DE GRUYTER Adv. Nonlinear Stud. 2018; 18(1): 1–15

Research Article

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Ground State for a Coupled Elliptic System with Critical Growth

DOI: 10.1515/ans-2017-6019
Received July 26, 2016; accepted May 4, 2017

Abstract: We study the following coupled elliptic system with critical nonlinearities:

$$\begin{cases} -\Delta u + u = f(u) + \beta h(u)K(v), & x \in \mathbb{R}^N, \\ -\Delta v + v = g(v) + \beta H(u)k(v), & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $\beta > 0$; f, g are differentiable functions with critical growth; and H, K are primitive functions of h and k, respectively. Under some assumptions on f, g, h and k, we obtain the existence of a positive ground state solution of this system for $N \geq 2$.

Keywords: Nonlinear Schrödinger System, Critical Exponent, Positive Solution, Ground State

MSC 2010: 35B09, 35J47

Communicated by: Zhi-Qiang Wang

1 Introduction

In recent years, many authors have studied the existence of solutions of the following nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{2q-2} u + \beta |v|^q |u|^{q-2} u, & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{2q-2} v + \beta |u|^q |v|^{q-2} v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2, \beta$ are constants, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary or $\Omega = \mathbb{R}^N$. System (1.1) arises in many physical problems, especially in nonlinear optics and Bose–Einstein condensation. We refer to [12, 14] and the references therein for experimental results and physical background of this system.

A solution $(u_0, v_0) \in E := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of (1.1) is called positive if $u_0 > 0$, $v_0 > 0$ and nontrivial if $(u_0, v_0) \neq (0, 0)$. This solution is a ground state in the sense that $(u_0, v_0) \neq (0, 0)$ and its energy is minimal among the energy of all nontrivial solutions. For the case $q = 2$ and $N \leq 3$, if $\beta > 0$ is small enough, existence of one positive solution of (1.1) was proved in [15], and if $\beta \in (0, \beta_1] \cup [\beta_2, +\infty)$, where β_1, β_2 are positive constants determined in terms of λ_i, μ_i and N, existence of one positive solution of (1.1) was obtained in [1, 2, 4, 23]. For the general subcritical case $1 < q < \frac{2^*}{2} := \frac{N}{(N-2)^+}$, $N \in \mathbb{N}^+$ and $\beta > \beta_3$, existence of one positive solution of (1.1) was shown in [20, 22]. The β_3 mentioned previously is a nonnegative constant determined

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in terms of λ_i, μ_i, q and N. Moreover, $\beta_3 > 0$ if $q \geq 2$ and $\beta_3 = 0$ if $1 < q < 2$. We refer the readers to [5, 8, 13, 18, 19, 21, 24, 25] for other results on the existence and multiplicity of solutions to (1.1) with subcritical exponent $1 < q < \frac{2^*}{2}$. For the critical case, where $q = \frac{2^*}{2}$ and Ω is a bounded smooth domain in $\mathbb{R}^N$, existence of a positive solution was studied in [10, 11, 27], and existence of a sign-changing solution was proved in [9].

Partially motivated by [1, 2, 4, 20, 22, 23], in this paper, we consider a more general system with critical exponent when $\beta > 0$:

$$\begin{cases} -\Delta u + u = f(u) + \beta h(u)K(v), & x \in \mathbb{R}^N, \\ -\Delta v + v = g(v) + \beta H(u)k(v), & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are C^1 functions with critical growth; h, k are also C^1 functions, and $H(s) := \int_0^s h(t)dt$, $K(s) := \int_0^s k(t)dt$. Our purpose is to search for a positive ground state solution of (1.2). Denote $F(s) := \int_0^s f(t)dt$, $G(s) := \int_0^s g(t)dt$, and the integral $\int_{\mathbb{R}^N} \cdot\, dx$ by $\int \cdot$. It is well known that a solution of (1.2) corresponds to a critical point of the energy functional $I : E \to \mathbb{R}$ defined by

$$I(u, v) := \frac{1}{2}\int (|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2) - \int \big(F(u) + G(v) + \beta H(u)K(v)\big).$$

Set

$$\mathcal{N} := \{(u, v) \in E \setminus \{(0, 0)\} : \langle I'(u, v), (u, v)\rangle = 0\}.$$

Then a positive ground state of (1.2) is a positive solution of the minimization problem

$$m := \inf_{\mathcal{N}} I(u, v).$$

To find the minimizer, for the case $N \geq 3$, we assume that

(f1) $\lim_{s\to 0^+} \frac{f(s)}{s} = \lim_{s\to 0^+} \frac{g(s)}{s} = 0$.
(f2) $\limsup_{s\to+\infty} \frac{f(s)}{s^{2^*-1}} \leq 1$ and $\limsup_{s\to+\infty} \frac{g(s)}{s^{2^*-1}} \leq 1$.
(f3) $f'(s)s - f(s) > 0$ and $g'(s)s - g(s) > 0$ if $s > 0$.
(f4) There are $\lambda > 0$ and $2 < r < 2^*$ such that $f(s) \geq \lambda s^{r-1}$ and $g(s) \geq \lambda s^{r-1}$ for every $s \geq 0$.
(h1) $|h(s)K(t)| \leq |s|^{2^*-1} + |t|^{2^*-1}$ and $|H(s)k(t)| \leq |s|^{2^*-1} + |t|^{2^*-1}$ for $|st| \geq C_0 > 0$.
(h2) There exist $C_1 > 0$, $C_2 > 0$ and $1 < p \leq \min\{\frac{2^*}{2}, 2\}$ such that $\lim_{s\to 0^+} \frac{h'(s)}{s^{p-2}} = C_1$ and $\lim_{s\to 0^+} \frac{k'(s)}{s^{p-2}} = C_2$.
(h3) $H(s)K(t) > 0$ and $h(s)K(t)s + H(s)k(t)t - 2H(s)K(t) > 0$ for $s > 0$, $t > 0$.

Let S and C_r be the best constants satisfying

$$S\Big(\int |u|^{2^*}\Big)^{\frac{2}{2^*}} \leq \int |\nabla u|^2 \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N),$$

and

$$C_r\Big(\int |u|^r\Big)^{\frac{2}{r}} \leq \int (|\nabla u|^2 + |u|^2) \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

The first result of the current paper is as follows.

Theorem 1.1. *Assume that* (f1), (f2), (f3), (f4), (h1), (h2), *and* (h3) *hold. If* $N \geq 3$ *and*

$$\lambda > \Big(N\Big(\frac{N}{N-2}\Big)^{\frac{N-2}{2}}(1+\beta)^{\frac{N-2}{2}} S^{-\frac{N}{2}}\Big)^{\frac{r-2}{2}} \Big(\frac{r-2}{2r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}},$$

then problem (1.2) *has a ground state solution* (u, v) *for all* $\beta > 0$*. Moreover, if* $1 < p < 2$*, then* $u > 0$, $v > 0$*; if* $p = 2$*, then there is* $\beta^* > 0$ *such that* $u > 0$, $v > 0$ *if* $\beta > \beta^*$.

Remark 1.2. Throughout this paper, the constants λ and r are defined as in condition (f4) and p is defined as in (h2).

For the case of dimension $N = 2$, according to a Trudinger–Moser type inequality [7, Lemma 2.1], we replace the growth assumptions (f2) and (h1) with the following two conditions, respectively:

(f2') $\lim_{s\to+\infty}\frac{f(s)}{e^{\alpha s^2}} = \lim_{s\to+\infty}\frac{g(s)}{e^{\alpha s^2}} = 0(+\infty)$ if $\alpha > 4\pi(\alpha < 4\pi)$.

(h1') $|h(s)K(t)| \le C(e^{\alpha s^2}-1) + C(|t|(e^{\alpha t^2}-1))^{\frac{r_1-1}{r_1}}$ and $|H(s)k(t)| \le C(e^{\alpha t^2}-1) + C(|s|(e^{\alpha s^2}-1))^{\frac{r_2-1}{r_2}}$ if $r_1, r_2 > 2$ and $\alpha > 4\pi$.

We get the following result:

Theorem 1.3. *Assume that* (f1), (f2'), (f3), (f4), (h1'), (h2) *and* (h3) *hold. If* $N = 2$ *and*

$$\lambda > \Big(\frac{r-2}{r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}},$$

then problem (1.2) *has a ground state solution* (u, v) *for all* $\beta > 0$. *Moreover, if* $1 < p < 2$, *then* $u > 0$, $v > 0$; *if* $p = 2$, *then there is* $\beta^{**} > 0$ *such that* $u > 0$, $v > 0$ *if* $\beta > \beta^{**}$.

Our results complement those in [20, 22] in the sense that we address the critical exponent problem. Moreover, it is easy to verify that (1.1) is a special case of (1.2) if $\mu_1 > 0$, $\mu_2 > 0$ are large enough and $1 < q < \frac{2^*}{2}$. Our results indicates that (1.1) has a positive ground state for all $\beta > 0$ if $1 < q < 2$ and for $\beta \ge \beta^*$ (or β^{**}) if $2 \le q < \frac{2^*}{2}$, which is consistent with [20, Theorem 2.3] and [22, Corollary 1].

To prove Theorem 1.1 and Theorem 1.3, we first need to estimate certain upper bound for the minimum level m since we are dealing with a problem with critical exponent. However, it is too difficult to get this bound directly. Fortunately, when we consider the minimization problem

$$A := \inf_{\mathcal{M}} \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2), \tag{1.3}$$

where

$$\mathcal{M} := \begin{cases} \Big\{(u, v) \in E \setminus \{(0, 0)\} : \int T(u, v) = 1\Big\}, & \text{if } N \ge 3,\\ \Big\{(u, v) \in E \setminus \{(0, 0)\} : \int T(u, v) = 0\Big\}, & \text{if } N = 2,\end{cases}$$

and

$$T(u, v) := F(u) + G(v) + \beta H(u)K(v) - \frac{|u|^2}{2} - \frac{|v|^2}{2},$$

we are able to estimate an upper bound of the minimum level A relating to the best constant of critical Sobolev imbedding, which allows us to prove that A is attained. Then by translation invariance we get that m is attained and (1.2) has a ground state solution. After that, we introduce the following single equations:

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.4}$$

and

$$-\Delta u + u = g(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.5}$$

which play an important role in the proof of our results. Under the assumptions (f1)–(f4) for $N \ge 3$ and (f1), (f2'), (f3), (f4) for $N = 2$, Alves and Souto [3] proved that both (1.4) and (1.5) have a positive ground state solution respectively if

$$\lambda > \Big(N\Big(\frac{N}{N-2}\Big)^{\frac{N-2}{2}} S^{-\frac{N}{2}}\Big)^{\frac{r-2}{2}} \Big(\frac{r-2}{2r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}} \quad \text{for } N \ge 3,$$

and

$$\lambda > \Big(\frac{r-2}{r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}} \quad \text{for } N = 2.$$

Denote the positive ground state solutions mentioned previously by U_1 and U_2, respectively. It is clear that $(U_1, 0)$ and $(0, U_2)$ solve (1.2). However, we are going to find a solution with both components being positive. We need to compare the minimal level m with the energy of $(U_1, 0)$ and $(0, U_2)$, which enables us to get sufficient conditions for existence of positive ground state solution of (1.2).

The rest of this paper is organized as follows. Section 2 is devoted to recall and fix some notations. The cases of dimension $N \ge 3$ and dimension $N = 2$ are treated in Section 3 and Section 4, respectively.

2 Some Notations

We need to define the following minimax value:

$$b := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

Denote the Pohozaev identity set by

$$\mathcal{P} := \Big\{(u,v) \in E \setminus \{(0,0)\} : (N-2)\int(|\nabla u|^2 + |\nabla v|^2) = 2N\int T(u,v)\Big\}.$$

Set $d := \inf_{\mathcal{P}} I(u,v)$. The values b and d are important in proving that a minimizing solution of (1.3) corresponds to a ground state of (1.2) (see the following Lemma 3.3 and Lemma 3.9).

Let

$$I_1(u) := \int |\nabla u|^2 + |u|^2 - \int F(u),$$

and

$$I_2(u) := \int |\nabla u|^2 + |u|^2 - \int G(u).$$

It is well known that $I_i(u)$, where $i = 1, 2$, are the energy functionals associated with (1.4) and (1.5), respectively. Let

$$B_i = I_i(U_i),$$

then we will show that

$$m < \min\{B_1, B_2\}$$

in Section 3, which implies that both components of the ground state solution are nontrivial.

Due to the fact that (1.2) is an autonomous cooperative system, under the Schwartz symmetrization progress we can minimize (1.3) on the subspace $E_r := H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ formed by radially symmetric functions. Moreover, since we seek positive solutions of (1.2), we may assume that f, g, h and k are odd functions.

3 The Case of Dimension $N \geq 3$

In this case, it is important to observe that the $\mathcal{M}$ defined in Section 1 is a C^1 manifold since we will use Ekeland's Variational Principle in the following proof (f3), we have

$$f(s)s - 2F(s) = \int_0^s (f'(s)s - f(s)) > 0, \quad g(s)s - 2G(s) = \int_0^s (g'(s)s - g(s)) > 0$$

for all $s > 0$. Then for all s, it holds

$$f(s)s - 2F(s) \geq 0, \quad g(s)s - 2G(s) \geq 0. \tag{3.1}$$

Let $J(u,v) := \int T(u,v)$, then from (3.1) and (h3),

$$\begin{aligned}\langle J'(u,v),(u,v)\rangle &= \int (f(u)u + g(v)v + \beta(h(u)K(v)u + H(u)k(v)v) - u^2 - v^2)\\ &\geq \int (2F(u) + 2G(v) + 2\beta H(u)K(v) - u^2 - v^2)\\ &\geq 2J(u,v) = 2 \neq 0,\end{aligned}$$

that is, $J'(u,v) \neq 0$ for every $(u,v) \in \mathcal{M}$. So the conclusion follows.

The following lemmas will be used in the sequel:

Lemma 3.1. *Any minimizing sequence* $\{(u_n, v_n)\}$ *of* (1.3) *is bounded in* E_r.

Proof. If $\{(u_n, v_n)\}$ is a minimizing sequence of (1.3), we have

$$\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \to A$$

and

$$\int(F(u_n) + G(v_n) + \beta H(u_n)K(v_n)) = \int\Big(\frac{|u_n|^2}{2} + \frac{|v_n|^2}{2}\Big) + 1. \tag{3.2}$$

Using the growth assumptions on f, g, h and k, we know that there is $C > 0$ such that

$$\begin{aligned} &F(s) \le \frac{1}{8}s^2 + Cs^{2^*}, \quad G(s) \le \frac{1}{8}s^2 + Cs^{2^*},\\ &H(s)K(t) \le C(s^{2^*} + t^{2^*}) + \frac{1}{8\beta}(s^2 + t^2). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we get

$$C_1\int(|u_n|^{2^*} + |v_n|^{2^*}) \ge \frac{1}{4}\int(|u_n|^2 + |v_n|^2) + 1$$

for some $C_1 > 0$. On the other hand, the definition of S indicates that

$$\int(|u_n|^{2^*} + |v_n|^{2^*}) \le S^{-\frac{2^*}{2}}\Big(\int|\nabla u_n|^2\Big)^{\frac{2^*}{2}} + S^{-\frac{2^*}{2}}\Big(\int|\nabla v_n|^2\Big)^{\frac{2^*}{2}}.$$

Thus, $\int(|u_n|^2 + |v_n|^2)$ is bounded, and so $\{(u_n, v_n)\}$ is bounded in E_r. □

Lemma 3.2. $A > 0$.

Proof. Notice that $A \ge 0$. To obtain a contradiction, assume that $A = 0$ and let $\{(u_n, v_n)\}$ be a minimizing sequence of (1.3) in E_r such that

$$\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \to 0,$$

and

$$\int(F(u_n) + G(v_n) + \beta H(u_n)K(v_n)) = \int\Big(\frac{|u_n|^2}{2} + \frac{|v_n|^2}{2}\Big) + 1.$$

Using a similar argument as in the proof of Lemma 3.1, it is not difficult to verify that

$$C_1\int(|u_n|^{2^*} + |v_n|^{2^*}) \ge \frac{1}{4}\int(|u_n|^2 + |v_n|^2) + 1 \ge 1 \quad \text{for some } C_1 > 0. \tag{3.4}$$

However, since $(u_n, v_n) \to 0$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, it holds

$$\int(|u_n|^{2^*} + |v_n|^{2^*}) \to 0,$$

which contradicts (3.4). Thus, $A > 0$. □

Lemma 3.3. *The following holds:*

$$b \ge \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}}(2A)^{\frac{N}{2}}.$$

Proof. We first claim that $\gamma([0, 1]) \cap \mathcal{P} \ne \emptyset$ for each $\gamma \in \Gamma$. Define

$$\phi(u, v) := \frac{N-2}{2}\int(|\nabla u|^2 + |\nabla v|^2) - N\int T(u, v) = NI(u, v) - \int(|\nabla u|^2 + |\nabla v|^2).$$

Then there is $\rho_0 > 0$ small enough such that $\phi(u, v) > 0$ for all (u, v) satisfying $0 < \|(u, v)\| \le \rho_0$. For any $\gamma \in \Gamma$, it is clear that $\phi(\gamma(0)) = 0$ and $\phi(\gamma(1)) \le NI(\gamma(1)) < 0$. Thus, there is $t_0 \in (0, 1)$ such that $\|\gamma(t_0)\| > \rho_0$ and $\phi(\gamma(t_0)) = 0$, which implies that $\gamma(t_0) \in \mathcal{P}$ and

$$d \le I(\gamma(t_0)) \le \max_{t\in[0,1]} I(\gamma(t)).$$

Hence,

$$d \le I(\gamma(t_0)) \le \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) = b.$$

On the other hand, there is a one-to-one correspondence $\Phi : \mathcal{M} \to \mathcal{P}$ such that

$$\Phi(u,v)(x) := \Big(u\Big(\frac{x}{t_u}\Big), v\Big(\frac{x}{t_u}\Big)\Big), \quad t_u := \sqrt{\frac{N-2}{2N}\Big(\int(|\nabla u|^2 + |\nabla v|^2)\Big)^{\frac{1}{2}}}.$$

Then

$$\begin{aligned} d &= \inf_{\mathcal{P}} I(u,v) = \inf_{\mathcal{M}} I(\Phi(u,v)(x)) \\ &= \inf_{\mathcal{M}} \frac{1}{2} t_u^{N-2} \int (|\nabla u|^2 + |\nabla v|^2) - t_u^N \int T(u,v) \\ &= \inf_{\mathcal{M}} \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} \Big(\int |\nabla u|^2 + |\nabla v|^2\Big)^{\frac{N}{2}} \\ &= \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}. \end{aligned}$$

Consequently,

$$b \ge d = \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}. \qquad \square$$

Applying Ekeland's Variational Principle [26], we obtain that there are sequences $\{(u_n, v_n)\} \subset \mathcal{M} \cap E_r$ and $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \to A,$$

and

$$L'(u_n, v_n) - \lambda_n J'(u_n, v_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N), \tag{3.5}$$

where $L(u,v) := \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2)$.

Lemma 3.4. *Let $\{\lambda_n\}$ be the sequence obtained in* (3.5)*, then* $\limsup_{n\to\infty} \lambda_n \le A$.

Proof. From (3.1), (3.5) and condition (h3), we get

$$\begin{aligned} \int(|\nabla u_n|^2 + |\nabla v_n|^2) &= \lambda_n \int (f(u_n)u_n + g(v_n)v_n + \beta h(u_n)K(v_n)u_n + \beta H(u_n)k(v_n)v_n - u_n^2 - v_n^2) + o(1) \\ &\ge \lambda_n \int (2F(u_n) + 2G(v_n) + 2\beta H(u_n)K(v_n) - u_n^2 - v_n^2) + o(1) \\ &= 2\lambda_n + o(1). \end{aligned}$$

Taking into account $\frac{1}{2}\int |\nabla u_n|^2 + |\nabla v_n|^2 \to A$, we deduce that $\limsup_{n\to\infty} \lambda_n \le A$. $\square$

According to the Concentration Compactness Principle of Lions [16, 17], for the minimizing sequence $\{(u_n, v_n)\}$ of (1.3), there are positive finite measures $\mu^{(1)}$, $\mu^{(2)}$, $\nu^{(1)}$, $\nu^{(2)}$ and sequences $\{\mu_i^{(1)}\}$, $\{\nu_i^{(1)}\}$, $\{\mu_i^{(2)}\}$, $\{\nu_i^{(2)}\}$ such that

$$\begin{aligned} |\nabla u_n|^2 &\rightharpoonup \mu^{(1)} \ge |\nabla u|^2 + \Sigma\delta_{x_i}\mu_i^{(1)}, & |\nabla v_n|^2 &\rightharpoonup \mu^{(2)} \ge |\nabla v|^2 + \Sigma\delta_{x_i}\mu_i^{(2)}, \\ |u_n|^{2^*} &\rightharpoonup \nu^{(1)} = |u|^{2^*} + \Sigma\delta_{x_i}\nu_i^{(1)}, & |v_n|^{2^*} &\rightharpoonup \nu^{(2)} = |v|^{2^*} + \Sigma\delta_{x_i}\nu_i^{(2)}, \\ \mu_i^{(1)} &\ge S(\nu_i^{(1)})^{\frac{2}{2^*}}, & \mu_i^{(2)} &\ge S(\nu_i^{(2)})^{\frac{2}{2^*}}. \end{aligned}$$

Let $\mu_i := \mu_i^{(1)} + \mu_i^{(2)}$, $\nu_i := \nu_i^{(1)} + \nu_i^{(2)}$, $\mu := \mu^{(1)} + \mu^{(2)}$, $\nu := \nu^{(1)} + \nu^{(2)}$. Then

$$|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup \mu \ge |\nabla u|^2 + |\nabla v|^2 + \Sigma\delta_{x_i}\mu_i, \quad |u_n|^{2^*} + |v_n|^{2^*} \rightharpoonup \nu = |u|^{2^*} + |v|^{2^*} + \Sigma\delta_{x_i}\nu_i. \tag{3.6}$$

Moreover, it is easy to verify that

$$\mu_i \ge S\nu_i^{\frac{2}{2^*}}. \tag{3.7}$$

Indeed, let ψ be a smooth function with compact support satisfying $0 \le \psi(x) \le 1$, $\psi(x) = 1$ in $B_1(0)$ and $\psi(x) = 0$ in $\mathbb{R}^N \setminus B_2(0)$. Set $\psi_\epsilon(x) := \psi(\frac{x-x_i}{\epsilon})$ for $\epsilon > 0$. Then

$$\int |\nabla \psi_\epsilon u_n|^2 + \int |\nabla \psi_\epsilon v_n|^2 \ge S\Big(\int |\psi_\epsilon u_n|^{2^*}\Big)^{\frac{2}{2^*}} + S\Big(\int |\psi_\epsilon v_n|^{2^*}\Big)^{\frac{2}{2^*}} \ge S\Big(\int |\psi_\epsilon u_n|^{2^*} + \int |\psi_\epsilon v_n|^{2^*}\Big)^{\frac{2}{2^*}}.$$

By letting $\epsilon \to 0$, we obtain (3.7).

Lemma 3.5. *If $\nu_i > 0$ for some index i, then $A \ge 2^{-\frac{2}{N}}(1+\beta)^{-\frac{N-2}{N}} S$.*

Proof. Let $\{(u_n, v_n)\} \subset E_r$ be a minimizing sequence of (1.3) to A. Then from (3.5) we infer that

$$\begin{aligned}\int (\nabla u_n \nabla(u_n\psi_\epsilon) + \nabla v_n \nabla(v_n \psi_\epsilon)) &= \lambda_n \int (f(u_n)u_n + g(v_n)v_n)\psi_\epsilon \\ &\quad + \lambda_n \int (\beta h(u_n)K(v_n)u_n + \beta H(u_n)k(v_n)v_n - u_n^2 - v_n^2)\psi_\epsilon + o(1). \end{aligned} \tag{3.8}$$

The growth assumptions on f, g, h and k imply that, for any $\eta > 1$, there is $C > 0$ such that

$$\begin{aligned} &sf(s) \le \eta s^{2^*} + \frac{s^2}{4} + C|s|^{q_1}, \quad sg(s) \le \eta s^{2^*} + \frac{s^2}{4} + C|s|^{q_2}, \\ &h(s)K(t)s + H(s)k(t)t \le \eta(s^{2^*} + t^{2^*}) + C(|s|^{q_3} + |t|^{q_4}) + \frac{1}{4\beta}(s^2 + t^2) \end{aligned} \tag{3.9}$$

for some $2 < q_1, q_2, q_3, q_4 < 2^*$. From (3.8) and (3.9), we get

$$\begin{aligned} &\int (|\nabla u_n|^2 + |\nabla v_n|^2)\psi_\epsilon + \int (u_n \nabla u_n + v_n \nabla v_n)\nabla \psi_\epsilon \\ &\quad \le \eta(1+\beta)\lambda_n \int (|u_n|^{2^*} + |v_n|^{2^*})\psi_\epsilon + C\lambda_n \int (|u_n|^{q_1} + |v_n|^{q_2} + \beta|u_n|^{q_3} + \beta|v_n|^{q_4})\psi_\epsilon \\ &\qquad + \frac{\lambda_n}{2}\int (|u_n|^2 + |v_n|^2)\psi_\epsilon. \end{aligned}$$

Letting $\epsilon \to 0$ and using Lemma 3.4, we have $\mu_i \le A\eta(1+\beta)\nu_i$ for all $\eta > 1$. So that $\mu_i \le A(1+\beta)\nu_i$, which together with (3.7) gives

$$S\nu_i^{\frac{2}{2^*}} \le \mu_i \le A(1+\beta)\nu_i.$$

Then it follows

$$\nu_i \ge \Big(\frac{S}{A(1+\beta)}\Big)^{\frac{N}{2}}. \tag{3.10}$$

On the other hand, since

$$\begin{aligned}\int (|u_n|^{2^*} + |v_n|^{2^*}) &\le S^{-\frac{2^*}{2}}\Big(\int |\nabla u_n|^2\Big)^{\frac{2^*}{2}} + S^{-\frac{2^*}{2}}\Big(\int |\nabla v_n|^2\Big)^{\frac{2^*}{2}} \\ &\le S^{-\frac{2^*}{2}}\Big(\int (|\nabla u_n|^2 + |\nabla v_n|^2)\Big)^{\frac{2^*}{2}},\end{aligned}$$

we have

$$\int (|u_n|^{2^*} + |v_n|^{2^*})\psi_\epsilon \le S^{-\frac{2^*}{2}}\Big(\int (|\nabla u_n|^2 + |\nabla v_n|^2)\Big)^{\frac{2^*}{2}}.$$

Letting $n \to \infty$ and $\epsilon \to 0$ gives

$$\nu_i \le S^{-\frac{2^*}{2}}(2A)^{\frac{2^*}{2}}. \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$A \ge 2^{-\frac{2}{N}}(1+\beta)^{-\frac{N-2}{N}} S. \qquad \square$$

Lemma 3.6. *If*

$$\lambda > \Big(N\Big(\frac{N}{N-2}\Big)^{\frac{N-2}{2}}(1+\beta)^{\frac{N-2}{2}} S^{-\frac{N}{2}}\Big)^{\frac{r-2}{2}}\Big(\frac{r-2}{2r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}}, \tag{3.12}$$

then

$$b < \frac{1}{N}\Big(\frac{N-2}{N}\Big)^{\frac{N-2}{2}}(1+\beta)^{-\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{3.13}$$

Proof. If we take $h_0 \in H_r^1(\mathbb{R}^N)$ such that

$$|h_0|_r^2 = C_r^{-1}, \quad \|h_0\| = 1.$$

Then

$$b \le \max_{t\ge 0} I(th_0, 0) \le \max_{t\ge 0}\Big(\frac{t^2}{2}\int(|\nabla h_0|^2 + |h_0|^2) - \frac{\lambda t^r}{r}\int |h_0|^r\Big) = \frac{r-2}{2r}\lambda^{-\frac{2}{r-2}} C_r^{\frac{r}{r-2}}. \tag{3.14}$$

Substituting (3.12) into (3.14), we get (3.13). □

Lemma 3.7. *Let* (u, v) *be the weak limit of the minimizing sequence of* (1.3)*, then* $(u, v) \ne (0, 0)$.

Proof. Assume that $(u, v) = (0, 0)$. Notice that $\{(u_n, v_n)\} \subset E_r$. By a lemma due to Strauss [6, Radial Lemma A.II], (u_n, v_n) is bounded in $L^\infty(|x| \ge R)$ for any $R > 0$, which indicates that (u_n, v_n) converge strongly to (u, v) in $L^q(|x| > R)$ for all $q > 2$. In particular,

$$(u_n, v_n) \to (0, 0) \quad \text{in } L^{2^*}(|x| > R) \times L^{2^*}(|x| > R) \text{ for all } R > 0. \tag{3.15}$$

Now we show that

$$(u_n, v_n) \to (0, 0) \quad \text{in } L^{2^*}_{\text{loc}}(\mathbb{R}^N) \times L^{2^*}_{\text{loc}}(\mathbb{R}^N). \tag{3.16}$$

Otherwise, there is positive finite measure ν_0 such that

$$|u_n|^{2^*} + |v_n|^{2^*} \rightharpoonup \nu_0.$$

Then by Lemma 3.5 and Lemma 3.3, we get

$$b > \frac{1}{N}\Big(\frac{N-2}{N}\Big)^{\frac{N-2}{2}}(1+\beta)^{-\frac{N-2}{2}} S^{\frac{N}{2}},$$

which contradicts Lemma 3.6. Thus, $\nu_0 = 0$ and (3.16) holds. (3.15) and (3.16) tell us

$$(u_n, v_n) \to (0, 0) \quad \text{in } L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N),$$

which may lead to a contradiction by repeating the same arguments used in the proof of Lemma 3.2 (see inequality (3.4)). Thus, $(u, v) \ne (0, 0)$. □

Lemma 3.8. *A is attained at some* $(u, v) \ne (0, 0)$*, with* $u \ge 0$*,* $v \ge 0$.

Proof. Let $\{(u_n, v_n)\} \subset E_r \cap \mathcal{M}$ and $\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|)^2 \to A$. Applying Lemma 3.1, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$, then it follows

$$L(u, v) = \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2) \le \liminf_{n\to\infty}\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \to A. \tag{3.17}$$

Since $\{(u_n, v_n)\} \subset E_r$, by [6, Radial Lemma A.II], we know that $\{(u_n, v_n)\}$ is bounded in $L^\infty(|x| > R)$, and then

$$F(u_n) \to F(u), \quad G(v_n) \to G(v), \quad H(u_n)K(v_n) \to H(u)K(v) \quad \text{in } L^1(|x| \ge R) \times L^1(|x| \ge R).$$

On the other hand, from Lemma 3.3, Lemma 3.5 and Lemma 3.6, and an argument analogous to that used in the proof of Lemma 3.7, we obtain that $\nu_i = 0$ for every i, where ν_i is defined in (3.6). Then it follows

$$(u_n, v_n) \to (u, v) \quad \text{in } L^{2^*}_{\text{loc}}(\mathbb{R}^N) \times L^{2^*}_{\text{loc}}(\mathbb{R}^N).$$

So we deduce that

$$F(u_n) \to F(u), \quad G(v_n) \to G(v), \quad H(u_n)K(v_n) \to H(u)K(v) \quad \text{in } L^1(B_R(0)) \times L^1(B_R(0)).$$

Thus,

$$F(u_n) \to F(u), \quad G(v_n) \to G(v), \quad H(u_n)K(v_n) \to H(u)K(v) \quad \text{in } L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N). \tag{3.18}$$

Notice that

$$\int (F(u_n) + G(v_n) + \beta H(u_n)K(v_n)) = \int \Big(\frac{|u_n|^2}{2} + \frac{|v_n|^2}{2}\Big) + 1,$$

which together with (3.17) and (3.18) gives

$$\int (F(u) + G(v) + \beta H(u)K(v)) \geq \int \Big(\frac{|u|^2}{2} + \frac{|v|^2}{2}\Big) + 1,$$

that is, $\int T(u, v) \geq 1$. If $(u, v) \notin \mathcal{M}$, then $\int T(u, v) > 1$. Define a function $l : [0, 1] \to \mathbb{R}$ by $l(t) = \int T(tu, tv)$. It is clear that $l(t) < 0$ for $t > 0$ small enough and $l(1) = \int T(u, v) > 1$. Hence, there is $t_0 \in (0, 1)$ such that $l(t_0) = 0$, which indicates $t_0(u, v) \in \mathcal{M}$, and

$$\frac{t_0^2}{2} \int (|\nabla u|^2 + |\nabla v|^2) \geq A. \tag{3.19}$$

However, from (3.17) and $t_0 \in (0, 1)$, we know that

$$\frac{t_0^2}{2} \int (|\nabla u|^2 + |\nabla v|^2) < \frac{1}{2} \int (|\nabla u|^2 + |\nabla v|^2) \leq A,$$

which contradicts (3.19). Therefore, $(u, v) \in \mathcal{M}$ and $\frac{1}{2} \int (|\nabla u|^2 + |\nabla v|^2) = A > 0$.

It is obvious that $(u, v) \neq (0, 0)$. Now it remains to prove that $u \geq 0$ and $v \geq 0$. Indeed, since

$$\int \big(|\nabla |u||^2 + |\nabla |v||^2\big) \leq \int (|\nabla u|^2 + |\nabla v|^2) = 2A, \qquad \int T(|u|, |v|) = \int T(u, v) = 1,$$

we may assume that $u \geq 0$ and $v \geq 0$. □

Lemma 3.9. *The following holds:*

$$m = b = \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}.$$

Proof. For every $(u, v) \in \mathcal{N}$, it holds

$$b \leq \max_{t \geq 0} I(tu, tv) = I(u, v).$$

Thus,

$$b \leq \inf_{(u,v)\in\mathcal{N}} \max_{t\geq 0} I(tu, tv) = \inf_{(u,v)\in\mathcal{N}} I(u, v) = m.$$

Then according to Lemma 3.3, we get

$$m \geq b \geq \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}.$$

The proof will be accomplished once we show that

$$m \leq \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}.$$

Let (u, v) be such that

$$\int (|\nabla u|^2 + |\nabla v|^2) = A, \qquad \int T(u, v) = 1.$$

It follows from the Lagrange multiplier rule that there is $\theta > 0$ such that (u, v) is a nontrivial solution of

$$\begin{cases} -\Delta u = \theta(f(u) + \beta h(u)K(v) - u), \\ -\Delta v = \theta(g(v) + \beta h(u)K(v) - v). \end{cases}$$

Define $(u_\theta, v_\theta) := (u(\frac{x}{\sqrt{\theta}}), v(\frac{x}{\sqrt{\theta}}))$, then (u_θ, v_θ) solves (1.2) and

$$\int (|\nabla u_\theta|^2 + |\nabla v_\theta|^2) = 2\theta^{\frac{N-2}{2}} A, \qquad \int T(u_\theta, v_\theta) = \theta^{\frac{N}{2}}.$$

Moreover, the Pohozaev identity gives

$$\theta = \frac{N-2}{N} A.$$

Thus,

$$m \le I(u_\theta, v_\theta) = \theta^{\frac{N-2}{2}} A - \theta^{\frac{N}{2}} = \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} (2A)^{\frac{N}{2}}.$$

The conclusion follows. □

Lemma 3.10. *If* $1 < p < 2$, *then* $m < \min\{B_1, B_2\}$ *for all* $\beta > 0$. *Furthermore, if* $p = 2$, *then there is* $\beta^* > 0$ *such that* $m < \min\{B_1, B_2\}$ *if* $\beta > \beta^*$.

Proof. Let u be a positive ground state of (1.4) satisfying $I_1(u) = B_1$. Then there is $t(s) > 0$ such that $I(tu, tsu) = \max_{t\ge 0} I(tu, tsu)$, that is,

$$t(1+s^2)\int(|\nabla u|^2 + |u|^2) = \int(f(tu)u + g(tsu)su + \beta h(tu)K(tsu)u + \beta H(tu)k(tsu)su).$$

Denote

$$\begin{aligned} L(t,s) := {}& t(1+s^2)\int(|\nabla u|^2 + |u|^2) \\ & - \int(f(tu)u + g(tsu)su + \beta h(tu)K(tsu)u + \beta H(tu)k(tsu)su). \end{aligned}$$

Calculating directly, we have

$$L_t'(1,0) = \int(f(u)u - f'(u)u^2) > 0.$$

Applying the Implicit Function Existence Theorem, we deduce that there is a C^1 function $t(s) : (0,\epsilon) \to \mathbb{R}$ such that $t(0) = 1$ and $\lim_{s\to 0^+} t(s) = 1$. Moreover, if $1 < p < 2$, we deduce that

$$\lim_{s\to 0^+} \frac{t'(s)}{s^{p-1}} = -\frac{\beta}{\int(f'(u)u^2 - f(u)u)} L_1 := -\beta L_0,$$

where

$$L_1 := \lim_{s\to 0^+} \frac{\int(h(tu)k(tsu)u^2 + H(tu)k'(tsu)tsu^2 + H(tu)k(tsu)u)}{s^{p-1}}.$$

According to (h1) and (h2), we know that L_0 is a positive constant. Then

$$t'(s) = -\beta L_0 s^{p-1}(1 + o(1)), \quad \text{as } s \to 0^+,$$

$$t(s) = 1 - \frac{\beta L_0}{p} s^p (1 + o(1)), \quad \text{as } s \to 0^+.$$

Therefore, by the Mean Value Theorem, we have for $s > 0$ small enough,

$$\begin{aligned} m &\le I(tu, tsu) \\ &= \frac{1}{2}\int(f(tu)tu - 2F(tu) + g(tsu)tsu - 2G(tsu)) \\ &\qquad + \frac{1}{2}\beta\int(h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu)) \\ &= \frac{1}{2}\int\Big(f(u)u - 2F(u) - (f'(u)u^2 - f(u)u)\frac{\beta L_0}{p}s^p\Big) + o(s^p) + o(s^2) \\ &\qquad + \frac{1}{2}\beta\int(h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu)) \\ &= \frac{1}{2}\int(f(u)u - 2F(u)) + o(s^p) + o(s^2) \\ &\qquad - \frac{1}{2}\beta\Big(\frac{L_1}{p}s^p - \int(h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu))\Big). \end{aligned}$$

Let

$$L_2 := \lim_{s\to 0^+} \frac{\int (h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu))}{s^p}. \tag{3.20}$$

From (h2) and the continuity of $k(s)$, we have

$$\begin{aligned} L_2 &= \lim_{s\to 0^+} \frac{\int (h(u)K(su)u + H(u)k(su)su - 2H(u)K(su))}{s^p} \\ &= \lim_{s\to 0^+} \frac{\int (h(u)k(su)u^2 + H(u)k'(su)su^2 - 2H(u)k(su)u)}{ps^{p-1}} \\ &= \frac{L_1}{p} - \lim_{s\to 0^+} \frac{3\int H(u)k(su)u}{ps^{p-1}}. \end{aligned} \tag{3.21}$$

Denote

$$\beta_0 := \lim_{s\to 0^+} \frac{3\int H(u)k(su)u}{ps^{p-1}} > 0,$$

then

$$\int (h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu)) = \frac{L_1}{p} s^p - \beta_0 s^p + o(s^p)$$

for $s > 0$ small enough. Thus,

$$\begin{aligned} m &\le \frac{1}{2} \int (f(u)u - 2F(u)) + o(s^p) - \beta_0 s^p \\ &< \frac{1}{2} \int (f(u)u - 2F(u)) \\ &= I(u, 0) = B_1 \end{aligned}$$

for $s > 0$ small enough. Analogously, if $p = 2$,

$$\lim_{s\to 0^+} \frac{t'(s)}{s^{p-1}} = -\beta L_0 + \frac{2\int (|\nabla u|^2 + |u|^2)}{\int (f'(u)u^2 - f(u)u)} := -L_3$$

and

$$\begin{aligned} t'(s) &= -L_3 s^{p-1}(1 + o(1)), \quad \text{as } s \to 0^+, \\ t(s) &= 1 - \frac{L_3}{p} s^p (1 + o(1)), \quad \text{as } s \to 0^+. \end{aligned}$$

Then for $s > 0$ small enough,

$$\begin{aligned} m &\le I(tu, tsu) \\ &= \frac{1}{2} \int (f(u)u - 2F(u)) + o(s^p) + \frac{2s^p}{p} \int (|\nabla u|^2 + |u|^2) \\ &\quad - \frac{1}{2}\beta\Big(\frac{L_1}{p} s^p - \int (h(tu)K(tsu)tu + H(tu)k(tsu)tsu - 2H(tu)K(tsu))\Big). \end{aligned}$$

Denote

$$\beta_1^* := \frac{2\int |\nabla u|^2 + |u|^2}{p\beta_0}.$$

The previous inequality together with (3.20) and (3.21) implies that if $\beta > \beta_1^*$, then

$$m \le \frac{1}{2} \int (f(u)u - 2F(u)) - Cs^p + o(s^p) < I(u, 0) = B_1$$

for $s > 0$ small enough. Similarly, we can prove that $m < B_2$ for all $\beta > 0$ if $1 < p < 2$, and $m < B_2$ for some $\beta_2^* > 0$ and $\beta > \beta_2^*$ if $p = 2$. We finish the proof by letting $\beta^* := \max\{\beta_1^*, \beta_2^*\}$. □

Proof of Theorem 1.1. It is a direct result of Lemma 3.8, Lemma 3.9 and Lemma 3.10. □

4 The Case of Dimension $N = 2$

In this case, the Pohozaev identity shows that any solution of (1.2) should satisfy $\int T(u,v) = 0$. Thus,

$$A = \inf\Big\{\frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2) : \int T(u,v) = 0,\ (u,v) \neq (0,0)\Big\} = \inf_{\mathcal{P}} \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2).$$

Define the following minimax value which will be used in the sequel:

$$c := \inf_{(u,v)\in E\setminus\{(0,0)\}} \max_{t\geq 0} I(tu, tv).$$

Now we give a compactness lemma.

Lemma 4.1. *Assume that* (f1), (f2'), (h1'), (h2) *hold. Let* $\{(u_n, v_n)\}$ *be a sequence in* E_r *such that*

$$\sup_n \int(|\nabla u_n^2 + |\nabla v_n|^2) = d_0 < 1 \quad \text{and} \quad \sup_n \int(|u_n|^2 + |v_n|^2) = a_0 < \infty.$$

Then

$$\int F(u_n) \to \int F(u), \quad \int G(v_n) \to \int G(v), \quad \int H(u_n)K(v_n) \to \int H(u)K(v)$$

when $(u_n, v_n) \rightharpoonup (u, v)$ *in* E_r.

Proof. We can assume that there is $(u,v) \in E_r$ such that

$$(u_n, v_n) \rightharpoonup (u,v) \text{ in } E, \quad (u_n, v_n) \to (u,v) \text{ on } \mathbb{R}^2 \times \mathbb{R}^2, \quad \lim_{|x|\to\infty}(u_n(x) + v_n(x)) = 0.$$

Using a Trudinger–Moser type inequality [7, Lemma 2.1], we know that for $d_1 \in (0,1)$ and $d_2 > 0$, there is $C(d_1, d_2)$ such that

$$\sup_{u\in Q_0} \int(e^{4\pi u^2} - 1) \leq C(d_1, d_2),$$

where $Q_0 := \{u \in H^1(\mathbb{R}^2) : \int|\nabla u|^2 \leq d_1 \text{ and } \int |u|^2 \leq d_2\}$. Choose ϵ small enough such that $d_1 := \frac{d_0}{(1-\epsilon)^2} \in (0,1)$ and set $\alpha := \frac{4\pi}{(1-\epsilon)^2} > 4\pi$, then we have

$$\int(e^{\alpha u_n^2} - 1) = \int\Big(e^{4\pi\big(\frac{u_n^2}{(1-\epsilon)^2}\big)} - 1\Big) \leq C(d_1, d_2). \tag{4.1}$$

Let $Q(s) := e^{\alpha s^2} - 1$, $R_1(s) := |s|(e^{\alpha_1 s^2} - 1) + |s|^{r_1}$, and $R_2(s) := |s|(e^{\alpha_1 s^2} - 1) + |s|^{r_2}$, where $r_1, r_2 > 2$. From (f1), (f2'), (h1'), (h2) and (4.1), we deduce that

$$\begin{aligned}
&\lim_{s\to 0}\frac{F(s)}{Q(s)} = \lim_{s\to+\infty}\frac{F(s)}{Q(s)} = \lim_{s\to 0}\frac{G(s)}{Q(s)} = \lim_{s\to+\infty}\frac{G(s)}{Q(s)} = 0,\\
&\lim_{s\to 0}\frac{R_1(s)}{Q(s)} = \lim_{s\to+\infty}\frac{R_1(s)}{Q(s)} = \lim_{s\to 0}\frac{R_2(s)}{Q(s)} = \lim_{s\to+\infty}\frac{R_2(s)}{Q(s)} = 0, \quad \text{for } \alpha_1 < \alpha,\\
&\sup_n \int |Q(u_n)| < +\infty, \quad \sup_n \int |Q(v_n)| < +\infty,
\end{aligned}$$

and as $n \to \infty$,

$$F(u_n) \to F(u), \quad G(u_n) \to G(u), \quad R_1(u_n) \to R_1(u), \quad R_2(v_n) \to R_2(v) \quad \text{a.e. on } \mathbb{R}^2.$$

Using the Compactness Lemma of Strauss [6, Theorem A.I], we get

$$\int F(u_n) \to \int F(u), \quad \int G(v_n) \to \int G(v), \quad \int R_1(u_n) \to \int R_1(u), \quad \int R_2(v_n) \to \int R_2(v).$$

Choosing α_1 such that $4\pi < \alpha_1 < \alpha$ and taking (h1') and (h2) into account, we have that there are $C_1, C_2 > 0$, $r_1, r_2 > 2$ such that

$$\begin{aligned}
H(u_n)K(v_n) &\leq C_1\big(|u_n|(e^{\alpha_1 u_n^2} - 1) + |u_n|^{r_1}\big) + C_2\big(|v_n|(e^{\alpha_1 v_n^2} - 1) + |v_n|^{r_2}\big)\\
&= C_1 R(u_n) + C_2 R(v_n).
\end{aligned}$$

Thus, we obtain $\int H(u_n)K(v_n) \to \int H(u)K(v)$. □

Lemma 4.2. *$A \leq c$.*

Proof. For each $(u, v) \in E \setminus \{(0,0)\}$, we set

$$l(t) := \int T(tu, tv) = \int \Big(F(tu) + G(tv) + \beta H(tu)K(tv) - \frac{t^2u^2}{2} - \frac{t^2u^2}{2}\Big).$$

It is clear that $l(t) < 0$ for $t > 0$ small and $l(t) > 0$ for $t > 0$ large enough. Thus, there is $t_0 > 0$ such that $l(t_0) = 0$, which indicates $t_0(u, v) \in \mathcal{M}$. Then

$$A \leq I(t_0u, t_0v) \leq \max_{t\geq 0} I(tu, tv),$$

which implies $A \leq c$. □

Lemma 4.3. *$A > 0$.*

Proof. To obtain a contradiction, suppose that $A = 0$. Let $\{(u_n, v_n)\}$ be such that

$$\frac{1}{2}\int (|\nabla u_n|^2 + |\nabla v_n|^2) \to A = 0, \qquad \int T(u_n, v_n) = 0.$$

For each $\theta_n > 0$, let $(\xi_n(x), \zeta_n(x)) = (u_n(\frac{x}{\theta_n}), v_n(\frac{x}{\theta_n}))$, then

$$\int (|\nabla \xi_n|^2 + |\nabla \zeta_n|^2) = \int (|\nabla u_n|^2 + |\nabla v_n|^2), \qquad \int T(\xi_n, \zeta_n) = 0,$$

and

$$\int (|\xi_n|^2 + |\zeta_n|^2) = \theta_n^2 \int (|u_n|^2 + |v_n|^2).$$

Choose

$$\theta_n^2 = \frac{1}{\int (|u_n|^2 + |v_n|^2)},$$

then we have

$$\frac{1}{2}\int (|\nabla \xi_n|^2 + |\nabla \zeta_n|^2) \to A = 0, \qquad \int (|\xi_n|^2 + |\zeta_n|^2) = 1, \qquad \int T(\xi_n, \zeta_n) = 0.$$

According to Lemma 4.1, we get

$$\int F(\xi_n) \to \int F(\xi), \qquad \int G(\zeta_n) \to \int G(\zeta), \qquad \int H(\xi_n)K(\xi_n) \to \int H(\xi)K(\zeta).$$

Note that $\int T(\xi_n, \zeta_n) = 0$, which implies

$$\int (F(\xi_n) + G(\zeta_n) + \beta H(\xi_n)K(\zeta_n)) = \frac{1}{2}.$$

Thus,

$$\int (F(\xi) + G(\zeta) + \beta H(\xi)K(\zeta)) = \frac{1}{2},$$

and $(\xi, \zeta) \neq (0, 0)$. However,

$$\int (|\nabla \xi|^2 + |\nabla \zeta|^2) \leq \liminf_{n\to\infty} \int (|\nabla \xi_n|^2 + |\nabla \zeta_n|^2) = A = 0,$$

which is a contradiction. Thus, $A > 0$. □

In much the same way as in the proof of Lemma 3.6 and Lemma 3.9, we deduce the following two lemmas:

Lemma 4.4. *If*

$$\lambda > \Big(\frac{r-2}{r}\Big)^{\frac{r-2}{2}} C_r^{\frac{r}{2}},$$

then $c < \frac{1}{2}$.

Lemma 4.5. $m = b = A$.

Proof of Theorem 1.3. Let $\{(u_n, v_n)\} \subset E_r$ be a minimizing sequence such that

$$\frac{1}{2}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \to A, \qquad \int T(u_n, v_n) \to 0. \tag{4.2}$$

Arguing as in the proof of Lemma 4.3, we may assume that

$$\int(|u_n|^2 + |v_n|^2) = 1. \tag{4.3}$$

From (4.2), Lemma 4.2 and Lemma 4.4, we have

$$\limsup_{n\to\infty}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \le 2A \le 2c < 1.$$

Then using Lemma 4.1, we get

$$\int F(u_n) \to \int F(u), \qquad \int G(v_n) \to \int G(v), \qquad \int H(u_n)K(v_n) \to \int H(u)K(v), \tag{4.4}$$

where (u, v) is the weak limit of (u_n, v_n). Comparing (4.4) together with (4.2) and (4.3) indicates

$$\int(F(u) + G(v) + \beta H(u)K(v)) = \frac{1}{2}.$$

Thus, $(u, v) \neq (0, 0)$. Moreover,

$$\int(|\nabla u|^2 + |\nabla v|^2) \le \liminf_{n\to\infty}\int(|\nabla u_n|^2 + |\nabla v_n|^2) \le A.$$

It remains to show that $\int T(u, v) = 0$. Since

$$\int(|u|^2 + |v|^2) \le \liminf_{n\to\infty}(|u_n|^2 + |v_n|^2) = 1,$$

we get

$$\int T(u, v) \ge 0.$$

If $\int T(u, v) > 0$, let us consider the function $l(t)$ defined in Lemma 4.2. Observe that $l(t) < 0$ for $t > 0$ small enough, and $l(1) = \int T(u, v) > 0$. Then there is $t_0 \in (0, 1)$ such that $l(t_0) = 0$, which is equivalent to

$$\int T(t_0 u, t_0 v) = 0.$$

Hence, $(t_0 u, t_0 v) \in \mathcal{M}$ and

$$\frac{1}{2}\int(|\nabla(t_0 u)|^2 + |\nabla(t_0 v)|^2) \ge A. \tag{4.5}$$

However, since $t_0 \in (0, 1)$, it is clear that

$$\frac{1}{2}\int(|\nabla(t_0 u)|^2 + |\nabla(t_0 v)|^2) < \frac{1}{2}\int(|\nabla u|^2 + |\nabla v|^2) \le A,$$

which conflicts with (4.5). So $\int T(u, v) = 0$, and A is attained. Furthermore, since

$$\int(|\nabla|u||^2 + |\nabla|v||^2) \le \int(|\nabla u|^2 + |\nabla v|^2) = 2A, \qquad \int T(|u|, |v|) = \int T(u, v) = 0,$$

we may assume that $u \ge 0$ and $v \ge 0$. Using Lemma 4.5, we know that (u, v) is a ground state solution of (1.2). Arguing analogously as in the proof of Lemma 3.10, we get that if $1 < p < 2$, then $u > 0$, $v > 0$ for all $\beta > 0$, and if $p = 2$, then there exists $\beta^{**} > 0$ such that $u > 0$, $v > 0$ for $\beta > \beta^{**}$. □

Funding: Supported partially by NNSF of China (11371091, 10831005).

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