if $c = 1$. All non-negative minimizers of $m(1)$ must be of the form*

$$\Big\{\frac{\lambda^N v_{q^*}(\lambda|x - x_0|)}{|v_{q^*}|_{L^1}} : \lambda \in \mathbb{R}^+,\ x_0 \in \mathbb{R}^N\Big\}. \tag{2.10}$$

Proof. (i) It follows easily by some scaling arguments.

(ii) From (i), we have $m(c) = 0$ for any $c \le 1$. Thus, if u is a minimizer of $m(c)$ with $c < 1$, we obtain from inequality (2.2) that

$$0 = m(c) \ge (1 - c^{\frac{2}{N}}) \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

This implies that $u \equiv 0$, which is a contradiction.

If $c = 1$, one can easily check that any u_0 satisfying (2.10) is a minimizer of $m(1)$. On the other hand, if $u \geq 0$ is a non-negative minimizer of $m(1)$, we get that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{Na_{q^*}}{N+1} \int_{\mathbb{R}^N} |u|^{2+\frac{2}{N}} dx,$$

which indicates that u is an optimizer of the Gagliardo–Nirenberg inequality (2.2). Hence, u must be of the form (2.10). □

Lemma 2.5. *Let $\{u_n\} \subset \mathcal{D}^{2,1}(\mathbb{R}^N)$ be a non-negative minimizing sequence of $m(1)$, and $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1$. Then, there exists $\{y_n\} \subset \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$, such that*

$$\lim_{n\to\infty} u_n(x + y_n) = \frac{\lambda_0^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda_0|x - x_0|) \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N) \quad \text{with } \lambda_0 = \left(\frac{|v_{q^*}|_{L^1}}{N}\right)^{\frac{1}{N+2}}.$$

Proof. From the definition of $m(1)$ and $\{u_n\}$, one has

$$\int_{\mathbb{R}^N} |u_n|^{2+\frac{2}{N}} dx = \frac{N+1}{Na_{q^*}} + o(1). \tag{2.11}$$

We will prove the compactness of $\{u_n\}$ by Lions' concentration-compactness principle [18, 19].

(I) We first rule out the possibility of *vanishing*: If

$$\limsup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n| dx = 0 \quad \text{for any } R > 0,$$

then $u_n \xrightarrow{n} 0$ in $L^q(\mathbb{R}^N)$ for any $1 < q < 2^*$, which contradicts (2.11).

(II) Now, assume that *dichotomy* occurs, i.e., for some $c_1 \in (0, 1)$, there exist $R_0, R_n > 0$, $\{y_n\} \subset \mathbb{R}^N$ and sequences $\{u_{1n}\}$, $\{u_{2n}\}$ such that

$$\begin{aligned}
&\operatorname{supp} u_{1n} \subset B_{R_0}(y_n), \quad \operatorname{supp} u_{2n} \subset B^c_{R_n}(y_n),\\
&\operatorname{dist}(\operatorname{supp} u_{1n}, \operatorname{supp} u_{2n}) \to +\infty \quad \text{as } n \to \infty,\\
&\left|\int_{\mathbb{R}^N} |u_{1n}| dx - c_1\right| \leq \varepsilon, \quad \left|\int_{\mathbb{R}^N} |u_{2n}| dx - (1 - c_1)\right| \leq \varepsilon,\\
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_{1n}|^2 dx + \int_{\mathbb{R}^N} |\nabla u_{2n}|^2 dx\right) \geq 0.
\end{aligned}$$

Let $\tilde{u}_{1n}(x) = u_{1n}(x + y_n)$. Then there is $u \in \mathcal{D}^{2,1}(\mathbb{R}^N)$ such that

$$\tilde{u}_{1n}(x) \xrightarrow{n} u \not\equiv 0 \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } 1 \leq p < 2^*. \tag{2.12}$$

Noting that

$$0 = m(1) = F(u_{1n}) + F(u_{2n}) + o_n(1) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, and letting $n \to \infty$ and then $\varepsilon \to 0$, we then obtain from (2.12) that

$$\int_{\mathbb{R}^N} |u| dx = c_1 < 1 \quad \text{and} \quad 0 \leq F(u) \leq m(1) = 0.$$

This together with (2.2) implies that $u = 0$, which is a contradiction.

(III) The above discussions indicate that *compactness* occurs, i.e., for any $\varepsilon > 0$, there exist $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that, for n large enough,

$$\int_{B_R(y_n)} |u_n| dx \geq 1 - \varepsilon.$$

Thus, there exists $u \in \mathcal{D}^{2,1}(\mathbb{R}^N)$ such that

$$u_n(x + y_n) \to u \text{ in } L^1(\mathbb{R}^N).$$

Therefore, we have $\int_{\mathbb{R}^N} |u| dx = 1$ and

$$m(1) = \lim_{n\to\infty} F(u_n) \geq F(u) \geq m(1).$$

This indicates that $u \geq 0$ is a minimizer of $m(1)$ and

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1. \tag{2.13}$$

Moreover, we obtain from Lemma 2.4 (ii) that

$$u(x) = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda |x - x_0|),$$

where

$$\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N}\right)^{\frac{1}{N+2}}$$

follows directly from (2.13). This finishes the proof of the lemma. □

Based on the above two lemmas, we are ready to prove that when $a = a_{q^*}$, there exist minimizers of $d_a(q^*)$ for any dimension $N \geq 1$.

Lemma 2.6. *Assume that $V(x)$ satisfies* (1.15). *Then $d_{a_{q^*}}(q^*)$ has at least one minimizer if $a = a_{q^*}$.*

Proof. Let $\{u_n\}$ be a minimizing sequence of $d_{a_{q^*}}(q^*)$. Similar to (2.3), one can deduce from (2.2) that $\{u_n\}$ is bounded in H. Now, we claim that

$$\int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx \text{ is also bounded uniformly as } n \to \infty. \tag{2.14}$$

Otherwise, if

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx = +\infty,$$

then, since $\{u_n\}$ is a minimizing sequence of $d_{a_{q^*}}(q^*)$, it follows from above that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx = +\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx}{\int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx} = \frac{N a_{q^*}}{N+1}.$$

Setting

$$w_n = \varepsilon_n^N u_n^2(\varepsilon_n x) \quad \text{with } \varepsilon_n = \left(\int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx\right)^{-\frac{1}{N+2}}, \tag{2.15}$$

we have

$$\int_{\mathbb{R}^N} |w_n| dx = \int_{\mathbb{R}^N} |u_n|^2 dx = 1,$$

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \varepsilon_n^{N+2} \int_{\mathbb{R}^N} |\nabla u_n^2|^2 dx = 1,$$

and

$$\int_{\mathbb{R}^N} |w_n|^{2+\frac{2}{N}} dx = \varepsilon_n^{N+2} \int_{\mathbb{R}^N} |u_n|^{4+\frac{4}{N}} dx \xrightarrow{n} \frac{N+1}{N a_{q^*}}.$$

Thus,

$$E_{q^*}^{a_{q^*}}(u_n) = \frac{1}{4}\varepsilon_n^{-(N+2)}F(w_n) + \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + V(x)u_n^2)dx = d_{a_{q^*}}(q^*) + o(1).$$

Consequently,

$$F(w_n) = 4\varepsilon_n^{N+2}\Big[d_{a_{q^*}}(q^*) - \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + V(x)u_n^2)dx + o(1)\Big] \xrightarrow{n} 0 = m(1).$$

This implies that $\{w_n\}$ is a minimizing sequence of $m(1)$ and $\int_{\mathbb{R}^N}|\nabla w_n|^2dx = 1$. It then follows from Lemma 2.5 that there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$w_n(\cdot + y_n) \xrightarrow{n} w_0 = \frac{\lambda_0^N}{|v_{q^*}|_{L^1}}v_{q^*}(\lambda_0 x) \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N).$$

However, it follows from (2.15) that

$$\liminf_{n\to\infty}\varepsilon_n^2\int_{\mathbb{R}^N}|\nabla u_n|^2dx = \liminf_{n\to\infty}\int_{\mathbb{R}^N}|\nabla w_n^{\frac{1}{2}}|^2dx \ge \int_{\mathbb{R}^N}|\nabla w_0^{\frac{1}{2}}|^2dx \ge C > 0.$$

This indicates that

$$\int_{\mathbb{R}^N}|\nabla u_n|^2dx \ge C\varepsilon_n^{-2} \xrightarrow{n} +\infty,$$

which contradicts the fact that $\{u_n\}$ is bounded in H. Thus, claim (2.14) is proved. Furthermore, similarly to the argument of Lemma 2.2 (i), one can obtain that $u_n \xrightarrow{n} u$ in X with u being a minimizer of $d_{a_{q^*}}(q^*)$. □

Proof of Theorem 1.2. Lemmas 2.2 and 2.6 imply the theorem. □

3 Case $a < a_{q^*}$: Proof of Theorem 1.3

The aim of this section is to prove that when $a < a_{q^*}$ is fixed, all minimizers of (1.1) are compact in the space X as $q \nearrow q^*$, which yields the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $\eta(x) \in C_0^\infty(\mathbb{R}^N)$ with $|\eta(x)|_{L^2}^2 = 1$. We can find a constant $C > 0$ independent of q such that

$$d_a(q) \le E_q^a(\eta) \le C < \infty.$$

Assume that u_q is a non-negative minimizer of (1.1). We deduce from (1.10) that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_q|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_q^2|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)|u_q|^2dx &= d_a(q) + \frac{a}{q+2}\int_{\mathbb{R}^N}|u_q|^{q+2}dx\\ &\le C + \frac{a}{q+2}\frac{1}{\lambda_q a_q}|\nabla u_q^2|_{L^2}^{\frac{2q}{q^*}}.\end{aligned} \tag{3.1}$$

This implies that

$$\frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_q^2|^2dx \le C + \frac{a}{q+2}\frac{1}{\lambda_q a_q}|\nabla u_q^2|_{L^2}^{\frac{2q}{q^*}}.$$

We claim that

$$\limsup_{q\nearrow q^*}\int_{\mathbb{R}^N}|\nabla u_q^2|^2dx \le C < \infty. \tag{3.2}$$

For otherwise, we have

$$\int_{\mathbb{R}^N}|\nabla u_q^2|^2dx =: M_q \to \infty \quad \text{as } q \nearrow q^*.$$

On the one hand, we know from (3.1) that

$$M_q \le C + \frac{a}{q+2}\frac{4}{\lambda_q a_q}|\nabla u_q^2|_{L^2}^{\frac{2q}{q^*}} \le \frac{1}{2}\Big(1 - \frac{a}{a_{q^*}}\Big)M_q^{\frac{q}{q^*}} + \frac{a}{q+2}\frac{4}{\lambda_q a_q}M_q^{\frac{q}{q^*}},$$

i.e.,

$$M_q \le \Big[\frac{1}{2}\Big(1 - \frac{a}{a_{q^*}}\Big) + \frac{a}{q+2}\frac{4}{\lambda_q a_q}\Big]^{\frac{q^*}{q^*-q}}. \tag{3.3}$$

On the other hand, from the definitions of λ_q and a_q in (1.11) we get that

$$\frac{1}{q+2}\frac{1}{\lambda_q} \to \frac{1}{4} \quad \text{and} \quad a_q \to a_{q^*} \quad \text{as } q \nearrow q^*. \tag{3.4}$$

Thus,

$$\lim_{q\nearrow q^*}\Big[\frac{1}{2}\Big(1 - \frac{a}{a_{q^*}}\Big) + \frac{a}{q+2}\frac{4}{\lambda_q a_q}\Big] = \Big[\frac{1}{2}\Big(1 - \frac{a}{a_{q^*}}\Big) + \frac{a}{a_{q^*}}\Big] < 1.$$

This together with (3.3) implies that

$$M_q \le \Big[\frac{1}{2}\Big(1 - \frac{a}{a_{q^*}}\Big) + \frac{a}{q+2}\frac{4}{\lambda_q a_q}\Big]^{\frac{q^*}{q^*-q}} \to 0 \quad \text{as } q \nearrow q^*,$$

which is impossible. Hence, (3.2) is obtained and it is easy to further show that

$$\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_q|^2 dx + V(x)|u_q|^2)dx \le C < \infty.$$

This means that $\{u_q\}$ is bounded in $H^1(\mathbb{R}^N)$. As a consequence, there exist a subsequence, still denoted by $\{u_q\}$, and $0 \le u_0 \in H$ such that

$$u_q \rightharpoonup u_0 \text{ in } H; \quad u_q \to u_0 \text{ in } L^p(\mathbb{R}^N) \quad \text{for all } 2 \le p < 2^*.$$

By applying Lebesgue's dominated convergence theorem, one can obtain from above that

$$\lim_{q\nearrow q^*}\int_{\mathbb{R}^N}|u_q|^{q+2}dx = \int_{\mathbb{R}^N}|u_0|^{q^*+2}dx.$$

Therefore,

$$\begin{aligned}\lim_{q\nearrow q^*} d_a(q) &= \lim_{q\nearrow q^*}\Big[\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_q|^2 + V(x)|u_q|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_q^2|^2 dx - \frac{a}{q+2}\int_{\mathbb{R}^N}|u_q|^{q+2}dx\Big]\\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + V(x)|u_0|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_0^2|^2 dx - \frac{a}{q^*+2}\int_{\mathbb{R}^N}|u_0|^{q^*+2}dx\\ &= E_{q^*}(u_0) \ge d_a(q^*).\end{aligned} \tag{3.5}$$

On the other hand, for any $\varepsilon > 0$, there exists $u_\varepsilon \in X$ with $|u_\varepsilon|_{L^2}^2 = 1$ such that

$$E_{q^*}^a(u_\varepsilon) \le d_a(q^*) + \varepsilon.$$

Then,

$$\begin{aligned}\lim_{q\nearrow q^*} d_a(q) &\le \lim_{q\nearrow q^*} E_q(u_\varepsilon)\\ &\le \lim_{q\nearrow q^*}\Big\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_\varepsilon^2|^2 dx - \frac{a}{q+2}\int_{\mathbb{R}^N}|u_\varepsilon|^{q+2}dx\Big\}\\ &= E_{q^*}^a(u_\varepsilon) + \lim_{q\nearrow q^*}\Big\{\frac{a}{q^*+2}\int_{\mathbb{R}^N}|u_\varepsilon|^{q^*+2}dx - \frac{a}{q+2}\int_{\mathbb{R}^N}|u_\varepsilon|^{q+2}dx\Big\}\\ &\le d_a(q^*) + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, this inequality together with (3.5) gives that

$$\lim_{q\to q^*} d_a(q) = d_a(q^*) = E_{q^*}^a(u_0).$$

Hence, u_0 is a non-negative minimizer of $d_a(q^*)$ and $u_q \to u_0$ in X as $q \nearrow q^*$. □

4 Case $a > a_{q^*}$: Proof of Theorem 1.4

This section is devoted to proving Theorem 1.4 on the blow-up of minimizers of (1.1) as $q \nearrow q^*$ for the case of $a > a_{q^*}$. For this purpose, we introduce the functional

$$\widetilde{E}_q^a(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx - \frac{a}{q+2}\int_{\mathbb{R}^N} |u|^{q+2}\,dx$$

and consider the following minimization problem:

$$\widetilde{d}_a(q) = \inf_{u\in M} \widetilde{E}_q^a(u). \tag{4.1}$$

We remark that when $a > a_{q^*}$, it follows from (3.4) that

$$\lim_{q\nearrow q^*} \frac{4aq}{q^*\lambda_q a_q(q+2)} = \frac{a}{a_{q^*}} > 1. \tag{4.2}$$

We then obtain from Lemma 4.2 below that $\widetilde{d}_a(q) < 0$ if $q < q^*$ and is close to q^*. As a consequence, one can use [15, Lemma 1.1] to deduce that (4.1) has at least one minimizer.

4.1 Blow-up Analysis for the Minimizers of (4.1)

In this subsection, we study the following concentration phenomenon for the minimizers of (4.1) as $q \nearrow q^*$, which is crucial for proving Theorem 1.4.

Theorem 4.1. *Let $a > a_{q^*}$ and let u_q be a non-negative minimizer of* (4.1) *with $q < q^*$. For any sequence of $\{u_q\}$, there exist a subsequence, still denoted by $\{u_q\}$, and $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ such that the scaling*

$$w_q = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) \tag{4.3}$$

satisfies

$$\lim_{q\nearrow q^*} \varepsilon_q^N u_q^2(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}) = w_0^2 := \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x-x_0|) \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N), \tag{4.4}$$

where λ and ε_q are given by (1.16) *and $x_0 \in \mathbb{R}^N$. Moreover, there exist positive constants C, μ and R independent of q such that*

$$w_q(x) \le Ce^{-\mu|x|} \quad \text{for any } |x| > R \quad \text{as } q \nearrow q^*. \tag{4.5}$$

To prove this theorem, we first give the following energy estimate of $\widetilde{d}_a(q)$.

Lemma 4.2. *Let $a > a_{q^*}$ be fixed. Then,*

$$\widetilde{d}_a(q) = -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^* a_q\lambda_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}(1+o(1))\ (\to -\infty) \quad \text{as } q \nearrow q^*.$$

Proof. For any $u \in M$, we obtain from (1.10) that

$$\begin{aligned}\widetilde{E}_q^a(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx - \frac{a}{q+2}\int_{\mathbb{R}^N} |u|^{q+2}\,d\\ &\ge \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx - \frac{a}{(q+2)\lambda_q a_q}\Big(\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx\Big)^{\frac{q}{q^*}}.\end{aligned} \tag{4.6}$$

Setting

$$\int_{\mathbb{R}^N} |\nabla u^2|^2\,dx = t \quad \text{and} \quad g(t) = \frac{1}{4}t - \frac{a}{(q+2)\lambda_q a_q} t^{\frac{q}{q^*}}, \quad t \in (0,+\infty),$$

we know that $g(t)$ gets its minimum at a unique point

$$t_q = \Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}. \tag{4.7}$$

Hence,

$$g(t) \geq g(t_q) = -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}. \tag{4.8}$$

From (4.6) and (4.8), we know that

$$\widetilde{d}_a(q) \geq -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}.$$

This gives the lower bound of $\widetilde{d}_a(q)$. We next will prove the upper bound.

Let

$$u_\tau = \frac{\tau^{\frac{N}{2}}}{|v_q|_{L^1}^{1/2}}\sqrt{v_q(\tau x)}, \quad \tau > 0,$$

where $v_q(x)$ is the unique non-negative solution of (1.12). Then, $\int_{\mathbb{R}^N} u_\tau^2\,dx = 1$ and it follows from (1.14) that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_\tau^2|^2\,dx &= \frac{\tau^{N+2}}{|v_q|_{L^1}^2}\int_{\mathbb{R}^N} |\nabla v_q|^2\,dx = \frac{\theta_q\tau^{N+2}}{(1-\theta_q)|v_q|_{L^1}},\\
\int_{\mathbb{R}^N} u_\tau^{q+2}\,dx &= \frac{\tau^{\frac{Nq}{2}}}{|v_q|_{L^1}^{(q+2)/2}}\int_{\mathbb{R}^N} v_q^{\frac{q+2}{2}}\,dx = \frac{\tau^{\frac{Nq}{2}}}{(1-\theta_q)|v_q|_{L^1}^{q/2}},\\
\int_{\mathbb{R}^N} |\nabla u_\tau|^2\,dx &= \frac{\tau^2}{|v_q|_{L^1}}\int_{\mathbb{R}^N} |\nabla\sqrt{v_q}|^2\,dx \leq c_1\tau^2.
\end{aligned}$$

Taking

$$\tau = \Big(\frac{(1-\theta_q)t_q|v_q|_{L^1}}{\theta_q}\Big)^{\frac{1}{N+2}}$$

with t_q given by (4.7), we then have

$$\begin{aligned}
\frac{1}{4}\int_{\mathbb{R}^N} |\nabla u_\tau^2|^2\,dx - \frac{a}{q+2}\int_{\mathbb{R}^N} u_\tau^{q+2}\,dx &= \frac{\theta_q\tau^{N+2}}{4(1-\theta_q)|v_q|_{L^1}} - \frac{a}{q+2}\frac{\tau^{\frac{Nq}{2}}}{(1-\theta_q)|v_q|_{L^1}^{q/2}}\\
&= -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\tau|^2\,dx \leq c_0\tau^2 \leq c_1\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}\cdot\frac{2}{N+2}}.$$

Therefore,

$$\widetilde{E}_q^a(u_\tau) \leq -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}} + c_1\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}\cdot\frac{2}{N+2}}. \tag{4.9}$$

From (4.2) we see that

$$\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}} \to +\infty \quad \text{as } q \nearrow q^*.$$

This together with (4.9) implies that

$$\widetilde{d}_a(q) \leq \widetilde{E}_q^a(u_\tau) \leq -\frac{q^*-q}{4q}\Big(\frac{4aq}{q^*\lambda_q a_q(q+2)}\Big)^{\frac{q^*}{q^*-q}}(1+o(1)),$$

which gives the upper bound of $\widetilde{d}_a(q)$ and completes the proof of the lemma. □

Lemma 4.3. *Let $a > a_{q^*}$ be fixed and let u_q be a non-negative minimizer of $\widetilde{d}_a(q)$. Then*

$$\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \approx \frac{4a}{q+2} \int_{\mathbb{R}^N} u_q^{q+2} dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{\frac{q^*}{q^*-q}} := t_q \tag{4.10}$$

and

$$\frac{\int_{\mathbb{R}^N} |\nabla u_q|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx} \to 0 \quad \text{as } q \nearrow q^*. \tag{4.11}$$

Here $a \approx b$ means that $\frac{a}{b} \to 1$ as $q \nearrow q^*$.

Proof. From (4.6), we have

$$\begin{aligned}\widetilde{d}_a(q) = \widetilde{E}_q^a(u_q) &\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{(q+2)\lambda_q a_q} \left(\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx\right)^{\frac{q}{q^*}} := g(t) \\ &= \frac{1}{4} t - \frac{a}{(q+2)\lambda_q a_q} t^{\frac{q}{q^*}} \quad \text{with } t = \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx.\end{aligned} \tag{4.12}$$

We first prove that

$$\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{\frac{q^*}{q^*-q}} = t_q \quad \text{as } q \nearrow q^*. \tag{4.13}$$

For otherwise, if it is false, then in subsequence sense we have

$$\lim_{q \nearrow q^*} \frac{\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx}{t_q} = \gamma \in [0,1) \cup (1,+\infty).$$

If $\gamma \in [0,1)$, then,

$$\lim_{q \nearrow q^*} \frac{g(\gamma t_q)}{g(t_q)} = \lim_{q \nearrow q^*} \frac{\gamma t_q - \frac{4a}{(q+2)\lambda_q a_q} (\gamma t_q)^{\frac{q}{q^*}}}{t_q - \frac{4a}{(q+2)\lambda_q a_q} t_q^{q/q^*}} = \lim_{q \nearrow q^*} \frac{q^* \gamma^{\frac{q}{q^*}} - q\gamma}{q^* - q} = \gamma(-\ln \gamma + 1) \in [0,1). \tag{4.14}$$

Let $\delta := \gamma(-\ln \gamma + 1) \in [0,1)$. We then obtain from (4.12) and (4.14) that

$$\widetilde{d}_a(q) \geq (1 + o(1)) g(\gamma t_q) \geq \frac{1+\delta}{2} g(t_q) = -\frac{1+\delta}{2} \cdot \frac{q^* - q}{4q} \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{\frac{q^*}{q^*-q}}.$$

This contradicts Lemma 4.2. Similarly to the above argument, one can prove also that $\gamma \in (1,+\infty)$ cannot occur. Thus, (4.13) is proved.

We next try to prove (4.11). On the contrary, if it is false, then there exists $\beta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_q|^2 dx \geq \frac{\beta}{2} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx.$$

Therefore,

$$\begin{aligned}0 \geq \widetilde{d}_a(q) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{q+2} \int_{\mathbb{R}^N} |u_q|^{q+2} dx \\ &\geq \frac{1+\beta}{4} \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx - \frac{a}{(q+2)\lambda_q a_q} \left(\int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx\right)^{\frac{q}{q^*}} =: \widetilde{g}(s),\end{aligned} \tag{4.15}$$

by setting

$$\widetilde{g}(s) = \frac{1+\beta}{4} s - \frac{a}{(q+2)\lambda_q a_q} s^{\frac{q}{q^*}} \quad \text{with } s = \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx.$$

One can easily check that

$$\tilde{g}(s) \geq \tilde{g}(s_q) \quad \text{with } s_q = \left(\frac{4aq}{q^*\lambda_q a_q(q+2)(1+\beta)}\right)^{\frac{q^*}{q^*-q}}$$

and

$$\tilde{g}(s_q) = -\frac{(1+\beta)(q^*-q)}{4q}\left(\frac{4aq}{q^*\lambda_q a_q(q+2)(1+\beta)}\right)^{\frac{q^*}{q^*-q}} =: A.$$

However

$$\frac{A}{-\frac{(q^*-q)}{4q}\left(\frac{4aq}{q^*\lambda_q a_q(q+2)}\right)^{\frac{q^*}{q^*-q}}} = (1+\beta)\left(\frac{1}{1+\beta}\right)^{\frac{q^*}{q^*-q}} \to 0 \quad \text{as } q \nearrow q^*.$$

This together with (4.15) indicates that

$$\frac{\tilde{d}_a(q)}{-\frac{(q^*-q)}{4q}\left(\frac{4aq}{q^*\lambda_q a_q(q+2)}\right)^{\frac{q^*}{q^*-q}}} \to 0 \quad \text{as } q \nearrow q^*,$$

which contradicts Lemma 4.2. Thus, (4.11) is proved.

Finally, taking

$$\frac{\tilde{d}_a(q)}{t_q} = \frac{1}{2t_q}\int_{\mathbb{R}^N} |\nabla u_q|^2\,dx + \frac{1}{4t_q}\int_{\mathbb{R}^N} |\nabla u_q^2|^2\,dx - \frac{a}{(q+2)t_q}\int_{\mathbb{R}^N} |u_q|^{q+2}\,dx,$$

we obtain from (4.11) and (4.13) that

$$\frac{4a}{(q+2)t_q}\int_{\mathbb{R}^N} |u_q|^{q+2}\,dx \to 1 \quad \text{as } q \nearrow q^*.$$

This gives (4.10). The proof of this lemma is finished. □

Applying Lemmas 4.2 and 4.3, we end this subsection by proving Theorem 4.1.

Proof of Theorem 4.1. Set

$$\varepsilon_q = t_q^{-\frac{1}{N+2}} = t_q^{-\frac{2}{Nq^*}}$$

with t_q given by (4.7). It follows from (4.2) that $\lim_{q\nearrow q^*} \varepsilon_q = 0$. Let u_q be a non-negative minimizer of $\tilde{d}_a(q)$ and define

$$\tilde{w}_q = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x).$$

From Lemma 4.3, we have

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_q^2|^2\,dx = \varepsilon_q^{N+2}\int_{\mathbb{R}^N} |\nabla u_q^2|^2\,dx \approx t_q^{-1}t_q = 1, \tag{4.16}$$

$$\int_{\mathbb{R}^N} |\tilde{w}_q|^{q+2}\,dx = \varepsilon_q^{\frac{qN}{2}}\int_{\mathbb{R}^N} |u_q|^{q+2}\,dx \approx \frac{q^*+2}{4a_{q^*}}, \tag{4.17}$$

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_q|^2\,dx = \varepsilon_q^2\int_{\mathbb{R}^N} |\nabla u_q|^2\,dx = o(1)\varepsilon_q^{-N}. \tag{4.18}$$

Since u_q is a minimizer of $\tilde{d}_a(q)$, there exists $\mu_q \in \mathbb{R}$ such that

$$-\Delta u_q - \Delta(u_q^2)u_q = \mu_q u_q + a|u_q|^q u_q. \tag{4.19}$$

Therefore,

$$\begin{aligned}
\mu_q &= \int_{\mathbb{R}^N} |\nabla u_q|^2\,dx + \int_{\mathbb{R}^N} |\nabla u_q^2|^2\,dx - a\int_{\mathbb{R}^N} |u_q|^{q+2}\,dx \\
&= 4\tilde{d}_a(q) - \int_{\mathbb{R}^N} |\nabla u_q|^2\,dx + \frac{a(2-q)}{q+2}\int_{\mathbb{R}^N} |u_q|^{q+2}\,dx.
\end{aligned}$$

From Lemmas 4.2 and 4.3, we get that

$$\mu_q \varepsilon_q^{N+2} = \mu_q t_q^{-1} = -\frac{q^* - q}{4q} + o(1) + \frac{2-q}{4}(1 + o(1)) \to -\frac{1}{N} \quad \text{as } q \nearrow q^*. \tag{4.20}$$

Using (4.17) and [18, Lemma I.1], we see that there exist $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ and $R, \eta > 0$ such that

$$\liminf_{q \nearrow q^*} \int_{B_R(y_{\varepsilon_q})} |\widetilde{w}_q|^2 dx > \eta > 0.$$

Let

$$w_q = \widetilde{w}_q(x + y_{\varepsilon_q}) = \varepsilon_q^{\frac{N}{2}} u_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q}). \tag{4.21}$$

Then

$$\liminf_{q \nearrow q^*} \int_{B_R(0)} |w_q|^2 dx > \eta > 0. \tag{4.22}$$

From (4.19), we see that $w_q(x)$ satisfies

$$-\varepsilon_q^N \Delta w_q - \Delta(w_q^2) w_q = \mu_q \varepsilon_q^{N+2} w_q + a \varepsilon_q^{N+2-\frac{Nq}{2}} w_q^{q+1}. \tag{4.23}$$

Note that $N + 2 = \frac{Nq^*}{2}$. We can deduce that

$$\varepsilon_q^{N+2-\frac{Nq}{2}} = \varepsilon_q^{\frac{N(q^*-q)}{2}} = t_q^{-\frac{q^*-q}{q^*}} = \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{-1} = \frac{q^* \lambda_q a_q (q+2)}{4aq}.$$

Consequently,

$$\lim_{q \nearrow q^*} a \varepsilon_q^{N+2-\frac{Nq}{2}} = \lim_{q \nearrow q^*} \frac{q^* \lambda_q a_q}{q} = a_{q^*}. \tag{4.24}$$

Moreover, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we deduce from (4.18) and (4.21) that

$$\left|\varepsilon_q^N \int_{\mathbb{R}^N} \nabla w_q \nabla \varphi \, dx\right| \leq C \varepsilon_q^N \left(\int_{\mathbb{R}^N} |\nabla w_q|^2 dx\right)^{\frac{1}{2}} = o(1) \varepsilon_q^{\frac{N}{2}} \to 0 \quad \text{as } q \nearrow q^*.$$

By passing to a subsequence, it then follows from (4.20)–(4.24) that

$$w_q^2 \rightharpoonup w_0^2 \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N) \quad \text{as } q \nearrow q^*,$$

where $0 \leq w_0 \not\equiv 0$ satisfies

$$-\Delta(w_0^2) w_0 = -\frac{1}{N} w_0 + a_{q^*} w_0^{3+\frac{4}{N}},$$

i.e.

$$-\Delta(w_0^2) = -\frac{1}{N} + a_{q^*} (w_0^2)^{1+\frac{2}{N}}. \tag{4.25}$$

Using classical Pohozaev identities, we obtain that

$$\int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx = \int_{\mathbb{R}^N} w_0^2 dx \quad \text{and} \quad \int_{\mathbb{R}^N} (w_0^2)^{\frac{q^*+2}{2}} dx = \frac{N+1}{N a_{q^*}} \int_{\mathbb{R}^N} w_0^2 dx.$$

Recalling the Gagliardo–Nirenberg inequality (2.2), we then have

$$\frac{N a_{q^*}}{N+1} \leq \frac{\int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx \left(\int_{\mathbb{R}^N} w_0^2 dx\right)^{\frac{2}{N}}}{\int_{\mathbb{R}^N} (w_0^2)^{2+\frac{2}{N}}} = \frac{N a_{q^*}}{N+1} \left(\int_{\mathbb{R}^N} w_0^2 dx\right)^{\frac{2}{N}}. \tag{4.26}$$

This indicates that

$$\int_{\mathbb{R}^N} w_0^2 dx \geq 1.$$

On the other hand, there always holds that

$$\int_{\mathbb{R}^N} w_0^2 dx \le \liminf_{q \nearrow q^*} \int_{\mathbb{R}^N} w_q^2 dx = 1.$$

Consequently, we have

$$\int_{\mathbb{R}^N} w_0^2 dx = 1, \tag{4.27}$$

and thus

$$w_q \to w_0 \text{ in } L^2(\mathbb{R}^N) \quad \text{as } q \nearrow q^*.$$

It then follows from (4.16), (4.23) and (4.25) that

$$\liminf_{q \nearrow q^*} \int_{\mathbb{R}^N} |\nabla w_q^2|^2 dx = \int_{\mathbb{R}^N} |\nabla w_0^2|^2 dx = 1. \tag{4.28}$$

This means that

$$w_q^2 \to w_0^2 \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N).$$

Moreover, it follows from (4.26) and (4.27) that $w_0^2 \ge 0$ is an optimizer of (2.2), thus it must be of the form

$$w_0^2 = \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - x_0|),$$

where

$$\lambda = \left(\frac{|v_{q^*}|_{L^1}}{N}\right)^{\frac{1}{N+2}}$$

follows from (4.28). This completes the proof of (4.4).

To complete the proof, it remains to show (4.5). Indeed, from (4.23) and (4.24) we see that

$$-\Delta(w_q^2) \le c(x) w_q^2 \quad \text{with } c(x) = 2a_{q^*} w_q^{q-2}.$$

Similarly to the proof of [12, Theorem 1.1], one can use DeGiorgi–Nash–Moser theory as well as the comparison principle to deduce that there exist $C, \beta, R > 0$ independent of q such that

$$w_q^2(x) \le Ce^{-\beta|x|} \quad \text{for any } |x| > R \quad \text{as } q \nearrow q^*.$$

This gives (4.5) by taking $\mu = \frac{\beta}{2}$. □

4.2 Proof of Theorem 1.4

This subsection is devoted to proving Theorem 1.4 on the blow-up behavior of minimizers of (1.1) as $q \nearrow q^*$. We first give precise energy estimates of $d_a(q)$ in the following lemma.

Lemma 4.4. *Let $a > a_{q^*}$ be fixed and let $\bar{u}_q(x)$ be a non-negative minimizer of $d_a(q)$. Then,*

$$0 \le d_a(q) - \tilde{d}_a(q) \to 0 \quad \text{as } q \nearrow q^*,$$

and

$$\int_{\mathbb{R}^N} V(x) \bar{u}_q^2 dx \to 0 \quad \text{as } q \nearrow q^*. \tag{4.29}$$

Proof. Let $\varphi(x)$ be a cut-off function such that $\varphi(x) \equiv 1$ if $|x| < 1$ and $\varphi(x) \equiv 0$ if $|x| > 2$. As in Section 4.1, we still denote by u_q a non-negative minimizer of $\tilde{d}_a(q)$ and let w_q be given by (4.3). For any $x_0 \in \mathbb{R}^N$, we set

$$\tilde{u}_q(x) = A_q \varphi(x - x_0) \varepsilon_q^{-\frac{N}{2}} w_q\left(\frac{x - x_0}{\varepsilon_q}\right) = A_q \varphi(x - x_0) u_q(x - x_0 + \varepsilon_q y_{\varepsilon_q}),$$

where $A_q \geq 1$ such that $\int_{\mathbb{R}^N} \tilde{u}_q^2 dx \equiv 1$. Using the exponential decay of w_q in (4.5), we have

$$0 \leq A_q^2 - 1 = \frac{\int_{|\varepsilon_q x| \geq 1} (1 - \varphi^2(\varepsilon_q x)) w_q^2(x) dx}{\int_{\mathbb{R}^N} \varphi^2(\varepsilon_q x) w_q^2(x) dx} \leq C e^{-\frac{\mu}{\varepsilon_q}} \quad \text{as } q \nearrow q^*,$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)\tilde{u}_q^2(x)dx &= A_q^2 \int_{\mathbb{R}^N} V(\varepsilon x + x_0)\varphi^2(\varepsilon_q x) w_q^2 dx \\ &\to V(x_0) \int_{\mathbb{R}^N} w_0^2 dx = V(x_0) \quad \text{as } q \nearrow q^*\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |\tilde{u}_q|^{q+2} dx &= \varepsilon_q^{-\frac{Nq}{2}} A_q^{q+2} \int_{\mathbb{R}^N} \varphi^{q+2}(\varepsilon_q x)|w_q|^{q+2} dx \\ &= \varepsilon_q^{-\frac{Nq}{2}} \int_{\mathbb{R}^N} |w_q|^{q+2} dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \\ &= \int_{\mathbb{R}^N} |u_q|^{q+2} dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*.\end{aligned}$$

In much the same way as above, one can prove that

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_q^2|^2 dx = \int_{\mathbb{R}^N} |\nabla u_q^2|^2 dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*,$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_q|^2 dx = \int_{\mathbb{R}^N} |\nabla u_q|^2 dx + O(e^{-\frac{\mu}{\varepsilon_q}}) \quad \text{as } q \nearrow q^*.$$

Therefore, choosing $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, we deduce from the above estimates that

$$\begin{aligned}0 &\leq d_a(q) - \tilde{d}_a(q) \\ &\leq E_q^a(\tilde{u}_q(x)) - \tilde{E}_q^a(u_q(x)) \\ &= \tilde{E}_q^a(\tilde{u}_q(x)) - \tilde{E}_q^a(u_q(x)) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\tilde{u}_q^2(x)dx \\ &= \frac{1}{2}V(x_0) + O(e^{-\frac{\mu}{\varepsilon_q}}) + o(1) \to 0 \quad \text{as } q \nearrow q^*.\end{aligned}$$

Moreover, if $\bar{u}_q$ is a non-negative minimizer of $d_a(q)$, then

$$\int_{\mathbb{R}^N} V(x)\bar{u}_q^2 dx = d_a(q) - \tilde{E}_q^a(\bar{u}_q) \leq d_a(q) - \tilde{d}_a(q) \to 0 \quad \text{as } q \nearrow q^*. \qquad \square$$

Proof of Theorem 1.4. We still use $\bar{u}_q$ to denote a non-negative minimizer of $d_a(q)$. Applying Lemma 4.4, one can check that all the conclusions in Lemma 4.3 also holds for $\bar{u}_q$, i.e.,

$$\int_{\mathbb{R}^N} |\nabla \bar{u}_q^2|^2 dx \approx \frac{4a}{q+2} \int_{\mathbb{R}^N} \bar{u}_q^{q+2} dx \approx \left(\frac{4aq}{q^* \lambda_q a_q (q+2)}\right)^{\frac{q^*}{q^*-q}} = t_q$$

and

$$\frac{\int_{\mathbb{R}^N} |\nabla \bar{u}_q|^2 dx}{\int_{\mathbb{R}^N} |\nabla \bar{u}_q^2|^2 dx} \to 0 \quad \text{as } q \nearrow q^*.$$

Moreover, similarly to (4.22), one can prove that there exists $\{y_{\varepsilon_q}\} \subset \mathbb{R}^N$ such that the scaling

$$\bar{w}_q(x) := \varepsilon_q^{\frac{N}{2}} \bar{u}_q(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q})$$

satisfies

$$\liminf_{q \nearrow q^*} \int_{B_R(0)} |\bar{w}_q|^2 dx > \eta > 0.$$

Then, repeating the proof of Theorem 4.1, we can prove that

$$\bar{w}_q^2 \to w_0^2 := \frac{\lambda^N}{|v_{q^*}|_{L^1}} v_{q^*}(\lambda|x - x_0|) \text{ in } \mathcal{D}^{2,1}(\mathbb{R}^N) \quad \text{with } \lambda = \Big(\frac{|v_{q^*}|_{L^1}}{N}\Big)^{\frac{1}{N+2}}.$$

Moreover, by (4.29) we see that

$$\int_{\mathbb{R}^N} V(x)\bar{u}_q^2 dx = \int_{\mathbb{R}^N} V(\varepsilon_q x + \varepsilon_q y_{\varepsilon_q})\bar{w}_q(x)dx \to 0 \quad \text{as } q \nearrow q^*.$$

This further indicates that the sequence $\{\varepsilon_q y_{\varepsilon_q}\}$ satisfies

$$\varepsilon_q y_{\varepsilon_q} \to A = \{x : V(x) = 0\} \quad \text{as } q \nearrow q^*.$$

The proof of Theorem 1.4 is complete. □

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