Research Article

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Boundary Singularities of Solutions to Semilinear Fractional Equations

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Abstract: We prove the existence of a solution of $(-\Delta)^s u + f(u) = 0$ in a smooth bounded domain Ω with a prescribed boundary value μ in the class of Radon measures for a large class of continuous functions f satisfying a weak singularity condition expressed under an integral form. We study the existence of a boundary trace for positive moderate solutions. In the particular case where $f(u) = u^p$ and μ is a Dirac mass, we show the existence of several critical exponents p. We also demonstrate the existence of several types of separable solutions of the equation $(-\Delta)^s u + u^p = 0$ in $\mathbb{R}^N_+$.

Keywords: s-Harmonic Functions, Semilinear Fractional Equations, Boundary Trace

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and $s \in (0,1)$. Define the s-fractional Laplacian as

$$(-\Delta)^s u(x) := \lim_{\varepsilon\to 0} (-\Delta)^s_\varepsilon u(x),$$

where

$$(-\Delta)^s_\varepsilon u(x) := a_{N,s} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad a_{N,s} := \frac{\Gamma(N/2+s)}{\pi^{N/2}\Gamma(2-s)} s(1-s).$$

We denote by G^Ω_s and M^Ω_s the Green kernel and the Martin kernel, respectively, of $(-\Delta)^s$ in Ω. Denote by $\mathbb{G}^\Omega_s$ and $\mathbb{M}^\Omega_s$ the Green operator and the Martin operator, respectively (see Section 2 for more details). Further, for $\phi \geq 0$, denote by $\mathfrak{M}(\Omega,\phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi\, d|\tau| < \infty$, and by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$. Let $\rho(x)$ be the distance from x to $\partial\Omega$. For $\beta > 0$, set

$$\Omega_\beta := \{x \in \Omega : \rho(x) < \beta\}, \quad D_\beta := \{x \in \Omega : \rho(x) > \beta\}, \quad \Sigma_\beta := \{x \in \Omega : \rho(x) = \beta\}.$$

Definition 1.1. We say that a function $u \in L^1_{\mathrm{loc}}(\Omega)$ possesses an s-boundary trace on $\partial\Omega$ if there exists a measure $\mu \in \mathfrak{M}(\partial\Omega)$ such that

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} |u - \mathbb{M}^\Omega_s[\mu]|\, dS = 0.$$

The s-boundary trace of u is denoted by $\operatorname{tr}_s(u)$.

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Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$, $\mu \in \mathfrak{M}(\partial\Omega)$ and let $f \in C(\mathbb{R})$ be a nondecreasing function with $f(0) = 0$. In this paper, we study the boundary singularity problem for semilinear fractional equations of the form

$$\begin{cases} (-\Delta)^s u + f(u) = \tau & \text{in } \Omega, \\ \operatorname{tr}_s(u) = \mu, & \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.1}$$

We denote by $\mathbb{X}_s(\Omega) \subset C(\mathbb{R}^N)$ the space of test functions ξ satisfying
(i) $\operatorname{supp}(\xi) \subset \bar{\Omega}$,
(ii) $(-\Delta)^s \xi(x)$ exists for all $x \in \Omega$, and $|(-\Delta)^s \xi(x)| \le C$ for some $C > 0$,
(iii) there exist $\varphi \in L^1(\Omega, \rho^s)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)^s_\varepsilon \xi| \le \varphi$ a.e. in Ω for all $\varepsilon \in (0, \varepsilon_0]$.

Definition 1.2. Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is called a weak solution of (1.1) if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \rho^s)$ and

$$\int_\Omega (u(-\Delta)^s \xi + f(u)\xi)\, dx = \int_\Omega \xi \, d\tau + \int_\Omega \mathbb{M}_s^\Omega[\mu](-\Delta)^s \xi \, dx \quad \text{for all } \xi \in \mathbb{X}_s(\Omega). \tag{1.2}$$

The boundary value problem with measure data for semilinear elliptic equations

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

was first studied by Gmira and Véron in [18], and then the typical model, i.e. problem (1.3) with $f(u) = u^p$ ($p > 1$), has been intensively investigated by numerous authors (see [22–26] and references therein). They proved that if f is a continuous, nondecreasing function satisfying

$$\int_1^\infty [f(t) - f(-t)] s^{-1-p_c} \, dt < \infty,$$

where $p_c := \frac{N+1}{N-1}$, then problem (1.3) admits a unique weak solution. In particular, when $f(u) = u^p$ with $1 < p < p_c$ and $\mu = k\delta_0$ with $0 \in \partial\Omega$ and $k > 0$, there exists a unique solution u_k of (1.3). It was shown [22, 26] that the sequence $\{u_k\}$ is increasing and converges to a function u_∞ which is a solution of the equation in (1.3).

To our knowledge, few papers concerning boundary singularity problems for nonlinear fractional elliptic equations have been published in the literature. The earliest works in this direction are the papers [10, 17] by Felmer et al., which deal with the existence, nonexistence and asymptotic behavior of large solutions for equations involving fractional Laplacians. Afterwards, Abatangelo [1] presented a suitable setting for the study of fractional Laplacian equations in a measure framework and provided a fairly comprehensive description of large solutions which improve the results in [10, 17]. Recently, Chen, Alhomedan, Hajaiej and Markowich [9] investigated semilinear elliptic equations involving measures concentrated on the boundary by employing an approximate method.

In the present paper, we aim to establish the existence and uniqueness of weak solutions of (1.1). To this end, we develop a theory for linear equations associated to (1.1):

$$\begin{cases} (-\Delta)^s u = \tau & \text{in } \Omega, \\ \operatorname{tr}_s(u) = \mu, & \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.4}$$

An existence and uniqueness result for (1.4) is stated in the following proposition.

Proposition 1.3. *Assume $s \in (\frac{1}{2}, 1)$. Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. Then problem* (1.4) *admits a unique weak solution. The solution is given by*

$$u = \mathbb{G}_s^\Omega[\tau] + \mathbb{M}_s^\Omega[\mu]. \tag{1.5}$$

Moreover, there exists a positive constant $c = c(N, s, \Omega)$ such that

$$\|u\|_{L^1(\Omega)} \leq c(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}). \tag{1.6}$$

This proposition allows us to study the semilinear equation (1.1). We first deal with the case of L^1 data.

Theorem 1.4. *Assume $s \in (\frac{1}{2}, 1)$. Let $f \in C(\mathbb{R})$ be a nondecreasing function satisfying $tf(t) \geq 0$ for every $t \in \mathbb{R}$.*

(I) Existence and uniqueness: *For every $\tau \in L^1(\Omega, \rho^s)$ and $\mu \in L^1(\partial\Omega)$, problem* (1.1) *admits a unique weak solution u. Moreover,*

$$u = \mathbb{G}_s^\Omega[\tau - f(u)] + \mathbb{M}_s^\Omega[\mu] \quad \text{in } \Omega, \tag{1.7}$$

$$-\mathbb{G}_s^\Omega[\tau^-] - \mathbb{M}_s^\Omega[\mu^-] \leq u \leq \mathbb{G}_s^\Omega[\tau^+] + \mathbb{M}_s^\Omega[\mu^+] \quad \text{in } \Omega. \tag{1.8}$$

(II) Monotonicity: *The mapping $(\tau, \mu) \mapsto u$ is nondecreasing.*

Remark. The restriction $s \in (\frac{1}{2}, 1)$ is due to the fact that in this range of s one has $\operatorname{tr}_s(\mathbb{G}[\tau]) = 0$ for every $\tau \in \mathfrak{M}(\Omega, \rho^s)$ (see Proposition 2.11). We conjecture that this still holds if $s \in (0, \frac{1}{2}]$.

We reveal that, in measures framework, because of the interplay between the nonlocal operator $(-\Delta)^s$ and the nonlinearity term $f(u)$, the analysis is much more intricate and there are three critical exponents

$$p_1^* := \frac{N + 2s}{N}, \quad p_2^* := \frac{N + s}{N - s}, \quad p_3^* := \frac{N}{N - 2s}.$$

This yields substantial new difficulties and leads to disclose new types of results. The new aspects are both on the technical side and on the one of the new phenomena observed.

Theorem 1.5. *Assume $s \in (\frac{1}{2}, 1)$. Let $f \in C(\mathbb{R})$ be a nondecreasing function, $tf(t) \geq 0$ for every $t \in \mathbb{R}$ and*

$$\int_1^\infty [f(s) - f(-s)] s^{-1-p_2^*} \, ds < \infty. \tag{1.9}$$

(I) Existence and Uniqueness: *For every $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$, there exists a unique weak solution of* (1.1). *This solution satisfies* (1.7) *and* (1.8). *Moreover, the mapping $(\tau, \mu) \mapsto u$ is nondecreasing.*

(II) Stability: *Assume that $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^s)$ converges weakly to $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$. Let u and u_n be the unique weak solutions of* (1.1) *with data (τ, μ) and (τ_n, μ_n), respectively. Then $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^p(\Omega, \rho^s)$.*

If μ is a Dirac mass concentrated at a point on $\partial\Omega$, we obtain the behavior of the solution near that boundary point.

Theorem 1.6. *Under the assumption of Theorem 1.5, let $z \in \partial\Omega$, $k > 0$ and let $u_{z,k}^\Omega$ be the unique weak solution of*

$$\begin{cases} (-\Delta)^s u + f(u) = 0 & \text{in } \Omega, \\ \operatorname{tr}_s(u) = k\delta_z, & \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Then

$$\lim_{\Omega \ni x \to z} \frac{u_{z,k}^\Omega(x)}{M_s^\Omega(x, z)} = k. \tag{1.10}$$

We next assume that $0 \in \partial\Omega$. Let $0 < p < p_2^*$ and denote by u_k^Ω the unique weak solution of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega, \\ \operatorname{tr}_s(u) = k\delta_0, & \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.11}$$

By Theorem 1.5, $u_k^\Omega \leq kM_s^\Omega(\cdot, 0)$ and $k \mapsto u_k^\Omega$ is increasing. Hence, it is natural to investigate $\lim_{k\to\infty} u_k^\Omega$. This is accomplishable thanks to the study of separable solutions of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \mathbb{R}_+^N, \\ u = 0 & \text{in } \overline{\mathbb{R}_-^N}, \end{cases} \tag{1.12}$$

with $p > 1$. Denote by

$$S^{N-1} := \Big\{\sigma = (\cos\phi\sigma', \sin\phi) : \sigma' \in S^{N-2}, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\Big\}$$

the unit sphere in $\mathbb{R}^N$, and by $S_+^{N-1} := S^{N-1} \cap \mathbb{R}_+^N$ the upper hemisphere. Writing separable solution under the form $u(x) = u(r, \sigma) = r^{-2s/(p-1)}\omega(\sigma)$, with $r > 0$ and $\sigma \in S_+^{N-1}$, we obtain that ω satisfies

$$\begin{cases} \mathcal{A}_s\omega - \mathcal{L}_{s,\frac{2s}{p-1}}\omega + \omega^p = 0 & \text{in } S_+^{N-1}, \\ \omega = 0 & \text{in } \overline{S_-^{N-1}}, \end{cases} \tag{1.13}$$

where $\mathcal{A}_s$ is a nonlocal operator naturally associated to the s-fractional Laplace–Beltrami operator, and $\mathcal{L}_{s,2s/(p-1)}$ is a linear integral operator with kernel. In analyzing the spectral properties of $\mathcal{A}_s$, we prove the following theorem.

Theorem 1.7. *Let $N \geq 2$, $s \in (0, 1)$ and $p > p_1^*$.*
(I) *If $p_2^* \leq p < p_3^*$, there exists no positive solution of (1.13) belonging to $W_0^{s,2}(S_+^{N-1})$.*
(II) *If $p_1^* < p < p_2^*$, there exists a unique positive solution $\omega^* \in W_0^{s,2}(S_+^{N-1})$ of (1.13).*

As a consequence of this result, we obtain the behavior of u_k^Ω when $k \to \infty$.

Theorem 1.8. *Assume $s \in (\frac{1}{2}, 1)$. Let $\Omega = \mathbb{R}_+^N$ or let Ω be a bounded domain with C^2 boundary containing 0.*
(I) *If $p \in (p_1^*, p_2^*)$, then $u_\infty^\Omega := \lim_{k\to 0} u_k^\Omega$ is a positive solution of*

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.14}$$

(i) *If $\Omega = \mathbb{R}_+^N$, then*

$$u_\infty^{\mathbb{R}_+^N}(x) = |x|^{-\frac{2s}{p-1}}\omega^*(\sigma) \quad \text{with } \sigma = \frac{x}{|x|} \text{ for all } x \in \mathbb{R}_+^N.$$

(ii) *If Ω is a bounded C^2 domain with $\partial\Omega$ containing 0, then*

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2s}{p-1}} u_\infty^\Omega(x) = \omega^*(\sigma) \tag{1.15}$$

locally uniformly on S_+^{N-1}. In particular, there exists a positive constant c depending on N, s, p and the C^2 norm of $\partial\Omega$ such that

$$c^{-1}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \leq u_\infty^\Omega(x) \leq c\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \quad \text{for all } x \in \Omega.$$

(II) *Assume $p \in (0, p_1^*]$. Then $\lim_{k\to\infty} u_k^\Omega = \infty$ in Ω.*

The main ingredients of the present study are the following: estimates on the Green kernel and Martin kernel, theory for linear fractional equations in connection with the notion s-boundary trace as mentioned above, similarity transformation and the study of equation (1.13).

The paper is organized as follows: In Section 2, we present important properties of s-boundary trace, and prove Proposition 1.3. Theorems 1.4–1.6 and 1.8 are obtained in Section 3. Finally, in Appendix A, we discuss separable solutions of (1.12) and demonstrate Theorem 1.7.

2 Linear Problems

Throughout the present paper, we denote by $c, c', c_1, c_2, C, \dots$ positive constants that may vary from line to line. If necessary, the dependence of these constants will be made precise.

2.1 *s*-Harmonic Functions

We first recall the definition of *s-harmonic functions* (see [3, p. 46], [4, p. 230] and [6, p. 20]). Denote by (X_t, P^x) the standard rotation invariant $2s$-stable Lévy process in $\mathbb{R}^N$ (i.e. homogeneous with independent increments) with characteristic function

$$E^0 e^{i\xi X_t} = e^{-t|\xi|^{2s}}, \quad \xi \in \mathbb{R}^N,\ t \geq 0.$$

Denote by E^x the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. We assume that sample paths of X_t are right-continuous and have left-hand-side limits a.s. The process (X_t) is Markov with transition probabilities given by

$$P_t(x, A) = P^x(X_t \in A) = \mu_t(A - x),$$

where μ_t is the one-dimensional distribution of X_t with respect to P^0. It is well known that $(-\Delta)^s$ is the generator of the process (X_t, P^x).

For each Borel set $D \subset \mathbb{R}^N$, set $t_D := \inf\{t \geq 0 : X_t \notin D\}$, i.e. t_D is the first exit time from D. If D is bounded, then $t_D < \infty$ a.s. Denote

$$E^x u(X_{t_D}) = E^x\{u(X_{t_D}) : t_D < \infty\}.$$

Definition 2.1. Let u be a Borel measurable function in $\mathbb{R}^N$. We say that u is s-harmonic in Ω if for every bounded open set $D \Subset \Omega$,

$$u(x) = E^x u(X_{t_D}), \quad x \in D.$$

We say that u is singular s-harmonic in Ω if u is s-harmonic and $u = 0$ in Ω^c.

Put

$$\mathcal{D}_s := \Big\{u : \mathbb{R}^N \mapsto \mathbb{R} : \text{Borel measurable such that} \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|)^{N+2s}}\Big\}.$$

The following result follows from [5, Corollary 3.10 and Theorem 3.12] and [6, p. 20] (see also [20]).

Proposition 2.2. *Let $u \in \mathcal{D}_s$.*
(i) *u is s-harmonic in Ω if and only if $(-\Delta)^s u = 0$ in Ω in the sense of distributions.*
(ii) *u is singular s-harmonic in Ω if and only if u is s-harmonic in Ω and $u = 0$ in Ω^c.*

2.2 Green Kernel, Poisson Kernel and Martin Kernel

In what follows, the notation $f \sim g$ means that there exists a positive constant c such that $c^{-1}f < g < cf$ in the domain of the two functions or in a specified subset of this domain.

Denote by G_s^Ω the Green kernel of $(-\Delta)^s$ in Ω. Namely, for every $y \in \Omega$,

$$\begin{cases} (-\Delta)^s G_s^\Omega(\cdot, y) = \delta_y & \text{in } \Omega, \\ G_s^\Omega(\cdot, y) = 0 & \text{in } \Omega^c, \end{cases}$$

where δ_y is the Dirac mass at y. By combining [1, Lemma 3.2] and [14, Corollary 1.3]), we get the following proposition.

Proposition 2.3. (i) *G_s^Ω is continuous, symmetric and positive in $\{(x,y) \in \Omega \times \Omega : x \neq y\}$, and $G_s^\Omega(x,y) = 0$ if x or y belongs to Ω^c.*
(ii) *$(-\Delta)^s G_s^\Omega(x,\cdot) \in L^1(\Omega^c)$ for every $x \in \Omega$, and $(-\Delta)^s G_s^\Omega(x,y) \leq 0$ for every $x \in \Omega$ and $y \in \Omega^c$.*
(iii) *There holds*

$$G_s^\Omega(x,y) \sim \min\{|x-y|^{2s-N},\ \rho(x)^s\rho(y)^s|x-y|^{-N}\} \quad \text{for all } (x,y) \in \Omega \times \Omega,\ x \neq y. \tag{2.1}$$

The similarity constant in the above estimate depends only on Ω and s.

Denote by $\mathbb{G}_s^\Omega$ the associated Green operator

$$\mathbb{G}_s^\Omega[\tau] = \int_\Omega G_s^\Omega(\cdot,y)\,d\tau(y), \quad \tau \in \mathfrak{M}(\Omega,\rho^s).$$

Put

$$k_{s,\gamma} := \begin{cases} p_3^* & \text{if } \gamma \in \left[0, \frac{N-2s}{N}s\right), \\ \dfrac{N+s}{N-2s+\gamma} & \text{if } \gamma \in \left[\frac{N-2s}{N}s, s\right]. \end{cases} \tag{2.2}$$

Chen and Véron obtained the following estimate for the Green operator [11, Proposition 2.3 and Proposition 2.6].

Lemma 2.4. *Assume $\gamma \in [0,s]$ and let $k_{s,\gamma}$ be as in (2.2).*
(i) *There exists a constant $c = c(N,s,\gamma,\Omega) > 0$ such that*

$$\|\mathbb{G}_s^\Omega[\tau]\|_{M^{k_{s,\gamma}}(\Omega,\rho^s)} \leq c\|\tau\|_{\mathfrak{M}(\Omega,\rho^\gamma)} \quad \text{for all } \tau \in \mathfrak{M}(\Omega,\rho^\gamma). \tag{2.3}$$

(ii) *Assume $\{\tau_n\} \subset \mathfrak{M}(\Omega,\rho^\gamma)$ converges weakly to $\tau \in \mathfrak{M}(\Omega,\rho^\gamma)$. Then $\mathbb{G}_s^\Omega[\tau_n] \to \mathbb{G}_s^\Omega[\tau]$ in $L^p(\Omega,\rho^s)$ for any $p \in [1, k_{s,\gamma})$.*

Let P_s^Ω be the Poisson kernel of $(-\Delta)^s$ defined by (see [7])

$$P_s^\Omega(x,y) := -a_{N,-s}\int_\Omega \frac{G_s^\Omega(x,z)}{|z-y|^{N+2s}}\,dz \quad \text{for all } x \in \Omega,\ y \in \overline{\Omega}^c.$$

The relation between P_s^Ω and G_s^Ω is expressed in [1, Proposition 2] (see also [14, Theorem 1.4], [4, Lemma 2] and [14, Theorem 1.5]).

Proposition 2.5. (i) *$P_s^\Omega(x,y) = -(-\Delta)^s G_s^\Omega(x,y)$ for every $x \in \Omega$ and $y \in \overline{\Omega}^c$. Furthermore, P_s^Ω is continuous in $\Omega \times \overline{\Omega}^c$.*
(ii) *There holds*

$$P_s^\Omega(x,y) \sim \frac{\rho(x)^s}{\rho(y)^s(1+\rho(y))^s}\frac{1}{|x-y|^N} \quad \text{for all } x \in \Omega,\ y \in \overline{\Omega}^c.$$

The similarity constant in the above estimate depends only on Ω and s.

Denote by $\mathbb{P}_s^\Omega$ the corresponding operator defined by

$$\mathbb{P}_s^\Omega[\nu](x) = \int_{\overline{\Omega}^c} P_s^\Omega(x,y)\,d\nu(y), \quad \nu \in \mathfrak{M}(\overline{\Omega}^c).$$

Fix a reference point $x_0 \in \Omega$ and denote by M_s^Ω the Martin kernel of $(-\Delta)^s$ in Ω, i.e.

$$M_s^\Omega(x,z) = \lim_{\Omega \ni y \to z} \frac{G_s^\Omega(x,y)}{G_s^\Omega(x_0,y)} \quad \text{for all } x \in \mathbb{R}^N,\ z \in \partial\Omega.$$

By [15, Theorem 3.6], the Martin boundary of Ω can be identified with the Euclidean boundary $\partial\Omega$. Denote by $\mathbb{M}_s^\Omega$ the associated Martin operator

$$\mathbb{M}_s^\Omega[\mu] = \int_{\partial\Omega} M_s^\Omega(\cdot,z)\,d\mu(z), \quad \mu \in \mathfrak{M}(\partial\Omega).$$

The next result [4, 15, 19] is important in the study of s-harmonic functions, which give a unique presentation of s-harmonic functions in terms of the Martin kernel.

Proposition 2.6. (i) *The mapping $(x,z) \mapsto M_s^\Omega(x,z)$ is continuous on $\Omega \times \partial\Omega$. For any $z \in \partial\Omega$, the function $M_s^\Omega(\cdot, z)$ is singular s-harmonic in Ω with $M_s^\Omega(x_0,z) = 1$. Moreover, if $z, z' \in \partial\Omega$, $z \neq z'$ then*

$$\lim_{x\to z'} M_s^\Omega(x,z) = 0.$$

(ii) *There holds*

$$M_s^\Omega(x,z) \sim \rho(x)^s|x-z|^{-N} \quad \textit{for all } x \in \Omega,\ z \in \partial\Omega. \tag{2.4}$$

The similarity constant in the above estimate depends only on Ω and s.

(iii) *For every $\mu \in \mathfrak{M}^+(\partial\Omega)$, the function $\mathbb{M}_s^\Omega[\mu]$ is singular s-harmonic in Ω with $u(x_0) = \mu(\mathbb{R}^N)$. Conversely, if u is a nonnegative singular s-harmonic function in Ω then there exists a unique $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that $u = \mathbb{M}_s^\Omega[\mu]$ in $\mathbb{R}^N$.*

(iv) *If u is a nonnegative s-harmonic function in Ω, then there exists a unique $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that*

$$u(x) = \mathbb{M}_s^\Omega[\mu](x) + \mathbb{P}_s^\Omega[u](x) \quad \textit{for all } x \in \Omega.$$

Lemma 2.7. (i) *There exists a constant $c = c(N,s,\gamma,\Omega)$ such that*

$$\|\mathbb{M}_s^\Omega[\mu]\|_{M^{\frac{N+\gamma}{N-s}}(\Omega,\rho^\gamma)} \le c\|\mu\|_{\mathfrak{M}(\partial\Omega)} \quad \textit{for all } \mu \in \mathfrak{M}(\partial\Omega),\ \gamma > -s. \tag{2.5}$$

(ii) *If $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$, then $\mathbb{M}_s^\Omega[\mu_n] \to \mathbb{M}_s^\Omega[\mu]$ in $L^p(\Omega,\rho^\gamma)$ for every $1 \le p < \frac{N+\gamma}{N-s}$.*

Proof. (i) Using (2.4) and an argument similar to the one in the proof of [2, Theorem 2.5], we obtain (2.5).

(ii) By combining the fact that $M_s^\Omega(x,z) = 0$ for every $x \in \Omega^c$, $z \in \partial\Omega$ and Proposition 2.6 (i), we deduce that for every $x \in \mathbb{R}^N$ we have $M_s^\Omega(x,\cdot) \in C(\partial\Omega)$. It follows that $\mathbb{M}_s^\Omega[\mu_n] \to \mathbb{M}_s^\Omega[\mu]$ everywhere in Ω. Due to (i) and the Holder inequality, we deduce that $\{\mathbb{M}_s^\Omega[\mu_n]\}$ is uniformly integrable with respect to $\rho^\gamma\, dx$ for any $1 \le p \le \frac{N+\gamma}{N-s}$. By invoking Vitali's theorem, we obtain the convergence in $L^p(\Omega,\rho^\gamma)$. □

2.3 Boundary Trace

We recall that, for $\beta > 0$,

$$\Omega_\beta := \{x \in \Omega : \rho(x) < \beta\}, \quad D_\beta := \{x \in \Omega : \rho(x) > \beta\}, \quad \Sigma_\beta := \{x \in \Omega : \rho(x) = \beta\}.$$

The following geometric property of C^2 domains can be found in [26].

Proposition 2.8. *There exists $\beta_0 > 0$ such that the following statements hold:*

(i) *For every point $x \in \overline{\Omega}_{\beta_0}$, there exists a unique point $z_x \in \partial\Omega$ such that $|x - z_x| = \rho(x)$. This implies*

$$x = z_x - \rho(x)\mathbf{n}_{z_x}.$$

(ii) *The mappings $x \mapsto \rho(x)$ and $x \mapsto z_x$ belong to $C^2(\overline{\Omega}_{\beta_0})$ and $C^1(\overline{\Omega}_{\beta_0})$, respectively. Furthermore,*

$$\lim_{x\to z_x} \nabla\rho(x) = -\mathbf{n}_{z_x}.$$

Proposition 2.9. *Assume $s \in (0,1)$. Then there exist positive constants $c = c(N,\Omega,s)$ such that, for every $\beta \in (0,\beta_0)$,*

$$c^{-1} \le \beta^{1-s} \int_{\Sigma_\beta} M_s^\Omega(x,y)\, dS(x) \le c \quad \textit{for all } y \in \partial\Omega. \tag{2.6}$$

Proof. For $r_0 > 0$ fixed, by (2.4),

$$\int_{\Sigma_\beta \setminus B_{r_0}(y)} M_s^\Omega(x,y)\, dS(x) \le c_1\beta^s, \tag{2.7}$$

which implies

$$\lim_{\beta\to 0} \int_{\Sigma_\beta \setminus B_{r_0}(y)} M_s^\Omega(x,y)\,dS(x) = 0 \quad \text{for all } y \in \partial\Omega.$$

Note that for r_0 fixed the rate of convergence is independent of y.

In order to prove (2.6), we may assume that the coordinates are placed so that $y = 0$ and the tangent hyperplane to $\partial\Omega$ at 0 is $x_N = 0$ with the x_N axis pointing into the domain. For $x \in \mathbb{R}^N$, put $x' = (x_1, \dots, x_{N-1})$. Pick $r_0 \in (0, \beta_0)$ sufficiently small (depending only on the C^2 characteristic of Ω) so that

$$\frac{1}{2}(|x'|^2 + \rho(x)^2) \le |x|^2 \quad \text{for all } x \in \Omega \cap B_{r_0}(0).$$

Hence if $x \in \Sigma_\beta \cap B_{r_0}(0)$, then $\frac{1}{4}(|x'| + \beta) \le |x|$. Combining this inequality and (2.4) leads to

$$\int_{\Sigma_\beta \cap B_{r_0}(0)} M_s^\Omega(x,0)\,dS(x) \le c_2\beta^s \int_{\Sigma_{\beta,0}} (|x'| + \beta)^{-N}\,dS(x) \le c_2\beta^s \int_{|x'|<r_0} (|x'| + \beta)^{-N}\,dx' = c_3\beta^{s-1}.$$

Therefore, for $\beta < r_0$,

$$\beta^{1-s} \int_{\Sigma_\beta \cap B_{r_0}(0)} M_s^\Omega(x,0)\,dS(x) \le c_4. \tag{2.8}$$

By combining estimates (2.7) and (2.8), we obtain the second estimate in (2.6). The first estimate in (2.6) follows from (2.4). □

As a consequence, we get the following estimates.

Corollary 2.10. *Assume $s \in (0,1)$. For every $\mu \in \mathfrak{M}^+(\partial\Omega)$ and $\beta \in (0, \beta_0)$, there holds*

$$c^{-1}\|\mu\|_{\mathfrak{M}(\partial\Omega)} \le \beta^{1-s} \int_{\Sigma_\beta} \mathbb{M}_s^\Omega[\mu]\,dS \le c\|\mu\|_{\mathfrak{M}(\partial\Omega)},$$

with c being as in (2.6).

Proposition 2.11. *Assume $s \in (\frac{1}{2}, 1)$. Then there exists a constant $c = c(s, N, \Omega)$ such that for any $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and any $0 < \beta < \beta_0$,*

$$\beta^{1-s} \int_{\Sigma_\beta} \mathbb{G}_s^\Omega[\tau]\,dS \le c \int_\Omega \rho^s\,d|\tau|. \tag{2.9}$$

Moreover,

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} \mathbb{G}_s^\Omega[\tau]\,dS = 0. \tag{2.10}$$

Proof. Without loss of generality, we may assume that $\tau > 0$. Denote $v := \mathbb{G}_s^\Omega[\tau]$. We first prove (2.9). By Fubini's theorem and (2.4),

$$\int_{\Sigma_\beta} v(x)\,dS(x) \le c_5\Big(\int_\Omega \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2s-N}\,dS(x)\,d\tau(y) + \beta^s \int_\Omega \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x-y|^{-N}\,dS(x)\rho(y)^s\,d\tau(y)\Big) =: I_{1,\beta} + I_{2,\beta}.$$

Note that if $x \in \Sigma_\beta \cap B_{\frac{\beta}{2}}(y)$, then $\frac{\beta}{2} \le \rho(y) \le \frac{3\beta}{2}$. Therefore,

$$\begin{aligned}
\beta^{1-s} I_{1,\beta} &\le c_6\beta^{1-2s} \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2s-N}\,dS(x) \int_\Omega \rho(y)^s\,d\tau(y) \\
&\le c_6\beta^{1-2s} \int_0^{\beta/2} r^{2s-N} r^{N-2}\,dr \int_\Omega \rho(y)^s\,d\tau(y) \\
&\le c_7 \int_\Omega \rho(y)^s\,d\tau(y),
\end{aligned}$$

where the last inequality holds since $s > \frac{1}{2}$. On the other hand, we have

$$I_{2,\beta} \le c_7\beta^s \int_{\beta/2}^{\infty} r^{-N} r^{N-2}\,dr \int_{\Omega} \rho(y)^s\,d\tau(y) = c_8\beta^{s-1} \int_{\Omega} \rho(y)^s\,d\tau(y).$$

Combining the above estimates, we obtain (2.9).

Next we demonstrate (2.10). Given $\varepsilon \in (0, \|\tau\|_{\mathfrak{M}(\Omega,\rho^s)})$ and $\beta_1 \in (0,\beta_0)$, put $\tau_1 = \tau\chi_{\bar{D}_{\beta_1}}$ and $\tau_2 = \tau\chi_{\Omega_{\beta_1}}$. We can choose $\beta_1 = \beta_1(\varepsilon)$ such that

$$\int_{\Omega_{\beta_1}} \rho(y)^s\,d\tau(y) \le \varepsilon.$$

Thus the choice of β_1 depends on the rate at which $\int_{\Omega_\beta} \rho^s\,d\tau$ tends to zero as $\beta \to 0$.

Put $v_i := \mathbb{G}_s^{\Omega}[\tau_i]$. Then, for $0 < \beta < \frac{\beta_1}{2}$,

$$\int_{\Sigma_\beta} v_1(x)\,dS(x) \le c_9\beta^s\beta_1^{-N} \int_{\Omega} \rho(y)^s\,d\tau_1(y),$$

which yields

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} v_1(x)\,dS(x) = 0. \tag{2.11}$$

On the other hand, due to (2.9),

$$\beta^{1-s} \int_{\Sigma_\beta} v_2\,dS \le c_{10} \int_{\Omega} \rho^s\,d\tau_2 \le c_{11}\varepsilon \quad \text{for all } \beta < \beta_0. \tag{2.12}$$

From (2.11) and (2.12) we obtain (2.10). □

Lemma 2.12. *Assume $s \in (\frac{1}{2}, 1)$. Let $u, w \in \mathcal{D}_s$ be two nonnegative functions satisfying*

$$\begin{cases} (-\Delta)^s u \le 0 \le (-\Delta)^s w & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

If $u \le w$ in $\mathbb{R}^N$, then $(-\Delta)^s u \in \mathfrak{M}(\Omega,\rho^s)$ and there exists a measure $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} |u - \mathbb{M}_s^{\Omega}[\mu]|\,dS = 0. \tag{2.13}$$

Moreover, if $\mu = 0$, then $u = 0$.

Proof. By the assumption, there exists a nonnegative Radon measure τ on Ω such that $(-\Delta)^s u = -\tau$.

We first prove that $\tau \in \mathfrak{M}^+(\Omega,\rho^s)$. Define

$$\tilde{M}_s^{\Omega}(x,z) := \lim_{\Omega\ni y\to z} \frac{G_s^{\Omega}(x,y)}{\rho(y)^s}.$$

By [1, p. 5547], there is a positive constant $c = c(\Omega, s)$ such that

$$\tilde{M}_s^{\Omega}(x,z) \sim \rho(x)^s|x-z|^{-N} \quad \text{for all } x \in \Omega,\ z \in \partial\Omega,$$

where the similarity constant depends only on Ω and s. This implies

$$\begin{aligned} c_{12}^{-1} < c_{13}^{-1} \int_{\partial\Omega} \rho(x)|x-z|^{-N}\,dS(z) \\ \le \rho(x)^{1-s} \int_{\partial\Omega} \tilde{M}_s^{\Omega}(x,z)\,dS(z) \\ \le c_{13} \int_{\partial\Omega} \rho(x)|x-z|^{-N}\,dS(z) < c_{12} \quad \text{for all } x \in \Omega. \end{aligned}$$

We define

$$\mathbb{E}_s^\Omega[u](z) := \lim_{\Omega \ni x \to z} \frac{u(x)}{\int_{\partial\Omega} \tilde{M}_s^\Omega(x,y)\,dS(y)}, \quad z \in \partial\Omega.$$

For any $\beta \in (0,\beta_0)$, denote by τ_β the restriction of τ to D_β and by v_β the restriction of u on Σ_β. By [1, Theorem 1.4], there exists a unique solution v_β of

$$\begin{cases} (-\Delta)^s v_\beta = -\tau_\beta & \text{in } D_\beta, \\ \mathbb{E}_s^{D_\beta}[v_\beta] = 0 & \text{on } \Sigma_\beta, \\ v_\beta = u|_{D_\beta^c} & \text{in } D_\beta^c. \end{cases}$$

Moreover, the solution can be written as

$$v_\beta + \mathbb{G}_s^{D_\beta}[\tau_\beta] = \mathbb{P}_s^{D_\beta}[u|_{D_\beta^c}] \quad \text{in } D_\beta. \tag{2.14}$$

By the maximum principle [1, Lemma 3.9], $v_\beta = u$ and $\mathbb{P}_s^{D_\beta}[u|_{D_\beta^c}] \le w$ a.e. in $\mathbb{R}^N$. This, together with (2.14), implies that

$$\mathbb{G}_s^{D_\beta}[\tau_\beta] \le w \quad \text{in } D_\beta.$$

Letting $\beta \to 0$ yields $\mathbb{G}_s^\Omega[\tau] < \infty$. For fixed $x_0 \in \Omega$, by (2.1), $G_s^\Omega(x_0,y) > c\rho(y)^s$ for every $y \in \Omega$. Hence the finiteness of $\mathbb{G}_s^\Omega[\tau]$ implies that $\tau \in \mathfrak{M}^+(\Omega,\rho^s)$.

We next show that there exists a measure $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that (2.13) holds. Put $v = u + \mathbb{G}_s^\Omega[\tau]$; then v is nonnegative singular s-harmonic in Ω due to the fact that $\mathbb{G}_s^\Omega[\tau] = 0$ in Ω^c. By Proposition 2.2 and Proposition 2.6 (iii), there exists $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that $v = \mathbb{M}_s^\Omega[\mu]$ in $\mathbb{R}^N$. By Proposition 2.11, we obtain (2.13). If $\mu = 0$, then $v = 0$, and thus $u = 0$. □

Definition 2.13. A function u possesses an s-boundary trace on $\partial\Omega$ if there exists a measure $\mu \in \mathfrak{M}(\partial\Omega)$ such that

$$\lim_{\beta \to 0} \beta^{1-s} \int_{\Sigma_\beta} |u - \mathbb{M}_s^\Omega[\mu]|\,dS = 0. \tag{2.15}$$

The s-boundary trace of u is denoted by $\operatorname{tr}_s(u)$.

Remark. (i) The notation of s-boundary trace is well defined. Indeed, suppose that μ and μ' satisfy (2.15). Put $v = (\mathbb{M}_s^\Omega[\mu - \mu'])^+$. Clearly, $v \le \mathbb{M}_s^\Omega[|\mu| + |\mu'|]$, $v = 0$ in Ω^c and

$$\lim_{\beta \to 0} \beta^{1-s} \int_{\Sigma_\beta} |v|\,dS = 0.$$

By Kato's inequality [8, Theorem 1.2], $(-\Delta)^s v \le 0$ in Ω. Therefore, we deduce $v \equiv 0$ from Lemma 2.12. This implies $\mathbb{M}_s^\Omega[\mu - \mu'] \le 0$. By permuting the role of μ and μ', we obtain $\mathbb{M}_s^\Omega[\mu - \mu'] \ge 0$. Thus $\mu = \mu'$.

(ii) It is clear that for every $\mu \in \mathfrak{M}(\partial\Omega)$ one has $\operatorname{tr}_s(\mathbb{M}_s^\Omega[\mu]) = \mu$. Moreover, if $s > \frac{1}{2}$, by Proposition 2.11, for every $\tau \in \mathfrak{M}(\Omega,\rho^s)$ one has $\operatorname{tr}_s(\mathbb{G}_s^\Omega[\tau]) = 0$.

(iii) This kind of boundary trace was first introduced by Nguyen and Marcus [21] in order to investigate semilinear elliptic equations with Hardy potential. In the present paper, we prove that it is still an effective tools in the study of nonlocal fractional elliptic equations.

2.4 Weak Solutions of Linear Problems

Definition 2.14. Let $\tau \in \mathfrak{M}(\Omega,\rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is called a weak solution of (1.4) if $u \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta)^s \xi\,dx = \int_\Omega \xi\,d\tau + \int_\Omega \mathbb{M}_s^\Omega[\mu](-\Delta)^s \xi\,dx \quad \text{for all } \xi \in \mathbb{X}_s(\Omega). \tag{2.16}$$

Proof of Proposition 1.3. The uniqueness follows from [11, Proposition 2.4]. Let u be as in (1.5). By [11],

$$\int_\Omega (u - \mathbb{M}_s^\Omega[\mu])(-\Delta)^s \xi \, dx = \int_\Omega \mathbb{G}_s^\Omega[\tau](-\Delta)^s \xi \, dx = \int_\Omega \xi \, d\tau \quad \text{for all } \xi \in \mathbb{X}_s(\Omega).$$

This implies (2.16), and therefore u is the unique solution of (1.4). Since $s \in (\frac{1}{2}, 1)$, by Proposition 2.11, $\mathrm{tr}_s(u) = \mathrm{tr}_s(\mathbb{M}_s^\Omega[\mu]) = \mu$. Finally, estimate (1.6) follows from Lemma 2.4 and Lemma 2.7. □

3 Nonlinear Problems

In this section, we study the nonlinear problem (1.1). The definition of weak solutions of (1.1) is given in Definition 1.2.

3.1 Subcritical Absorption

Proof of Theorem 1.4. We prove this theorem in several steps.

Monotonicity. Let $\tau, \tau' \in L^1(\Omega, \rho^s)$, $\mu, \mu' \in L^1(\partial\Omega)$ and let u and u' be the weak solutions of (1.1) with data (τ, μ) and (τ', μ'), respectively. We will show that if $\tau \le \tau'$ and $\mu \le \mu'$, then $u \le u'$ in Ω. Indeed, by putting $v := (u - u')^+$, it is sufficient to prove that $v \equiv 0$. Since (1.7) holds, it follows that

$$|u| \le \mathbb{G}_s^\Omega[|\tau| + |f(u)|] + \mathbb{M}_s^\Omega[|\mu|] \quad \text{in } \Omega.$$

Similarly,

$$|u'| \le \mathbb{G}_s^\Omega[|\tau'| + |f(u')|] + \mathbb{M}_s^\Omega[|\mu'|] \quad \text{in } \Omega.$$

Therefore,

$$0 \le v \le |u| + |u'| \le \mathbb{G}_s^\Omega[|\tau| + |\tau'| + |f(u)| + |f(u')|] + \mathbb{M}_s^\Omega[|\mu| + |\mu'|] =: w.$$

By Kato's inequality, the assumption $\tau \le \tau'$ and the monotonicity of f, we obtain

$$(-\Delta)^s v \le \mathrm{sign}^+(u - u')(\tau - \tau') - \mathrm{sign}^+(u - u')(f(u) - f(u')) \le 0.$$

Therefore,

$$(-\Delta)^s v \le 0 \le (-\Delta)^s w \quad \text{in } \Omega.$$

Since $\mu \le \mu'$, it follows that $\mathrm{tr}_s(v) = 0$. By Lemma 2.12, $v = 0$, and thus $u \le u'$.

Existence. *Step 1: Assume that $\tau \in L^\infty(\Omega)$ and $\mu \in L^\infty(\partial\Omega)$.* Put

$$\hat{f}(t) := f(t + \mathbb{M}_s^\Omega[\mu]) - f(\mathbb{M}_s^\Omega[\mu]) \quad \text{and} \quad \hat{\tau} := \tau - f(\mathbb{M}_s^\Omega[\mu]).$$

Then $\hat{f}$ is nondecreasing and $t\hat{f}(t) \ge 0$ for every $t \in \mathbb{R}$ and $\hat{\tau} \in L^1(\Omega, \rho^s)$. Consider the problem

$$\begin{cases} (-\Delta)^s v + \hat{f}(v) = \hat{\tau} & \text{in } \Omega, \\ v = 0 & \text{in } \Omega^c. \end{cases} \tag{3.1}$$

By [12, Proposition 3.1], there exists a unique weak solution v of (3.1). This means $v \in L^1(\Omega)$, $\hat{f}(v) \in L^1(\Omega, \rho^s)$ and

$$\int_\Omega (v(-\Delta)^s \xi + \hat{f}(v)\xi) \, dx = \int_\Omega \xi \hat{\tau} \, dx \quad \text{for all } \xi \in \mathbb{X}_s(\Omega). \tag{3.2}$$

Put $u := v + \mathbb{M}_s^\Omega[\mu]$; then $u \in L^1(\Omega)$ and $f(u) \in L^1(\Omega, \rho^s)$. By (3.2), u satisfies (1.2).

Step 2: Assume that $0 \le \tau \in L^1(\Omega, \rho^s)$ *and* $0 \le \mu \in L^1(\partial\Omega)$. Let $\{\tau_n\} \subset C^1(\overline{\Omega})$ be a nondecreasing sequence converging to τ in $L^1(\Omega, \rho^s)$ and let $\{\mu_n\} \subset C^1(\partial\Omega)$ be a nondecreasing sequence converging to μ in $L^1(\partial\Omega)$.

Then $\{\mathbb{M}_s^\Omega[\mu_n]\}$ is nondecreasing, and by Lemma 2.7 (ii) it converges to $\mathbb{M}_s^\Omega[\mu]$ a.e. in Ω and in $L^p(\Omega,\rho^s)$ for every $1 \le p < p_2^*$. Let u_n be the unique solution of (1.1) with τ and μ replaced by τ_n and μ_n, respectively. By step 1 and the monotonicity of f, we derive that $\{u_n\}$ and $\{f(u_n)\}$ are nondecreasing. Moreover,

$$\int_\Omega (u_n(-\Delta)^s\xi + f(u_n)\xi)\,dx = \int_\Omega \xi\,d\tau_n + \int_\Omega \mathbb{M}_s^\Omega[\mu_n](-\Delta)^s\xi\,dx \quad \text{for all } \xi \in \mathbb{X}_s(\Omega). \tag{3.3}$$

Let $\eta \in C(\overline{\Omega})$ be the solution of

$$\begin{cases} (-\Delta)^s\eta = 1 & \text{in } \Omega, \\ \eta = 0 & \text{in } \Omega^c. \end{cases} \tag{3.4}$$

Then $c^{-1}\rho^s < \eta < c\rho^s$ in Ω for some $c > 1$. By choosing $\xi = \eta$ in (3.3), we get

$$\|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega,\rho^s)} \le c(\|\tau_n\|_{L^1(\Omega,\rho^s)} + \|\mu_n\|_{L^1(\partial\Omega)}) \le c'(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}).$$

Hence $\{u_n\}$ and $\{f(u_n)\}$ are uniformly bounded in $L^1(\Omega)$ and $L^1(\Omega,\rho^s)$, respectively. By the monotone convergence theorem, there exists $u \in L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^1(\Omega,\rho^s)$. By letting $n \to \infty$ in (3.3), we deduce that u satisfies (1.2), namely u is a weak solution of (1.1).

The uniqueness follows from the monotonicity.

Step 3: Assume that $\tau \in L^1(\Omega,\rho^s)$ and $\mu \in L^1(\partial\Omega)$. Let $\{\tau_n\} \subset C^1(\overline{\Omega})$ be a sequence such that $\{\tau_n^+\}$ and $\{\tau_n^-\}$ are nondecreasing and $\tau_n^\pm \to \tau^\pm$ in $L^1(\Omega,\rho^s)$. Let $\{\mu_n\} \subset C^1(\partial\Omega)$ be a sequence such that $\{\mu_n^+\}$ and $\{\mu_n^-\}$ are nondecreasing and $\mu_n^\pm \to \mu^\pm$ in $L^1(\partial\Omega)$. Let u_n be the unique weak solution of (1.1) with data (τ_n,μ_n). Then

$$u_n = \mathbb{G}_s^\Omega[\tau_n - f(u_n)] + \mathbb{M}_s^\Omega[\mu_n]. \tag{3.5}$$

Let $w_{1,n}$ and $w_{2,n}$ be the unique weak solutions of (1.1) with data (τ_n^+,μ_n^+) and $(-\tau_n^-,-\mu_n^-)$, respectively. Then

$$\|w_{i,n}\|_{L^1(\Omega)} + \|f(w_{i,n})\|_{L^1(\Omega,\rho^s)} \le c'(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}), \quad i = 1, 2. \tag{3.6}$$

Moreover, for any $n \in \mathbb{N}$, $w_{2,n} \le 0 \le w_{1,n}$ and

$$-\mathbb{G}_s^\Omega[\tau_n^-] - \mathbb{M}_s^\Omega[\mu_n^-] \le w_{2,n} \le u_n \le w_{1,n} \le \mathbb{G}_s^\Omega[\tau_n^+] + \mathbb{M}_s^\Omega[\mu_n^+],$$

it follows that

$$|u_n| \le w_{1,n} - w_{2,n} \quad \text{and} \quad |f(u_n)| \le f(w_{1,n}) - f(w_{2,n}). \tag{3.7}$$

This together with (3.6) implies

$$\|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega,\rho^s)} \le c''(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}). \tag{3.8}$$

Put $v_n := \mathbb{G}_s^\Omega[\tau_n - f(u_n)]$. By (3.8), the sequence $\{\tau_n - f(u_n\}$ is uniformly bounded in $L^1(\Omega,\rho^s)$. Hence by [11, Proposition 2.6], the sequence $\{v_n\}$ is relatively compact in $L^q(\Omega)$ for $1 \le q < \frac{N}{N-s}$. Consequently, up to a subsequence, $\{v_n\}$ converges in $L^q(\Omega)$ and a.e. in Ω to a function v. On the other hand, by Lemma 2.7 (ii), up to a subsequence, $\{\mathbb{M}_s^\Omega[\mu_n]\}$ converges in $L^q(\Omega,\rho^s)$ for $1 \le q < p_2^*$ and a.e. in Ω to $\mathbb{M}_s^\Omega[\mu]$. Due to (3.5), we deduce that $\{u_n\}$ converges a.e. in Ω to $u = v + \mathbb{M}_s^\Omega[\mu]$. Since f is continuous, $\{f(u_n)\}$ converges a.e. in Ω to $f(u)$.

By step 2, the sequences $\{w_{1,n}\}$, $\{f(w_{1,n})\}$, $\{-w_{2,n}\}$ and $\{-f(w_{2,n})\}$ are increasing and converge to w_1 in $L^1(\Omega)$, $f(w_1)$ in $L^1(\Omega,\rho^s)$, $-w_2$ in $L^1(\Omega)$ and $-f(w_2)$ in $L^1(\Omega,\rho^s)$, respectively. In the light of (3.7) and the generalized dominated convergence theorem, we obtain that $\{u_n\}$ and $\{f(u_n)\}$ converge to u and $f(u)$ in $L^1(\Omega)$ and $L^1(\Omega,\rho^s)$, respectively. By passing to the limit in (3.3), we derive that u satisfies (1.2).

The uniqueness follows from the monotonicity. □

Define

$$C(\overline{\Omega},\rho^{-s}) := \{\zeta \in C(\overline{\Omega}) : \rho^{-s}\zeta \in C(\overline{\Omega})\}.$$

This space is endowed with the norm

$$\|\zeta\|_{C(\overline{\Omega},\rho^{-s})} = \|\rho^{-s}\zeta\|_{C(\overline{\Omega})}.$$

We say that a sequence $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^s)$ converges weakly to a measure $\tau \in \mathfrak{M}(\Omega, \rho^s)$ if

$$\lim_{n\to\infty} \int_\Omega \zeta \, d\tau_n = \int_\Omega \zeta \, d\tau \quad \text{for all } \zeta \in C(\overline{\Omega}, \rho^{-s}).$$

Proof of Theorem 1.5. Monotonicity. The monotonicity can be proved by using an argument similar to the one in the proof of Theorem 1.4.

Existence. Let $\{\tau_n\} \subset C^1(\Omega)$ and $\{\mu_n\} \subset C^1(\partial\Omega)$ such that $\tau_n^\pm \to \tau^\pm$ weakly and $\mu_n^\pm \to \mu^\pm$ weakly. Then there is a positive constant c independent of n such that

$$\|\tau_n\|_{\mathfrak{M}(\Omega,\rho^s)} \le c\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} \quad \text{and} \quad \|\mu_n\|_{\mathfrak{M}(\partial\Omega)} \le c\|\mu\|_{\mathfrak{M}(\partial\Omega)}. \tag{3.9}$$

Let u_n, $w_{1,n}$ and $w_{2,n}$ as in the proof of Theorem 1.4. Then

$$|u_n| \le \max(w_{1,n}, -w_{2,n}) \le \mathbb{G}_s^\Omega[|\tau_n|] + \mathbb{M}_s^\Omega[|\mu_n|].$$

This, together with (2.3), (2.5) and (3.9), implies that

$$\|u_n\|_{M^{p_2^*}(\Omega,\rho^s)} \le c(\|\tau_n\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)}) \le c'(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}). \tag{3.10}$$

We have

$$\begin{aligned}
&\int_\Omega (w_{1,n}(-\Delta)^s\xi + f(w_{1,n})\xi)\, dx = \int_\Omega \xi \, d\tau_n^+ + \int_\Omega \mathbb{M}_s^\Omega[\mu_n^+](-\Delta)^s\xi \, dx, \\
&\int_\Omega (w_{2,n}(-\Delta)^s\xi + f(w_{2,n})\xi)\, dx = -\int_\Omega \xi \, d\tau_n^- - \int_\Omega \mathbb{M}_s^\Omega[\mu_n^-](-\Delta)^s\xi \, dx \quad \text{for all } \xi \in \mathbb{X}_s(\Omega).
\end{aligned}$$

From this

$$\int_\Omega [(w_{1,n} - w_{2,n}) + (f(w_{1,n}) - f(w_{2,n})\eta]\, dx = \int_\Omega \eta \, d|\tau_n| + \int_\Omega \mathbb{M}_s^\Omega[|\mu_n|]\, dx$$

follows. We infer from (3.7) and the estimate $c^{-1}\rho^s \le \eta \le c\rho^s$ that

$$\|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega,\rho^s)} \le c(\|\tau_n\|_{L^1(\Omega,\rho^s)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)}) \le c'(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}).$$

This implies that $\{u_n\}$ and $\{f(u_n)\}$ are uniformly bounded in $L^1(\Omega)$ and $L^1(\Omega, \rho^s)$, respectively. By an argument similar to the one in step 3 of the proof of Theorem 1.4, we deduce that, up to a subsequence, $\{u_n\}$ converges a.e. in Ω to a function u and $\{f(u_n)\}$ converges a.e. in Ω to $f(u)$. By the Hölder inequality, we infer that $\{u_n\}$ is uniformly integrable in $L^1(\Omega)$.

Next we prove that $\{f \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^s)$. Define $\tilde{f}(s) := f(|s|) - f(-|s|)$, $s \in \mathbb{R}$. Then $\tilde{f}$ is nondecreasing in $\mathbb{R}$ and $|f(s)| \le \tilde{f}(s)$ for every $s \in \mathbb{R}$. For $\ell > 0$ and $n \in \mathbb{N}$, set

$$A_n(\ell) := \{x \in \Omega : |u_n(x)| > \ell\}, \quad a_n(\ell) := \int_{A_n(\ell)} \rho^s \, dx.$$

We take an arbitrary Borel set $D \subset \Omega$ and estimate

$$\int_D |f(u_n)|\rho^s \, dx = \int_{D\cap A_n(\ell)} |f(u_n)|\rho^s \, dx + \int_{D\setminus A_n(\ell)} |f(u_n)|\rho^s \, dx \le \int_{A_n(\ell)} \tilde{f}(u_n)\rho^s \, dx + \tilde{f}(\ell)\int_D \rho^s \, dx. \tag{3.11}$$

On one hand, we have

$$\int_{A_n(\ell)} \tilde{f}(u_n)\rho^s \, dx = a_n(\ell)\tilde{f}(\ell) + \int_\ell^\infty a_n(s)\, d\tilde{f}(s).$$

From (3.10), we infer $a_n(s) \le \tilde{c}s^{-p_2^*}$, where $\tilde{c}$ is a positive constant independent of n. Hence, for any $l > \ell$,

$$a_n(\ell)\tilde{f}(\ell) + \int_\ell^l a_n(s)\, d\tilde{f}(s) \le \tilde{c}\ell^{-p_2^*}\tilde{f}(\ell) + \tilde{c}\int_\ell^l s^{-p_2^*}\, d\tilde{f}(s) \le \tilde{c}l^{-p_2^*}\tilde{f}(l) + \frac{\tilde{c}}{p_2^* + 1}\int_\ell^l s^{-1-p_2^*}\tilde{f}(s)\, ds. \tag{3.12}$$

By assumption (1.9), there exists a sequence $\{l_k\}$ such that $l_k \to \infty$ and $l_k^{-p_2^*}\tilde f(l_k) \to 0$ as $k \to \infty$. Taking $l = l_k$ in (3.12) and then letting $k \to \infty$, we obtain

$$a_n(\ell)\tilde f(\ell) + \int_\ell^\infty a_n(s)\, d\tilde f(s) \le \frac{\tilde c}{p_2^* + 1}\int_\ell^\infty s^{-1-p_2^*}\tilde f(s)\, ds. \tag{3.13}$$

From assumption (1.9), we see that the right-hand side of (3.13) tends to 0 as $\ell \to \infty$. Therefore, for any $\varepsilon > 0$, one can choose $\ell > 0$ such that the right-hand side of (3.13) is smaller than $\frac{\varepsilon}{2}$. Fix such ℓ; one then can choose $\delta > 0$ small such that if $\int_D \rho^s\, dx < \delta$, then $\tilde f(\ell)\int_D \rho^s\, dx < \frac{\varepsilon}{2}$. Therefore, from (3.11) we derive that

$$\int_D \rho^s\, dx < \delta \quad \text{implies} \quad \int_D |f(u_n)|\rho^s\, dx < \varepsilon.$$

This means $\{f \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^s)$.

By Vitali's convergence theorem, we deduce that, up to a subsequence, $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^1(\Omega, \rho^s)$. Since u_n satisfies (3.3), by passing to the limit, we deduce that u is a weak solution of (1.1).

Stability. Assume $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^s)$ converges weakly to $\tau \in \mathfrak{M}(\Omega, \rho^s)$, and $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$. Let u and u_n be the unique weak solution of (1.1) with data (τ, μ) and (τ_n, μ_n), respectively. Then by an argument similar to the one in the existence part, we deduce that $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^p(\Omega, \rho^s)$. □

Proposition 3.1. *Assume f is a continuous nondecreasing function on $\mathbb{R}$ satisfying $f(0) = 0$ and (1.9). Then for every $z \in \partial\Omega$,*

$$\lim_{\Omega \ni x \to z} \frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot\,, z))](x)}{M_s^\Omega(x, z)} = 0.$$

Proof. By (2.1),

$$G_s^\Omega(x, y) \le c_{14}\rho(x)^s|x - y|^{-N}\min\{\rho(y)^s, |x - y|^s\} \quad \text{for all } x \ne y.$$

Hence,

$$\frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot\,, z))](x)}{M_s^\Omega(x, z)} \le c_{15}|x - z|^N \int_\Omega |x - y|^{-N}\min\{|x - y|^s, |y - z|^s\} f(|y - z|^{s-N})\, dy. \tag{3.14}$$

Put

$$\mathcal{D}_1 := \Omega \cap B\Big(x, \frac{|x - z|}{2}\Big), \quad \mathcal{D}_2 := \Omega \cap B\Big(z, \frac{|x - z|}{2}\Big), \quad \mathcal{D}_3 := \Omega \setminus (\mathcal{D}_1 \cup \mathcal{D}_2), \tag{3.15}$$

$$I_i := |x - z|^N \int_{\mathcal{D}_i} |x - y|^{-N}\min\{|x - y|^s, |y - z|^s\} f(|y - z|^{s-N})\, dy, \quad i = 1, 2, 3.$$

Therefore, for every $y \in \mathcal{D}_1$, $|x - z| \le 2|y - z|$, we have

$$I_1 \le c_{16}|x - z|^N f(|x - z|^{s-N}) \int_{\mathcal{D}_1} |x - y|^{s-N}\, dy \le c_{17}|x - z|^{N+s} f(|x - z|^{s-N}).$$

Hence,

$$\lim_{x \to z} I_1 \le c_{17} \lim_{x \to z} |x - z|^{N+s} f(|x - z|^{s-N}) = 0. \tag{3.16}$$

We next estimate I_2. Thus, for every $y \in \mathcal{D}_2$, $|x - z| \le 2|x - y|$, we have

$$I_2 \le c_{18}\int_{\mathcal{D}_2} |y - z|^s f(|y - z|^{s-N})\, dy \le c_{37}\int_{|x-z|^{s-N}}^\infty t^{-1-p_2^*} f(t)\, dt.$$

Therefore, by (1.9),

$$\lim_{x \to z} I_2 \le c_{19} \lim_{x \to z} \int_{|x-z|^{s-N}}^\infty t^{-1-p_2^*} f(s)\, ds = 0. \tag{3.17}$$

Finally, we estimate I_3. Therefore, for every $y \in \mathcal{D}_3$, $|y - z| \le 3|x - y|$, we have

$$I_3 \le c_{20}|x - z|^N \int_{\mathcal{D}_3} |y - z|^{s-N} f(|y - z|^{s-N})\, dy \le c_{21}|x - z|^N \int_0^{|x-z|^{s-N}} t^{-\frac{N}{N-s}} f(t)\, dt. \tag{3.18}$$

Put

$$g_1(r) = \int_0^{r^{s-N}} t^{-\frac{N}{N-s}} f(t)\, dt, \quad g_2(r) = r^{-N}.$$

If $\lim_{r\to 0} g_1(r) < \infty$, then $\lim_{x\to z} I_3 = 0$ by (3.18). Otherwise, $\lim_{r\to 0} g_1(r) = \infty = \lim_{r\to 0} g_2(r)$. Therefore, by L'Hôpital's rule,

$$\lim_{r\to 0} \frac{g_1(r)}{g_2(r)} = \lim_{r\to 0} \frac{g_1'(r)}{g_2'(r)} = \lim_{r\to 0} \frac{N - s}{N} r^{N+s} f(r^{s-N}) = 0. \tag{3.19}$$

By combining (3.18) and (3.19), we obtain

$$\lim_{x\to z} I_3 \le c_{22} \lim_{x\to z} |x - z|^N \int_0^{|x-z|^{s-N}} t^{-\frac{N}{N-s}} f(t)\, dt = 0. \tag{3.20}$$

We deduce (3.14) by gathering (3.16), (3.17) and (3.20). □

Proof of Theorem 1.6. From Theorem 1.5 we get

$$kM_s^\Omega(x, z) - \mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot\,, z))](x) \le u_{k,z}^\Omega(x) \le kM_s^\Omega(x, z),$$

which implies

$$k - \frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot\,, z))](x)}{M_s^\Omega(x, z)} \le \frac{u_{k,z}^\Omega(x)}{M_s^\Omega(x, z)} \le k.$$

We derive (1.10) due to Proposition 3.1. □

3.2 Power Absorption

In this subsection, we assume that $0 \in \partial\Omega$. Let $0 < p < p_2^*$ and denote by u_k^Ω the unique solution of (1.11). By Theorem 1.5, $u_k^\Omega \le kM_s^\Omega(\cdot\,, 0)$ and $k \mapsto u_k^\Omega$ is increasing. Therefore, it is natural to investigate $\lim_{k\to\infty} u_k^\Omega$.

For any $\ell > 0$, put

$$T_\ell[u](y) := \ell^{\frac{2s}{p-1}} u(\ell y), \quad y \in \Omega_\ell := \ell^{-1}\Omega.$$

If u is a solution of (1.14) in Ω, then $T_\ell[u]$ is a solution of (1.14) in Ω_ℓ.

By Corollary A.9, the function

$$x \mapsto U(x) = \ell_{s,p}|x|^{-\frac{2s}{p-1}}, \quad x \ne 0,$$

where $\ell_{s,p}$ is a positive constant, is a radial singular solution of

$$(-\Delta)^s u + u^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Lemma 3.2. *Assume $p \in (p_1^*, p_2^*)$. Then there exists a positive constant C depending on N, s, p and the C^2 characteristic of Ω such that the following holds: if u is a positive solution of* (1.14) *satisfying $u \le U$ in Ω, then there holds*

$$u(x) \le C\rho(x)^s |x|^{-\frac{(p+1)s}{p-1}} \quad \textit{for all } x \in \Omega. \tag{3.21}$$

Proof. Let $P \in (\partial\Omega \setminus \{0\}) \cap B_1(0)$ and put

$$d = d(P) := \frac{1}{2}|P| < \frac{1}{2}.$$

Put

$$u_d(y) = T_d[u](y), \quad y \in \Omega_d := d^{-1}\Omega.$$

Then u_d is a solution of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega_d, \\ u = 0 & \text{in } (\Omega_d)^c. \end{cases} \tag{3.22}$$

Moreover,

$$u_d(y) \le T_d[U](y) = d^{\frac{2s}{p-1}} U(\,dy) = \ell_{s,p}|y|^{-\frac{2s}{p-1}} = U(y).$$

Put $P_d = d^{-1}P$ and let β_0 be the constant in Proposition 2.8. We may assume $\beta_0 \le \frac{1}{4}$. Let $\zeta_P \in C^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$ in $\mathbb{R}^N$, $\zeta = 0$ in $B_{\beta_0}(P_d)$ and $\zeta = 1$ in $\mathbb{R}^N \setminus B_{2\beta_0}(P_d)$. Let $\eta_d \in C(\overline{\Omega}_d)$ be the solution of (3.4) with Ω replaced by Ω_d. For $l > 0$, denote

$$V_{d,l} := \zeta_P U + l\eta_d.$$

We will compare u_d with $V_{d,l}$.

Step 1: We show that $V_{d,l}$ is a supersolution of (3.22) *for l large enough.* For $y \in \Omega_d \setminus B_{4\beta_0}(P_d)$, we have $\zeta_P(y) = 1$, and hence

$$\begin{aligned} (-\Delta)^s(\zeta_P U)(y) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(y)} \frac{U(y) - \zeta_P(z)U(z)}{|y-z|^{N+2s}}\,dz \\ &= (-\Delta)^s U(y) + \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(y)} \frac{U(z) - \zeta_P(z)U(z)}{|y-z|^{N+2s}}\,dz \\ &\ge (-\Delta)^s U(y) - \int_{B_{\frac{1}{2}}(P_d)} \frac{U(z)}{|y-z|^{N+2s}}\,dz \\ &\ge (-\Delta)^s U(y) - c_{26}, \end{aligned}$$

where $c_{26} = c_{26}(N, s, p, \beta_0)$. Since

$$(\Omega_d \cap B_{2\beta_0}(0)) \subset (\Omega_d \setminus B_{4\beta_0}(P_d)),$$

it follows that, for any $y \in \Omega_d \cap B_{2\beta_0}(0) \setminus \{0\}$,

$$\begin{aligned} (-\Delta)^s V_{d,l}(y) + (V_{d,l}(y))^p &= (-\Delta)^s(\zeta_P U)(y) + l(-\Delta)^s\eta_d(y) + (\zeta_P(y)U(y) + l\eta_d(y))^p \\ &\ge (-\Delta)^s U(y) - c_{26} + l + U(y)^p. \end{aligned}$$

Therefore, if we choose $l \ge c_{26}$, then

$$(-\Delta)^s V_{d,l} + (V_{d,l})^p \ge 0 \quad \text{in } \Omega_d \cap B_{2\beta_0}(0) \setminus \{0\}. \tag{3.23}$$

Next we see that there exists $c_{27} > 0$ such that

$$|(-\Delta)^s(\zeta_P U)| \le c_{27} \quad \text{in } \Omega_d \setminus B_{2\beta_0}(0).$$

Consequently,

$$(-\Delta)^s V_{d,l} = (-\Delta)^s(\zeta_P U) + l(-\Delta)^s\eta_d \ge -c_{27} + l.$$

Therefore, if we choose $l \ge c_{27}$, then

$$(-\Delta)^s V_{d,l} \ge 0 \quad \text{in } \Omega_d \setminus B_{2\beta_0}(0). \tag{3.24}$$

By combining (3.23) and (3.24), for $l \ge \max\{c_{26}, c_{27}\}$, we deduce that $V_{d,l}$ is a supersolution of (3.22).

Step 2: We show that $u_d \le V_{d,l}$ in Ω_d. By contradiction, we assume that there exists $x_0 \in \Omega_d$ such that

$$(u_d - V_{d,l})(x_0) = \max_{x\in\Omega_d}(u_d - V_{d,l}) > 0.$$

Then $(-\Delta)^s(u_d - V_{d,l})(x_0) \geq 0$. It follows that

$$0 \leq (-\Delta)^s(u_d - V_{d,l})(x_0) \leq -(u_d(x_0)^p - V_{d,l}(x_0)^p) < 0.$$

This contradiction implies that $u_d \leq V_{d,l}$ in Ω_d.

Step 3: End of proof. From step 2 we deduce that

$$u_d \leq l\eta_d \quad \text{in } \Omega_d \cap B_{\beta_0}(P_d).$$

We note that $\eta_d(y) \leq c\,\mathrm{dist}(y, \partial\Omega_d)^s$ for every $y \in \Omega_d$. Here the constant c depends on N, s and the C^2 characteristic of Ω_d. Since $d < \frac{1}{2}$, a C^2 characteristic of Ω_d can be taken as a C^2 characteristic of Ω. Therefore, the constant c can be taken independently of P. Consequently,

$$u_d(y) \leq lc\,\mathrm{dist}(y, \partial\Omega_d)^s \quad \text{for all } y \in \Omega_d \cap B_{\beta_0}(P_d).$$

This implies

$$u(x) \leq c'\rho(x)^s d^{-\frac{(p+1)s}{p-1}} \quad \text{for all } x \in \Omega \cap B_{d\beta_0}(P). \tag{3.25}$$

Put

$$\mathcal{F}_1 := \Omega_{\beta_0} \cap B_{\frac{1}{1+\beta_0}}(0) \cap \{x : \rho(x) \leq \beta_0|x|\}, \quad \mathcal{F}_2 := \Omega_{\beta_0} \cap B_{\frac{1}{1+\beta_0}}(0) \cap \{x : \rho(x) > \beta_0|x|\}.$$

If $x \in \mathcal{F}_1$, then let $P \in \partial\Omega \setminus \{0\}$ such that $\rho(x) = |x - P|$. It follows that

$$\frac{1}{2}(1 - \beta_0)|x| < d = \frac{1}{2}|P| \leq \frac{1}{2}(1 + \beta_0)|x| < \frac{1}{2}. \tag{3.26}$$

By combining (3.25) and (3.26), we get

$$u(x) \leq c'(1 - \beta_0)^{-\frac{(p+1)s}{p-1}}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}.$$

If $x \in \mathcal{F}_2$, then (3.21) follows from the assumption $u \leq U$. Thus (3.21) holds for every $x \in \Omega_{\beta_0} \cap B_{1/(1+\beta_0)}(0)$. If $x \in \Omega \setminus B_{1/(1+\beta_0)}(0)$, then by an argument similar to the one in step 1 and step 2 without similarity transformation, we deduce that there exist constants c and $\tilde{\beta} \in (0, 1/(2(1 + \beta_0)))$ depending on N, s, p and the C^2 characteristic of Ω such that (3.21) holds in $B_{\tilde{\beta}}(P) \cap \Omega$ for every $P \in \partial\Omega \setminus B_{1/(1+\beta_0)}(0)$. Finally, since $u \leq U$, inequality (3.21) holds in $D_{\tilde{\beta}/2} = \{x \in \Omega : \rho(x) > \tilde{\beta}/2\}$. Thus (3.21) holds in Ω. □

Lemma 3.3. *Let $p \in (0, p_2^*)$. There exists a constant $c = c(N, s, p, \Omega) > 0$ such that for any $x \in \Omega$ and $z \in \partial\Omega$ there holds*

$$\mathbb{G}_s^\Omega[M_s^\Omega(\cdot, z)^p](x) \leq \begin{cases} c\rho(x)^s|x - z|^{s-(N-s)p} & \text{if } \frac{s}{N-s} < p < p_2^*, \\ -c\rho(x)^s \ln|x - z| & \text{if } p = \frac{s}{N-s}, \\ c\rho(x)^s & \text{if } 0 < p < \frac{s}{N-s}. \end{cases} \tag{3.27}$$

Proof. We use an argument similar to the one in the proof of Proposition 3.1. It is easy to see that for every $x \in \Omega$ and $z \in \partial\Omega$,

$$\mathbb{G}_s^\Omega[M_s^\Omega(\cdot, z)^p](x) \leq c_{23}\rho(x)^s \int_\Omega |x - y|^{-N}|y - z|^{(s-N)p} \min\{|x - y|^s, |y - z|^s\}\,dy.$$

Let $\mathcal{D}_i$, $i = 1, 2, 3$, be as in (3.15) and put

$$J_i := \rho(x)^s \int_{\mathcal{D}_i} |x - y|^{-N}|y - z|^{(s-N)p} \min\{|x - y|^s, |y - z|^s\}\,dy.$$

Now, by proceeding as in the proof of Proposition 3.1, we deduce easily that there is a positive constant $c_{24} = c_{24}(N, s, p, \Omega)$ such that

$$J_i \leq c_{24}\rho(x)^s|x - z|^{s-(N-s)p}, \quad i = 1, 2, \tag{3.28}$$

and

$$J_3 \le c_{24}\rho(x)^s \int_{|x-z|/2}^{\operatorname{diam}(\Omega)} r^{s-1-(N-s)p}\,dr \le \begin{cases} c_{25}\rho(x)^s|x-z|^{s-(N-s)p} & \text{if } \frac{s}{N-s} < p < p_2^*, \\ -c_{25}\rho(x)^s \ln|x-z| & \text{if } p = \frac{s}{N-s}, \\ c_{25}\rho(x)^s & \text{if } 0 < p < \frac{s}{N-s}. \end{cases} \tag{3.29}$$

Combining (3.28) and (3.29) implies (3.27). □

Proposition 3.4. *Assume $p \in (p_1^*, p_2^*)$. Then $u_\infty^\Omega := \lim_{k\to 0} u_k^\Omega$ is a positive solution of* (1.14). *Moreover, there exists $c = c(N,s,p,\Omega) > 0$ such that*

$$c^{-1}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \le u_\infty^\Omega(x) \le c\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \quad \textit{for all } x \in \Omega. \tag{3.30}$$

Proof. We first claim that for any $k > 0$,

$$u_k^\Omega \le U \quad \text{in } \Omega. \tag{3.31}$$

Indeed, by (2.4),

$$u_k^\Omega(x) \le kM_s^\Omega(x,0) \le c_{28}k\rho(x)^s|x|^{-N} \le c_{28}k|x|^{s-N} \quad \text{for all } x \in \Omega.$$

Since $p < p_2^*$, it follows that

$$\lim_{\Omega\ni x\to 0} \frac{u_k^\Omega(x)}{U(x)} = 0.$$

By proceeding as in step 2 of the proof of Lemma 3.2, we deduce that $u_k^\Omega \le U$ in Ω.

Consequently, $u_\infty^\Omega := \lim_{k\to\infty} u_k^\Omega$ is a solution of (1.14) vanishing on $\partial\Omega \setminus \{0\}$ and satisfying $u_\infty^\Omega \le U$ in Ω. In the light of Lemma 3.2, we obtain the upper bound in (3.30).

Next we prove the lower bound in (3.30). By (2.4) and Lemma 3.3, for any $k > 0$ and $x \in \Omega$ we have

$$u_k^\Omega(x) \ge kM_s^\Omega(x,0) - k^p\mathbb{G}_s^\Omega[M_s^\Omega(\cdot,0)^p](x) \ge c_{29}^{-1}k\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}k^{p-1}|x|^{N+s-(N-s)p}).$$

For $x \in \Omega$, one can choose $r > 0$ such that $x \in \Omega \cap (B_{2r}(0) \setminus B_r(0))$. Choose

$$k = ar^{-\frac{N+s-(N-s)p}{p-1}},$$

where $a > 0$ will be made precise later on. Then

$$u_k^\Omega(x) \ge c_{31}a\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}(1 - c_{29}c_{30}a^{p-1}).$$

By choosing $a = (2c_{29}c_{30})^{-\frac{1}{p-1}}$, we deduce for any $x \in \Omega$ that there exists $k > 0$ depending on $|x|$ such that

$$u_k^\Omega(x) \ge c_{32}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}.$$

Since $u_\infty^\Omega \ge u_k^\Omega$ in Ω, we obtain the first inequality in (3.30). □

Proposition 3.5. *Assume $0 < p \le p_1^*$. There exist $k_0 = k_0(N,s,p)$ and $c = c(N,s,p,\Omega)$ such that the following holds: there exists a decreasing sequence of positive numbers $\{r_k\}$ such that $\lim_{k\to\infty} r_k = 0$ and for any $k > k_0$,*

$$u_k^\Omega(x) \ge \begin{cases} c\rho(x)^s|x|^{-N-s} & \textit{if } 0 < p < p_1^*, \\ c\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1} & \textit{if } p = p_1^*, \end{cases} \quad \textit{for all } x \in \Omega \setminus B_{r_k}(0). \tag{3.32}$$

Proof. For any $\ell > 0$, we have

$$u_\ell^\Omega(x) \ge \ell M_s^\Omega(x,0) - \ell^p\mathbb{G}_s^\Omega[M_s^\Omega(\cdot,0)^p](x) \quad \text{for all } x \in \Omega. \tag{3.33}$$

Case 1: $p \in (\frac{s}{N-s}, p_1^*)$. Put

$$k_1 := (2c_{29}c_{30})^{\frac{s}{N+2s-Np}}$$

and take $k > k_1$. For $\ell > 0$, put $r_\ell = \ell^{-1/s}$; then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$; then one can choose $\ell \in (\max(2^{-s}k, k_1), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{r_\ell/2}(0))$. From (3.33) (2.4) and (3.27) we get

$$\begin{aligned} u_\ell^\Omega(x) &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}|x|^{N+s-(N-s)p}) \\ &\geq c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}(1 - c_{29}c_{30}r_\ell^{N+2s-Np}) \\ &\geq (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\geq c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here the first estimate holds since $\frac{N}{N-s} < p < p_2^*$, and the third estimate holds since $p < p_1^*$ and $\ell > k_1$. Since $k > \ell$, we deduce that

$$u_k^\Omega(x) \geq c_{33}\rho(x)^s|x|^{-N-s} \quad \text{for all } x \in \Omega \setminus B_{r_k}(0). \tag{3.34}$$

Case 2: $p = \frac{s}{N-s}$. Put

$$k_2 = \Big(\frac{2c_{29}c_{30}(1+s)}{s}\Big)^{\frac{s}{N-sp}}$$

and take $k > k_2$. For $\ell > 0$, put $r_\ell = \ell^{-1/s}$; then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$; then one can choose $\ell \in (\max(2^{-s}k, k_2), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{r_\ell/2}(0))$. From (3.33), (2.4) and Lemma 3.3 we get

$$\begin{aligned} u_\ell^\Omega(x) &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 + c_{29}c_{30}\ell^{p-1}|x|^N \ln|x|) \\ &\geq c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}\Big(1 + c_{29}c_{30}r_\ell^{N+s-sp}\ln\Big(\frac{r_\ell}{2}\Big)\Big) \\ &\geq (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\geq c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here the third estimate holds since $\ell > k_2$ and $N - sp > 0$. Therefore, (3.34) holds.

Case 3: $p \in (0, \frac{s}{N-s})$. Put

$$k_3 = (2c_{29}c_{30})^{\frac{s}{N+s-sp}}$$

and take $k > k_3$. For $\ell > 0$, put $r_\ell = \ell^{-1/s}$; then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$; then one can choose $\ell \in (\max(2^{-s}k, k_3), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{r_\ell/2}(0))$. From (3.33), (2.4) and (3.27) we get

$$\begin{aligned} u_\ell^\Omega(x) &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}|x|^N) \\ &\geq c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}(1 - c_{29}c_{30}r_\ell^{N+s-sp}) \\ &\geq (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\geq c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here the third estimate holds since $\ell > k_3$ and $N + s - sp > 0$. Therefore, (3.34) holds.

Case 4: $p = p_1^*$. Put

$$k_4 = \exp\big((2c_{29}c_{30})^{\frac{s}{N+s-(N-s)p}}\big)$$

and take $k > k_4$. For $\ell > 0$, put $r_\ell = (\ell\ln(\ell))^{-1/s}$; then $\ell\ln(\ell) = r_\ell^{-s}$ and $\ell < r_\ell^{-s}$ when $\ell > 3$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$; then one can choose $\ell \in (\max(2^{-s}k, k_4), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{r_\ell/2}(0))$. From (3.33), (2.4) and (3.27) we get

$$\begin{aligned} u_\ell^\Omega(x) &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}|x|^{N+s-(N-s)p}) \\ &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}(\ell\ln(\ell))^{-\frac{N+s-(N-s)p}{s}}) \\ &= c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ln(\ell)^{-\frac{N+s-(N-s)p}{s}}) \\ &\geq (2c_{29})^{-1}\ell\rho(x)^s|x|^{-N} \\ &\geq c_{34}\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1}. \end{aligned}$$

Here the inequality holds since $p = p_1^*$, and the last estimate follows from the estimate

$$\ell = \frac{r_\ell^{-s}}{\ln(\ell)} > \frac{|x|^{-s}}{-s2^s \ln|x|}.$$

Since $u_k^\Omega(x) \geq u_\ell^\Omega(x)$, we derive

$$u_k^\Omega(x) \geq c_{34}\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1}.$$

By putting $k_0 := \max(k_1, k_2, k_3, k_4)$, we obtain (3.32). □

Proposition 3.6. *Assume* $0 < p \leq p_1^*$. *Then* $\lim_{k\to\infty} u_k^\Omega(x) = \infty$ *for every* $x \in \Omega$.

Proof. The proposition can be obtained by adapting the argument in the proof of [9, Theorem 1.2]. Let $r_0 > 0$ and put

$$\theta_k := \int\limits_{B_{r_0}(0)} u_k^\Omega(x)\,dx.$$

Then

$$\theta_k \geq c \int\limits_{(B_{r_0}\cap\Omega)\setminus B_{r_k}(0)} \rho(x)^s|x|^{N-s}(-\ln|x|)^{-1}\,dx,$$

which implies

$$\lim_{k\to\infty} \theta_k = \infty. \tag{3.35}$$

Fix $y_0 \in \Omega \setminus \overline{B}_{r_0}(0)$ and set $\delta := \frac{1}{2}\min\{\rho(y_0), |y_0| - r_0\}$. By [13, Lemma 2.4], there exists a unique classical solution w_k of the following problem:

$$\begin{cases} (-\Delta)^s w_k + w_k^p = 0 & \text{in } B_\delta(y_0), \\ w_k = 0 & \text{in } \mathbb{R}^N \setminus (B_\delta(y_0) \cup B_{r_0}(0)), \\ w_k = u_k^\Omega & \text{in } B_{r_0}(0). \end{cases}$$

By [13, Lemma 2.2],

$$u_k^\Omega \geq w_k \quad \text{in } B_\delta(y_0). \tag{3.36}$$

Next put $\tilde{w}_k := w_k - \chi_{B_{r_0}(0)} u_k$; then $\tilde{w}_k = w_k$ in $B_\delta(y_0)$. Moreover, for $x \in B_\delta(y_0)$,

$$\begin{aligned} (-\Delta)^s \tilde{w}_k(x) &= \lim_{\varepsilon\to 0} \int\limits_{B_\delta(y_0)\setminus B_\varepsilon(x)} \frac{w_k(x) - w_k(z)}{|z - x|^{N+2s}}\,dz + \lim_{\varepsilon\to 0} \int\limits_{B_\delta^c(y_0)\setminus B_\varepsilon(x)} \frac{w_k(x)}{|z - x|^{N+2s}}\,dz \\ &= \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{w_k(x) - w_k(z)}{|z - x|^{N+2s}}\,dz + \int\limits_{B_{r_0}(0)} \frac{u_k^\Omega(z)}{|z - x|^{N+2s}}\,dz \\ &\geq (-\Delta)^s w_k(x) + A\theta_k, \end{aligned}$$

where $A = (|y_0| + r_0)^{-N-2s}$. It follows that, for $x \in B_\delta(y_0)$,

$$(-\Delta)^s \tilde{w}_k(x) + \tilde{w}_k^p(x) \geq (-\Delta)^s w_k(x) + w_k^p(x) + A\theta_k = A\theta_k.$$

Therefore, $\tilde{w}_k \in C(\overline{B_\delta(y_0)})$ is a supersolution of

$$\begin{cases} (-\Delta)^s w + w^p = A\theta_k & \text{in } B_\delta(y_0), \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_\delta(y_0). \end{cases} \tag{3.37}$$

Let $\eta_0 \in C(\overline{B_\delta(y_0)})$ be the unique solution of

$$\begin{cases} (-\Delta)^s \eta_0 = 1 & \text{in } B_\delta(y_0), \\ \eta_0 = 0 & \text{in } \mathbb{R}^N \setminus B_\delta(y_0). \end{cases}$$

We can choose k large enough so that the function

$$\frac{\eta_0(A\theta_k)^{\frac{1}{p}}}{2\max_{\mathbb{R}^N}\eta_0}$$

is a subsolution of (3.37). By [13, Lemma 2.2], we obtain

$$\tilde{w}_k(x) \geq \frac{\eta_0(A\theta_k)^{\frac{1}{p}}}{2\max_{\mathbb{R}^N}\eta_0} \quad \text{for all } x \in B_\delta(y_0). \tag{3.38}$$

Put

$$\underline{c} := \min_{x\in B_\delta(y_0)} \frac{\eta_0}{2\max_{\mathbb{R}^N}\eta_0}.$$

Then we derive from (3.38) that

$$w_k(x) \geq \underline{c}(A\theta_k)^{\frac{1}{p}} \quad \text{for all } x \in B_\delta(y_0). \tag{3.39}$$

By combining (3.35), (3.36) and (3.39), we deduce that

$$\lim_{k\to\infty} u_k^\Omega(x) = \infty \quad \text{for all } x \in B_{\frac{\delta}{2}}(y_0).$$

This implies

$$\lim_{k\to\infty} u_k^\Omega(x) = \infty \quad \text{for all } x \in \Omega. \qquad \square$$

Theorem 3.7. *Assume $p \in (1, p_2^*)$ and that either $\Omega = \mathbb{R}_+^N := \{x = (x', x_N) : x_N > 0\}$ or $\partial\Omega$ is compact with $0 \in \partial\Omega$. Then, for any $k > 0$, there exists a unique solution u_k^Ω of problem* (1.11) *satisfying $u_k^\Omega \leq kM_s^\Omega(\cdot, 0)$ in Ω and*

$$\lim_{|x|\to 0} \frac{u_k^\Omega(x)}{M_s^\Omega(x, 0)} = k.$$

Moreover, the map $k \mapsto u_k^\Omega$ is increasing.

Proof. We divide the proof into two steps.

Step 1: Existence. For $R > 0$, we set $\Omega_R = \Omega \cap B_R$ and let $u := u_k^{\Omega_R}$ be the unique solution of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega_R, \\ \operatorname{tr}_s(u) = k\delta_0, & \\ u = 0 & \text{on } \Omega_R^c. \end{cases}$$

Then

$$u_k^{\Omega_R}(x) \leq kM_s^{\Omega_R}(x, 0) \quad \text{for all } x \in \Omega_R. \tag{3.40}$$

Since $R \mapsto M_s^{\Omega_R}(\cdot, 0)$ is increasing, it follows from (1.10) that $R \mapsto u_k^{\Omega_R}$ is increasing too with the limit u^* and there holds

$$u^*(x) \leq kM_s^\Omega(x, 0) \quad \text{for all } x \in \Omega.$$

From (3.40) we deduce that

$$u_k^{\Omega_R}(x) \leq ck|x|^{s-N} \qquad \text{for all } x \in \Omega_R,$$

where c depends only on N, s and the C^2 characteristic of Ω. Hence by the regularity up to the boundary [27], $\{u_k^{\Omega_R}\}$ is uniformly bounded in $C^s_{\mathrm{loc}}(\overline{\Omega} \setminus B_\varepsilon)$ and in $C^{2s+\alpha}_{\mathrm{loc}}(\Omega \setminus B_\varepsilon)$ for any $\varepsilon > 0$. Therefore, $\{u_k^{\Omega_R}\}$ converges locally uniformly, as $R \to \infty$, to $u^* \in C(\overline{\Omega} \setminus \{0\}) \cap C^{2s+\alpha}(\Omega)$. Thus u^* is a positive solution of (1.14). Moreover, by combining (1.10), (3.40) and the facts that $M_s^{\Omega_R} \uparrow M_s^\Omega$ and $u_k^{\Omega_R} \uparrow u_k^\Omega$, we deduce that $\operatorname{tr}_s(u^*) = k\delta_0$ and

$$\lim_{\Omega\ni x\to 0} \frac{u^*(x)}{M_s^\Omega(x, 0)} = k.$$

Step 2: Uniqueness. Suppose u and u' are two weak solutions of (1.14) satisfying $\max\{u,u'\} \le kM_s^\Omega(\cdot,0)$ in Ω and

$$\lim_{\Omega\ni x\to 0}\frac{u(x)}{M_s^\Omega(x,0)} = \lim_{\Omega\ni x\to 0}\frac{u'(x)}{M_s^\Omega(x,0)} = k. \tag{3.41}$$

Take $\varepsilon > 0$ and put $u_\varepsilon := (1+\varepsilon)u' + \varepsilon$, $v := (u-u_\varepsilon)_+$. Then by (3.41) there exists a smooth bounded domain $G \subset \Omega$ such that $v = 0$ in G^c and $\mathrm{tr}_s^G(v) = 0$. In the light of Kato's inequality, we derive $(-\Delta)^s v \le 0$ in G. Moreover, $v \le kM_s^\Omega(\cdot,0)$ in G. By Lemma 2.12, we obtain $v = 0$ in G, and therefore $u \le (1+\varepsilon)u' + \varepsilon$ in Ω. Letting $\varepsilon \to 0$ yields $u \le u'$ in Ω. By permuting the role of u and u', we derive $u = u'$ in Ω.

By an argument similar to the one in step 2, we can show that $k \mapsto u_k^\Omega$ is increasing. □

Proof of Theorem 1.8. We have two cases.

(i) Case 1: $p_1^* < p < p_2^*$. Since $\partial\Omega \in C^2$, there exist two open balls B and B' such that $B \subset \Omega \subset B'^c$ and $\partial B \cap \partial B' = \{0\}$. Since $M_s^B(x,0) \le M_s^\Omega(x,0) \le M_s^{B'^c}(x,0)$, it follows from Theorem 3.7 that

$$u_k^B \le u_k^\Omega \le u_k^{B'^c}, \tag{3.42}$$

where the first inequality holds in B and the second inequality holds in Ω.

Let $\mathcal{O}$ be B, Ω or B'^c. Because of the uniqueness, we have

$$T_\ell[u_k^{\mathcal{O}}] = u^{\mathcal{O}_\ell}_{k\ell^{\frac{2s}{p-1}+1-N}} \quad \text{for all } \ell > 0, \tag{3.43}$$

with $\mathcal{O}_\ell = \ell^{-1}\mathcal{O}$. By Theorem 3.7, the sequence $\{u_k^{\mathcal{O}}\}$ is increasing, and by (3.31) we have $u_k^{\mathcal{O}} \le U$. It follows that $\{u_k^{\mathcal{O}}\}$ converges to a function $u_\infty^{\mathcal{O}}$ which is a positive solution of (1.14) with Ω replaced by $\mathcal{O}$.

Step 1: $\mathcal{O} := \mathbb{R}_+^N$. Then $\mathcal{O}_\ell = \mathbb{R}_+^N$. Letting $k \to \infty$ in (3.43) yields

$$T_\ell[u_\infty^{\mathbb{R}_+^N}] = u_\infty^{\mathbb{R}_+^N} \quad \text{for all } \ell > 0.$$

Therefore, $u_\infty^{\mathbb{R}_+^N}$ is self-similar, and thus it can be written in the separable form

$$u_\infty^{\mathbb{R}_+^N}(x) = u_\infty^{\mathbb{R}_+^N}(r,\sigma) = r^{-\frac{2s}{p-1}}\omega(\sigma),$$

where $r = |x|$, $\sigma = \frac{x}{|x|} \in S^{N-1}$ and ω satisfies (1.13). Since $p_1^* < p < p_2^*$, it follows from Theorem 1.7 that $\omega = \omega^*$, the unique positive solution of (1.13). This means

$$u_\infty^{\mathbb{R}_+^N}(x) = r^{-\frac{2s}{p-1}}\omega^*(\sigma).$$

This implies (3.30).

Step 2: $\mathcal{O} := B$ *or* B'^c. In accordance with our previous notations, we set $B_\ell = \ell^{-1}B$ and $(B'^c)_\ell = \ell^{-1}B'^c$ for $\ell > 0$, and we have

$$T_\ell[u_\infty^B] = u_\infty^{B_\ell} \quad \text{and} \quad T_\ell[u_\infty^{B'^c}] = u_\infty^{(B'^c)_\ell} \tag{3.44}$$

and

$$u_\infty^{B_{\ell'}} \le u_\infty^{B_\ell} \le u_\infty^{\mathbb{R}_+^N} \le u_\infty^{(B'^c)_\ell} \le u_\infty^{(B'^c)_{\ell''}}, \quad 0 < \ell \le \ell', \ell'' \le 1.$$

When $\ell \to 0$, then $u_\infty^{B_\ell} \uparrow \underline{u}_\infty^{\mathbb{R}_+^N}$ and $u_\infty^{(B'^c)_\ell} \downarrow \overline{u}_\infty^{\mathbb{R}_+^N}$, where $\underline{u}_\infty^{\mathbb{R}_+^N}$ and $\overline{u}_\infty^{\mathbb{R}_+^N}$ are positive solutions of (3.31) in $\mathbb{R}_+^N$ such that

$$u_\infty^{B_\ell} \le \underline{u}_\infty^{\mathbb{R}_+^N} \le u_\infty^{\mathbb{R}_+^N} \le \overline{u}_\infty^{\mathbb{R}_+^N} \le u_\infty^{(B'^c)_\ell}, \quad 0 < \ell \le 1.$$

Furthermore, there also holds for $\ell, \ell' > 0$,

$$T_{\ell'\ell}[u_\infty^B] = T_{\ell'}[T_\ell[u_\infty^B]] = u_\infty^{B_{\ell\ell'}} \text{ and } T_{\ell'\ell}[u_\infty^{B'^c}] = T_{\ell'}[T_\ell[u_\infty^{B'^c}]] = u_\infty^{(B_c')_{\ell\ell'}}.$$

Letting $\ell \to 0$ and using (3.44) and the above convergence, we obtain

$$\underline{u}_\infty^{\mathbb{R}_+^N} = T_{\ell'}[\underline{u}_\infty^{\mathbb{R}_+^N}] \quad \text{and} \quad \overline{u}_\infty^{\mathbb{R}_+^N} = T_{\ell'}[\overline{u}_\infty^{\mathbb{R}_+^N}], \qquad \ell' > 0.$$

Again this implies that $\underline{u}_\infty^{\mathbb{R}_+^N}$ and $\overline{u}_\infty^{\mathbb{R}_+^N}$ are separable solutions of (1.12). Since $p_1^* < p < p_2^*$, by Theorem 1.7,

$$\underline{u}_\infty^{\mathbb{R}_+^N}(x) = \overline{u}_\infty^{\mathbb{R}_+^N}(x) = u_\infty^{\mathbb{R}_+^N}(x) = r^{-\frac{2s}{p-1}}\omega^*(\sigma) \quad \text{with } r = |x|,\ \sigma = \frac{x}{|x|},\ x \neq 0.$$

Step 3: End of the proof. From (3.42) and (3.44) there holds

$$u_\infty^{B_\ell} \le T_\ell[u_\infty^\Omega] \le u_\infty^{(B'^c)_\ell}, \quad 0 < \ell \le 1. \tag{3.45}$$

Since the left-hand side and the right-hand side of (3.45) converge to the same function $u_\infty^{\mathbb{R}_+^N}$, we obtain

$$\lim_{\ell\to 0} \ell^{\frac{2s}{p-1}} u_\infty^\Omega(\ell x) = |x|^{-\frac{2s}{p-1}}\omega^*\Big(\frac{x}{|x|}\Big),$$

and this convergence holds in any compact subset of Ω. Taking $|x| = 1$, we derive (1.15). Estimate (3.30) follows from Proposition 3.4.

(ii) Case 2: $0 < p \le p_1^*$. Then by Proposition 3.6, $\lim_{k\to\infty} u_k^\Omega(x) = \infty$ for every $x \in \Omega$. □

A Separable Solutions

A.1 Separable s-Harmonic Functions

We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$. Consider the following parametric representation of the unit sphere:

$$S^{N-1} = \Big\{\sigma = (\cos\phi\,\sigma', \sin\phi) : \sigma' \in S^{N-2},\ -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}\Big\}.$$

Hence $x_N = r\sin\phi$. We define the spherical fractional Laplace–Beltrami operator $\mathcal{A}_s$ by

$$\mathcal{A}_s\omega(\sigma) := \lim_{\varepsilon\to 0} \mathcal{A}_{s,\varepsilon}\omega(\sigma)$$

with

$$\mathcal{A}_{s,\varepsilon}\omega(\sigma) := a_{N,s} \int\int_{\mathbb{R}_+\times S^{N-1}\setminus B_\varepsilon(\vec{\sigma})} \frac{(\omega(\sigma) - \omega(\eta))\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma, \eta\rangle)^{\frac{N}{2}+s}}\,dS(\eta)\,d\tau,$$

where $\vec{\sigma} = (1, \sigma)$. If $u : (r, \sigma) \mapsto u(r, \sigma) = r^{-\beta}\omega(\sigma)$ is s-harmonic in $\mathbb{R}^N \setminus \{0\}$, it satisfies, at least formally,

$$\mathcal{A}_s\omega - \mathcal{L}_{s,\beta}\omega = 0 \quad \text{on } S^{N-1},$$

where $\mathcal{L}_{s,\beta}$ is the integral operator

$$\mathcal{L}_{s,\beta}\omega(\sigma) := a_{N,s} \int_0^\infty \int_{S^{N-1}} \frac{(\tau^{-\beta} - 1)\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma, \eta\rangle)^{\frac{N}{2}+s}}\,\omega(\eta)\,dS(\eta)\,d\tau,$$

whenever this integral is defined. We will see in the next two lemmas that the role of the exponent $\beta_0 = N$ is fundamental for the definition of $\mathcal{L}_{s,\beta}\omega$.

Lemma A.1. *If $N \ge 2$, $s \in (0, 1)$, $\beta < N$ and $(\sigma, \eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ such that $\langle\sigma, \eta\rangle \neq 1$, we define*

$$B_{s,\beta}(\sigma, \eta) := \int_0^\infty \frac{(\tau^{-\beta} - 1)\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma, \eta\rangle)^{\frac{N}{2}+s}}\,d\tau.$$

Then the following statements hold:
(i) $B_{s,\beta}(\sigma, \eta) < 0$ *if and only if* $\beta < N - 2s$.
(ii) $B_{s,\beta}(\sigma, \eta) = 0$ *if and only if* $\beta = N - 2s$.
(iii) $B_{s,\beta}(\sigma, \eta) > 0$ *if and only if* $\beta > N - 2s$.

Proof. Since $\beta < N$, the integral in (A.1) is absolutely convergent. We write

$$B_{s,\beta}(\sigma,\eta) = \int_0^1 \frac{(\tau^{-\beta}-1)\tau^{N-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau + \int_1^\infty \frac{(\tau^{-\beta}-1)\tau^{N-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau =: I + II.$$

By the change of variable $\tau \mapsto \tau^{-1}$,

$$II = -\int_0^1 \frac{(\tau^{-\beta}-1)\tau^{N-1+c_s}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau,$$

where $c_s = \beta + 2s - N$. Since

$$B_{s,\beta}(\sigma,\eta) = \int_0^1 \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau, \tag{A.1}$$

the claim follows. □

As a byproduct of (A.1) we have the following monotonicity formula.

Lemma A.2. *If $N \geq 2$ and $s \in (0,1)$, then for any $(\sigma,\eta) \in S^{N-1}\times S^{N-1}$ the mapping $\beta \mapsto B_{s,\beta}(\sigma,\eta)$ is continuous and increasing from $(N-2s, N)$ onto $(0,\infty)$.*

In the next result we analyze the behavior of $B_{s,\beta}(\sigma,\eta)$ when $\sigma-\eta \to 0$ on S^{N-1}.

Lemma A.3. *Assume $N \geq 2$, $s \in (0,1)$ and $\beta < N$ with $\beta \neq N-2s$.*
(I) *If $N \geq 3$, there exists $c = c(N,\beta,s) > 0$ such that*

$$|B_{s,\beta}(\sigma,\eta)| \leq c|\sigma-\eta|^{3-N-2s} \quad \text{for all } (\sigma,\eta) \in S^{N-1}\times S^{N-1}. \tag{A.2}$$

(II) *If $N = 2$, then one of the following statements holds:*
 (i) $s > \frac{1}{2}$ *and* (A.2) *holds with $N = 2$.*
 (ii) $s = \frac{1}{2}$ *and*

$$|B_{s,\beta}(\sigma,\eta)| \leq c(-\ln|\sigma-\eta|+1) \quad \text{for all } (\sigma,\eta) \in S^1\times S^1.$$

 (iii) $0 < s < \frac{1}{2}$ *and*

$$|B_{s,\beta}(\sigma,\eta)| \leq c \quad \text{for all } (\sigma,\eta) \in S^1\times S^1.$$

Proof. First, notice that the quantity

$$\int_0^{\frac{1}{2}} \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau$$

is uniformly bounded with respect to (σ,η). The only possible singularity in the expression given in (A.1) occurs when $\langle\sigma,\eta\rangle = 1$ and $\tau = 1$. We write $\langle\sigma,\eta\rangle = 1-\frac{1}{2}\kappa^2$ and $t = 1-\tau$. Hence,

$$(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s} = (t^2+(1-t)\kappa^2)^{\frac{N}{2}+s} \approx \kappa^{N+2s}\Big(1+\Big(\frac{t}{\kappa}\Big)^2\Big)^{\frac{N}{2}+s}$$

as $t \to 0$. Moreover,

$$(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s}) = ((1-t)^{-\beta}-1)((1-t)^{N-1}-(1-t)^{N-1+c_s}) = c_s\beta t^2 + O(t^3) \quad \text{as } t \to 0.$$

Hence,

$$\int_{\frac{1}{2}}^1 \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,d\tau = \int_0^{\frac{1}{2}} \frac{((1-t)^{-\beta}-1)((1-t)^{N-1}-(1-t)^{N-1+c_s})}{(t^2+(1-t)\kappa^2)^{\frac{N}{2}+s}}\,dt$$

$$\approx c_s\kappa^{3-N-2s}\int_0^{\frac{1}{2\kappa}} \frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}\,dx.$$

If $N = 2$ and $s < \frac{1}{2}$, then

$$\left|\kappa^{1-2s}\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{1+s}}\,dx\right| \le M$$

for some $M > 0$ independent of κ. If $N = 2$ and $s = \frac{1}{2}$, then

$$\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{1+\frac{1}{2}}}\,dx = \ln\Big(\frac{1}{\kappa}\Big)(1+o(1)),$$

and if $N = 3$ or $N = 2$ and $s > \frac{1}{2}$, then

$$\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}\,dx \to \int_0^{\infty}\frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}\,dx$$

as $\kappa \to 0$. Since $\sigma, \eta \in S^{N-1}$, there holds $\kappa^2 = 2(1 - \langle\sigma,\eta\rangle) = |\sigma - \eta|^2$. Thus the claim follows. □

Proposition A.4. *Assume $N \ge 2$, $s \in (0, 1)$ and $\beta < N$ with $\beta \ne N - 2s$. Then $\omega \mapsto \mathcal{L}_{s,\beta}\omega$ is a continuous linear operator from $L^q(S^{N-1})$ into $L^r(S^{N-1})$ for any $1 \le q, r \le \infty$ such that*

$$\frac{1}{r} > \frac{1}{q} - \frac{2(1-s)}{N-1}. \tag{A.3}$$

Furthermore, $\mathcal{L}_{s,\beta}$ is a positive (resp. negative) operator if $\beta < N - 2s$ (resp. $N - 2s < \beta < N$).

Proof. By Lemma A.3, for any $\eta \in S^{N-1}$, we have $B_{s,\beta}(\cdot,\eta) \in L^a(S^{N-1})$ for all $1 < a < \frac{N-1}{N+2s-3}$ if $N \ge 3$ or $N = 2$ and $s > \frac{1}{2}$. Furthermore,

$$B_{s,\beta}(\cdot,\eta) \in \bigcap_{1\le a<\infty} L^a(S^1)$$

if $N = 2$ and $s = \frac{1}{2}$, and $B_{s,\beta}(\cdot,\eta)$ is uniformly bounded on S^1 if $N = 2$ and $0 < s < \frac{1}{2}$. The continuity result follows from Young's inequality and the sign assertion from Lemma A.1. □

The above calculations justify the name of the fractional Laplace–Beltrami operator given to $\mathcal{A}_s$ since we have the following relation.

Lemma A.5. *Assume $N \ge 2$ and $s \in (0, 1)$. Then*

$$\mathcal{A}_s\omega(\sigma) = b_{N,s}CPV\int_{S^{N-1}}\frac{(\omega(\sigma)-\omega(\eta))}{|\sigma-\eta|^{N-1+2s}}\,dS(\eta) + \mathcal{B}_s\omega(\sigma),$$

where $\mathcal{B}_s$ is a bounded linear operator from $L^q(S^{N-1})$ into $L^r(S^{N-1})$ for q and r satisfying (A.3) *and*

$$b_{N,s} := 2a_{N,s}\int_0^{\infty}\frac{dx}{(x^2+1)^{\frac{N}{2}+s}}.$$

Proof. If $(\sigma,\eta) \in S^{N-1} \times S^{N-1}$, we set $\langle\sigma,\eta\rangle = 1 - \frac{1}{2}\kappa^2$. Then

$$\int_0^{\infty}\frac{\tau^{N-1}\,d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} = \int_0^{1}\frac{(\tau^{N-1}+\tau^{2s-1})\,d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}.$$

Then we put $t = 1 - \tau$. Hence when $t \to 0$, we have after some straightforward computation

$$\frac{(\tau^{N-1}+\tau^{2s-1})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} = \frac{(2-(N+2s-2)t+O(t^2))\big(1+\frac{(N+2s)t\kappa^2}{2(t^2+\kappa^2)}+O((\frac{t\kappa^2}{t^2+2\kappa^2})^2)\big)}{(t^2+\kappa^2)^{\frac{N}{2}+s}} = \frac{2+2t+O(t^2)}{(t^2+\kappa^2)^{\frac{N}{2}+s}}.$$

This implies

$$\int_0^1 \frac{(\tau^{N-1}+\tau^{2s-1})\,d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} = 2\kappa^{1-N-2s}\int_0^{\frac{1}{\kappa}} \frac{dx}{(x^2+1)^{\frac{N}{2}+s}} + 2\kappa^{2-N-2s}\int_0^{\frac{1}{\kappa}} \frac{x\,dx}{(x^2+1)^{\frac{N}{2}+s}} + O(\kappa^{3-N-s})\int_0^{\frac{1}{\kappa}} \frac{x^2\,dx}{(x^2+1)^{\frac{N}{2}+s}}$$
$$= 2\kappa^{1-N-2s}\int_0^{\infty} \frac{dx}{(x^2+1)^{\frac{N}{2}+s}} + O(1) + O(\kappa^{3-N-s})\int_0^{\frac{1}{\kappa}} \frac{x^2\,dx}{(x^2+1)^{\frac{N}{2}+s}}.$$

Since $\kappa = |\sigma-\eta|$, the claim follows from Proposition A.4 and the kernel estimate in Lemma A.3. □

Lemma A.6. *Under the assumption of Lemma A.5, there holds*

$$\Big|\int_{S^{N-1}} \omega\mathcal{L}_{s,\beta}\omega\,dS\Big| \le c_{35}\int_{S^{N-1}} \omega^2\,dS \quad \text{for all } \omega\in L^2(S^{N-1}), \tag{A.4}$$

where

$$c_{35} = \int_0^1 \Big(\int_{S^{N-1}} \frac{dS(\eta)}{(1+\tau^2-2\tau\langle\mathbf{e}_N,\eta\rangle)^{\frac{N}{2}+s}}\Big)(\tau^{-\beta}-1)|\tau^{N-1}-\tau^{N-1+c_s}|\,d\tau.$$

Proof. There holds, by the Cauchy–Schwarz inequality,

$$\Big|\int_{S^{N-1}} \omega\mathcal{L}_{s,\beta}\omega\,dS\Big| \le \int_0^1 \Big(\int_{S^{N-1}}\int_{S^{N-1}} \frac{|\omega(\eta)|\,|\omega(\sigma)|\,dS(\eta)\,dS(\sigma)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\Big)(\tau^{-\beta}-1)|\tau^{N-1}-\tau^{N-1+c_s}|\,d\tau$$
$$\le \int_0^1 \Big(\int_{S^{N-1}}\int_{S^{N-1}} \frac{\omega^2(\eta)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\,dS(\eta)\,dS(\sigma)\Big)\times(\tau^{-\beta}-1)|\tau^{N-1}-\tau^{N-1+c_s}|\,d\tau$$
$$\le \int_{S^{N-1}} \Big(\int_0^1\Big(\int_{S^{N-1}} \frac{dS(\sigma)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\Big)(\tau^{-\beta}-1)|\tau^{N-1}-\tau^{N-1+c_s}|\,d\tau\Big)\omega^2(\eta)\,dS(\eta).$$

Since by invariance by rotation we have

$$\int_{S^{N-1}} \frac{dS(\sigma)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} = \int_{S^{N-1}} \frac{dS(\sigma)}{(1+\tau^2-2\tau\langle\mathbf{e}_N,\sigma\rangle)^{\frac{N}{2}+s}},$$

we derive (A.4). □

We denote the upper hemisphere of the unit sphere in $\mathbb{R}^N$ by $S_+^{N-1} = S^{N-1}\cap\mathbb{R}_+^N$.

Proposition A.7. *Let $N\ge 2$, $s\in(0,1)$ and $N-2s<\beta<N$. Then there exist a unique $\lambda_{s,\beta}>0$ and a unique (up to a homothety) positive $\psi_1\in W_0^{s,2}(S_+^{N-1})$ such that*

$$\mathcal{A}_s\psi_1 = \lambda_{s,\beta}\mathcal{L}_{s,\beta}\psi_1 \quad \text{in } S_+^{N-1}. \tag{A.5}$$

Furthermore, the mapping $\beta\mapsto\lambda_{s,\beta}$ is continuous and decreasing from $(N-2s,N)$ onto $(0,\infty)$. Finally, $\lambda_{s,\beta}=1$ if and only if $\beta=N-s$ and $\psi_1(\sigma)=(\sin\phi)^s$.

Proof. We first notice that

$$\int_{S_+^{N-1}} \omega\mathcal{A}_s\omega\,dS = \frac{1}{2}\int_{S_+^{N-1}}\int_0^\infty\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))^2}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}\,dS(\eta)\,d\tau\,dS(\sigma) \tag{A.6}$$

for any $\omega\in C_0^1(S_+^{N-1})$. By Lemma A.5 and (A.3) with $r=q=2$,

$$\int_{S_+^{N-1}}\int_0^\infty\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))^2}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}\,dS(\eta)\,d\tau\,dS(\sigma) \le c_{36}\|\omega\|^2_{W_0^{s,2}(S_+^{N-1})} + c_{37}\|\omega\|^2_{L^2(S_+^{N-1})},$$

where

$$\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)} = \int_{S^{N-1}_+}\int_{S^{N-1}_+} \frac{(\omega(\sigma)-\omega(\eta))^2}{|\eta-\sigma|^{N-1+2s}}\,dS(\eta)\,dS(\sigma).$$

Since by the Poincaré inequality [16] there holds

$$\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)} \geq c_{38}\|\omega\|^2_{L^2(S^{N-1}_+)},$$

we obtain that the right-hand side of (A.6) is bounded from above by

$$\Big(\frac{1}{2}c_{36} + \frac{c_{37}}{2c_{38}}\Big)\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)}.$$

Next we use the expansion estimates in Lemma A.5 to obtain that

$$\frac{\tau^{N-1}+\tau^{2s-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} \geq \frac{1}{(t^2+\kappa^2)^{\frac{N}{2}+s}} \quad \text{for all } t = 1-\tau \in (0,\varepsilon_0) \text{ and all } (\sigma,\eta)\in S^{N-1}_+\times S^{N-1}_+,$$

where $\kappa = |\sigma-\eta| \leq 2$. Hence,

$$\int_0^\infty \frac{\tau^{N-1}\,d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} \geq \int_0^{\varepsilon_0} \frac{dt}{(t^2+\kappa^2)^{\frac{N}{2}+s}} = \kappa^{1-N-2s}\int_0^{\frac{\varepsilon_0}{2}} \frac{dt}{(t^2+1)^{\frac{N}{2}+s}}.$$

Therefore,

$$\int_{S^{N-1}_+} \omega\mathcal{A}_s\omega\,dS \geq \int_0^{\frac{\varepsilon_0}{2}} \frac{dt}{2(t^2+1)^{\frac{N}{2}+s}}\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)}.$$

Finally, we obtain

$$\frac{1}{c_{39}}\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)} \leq \int_{S^{N-1}_+}\int_0^\infty\int_{S^{N-1}_+} \frac{(\omega(\sigma)-\omega(\eta))^2}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}\,dS(\eta)\,d\tau\,dS(\sigma) \leq c_{39}\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)}.$$

We consider the following bilinear form in $W^{s,2}_0(S^{N-1}_+)$:

$$\mathbb{A}(\omega,\zeta) := \int_{S^{N-1}_+}\int_0^\infty\int_{S^{N-1}_+} \frac{(\omega(\sigma)-\omega(\eta))\zeta(\sigma)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}\,dS(\eta)\,d\tau\,dS(\sigma).$$

Then $\mathbb{A}$ is symmetric and there hold

$$\mathbb{A}(\omega,\omega) = \int_{S^{N-1}_+} \omega\mathcal{A}_s\omega\,dS \geq \frac{1}{2c_{39}}\|\omega\|^2_{W^{s,2}_0(S^{N-1}_+)}$$

and

$$|\mathbb{A}(\omega,\zeta)| \leq \Big(\int_{S^{N-1}_+}\omega\mathcal{A}_s\omega\,dS\Big)^{\frac{1}{2}}\Big(\int_{S^{N-1}_+}\zeta\mathcal{A}_s\zeta\,dS\Big)^{\frac{1}{2}} \leq \frac{c_{39}}{2}\|\omega\|_{W^{s,2}_0(S^{N-1}_+)}\|\zeta\|_{W^{s,2}_0(S^{N-1}_+)}.$$

By the Riesz theorem, for any $L \in W^{-s,2}(S^{N-1}_+)$ there exists $\omega_L \in W^{s,2}_0(S^{N-1}_+)$ such that

$$\mathbb{A}(\omega_L,\zeta) = L(\zeta) \quad \text{for all } \zeta \in W^{s,2}_0(S^{N-1}_+).$$

We set $\omega_L = \mathcal{A}_s^{-1}(L)$. It is clear that $\mathcal{A}_s^{-1}$ is positive, and since the embedding of $W^{s,2}_0(S^{N-1}_+)$ into $L^2(S^{N-1}_+)$ is compact by the Rellich–Kondrachov theorem [16], $\mathcal{A}_s^{-1}$ is a compact operator. Hence the operator

$$\omega \mapsto \mathcal{A}_s^{-1}\circ\mathcal{L}_{s,\beta}\omega$$

is a compact positive operator (here we use the fact that $\beta > N - 2s$, which makes $\mathcal{B}_{s,\beta}$ positive). By the Krein–Rutman theorem, there exist $\mu > 0$ and $\psi_1 \in W_0^{s,2}(S_+^{N-1})$, $\psi_1 \geq 0$, such that

$$\mathcal{A}_s^{-1} \circ \mathcal{L}_{s,\beta}\psi_1 = \mu\psi_1.$$

The function ψ_1 is the unique positive eigenfunction and μ is the only positive eigenvalue with positive eigenfunctions. Furthermore, μ is the spectral radius of $\mathcal{A}_s^{-1} \circ \mathcal{B}_{s,\beta}$. If we set $\lambda_{s,\beta} = \mu^{-1}$, we obtain (A.5). It is also classical that $\lambda_{s,\beta}$ can be defined by

$$\lambda_{s,\beta} := \inf\Big\{ \int_{S_+^{N-1}} \omega\mathcal{A}_s\omega \, dS : \omega \in W_0^{s,2}(S_+^{N-1}),\ \omega \geq 0,\ \int_{S_+^{N-1}} \omega\mathcal{L}_{s,\beta}\omega \, dS = 1\Big\}. \tag{A.7}$$

Using (A.1), Lemma A.2 and the monotone convergence theorem, we derive that the mapping

$$\beta \mapsto \int_{S_+^{N-1}} \omega\mathcal{L}_{s,\beta}\omega \, dS$$

is increasing and continuous. This implies that $\beta \mapsto \lambda_{s,\beta}$ is decreasing and continuous. Since

$$\int_{S_+^{N-1}} \omega\mathcal{L}_{s,\beta}\omega \, dS \to \infty$$

when $\beta \uparrow N$, expression (A.7) implies that $\lambda_{s,\beta} \to 0$ when $\beta \uparrow N$. Next, if $\omega \geq 0$ is an element of $W_0^{s,2}(S_+^{N-1})$ such that

$$\int_{S_+^{N-1}} \omega\mathcal{L}_{s,\beta}\omega \, dS = 1,$$

we derive from the Poincaré inequality [16] and (A.4) that

$$\|\omega\|^2_{W_0^{s,2}(S_+^{N-1})} \geq c_{38}\|\omega\|^2_{L^2(S_+^{N-1})} \geq \frac{c_{38}}{c_{35}}.$$

Since $c_{35} \to 0$ when $\beta \downarrow N - 2s$, we infer that $\lim_{\beta\to N-2s} \lambda_{s,\beta} = \infty$. Consequently, the mapping $\beta \mapsto \lambda_{s,\beta}$ is a decreasing homeomorphism from $(N - 2s, N)$ onto $(0, \infty)$ and there exists a unique $\beta_s \in (N - 2s, N)$ such that $\lambda_{s,\beta_s} = 1$. The following expression of the Martin kernel in $\mathbb{R}_+^N$ is classical:

$$M_s^{\mathbb{R}_+^N}(x, y) = c_{N,s}x_N^s|x - y|^{-N} \quad \text{for all } x \in \mathbb{R}_+^N,\ y \in \partial\mathbb{R}_+^N.$$

Hence if $y = 0$, it is a separable singular s-harmonic function expressed in spherical coordinates with $x = (r, \sigma)$ by

$$M_s^{\mathbb{R}_+^N}((r, \sigma), 0) = c_{N,s}r^{s-N}(\sin\phi)^s.$$

This means that the function $\sigma \mapsto \omega(\sigma) = (\sin\phi)^s$, which vanishes on $\overline{S_-^{N-1}}$ and belongs to

$$W_0^{s,2}(S_+^{N-1}) \cap L^\infty(S_+^{N-1}),$$

satisfies

$$\mathcal{A}_s\omega - \mathcal{L}_{s,N-s}\omega = 0.$$

The uniqueness of the positive eigenfunction implies that this function is ψ_1 and $\beta = N - s$. □

A.2 The Nonlinear Problem

A.2.1 Separable Solutions in $\mathbb{R}^N$

If we look for separable positive solutions of

$$(-\Delta)^s u + u^p = 0 \quad \text{in } \mathbb{R}^N \tag{A.8}$$

under the form $u(x) = r^{-2s/(p-1)}\omega(\sigma)$, where $x = (r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$, then ω satisfies

$$\mathcal{A}_s\omega - \mathcal{L}_{s,\frac{2s}{p-1}}\omega + \omega^p = 0 \quad \text{in } S^{N-1}. \tag{A.9}$$

Proposition A.8. *Assume $N \geq 2$ and $s \in (0, 1)$.*
(i) *If $p \geq p_3^*$, then there exists no positive solution of* (A.9).
(ii) *If $p_1^* < p < p_3^*$, then the unique positive solution of* (A.9) *is a constant function with value*

$$\ell_{s,p} = (c_{35})^{\frac{1}{p-1}},$$

where c_{35} is the constant defined in Lemma A.6.

Proof. If $p \geq p_3^*$, we assume that there exists a solution $\omega \geq 0$ of (A.9). Then ω satisfies

$$\int_{S^{N-1}} \omega\mathcal{A}_s\omega\, dS - \int_{S^{N-1}} \omega\mathcal{L}_{s,\frac{2s}{p-1}}\omega\, dS + \int_{S^{N-1}} \omega^{p+1}\, dS = 0.$$

Since $p \geq p_3^*$, we have $c_s \leq 0$, which implies

$$\int_{S^{N-1}} \omega\mathcal{L}_{s,\frac{2s}{p-1}}\omega\, dS \leq 0.$$

Then $\omega = 0$ since the two other integrals are nonnegative.

Next, if $p_1^* < p < p_3^*$, it is clear that if ω is a constant nonnegative solution of (A.9), then we have

$$\omega \int_0^1 \int_{S^{N-1}} \frac{(\tau^{-\frac{2s}{p-1}-1})(\tau^{N-1} - \tau^{N-1+c_s})}{(1 + \tau^2 - 2\tau\langle\sigma, \eta\rangle)^{\frac{N}{2}+s}}\, dS(\eta)\, d\tau = \omega^p \quad \text{for all } \sigma \in S^{N-1}.$$

Using invariance by rotation of the integral term on S^{N-1}, we derive the claim. Conversely, assume ω is any bounded nonconstant positive solution; then it belongs to $C^2(S^{N-1})$ by [27]. Let $\sigma_0 \in S^{N-1}$, where ω is maximal. Then $\mathcal{A}_s\omega(\sigma_0) \geq 0$, and thus

$$\omega^p(\sigma_0) \leq \mathcal{L}_{s,\frac{2s}{p-1}}\omega(\sigma_0) \leq \omega(\sigma_0) \int_0^1 \int_{S^{N-1}} \frac{(\tau^{-\frac{2s}{p-1}} - 1)(\tau^{N-1} - \tau^{N-1+c_s})}{(1 + \tau^2 - 2\langle\sigma_0, \eta\rangle)^{\frac{N}{2}+s}}\, dS(\eta)\, d\tau = c_{35}\omega(\sigma_0).$$

Hence $\omega(\sigma_0) < \ell_{s,p}$. Similarly, $\min_{S^{N-1}} \omega > \ell_{s,p}$, which is a contradiction. □

Corollary A.9. *Assume $N \geq 2$, $s \in (0, 1)$ and $p_1^* < p < p_3^*$. Then the only positive separable solution u of* (A.8) *in $\mathbb{R}^N \setminus \{0\}$ is*

$$x \mapsto U(x) = \ell_{s,p}|x|^{-\frac{2s}{p-1}}.$$

A.2.2 Separable Solutions in $\mathbb{R}_+^N$

If we consider separable solutions $x \mapsto u(x) = r^{-\frac{2s}{p-1}}\omega(\sigma)$ of problem (1.12), then ω satisfies (1.13).

Proof of Theorem 1.7. We prove this in three steps.

Step 1: Nonexistence. Assume that such a solution $\omega \geq 0$ exists; then

$$\int_{S_+^{N-1}} \omega\mathcal{A}_s\omega\, dS - \int_{S_+^{N-1}} \omega\mathcal{L}_{s,\frac{2s}{p-1}}\omega\, dS + \int_{S_+^{N-1}} \omega^p\, dS = 0.$$

Hence,

$$(\lambda_{s,\frac{2s}{p-1}} - 1) \int_{S_+^{N-1}} \omega\mathcal{L}_{s,\frac{2s}{p-1}}\omega\, dS + \int_{S_+^{N-1}} \omega^p\, dS \leq 0.$$

If $\lambda_{s,2s/(p-1)} \geq 1$, equivalently $p \geq p_2^*$, the only nonnegative solution is the trivial one.

Step 2: Existence. Consider the following functional with domain $W_0^{s,2}(S_+^{N-1}) \cap L^{p+1}(S_+^{N-1})$:

$$\omega \mapsto \mathcal{J}(\omega) := \int_{S_+^{N-1}} \omega \mathcal{A}_s \omega \, dS + \frac{1}{p+1} \int_{S_+^{N-1}} |\omega|^{p+1} \, dS - \int_{S_+^{N-1}} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega \, dS.$$

Because of Lemma A.6, $\mathcal{J}(\omega) \to \infty$ when $\|\omega\|_{W_0^{s,2}(S_+^{N-1})} + \|\omega\|_{L^{p+1}(S_+^{N-1})} \to \infty$. Furthermore, for $\varepsilon > 0$, we have

$$\mathcal{J}(\varepsilon\psi_1) = \varepsilon^2(\lambda_{s,\frac{2s}{p-1}} - 1) \int_{S_+^{N-1}} \psi_1 \mathcal{L}_{s,\frac{2s}{p-1}} \psi_1 \, dS + \frac{\varepsilon^{p+1}}{p+1} \int_{S_+^{N-1}} |\psi_1|^{p+1} \, dS.$$

This implies that $\inf \mathcal{J}(\omega) < 0$ if $\lambda_{s,2s/(p-1)} < 1$, and thus the infimum of $\mathcal{J}$ in $W_0^{s,2}(S_+^{N-1}) \cap L_+^{p+1}(S_+^{N-1})$ is achieved by a nontrivial nonnegative solution of (1.13).

Step 3: Uniqueness. (i) *Existence of a maximal solution.* By [27], any solution ω is smooth. Hence, at its maximum σ_0 it satisfies $\mathcal{A}_s\omega(\sigma_0) \geq 0$, and thus

$$\omega(\sigma_0)^p \leq \mathcal{L}_{s,\frac{2s}{p-1}} \omega(\sigma_0) \leq \omega(\sigma_0)c_{35}.$$

This implies that $\sup \omega \leq \ell_{s,p}$. From this inequality the set $\mathcal{E} \subset W_0^{s,2}(S_+^{N-1})$ of positive solutions of (1.13) is bounded in $W_0^{s,2}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$, and thus in $C^s(S^{N-1}) \cap C^2(S_+^{N-1})$ by [27]. We put $\overline{\omega}(\sigma) = \sup\{\omega(\sigma) : \omega \in \mathcal{E}\}$. There exists a countable dense set $\mathcal{S} := \{\sigma_n\} \subset S_+^{N-1}$ and a sequence of functions $\{\omega_n\} \subset \mathcal{E}$ such that

$$\lim_{n\to\infty} \omega_n(\sigma_k) = \overline{\omega}(\sigma_k).$$

Furthermore, this sequence $\{\omega_n\}$ can be constructed such that $\{\omega_n(\sigma_k)\}$ is nondecreasing for any k. Finally, by a local compactness estimate, $\{\omega_n\}$ converges to $\overline{\omega}$ in $C^{s-\delta}(S^{N-1}) \cap C^2(S_+^{N-1})$ for any $\delta \in (0, s)$ and weakly in $W_0^{s,2}(S_+^{N-1})$. This implies that $\overline{\omega}$ belongs to $\mathcal{E}$. It follows from [27, Theorem 1.2] that any $\omega \in \mathcal{E}$ satisfies

$$\omega(\sigma) \leq c_{40}(\operatorname{dist}(\sigma, \partial S_+^{N-1}))^s = c_{40}\phi^s \quad \text{for all } \sigma \in S_+^{N-1}. \tag{A.10}$$

(ii) *Existence of a minimal solution.* From Theorem 3.7 follows that $u_k^{\mathbb{R}_+^N} \uparrow u_\infty^{\mathbb{R}_+^N}$ and $u_\infty^{\mathbb{R}_+^N}$ is self-similar and it is the minimal solution of (1.14) in $\mathbb{R}_+^N$ which satisfies

$$\lim_{x\to 0} \frac{u_\infty^{\mathbb{R}_+^N}(x)}{M_s^{\mathbb{R}_+^N}(x, 0)} = \infty.$$

Hence $u_\infty^{\mathbb{R}_+^N}(r, \sigma) = r^{-\frac{2s}{p-1}} \underline{\omega}(\sigma)$ and $\underline{\omega}$ is the minimal positive solution of (1.13). Furthermore, it follows from (3.30) that

$$\underline{\omega}(\sigma) \geq c_{41}(\operatorname{dist}(\sigma, \partial S_+^{N-1}))^s = c_{41}\phi^s \quad \text{for all } \sigma \in S_+^{N-1} \tag{A.11}$$

if $\phi = \phi(\sigma)$ is the latitude of σ.

(iii) *End of the uniqueness proof.* By combining (A.10) and (A.11), we infer that there exists $K > 1$ such that

$$\overline{\omega} \leq K\underline{\omega} \quad \text{in } S_+^{N-1}.$$

Assume $\overline{\omega} \neq \underline{\omega}$; then

$$\omega_1 := \underline{\omega} - \frac{1}{2K}(\overline{\omega} - \underline{\omega})$$

is a positive supersolution (by convexity) of (1.13). Moreover,

$$\omega_2 := \Big(\frac{1}{2} + \frac{1}{2K}\Big)\underline{\omega}$$

is a positive subsolution of (1.13) smaller than ω_1, hence also than $\underline{\omega}$. It follows by classical construction that there exists a solution $\tilde{\omega}$ of (1.13) which satisfies $\omega_2 \leq \tilde{\omega} \leq \omega_1$, which contradicts the minimality of $\underline{\omega}$. □

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