Research Article

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Symmetric and Asymmetric Solutions of p-Laplace Elliptic Equations in Hollow Domains

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Abstract: In the present paper, we study the p-Laplace equation in a hollow symmetric bounded domain. Let H and G be closed subgroups of the orthogonal group such that $H \subsetneq G$. Then we prove the existence of a positive solution which is H-invariant and G-non-invariant. Furthermore, we give several examples of H, G and Ω, and find symmetric and asymmetric solutions.

Keywords: p-Laplace Equation, Group Invariant Solution, Least Energy Solution, Positive Solution, Variational Method

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1 Introduction and Problems

We study the existence of symmetric and asymmetric positive solutions for the p-Laplace elliptic equation

$$-\Delta_p u = f(x, u), \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p \geq 2$, Ω is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary $\partial\Omega$, and $f(x, u)$ is a continuous function on $\overline{\Omega} \times [0, \infty)$. In the present paper, we consider a domain Ω which has a symmetry and a hole like an annulus. We denote the orthogonal group by $O(N)$, which is the set of all $N \times N$ orthogonal matrices. Let G be a closed subgroup of $O(N)$. We call Ω a *G-invariant domain* if $g(\Omega) = \Omega$ for all $g \in G$. We call $f(x, u)$ a *G-invariant function* if $f(gx, u) = f(x, u)$ for all $g \in G$, $x \in \Omega$ and $u \in [0, \infty)$. We call a solution $u(x)$ of (1.1) a *G-invariant solution* if $u(gx) = u(x)$ for $g \in G$ and $x \in \Omega$. We define the Lagrangian functional $I(u)$ for (1.1) by

$$I(u) := \int_\Omega \Big(\frac{1}{p}|\nabla u|^p - F(x, u)\Big)\, dx, \quad F(x, u) := \int_0^u f(x, s)\, ds.$$

We denote the Fréchet derivative of $I(u)$ by $I'(u)$, which is computed as follows:

$$I'(u)v = \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v - f(x, u)v)\, dx \quad \text{for } u, v \in W_0^{1,p}(\Omega).$$

Here $W_0^{1,p}(\Omega)$ is the usual Sobolev space. We put

$$J(u) := I'(u)u = \int_\Omega (|\nabla u|^p - f(x, u)u)\, dx.$$

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We define the *Nehari manifold* $\mathcal{N}$ and the *minimum energy* I_0 by

$$\begin{aligned}\mathcal{N} &:= \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : J(u) = 0\},\\ I_0 &:= \inf\{I(u) : u \in \mathcal{N}\}.\end{aligned} \tag{1.2}$$

We call $u(x)$ a *minimum energy solution* if $u \in \mathcal{N}$ and $I(u) = I_0$. A minimum energy solution is not necessarily unique. We shall prove in Section 4 that $I_0 > 0$ and that there exists a minimum energy solution which becomes a positive solution of (1.1) after replacing u by $-u$, if necessary.

For a closed subgroup G of $O(N)$, we put

$$W_0^{1,p}(\Omega, G) := \{u \in W_0^{1,p}(\Omega) : u(gx) = u(x),\ g \in G,\ x \in \Omega\},$$

$$\mathcal{N}(G) := \mathcal{N} \cap W_0^{1,p}(\Omega, G), \tag{1.3}$$

$$I_G := \inf\{I(u) : u \in \mathcal{N}(G)\}. \tag{1.4}$$

We call u a *G-minimum solution* if $u \in \mathcal{N}(G)$ and $I(u) = I_G$. Such a minimizer exists and becomes a G-invariant positive solution of (1.1); this will be proved in Section 4. To avoid confusion, we call a usual minimum energy solution a *global minimum solution*. Since a G-minimum solution exists, we are interested in a G-non-invariant positive solution. The purpose of the present paper is to solve the two problems below.

Problem A. Find G and Ω such that G is a closed subgroup of $O(N)$, Ω is a G-invariant bounded domain, and no global minimum solution is G-invariant.

Problem B. Find H, G and Ω such that H and G are closed subgroups of $O(N)$ satisfying $H \subsetneq G \subset O(N)$, Ω is a G-invariant bounded domain, and no H-minimum solution is G-invariant.

Comparing the two problems above, we see that Problem A is included in Problem B. Indeed, we put $H := \{e\}$, with e being the unit matrix. Then a global minimum solution is equal to an H-minimum solution. Therefore, an answer to Problem A follows from that to Problem B with $H = \{e\}$.

The goal of the present paper is to solve Problem B. We shall explain the reason why we have an interest in this problem. We start with $G = O(N)$ and introduce the result by Coffman [6]. Let $A_N(a, \varepsilon)$ be an annulus defined by

$$A_N(a, \varepsilon) := \{x \in \mathbb{R}^N : a < |x| < a + \varepsilon\}.$$

We consider the Emden–Fowler equation

$$-\Delta u = u^p, \quad u > 0 \quad \text{in } A_N(a, \varepsilon), \qquad u = 0 \quad \text{on } \partial A_N(a, \varepsilon). \tag{1.5}$$

Two solutions $u(x)$ and $v(x)$ are called *equivalent* if $u(gx) = v(x)$ for a certain $g \in O(N)$. Coffman [6], Li [15] and Byeon [3] proved the next result.

Proposition 1.1. *The number of non-equivalent positive solutions for* (1.5) *diverges to infinity as* $\varepsilon \to +0$. *Furthermore, no global minimum solution is radially symmetric when* ε *is small enough.*

For more results on the annulus, we refer to [1, 5, 9, 10, 16–19, 21, 22, 26, 29, 30]. When Ω is not an annulus, the existence of multiple positive solutions is proved in [4].

To explain the reason why we study Problems A and B, we shall propose some problems. First, we consider a hollow regular polygon, which is a domain enclosed by a small and a large regular n-gon with a common center and with their sides parallel to each other. To give a strict definition, let P_n be an interior of a regular n-gon centered at the origin. For $\varepsilon > 0$, we define

$$(1 + \varepsilon)P_n := \{(1 + \varepsilon)x : x \in P_n\}.$$

Remove $\overline{P}_n$ from $(1 + \varepsilon)P_n$ and define

$$\mathrm{HP}_n(\varepsilon) := (1 + \varepsilon)P_n \setminus \overline{P}_n. \tag{1.6}$$

This is a *hollow regular n-gon*. Define

$$G_n := \left\{\rho\left(\frac{2j\pi}{n}\right) : j = 0, 1, \ldots, n-1\right\}, \quad \rho(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.7}$$

Then $\mathrm{HP}_n(\varepsilon)$ is G_n-invariant. Suppose that $f(x, u)$ is also G_n-invariant. Observing Proposition 1.1, we consider the problem below.

Problem 1.2. Let $\Omega = \mathrm{HP}_n(\varepsilon)$ be the hollow regular n-gon, with $\varepsilon > 0$ small enough. Does (1.1) have a positive solution without G_n-invariance?

For the homogeneous nonlinear term $f(x, u) = u^q$, we solved the above problem in the affirmative for $p = 2$ (see [14]) and for $p \geq 2$ (see [13]). In the corresponding papers, we used the functional $R(u)$ defined by

$$R(u) := \left(\int_\Omega |\nabla u|^2\,dx\right)\left(\int_\Omega |u|^{q+1}\,dx\right)^{-2/(q+1)}.$$

Then a global minimum solution is defined by a minimizer of $R(u)$ in $\mathcal{N}$. In equation (1.1), the nonlinear term $f(x, u)$ is not assumed to have a homogeneity like $f(u) = u^q$. Therefore, we cannot use $R(u)$ in the present paper, and hence we employ the Lagrangian functional $I(u)$ and develop a new method. Another aim of the present paper is to extend the results in [13, 14] to a more general equation (1.1).

We shall propose a new problem. Let L be one of the axes of reflection symmetry for a regular n-gon P_n. Denote the reflection group with respect to L by H_L. We call u an *L-solution* if u is an H_L-minimum solution for (1.1) with $\Omega = \mathrm{HP}_n(\varepsilon)$. Recall that two solutions are called equivalent if they are transformed by an orthogonal matrix. We consider the next problem for $\Omega = \mathrm{HP}_n(\varepsilon)$.

Problem 1.3. Find $\mathrm{HP}_n(\varepsilon)$, L_1 and L_2 such that L_1 and L_2 are axes of symmetry for P_n and no L_1-solution is equivalent to any L_2-solution.

For the annulus, we consider an H-invariant non-radial solution for a closed subgroup H of $O(N)$.

Problem 1.4. Let Ω be the annulus $A_N(a, \varepsilon)$. Find all closed subgroups H of $O(N)$ such that no H-minimum solution is radially symmetric.

An answer to the problem above follows from that to Problem B by putting $G = O(N)$. To state a new problem, we consider $\mathrm{HP}_3(\varepsilon)$ given by (1.6) with $n = 3$, that is, it is a hollow equilateral triangle. Then $\mathrm{HP}_3(\varepsilon)$ has three axes of reflection symmetry. Let L be one of these axes.

Problem 1.5. Let $\Omega = \mathrm{HP}_3(\varepsilon)$ be the hollow equilateral triangle. Does (1.1) have a positive solution which is reflectionally symmetric with respect to L and not invariant under the rotation of 120°?

We shall consider a cube and a cross polytope in $\mathbb{R}^N$. Let C_N be the following N-dimensional cube:

$$C_N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : |x_i| < 1 \text{ for } 1 \leq i \leq N\}.$$

We define a *hollow cube* $\mathrm{HC}_N(\varepsilon)$ by

$$\mathrm{HC}_N(\varepsilon) := (1+\varepsilon)C_N \setminus \overline{C}_N. \tag{1.8}$$

Let E be the union of all edges of C_N. Let $\mathrm{SC}_N(\varepsilon)$ be an ε-neighborhood of E. This is a *skeleton cube*. We denote a regular cross polytope in $\mathbb{R}^N$ by CP_N. Then CP_N is a convex hull of all points P_i^+ and P_i^- with $i = 1, 2, \ldots, N$, where

$$P_i^\pm := (0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

is a point in $\mathbb{R}^N$ whose i-th element is ± 1 and the others are 0. Therefore, CP_N is defined by

$$\mathrm{CP}_N := \left\{\sum_{i=1}^N t_i^+ P_i^+ + \sum_{i=1}^N t_i^- P_i^- : t_i^\pm \geq 0,\ \sum_{i=1}^N t_i^+ + \sum_{i=1}^N t_i^- < 1\right\} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : |x_1| + \cdots + |x_N| < 1\}.$$

Define

$$\mathrm{HCP}_N(\varepsilon) := (1+\varepsilon)\mathrm{CP}_N \setminus \overline{\mathrm{CP}}_N, \tag{1.9}$$

which is a *hollow cross polytope*. Let $\mathrm{SCP}_N(\varepsilon)$ be an ε-neighborhood of the union of all edges of CP_N. This is a *skeleton cross polytope*.

Let H be the set of all orthogonal matrices $g \in O(N)$ whose each element is equal to 0 or 1, that is, H is the set of all $N \times N$ permutation matrices. Let G be the set of $g \in O(N)$ whose each element is equal to 0, 1 or -1. We define Ω by $\mathrm{HC}_N(\varepsilon)$, $\mathrm{SC}_N(\varepsilon)$, $\mathrm{HCP}_N(\varepsilon)$, $\mathrm{SCP}_N(\varepsilon)$ or $A_N(a,\varepsilon)$ for $\varepsilon > 0$ small. It is easy to verify that Ω is G-invariant because C_N and CP_N are G-invariant. The group H is isomorphic to the symmetric group S_N, which is the set of all permutations

$$\sigma = \begin{pmatrix} 1 & \cdots & N \\ \sigma(1) & \cdots & \sigma(N) \end{pmatrix}.$$

A function $u(x)$ is H-invariant if and only if

$$u(x_1,\dots,x_N) = u(x_{\sigma(1)},\dots,x_{\sigma(N)}) \quad \text{for } \sigma \in S_N,\ x \in \Omega. \tag{1.10}$$

A function $u(x)$ is G-invariant if and only if

$$u(x_1,\dots,x_N) = u(\tau_1 x_{\sigma(1)},\dots,\tau_N x_{\sigma(N)}) \tag{1.11}$$

for $\sigma \in S_N$, $\tau_1,\dots,\tau_N \in \{1,-1\}$ and $x \in \Omega$.

Problem 1.6. Let Ω be one of the following: $\mathrm{HC}_N(\varepsilon)$, $\mathrm{SC}_N(\varepsilon)$, $\mathrm{HCP}_N(\varepsilon)$, $\mathrm{SCP}_N(\varepsilon)$ or $A_N(a,\varepsilon)$, with $\varepsilon > 0$ being small. Does there exist a positive solution $u(x)$ which satisfies (1.10) and does not satisfy (1.11)?

All the problems in this section are reduced to Problem B. Therefore, we shall solve Problem B. This paper is organized in five sections. In Section 2, we state the main results. In Section 3, by using the main theorems, we solve all the problems that appear in Section 1. In Section 4, we prove the existence of a global minimum solution and a G-minimum solution. In Section 5, we prove the main theorems.

2 Main Results

In this section, we state the main results. For a closed subgroup G of $O(N)$, we define the *orbit* of G through $x \in \mathbb{R}^N$ by

$$G(x) := \{gx : g \in G\}. \tag{2.1}$$

We suppose the two assumptions below.

Assumption 2.1. Let G, H and D be such that G and H are closed subgroups of $O(N)$, D is a G-invariant bounded domain in $\mathbb{R}^N$, $0 \notin \overline{D}$, $H \subsetneq G$ and $H(x) \subsetneq G(x)$ for $x \in \overline{D}$. Here $H(x)$ and $G(x)$ are orbits defined by (2.1).

Assumption 2.2. Let G be a closed subgroup of $O(N)$ and let D be a G-invariant bounded domain in $\mathbb{R}^N$ whose closure does not contain the origin. Suppose that $p \geq 2$ and assume the following conditions:

(i) $f(x,u)$ is a G-invariant continuous function on $\overline{D} \times [0,\infty)$ such that $f(x,0) = 0$ and $f(x,u) > 0$ for $u > 0$ and $x \in \overline{D}$.

(ii) $f(x,u)$ has a continuous partial derivative $f_u(x,u)$, and there exists a constant $q \in (p,\infty)$ (hence $q > p \geq 2$) such that

$$\frac{\partial}{\partial u}\left(\frac{f(x,u)}{u^{q-1}}\right) = \frac{f_u(x,u)u - (q-1)f(x,u)}{u^q} > 0 \tag{2.2}$$

for $u > 0$ and $x \in \overline{D}$.

(iii) There exist constants $r, C > 0$ such that $|f_u(x,u)| \leq C(u^{r-2}+1)$ for $u \geq 0$ and $x \in \overline{D}$, where r satisfies $p < r < Np/(N-p)$ when $p < N$, and $p < r < \infty$ when $p \geq N$.

As a consequence of the assumption above, we have $f_u(x, 0) = 0$.

Example 2.3. Some examples of $f(x, u)$ satisfying Assumption 2.2 are

$$a(x)u^{r-1}, \quad a(x)u^{r-1} + b(x)u^{s-1}, \quad a(x)\frac{u^{r-1}}{1+u^t},$$
$$a(x)u^{r-1}\log(1+u), \quad a(x)u^{r-1}\tanh u, \quad a(x)u^{r-1}\arctan u,$$

where we assume that $r, s \in (p, Np/(N-p))$ for $p < N$ and $r, s \in (p, \infty)$ for $p \geq N$. We assume also $0 < t < r - p$, and that $a(x)$, $b(x)$ are G-invariant positive continuous functions on $\overline{D}$. The sum of these functions also satisfies Assumption 2.2.

We shall give a domain Ω as a G-invariant subdomain of D. We extend $f(x, u)$ as an odd function of u, i.e., $f(x, -u) := -f(x, u)$. Observe the definition of I_G in (1.4). To give an affirmative answer to Problem B, it is enough to show that $I_H < I_G$. Indeed, this inequality ensures that no H-minimum solution is G-invariant. We denote the first eigenvalue of the Dirichlet p-Laplacian in the domain Ω by $\lambda_1(\Omega)$, i.e.,

$$-\Delta_p u = \lambda_1(\Omega)u^{p-1} \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We state the first main result, which is an answer to Problem B.

Theorem 2.4. *Let Assumptions 2.1 and 2.2 hold. Then there exists a constant $\Lambda > 0$ such that if Ω is a G-invariant subdomain of D which satisfies $\lambda_1(\Omega) > \Lambda$, then we have that $I_H < I_G$. Therefore, no H-minimum solution is G-invariant.*

Observe the definitions of $\mathcal{N}(G)$ and I_G given in (1.3) and (1.4), respectively. The inclusion $H \subset G$ implies $\mathcal{N}(G) \subset \mathcal{N}(H)$, which ensures that $I_H \leq I_G$. Consequently, the stronger symmetry has higher energy. The essential point of Theorem 2.4 is the strict inequality $I_H < I_G$. The theorem above guarantees the existence of at least two positive solutions: one is an H-invariant G-non-invariant positive solution and the other is a G-invariant positive solution.

It is well known that the first eigenvalue $\lambda_1(\Omega)$ diverges to infinity as the volume of Ω tends to 0. Therefore, we have the corollary below.

Corollary 2.5. *Under Assumptions 2.1 and 2.2, there exists a constant $\varepsilon > 0$ such that if Ω is a G-invariant subdomain of D satisfying $|\Omega| < \varepsilon$, then $I_H < I_G$. Here $|\Omega|$ denotes the volume of Ω.*

Put $H = \{e\}$, with e being the unit matrix. Then an H-minimum solution coincides with a global minimum solution. Furthermore, the assumption $H(x) = \{x\} \subsetneq G(x)$ is equivalent to a condition that $G(x)$ has at least two points. In other words, there do not exist any fixed points of G in $\overline{D}$, where x is called a fixed point of G if $gx = x$ for all $g \in G$. Therefore, Theorem 2.4 is reduced to the result below, which is an answer to Problem A.

Theorem 2.6. *Let Assumption 2.2 hold and assume that for each $x \in \overline{D}$, $G(x)$ includes at least two points. Then there exists a constant $\Lambda > 0$ such that if Ω is a G-invariant subdomain of D which satisfies $\lambda_1(\Omega) > \Lambda$, then we have that $I_0 < I_G$. Therefore, no global minimum solution is G-invariant.*

In the theorems above, we first fix D and then define the subset Ω of D. This is complicated. We shall explain the reason why we need this procedure. To this end, we consider the equation,

$$-\Delta u = u^q, \quad u > 0 \quad \text{in } \Omega(a, b), \qquad u = 0 \quad \text{on } \partial\Omega(a, b), \tag{2.3}$$

where $\Omega(a, b)$ is the annulus $a < |x| < b$ in $\mathbb{R}^N$ and q is assumed to satisfy $1 < q < (N+2)/(N-2)$ for $N \geq 3$ and $1 < q < \infty$ for $N = 2$. Then Dancer [8] proved the following result.

Lemma 2.7 ([8]). *For any $b > 0$, there exists an $\varepsilon > 0$, depending on b, such that if $0 < a < \varepsilon$, then (2.3) has a unique positive solution. Moreover, it becomes radially symmetric. In particular, a global minimum solution is radially symmetric.*

Let $B(b)$ denote a ball centered at the origin with radius $b > 0$. Since $\Omega(a, b) \subset B(b)$, we have that

$$\lambda_1(\Omega(a, b)) \geq \lambda_1(B(b)).$$

As $b \to 0$, the right-hand side diverges to infinity and so does $\lambda_1(\Omega(a,b))$. Putting $H := \{e\}$, with e being the unit matrix, and $G := O(N)$, we compare Lemma 2.7 and Theorem 2.4. Let $\Lambda > 0$ be any number. If we do not set D, then we can choose $b > 0$ sufficient small so that $\lambda_1(\Omega(a,b)) > \Lambda$. Lemma 2.7 says that a global minimum solution is radially symmetric when $a > 0$ is small enough. Hence, the conclusion of Theorem 2.4 does not hold. This is caused by the assumption that the radius $a > 0$ of the inner hole of $\Omega(a,b)$ is very small. We first fix D and then define the subset Ω of D in Theorem 2.4. This procedure prevents the inner hole of $\Omega(a,b)$ from shrinking.

Let H be a closed subgroup of $O(N)$. Then it is a linear isometric transformation group on $\mathbb{R}^N$. Hence, it becomes a transformation group on the unit sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}.$$

H is said to be *transitive* on S^{N-1} if $H(x) = S^{N-1}$ for $x \in S^{N-1}$, where $H(x)$ is the orbit defined by (2.1). All the transitive Lie groups have been listed by Montgomery and Samelson [23], and Borel [2] (see also [11, p. 186, Theorem 2.6] or [25, p. 267, Theorem 3]).

Theorem 2.8 ([2, 23]). *Let $N \geq 2$ and H be a connected closed subgroup of* $\mathrm{SO}(N)$. *Here* $\mathrm{SO}(N)$ *is the special orthogonal group (rotation group). Then H is transitive on S^{N-1} if and only if H is locally isomorphic to one of the following groups:*
- $\mathrm{SO}(N)$,
- $\mathrm{SU}(m)$, $U(m)$ *if* $N = 2m$,
- $\mathrm{Sp}(m)$, $\mathrm{Sp}(m)\mathrm{Sp}(1)$, $\mathrm{Sp}(m)U(1)$ *if* $N = 4m$,
- $\mathrm{Spin}(9)$ *if* $N = 16$,
- $\mathrm{Spin}(7)$ *if* $N = 8$,
- G_2 *if* $N = 7$.

Let H be a closed subgroup of $O(N)$ which is not necessarily connected. Then it is transitive on S^{N-1} if and only if the connected component of H, including the unit matrix, is locally isomorphic to one of the Lie groups listed in Theorem 2.8.

Let $\Omega = A_N(a,\varepsilon)$ be the annulus for ε small enough. Let H be a closed subgroup of $O(N)$ and put $G := O(N)$. Then $G(x) = S^{N-1}$ for $x \in S^{N-1}$. If H is not transitive on S^{N-1}, then $H(x) \subsetneq G(x)$ for $x \in S^{N-1}$, and so $H(x) \subsetneq G(x)$ for $x \neq 0$. Theorem 2.4 says that if H is not transitive, then no H-minimum solution is G-invariant (i.e., radially symmetric). On the other hand, if H is transitive, any H-invariant function is radially symmetric. Consequently, we have the next result.

Theorem 2.9. *Let Assumption 2.2 hold with $G = O(N)$ and $D = A_N(a,1)$. Let $\Omega = A_N(a,\varepsilon)$ be the annulus with $\varepsilon > 0$ small enough. Then the following are equivalent:*
(i) *No H-minimum solution is radially symmetric.*
(ii) *H is not transitive on S^{N-1}.*

The theorem above is a complete answer to Problem 1.4. We give an application of this theorem to a non-transitive group.

Example 2.10. Let

$$G := \mathrm{SO}(m) \times \mathrm{SO}(N-m) = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} : g \in \mathrm{SO}(m),\ h \in \mathrm{SO}(N-m) \right\}.$$

Clearly, G is not transitive. Therefore, for small $\varepsilon > 0$, a G-minimum solution $u(x)$ of (1.1) on $A_N(a,\varepsilon)$ is not radially symmetric, that is, $u = u(|x'|, |x''|)$ for $x' \in \mathbb{R}^m$, $x'' \in \mathbb{R}^{N-m}$, with $x = (x', x'') \in A_N(a,\varepsilon)$ but $u \neq u(|x|)$.

Since any finite subgroup is not transitive, we have the next result.

Corollary 2.11. *Let the assumptions of Theorem 2.9 hold. If H is a finite subgroup of $O(N)$, then for small $\varepsilon > 0$, no H-minimum solution of* (1.1) *on $A_N(a,\varepsilon)$ is radially symmetric.*

3 Answers to the Problems

By applying the main theorems stated in Section 2, we give an affirmative answer to all the problems given in Section 1. Throughout this section, we assume that $f(x, u)$ satisfies Assumption 2.2. In order to use Theorem 2.4, we have only to verify that G and H satisfy Assumption 2.1.

We recall the definition of equivalence given in Section 1. Solutions u and v are called equivalent if $u(gx) = v(x)$ for $x \in \Omega$, with a certain $g \in O(N)$. Define $\mathrm{HP}_n(\varepsilon)$ and G_n by (1.6) and (1.7), respectively. Then $\Omega := \mathrm{HP}_n(\varepsilon)$ is G_n-invariant. If m is a divisor of n, then G_m is a subgroup of G_n and hence Ω is G_m-invariant. The next result gives an answer to Problem 1.2.

Theorem 3.1. *Put $\Omega = \mathrm{HP}_n(\varepsilon)$ with $\varepsilon > 0$ small enough. Let k and m be any two divisors of n satisfying $1 \le k < m \le n$. Let u_k and u_m be a G_k- and G_m-minimum solution, respectively. Then u_k is not equivalent to u_m. Moreover, if k is a divisor of m, then $I(u_k) < I(u_m)$. Therefore,* (1.1) *has at least d non-equivalent solutions, where d is a number of all divisors of n.*

In the theorem above, for $k = 1$ and $m = n$, u_1 is a global minimum solution and u_n is a G_n-minimum solution. By Theorem 3.1, $I_0 = I(u_1) < I(u_n) = I_{G_n}$, and hence a global minimum solution is not G_n-invariant. Problem 1.2 is solved, in the affirmative.

Proof of Theorem 3.1. We use a method we developed in [14]. Let k and m be any two divisors of n such that $k < m$. Let u_k and u_m be a G_k- and G_m-minimum solution, respectively. We divide the proof into two cases.

(i) Assume that k is a divisor of m. Then $G_k \subsetneq G_m$ and $G_k(x) \subsetneq G_m(x)$ in $\overline{D}$, where $D := \mathrm{HP}_n(1)$. Hence, $\Omega = \mathrm{HP}_n(\varepsilon) \subset D$ for $0 < \varepsilon < 1$. By Theorem 2.4, $I_{G_k} < I_{G_m}$, i.e., $I(u_k) < I(u_m)$. Since any orthogonal transformation leaves $I(\cdot)$ invariant, u_k is not equivalent to u_m.

(ii) Assume that k is not a divisor of m. Suppose on the contrary that u_k is equivalent to u_m. Then $u_k(x) = u_m(h_0 x)$ for a certain $h_0 \in H$, where

$$H := \{h \in O(2) : h(\Omega) = \Omega\}.$$

Then u_k is $h_0^{-1} G_m h_0$-invariant because for $g \in G_m$,

$$u_k(h_0^{-1} g h_0 x) = u_m(g h_0 x) = u_m(h_0 x) = u_k(x).$$

Therefore, u_k is both G_k- and $h_0^{-1} G_m h_0$-invariant, and hence it is invariant under the group $\langle G_k \cup h_0^{-1} G_m h_0 \rangle$, where $\langle A \rangle$ denotes a group generated by A. By [14, Lemma 4.2], we have that $h^{-1} G_m h = G_m$ for any $h \in O(2)$. Accordingly, u_k is $\langle G_k \cup G_m \rangle$-invariant. Let r be the least common multiple of k and m. Then there exist $\mu, \nu \in \mathbb{Z}$ such that $\mu/k + \nu/m = 1/r$. This shows that $\langle G_k \cup G_m \rangle = G_r$. Therefore, u_k is G_r-invariant. However, k is a divisor of r, which contradicts (i). Therefore, u_k is not equivalent to u_m. The proof is complete. □

Theorem 3.1 remains valid for the two-dimensional annulus $\Omega = A_2(a, \varepsilon)$. Hence, we have the next result.

Corollary 3.2. *Let $\Omega = A_2(a, \varepsilon)$ be the two-dimensional annulus. Then the number of non-equivalent positive solutions for* (1.1) *diverges to infinity as $\varepsilon \to +0$.*

Let us consider Problem 1.3. Let P_n be a regular n-gon. We call P_n an even (resp. odd) polygon if n is even (resp. odd). Instead of a middle point of a side, we say a midpoint for simplicity. For an odd polygon, an axis of reflection symmetry goes through one vertex and one midpoint. For an even polygon, an axis of symmetry passes through either two vertices or two midpoints. Recall that for an axis of symmetry L, we call $u(x)$ an L-solution if it is an H_L-minimum solution, where H_L is a reflection group with respect to L. Let n be odd and L_1 and L_2 be two axes of symmetry for P_n. Choose a rotation matrix $g \in \mathrm{SO}(2)$ satisfying $g(L_2) = L_1$. Then g leaves P_n invariant. Let $u(x)$ be an L_1-solution. Then $v(x) := u(gx)$ is an L_2-solution. Hence, the L_1-solution u is equivalent to the L_2-solution v. Consequently, for an odd polygon, we cannot find L_1 and L_2 satisfying the condition of Problem 1.3. Consider an even polygon. Choose the center of P_n as the coordinate origin 0. For a point $x \neq 0$, we denote the line through x and the origin by L_x. For a vertex v and a midpoint m, L_v and L_m are axes of symmetry. For any two vertices v and v', an L_v-solution is equivalent to a certain $L_{v'}$-solution for the same reason given in the odd polygon case. Similarly, an L_m-solution is equivalent to an $L_{m'}$-solution for the midpoints m and m'. Thus, our target is L_v and L_m in an even polygon. Our answer to Problem 1.3 is as follows.

Theorem 3.3. *Let $\Omega := \mathrm{HP}_{4n}(\varepsilon)$ be a hollow regular $4n$-gon, where $\varepsilon > 0$ is small enough and n is a positive integer. Then no L_v-solution is equivalent to any L_m-solution for any vertex v and any midpoint m.*

The theorem above comes from Theorem 2.4 with the help of elementary Euclidean geometry. To prove the theorem, we first prove the following lemma.

Lemma 3.4. *Let P_n be an even polygon. Then the following assertions hold:*
(i) *If n is even and not a multiple of 4, then there exist a vertex v and a midpoint m such that L_v is perpendicular to L_m.*
(ii) *If n is a multiple of 4, then for any vertex v and any midpoint m, L_v is not perpendicular to L_m.*

Proof. We give a rigorous proof by using the complex plane. Put P_n in the complex plane so that all its vertices lie on the unit circle $|z| = 1$ and the point $z = 1$ is one of them. We consider P_{4n} with a positive integer n. Then all the vertices are written as $V_k = e^{2k\pi i/4n}$, with $k = 0, 1, 2, \dots, 4n - 1$. Note that V_0 is the point $z = 1$. Denote the origin by O. If V is a point on the unit circle such that $\overline{OV} \perp \overline{OV_0}$, i.e., the line segment $\overline{OV}$ is perpendicular to the x-axis, then V is equal to either $e^{\pi i/2}$ or $e^{3\pi i/2}$. Such points are achieved at $V_k = e^{2k\pi i/4n}$ for $k = n$ and $k = 3n$. These are vertices of P_{4n}. Therefore, for any midpoint M of P_{4n}, the segment $\overline{OM}$ is not perpendicular to $\overline{OV_0}$. Thus, assertion (ii) holds.

Consider P_{2n} with an odd integer $n \geq 3$. Then all the vertices are represented as $V_k = e^{2\pi ki/2n}$, with $k = 0, 1, 2, \dots, 2n - 1$. Denote all the midpoints of P_{2n} by M_k, with $k = 0, 1, \dots, 2n - 1$. Since a midpoint does not lie on the unit circle, we put $N_k := M_k/|M_k|$ so that $N_k = e^{2\pi(2k+1)i/4n}$, with $k = 0, 1, \dots, 2n - 1$. Recall that n is odd. The points N_k, with $2k + 1 = n, 3n$ lie on the y-axis. This proves assertion (i). The proof is complete. □

For two straight lines in the plane $\mathbb{R}^2$, we have the next lemma.

Lemma 3.5. *Let L_1 and L_2 be two different lines passing through the origin, which are not perpendicular to each other. Let G be a group generated by reflections with respect to L_1 and L_2. Then $G(x)$ includes at least three points for $x \neq 0$, where $G(x)$ is the orbit defined by* (2.1).

Proof. The lemma follows from elementary geometry, nevertheless we give a proof. Let x be a point that does not lie on L_1 and L_2. The reflection with respect to L_1 (resp. L_2) moves x to another point x_1 (resp. x_2). Then $x \neq x_i$, $i = 1, 2$, and $x_1 \neq x_2$. Thus, we have at least three points.

Let $x \in L_1$ and $x \neq 0$. Then it is not on L_2. A reflection with respect to L_2 moves x to another point x'. Then x' does not lie on the line L_1 because L_1 and L_2 are not perpendicular to each other. The reflection by L_1 moves x' to another point x''. Since $x \in L_1$, x'' is not equal to x. Hence, we have at least three points, x, x' and x'', which are different from each other. In the case where $x \in L_2$, the argument above remains valid. The proof is complete. □

Throughout this section, $|A|$ stands for the cardinal number of a set A. In the lemma above, the conclusion $|G(x)| \geq 3$ follows from the assumption that two lines are not perpendicular. Indeed, if $L_1 \perp L_2$, then $|G(x)| = 2$ for $x \in (L_1 \cup L_2) \setminus \{0\}$. In Lemma 3.5, the estimate $|G(x)| \geq 3$ is optimal. Indeed, there exists an example of L_1, L_2 and x where we have $|G(x)| = 3$. We choose the x-axis as L_1, and take L_2 so that the angle between L_1 and L_2 is 60°. Choose a point (complex number) $z = 1$ on L_1. Move it by all compositions of reflections by L_1 and L_2. Then $G(1) = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ and $|G(1)| = 3$. Thus, 3 is optimal.

Some readers may have an interest in $|G(x)|$. Let us investigate it, which is of independent interest. Assume that L_1 and L_2 pass through the origin. Let α be the angle between L_1 and L_2. Then the following assertions hold:

(i) $|G(x)| = \infty$ for any $x \neq 0$ if and only if α/π is an irrational number.
(ii) $|G(x)| < \infty$ for any $x \neq 0$ if and only if α/π is a rational number.
(iii) $|G(x)| = 3$ if and only if $\alpha = \pi/3$ (or $2\pi/3$) and $x \in (L_1 \cup L_2 \cup L_3) \setminus \{0\}$. Here L_3 is a line through the origin whose angle between L_i is $\pi/3$ for $i = 1, 2$.

To show the claims above, we choose L_1 as the x-axis. Let f_i be a function in the complex plane which is defined by the reflection with respect to L_i for $i = 1, 2$. Then $f_1(z) = \bar{z} = re^{-i\theta}$ and $f_2(z) = re^{i(2\alpha-\theta)}$ for $z = re^{i\theta}$. Since G is all multiple compositions of f_1 and f_2, we have

$$G(z) = \{re^{i(2n\alpha+\theta)} : n \in \mathbb{Z}\} \cup \{re^{i(2n\alpha-\theta)} : n \in \mathbb{Z}\}$$

for $z = re^{i\theta}$. This expression proves (i), (ii) and (iii).

We consider a reflection group with respect to a straight line. It is easy to verify that

$$\{g \in O(2) : \det g = -1\} = \{\tilde{\rho}(\theta) : \theta \in \mathbb{R}\}, \quad \tilde{\rho}(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

From a direct computation, it follows that $\tilde{\rho}(\theta)^2 = e$, with e being the unit matrix. Hence, $\tilde{\rho}(\theta)$ is a reflection matrix. In fact, it represents a reflection with respect to a line whose angle between the x-axis is $\theta/2$. Therefore, a subgroup H of $O(2)$ is a reflection group with respect to a line if and only if $H = \{e, g\}$, with a matrix $g \in O(2)$ satisfying $\det g = -1$. We shall show Theorem 3.3.

Proof of Theorem 3.3. Let $\Omega := \mathrm{HP}_{4n}(\varepsilon)$. Recall that the center of P_{4n} is the coordinate origin 0 and for a point $x \neq 0$, L_x denotes a line through x and 0. Let v be any vertex and m be any midpoint of P_{4n}. Let u_v and u_m be an L_v-solution and an L_m-solution, respectively. That is, u_v is a minimum energy solution in all positive solutions having reflection symmetry with respect to L_v. Denote the reflection group with respect to L_v or L_m by H_v or H_m, respectively. We shall show that u_v is not equivalent to u_m. Suppose on the contrary that they are equivalent. Then $u_v(x) = u_m(kx)$ for a certain $k \in K$, where

$$K := \{k \in O(2) : k(\Omega) = \Omega\}.$$

By using the same method as in the proof of Theorem 3.1, we see that u_v is $k^{-1}H_m k$-invariant. Since H_m is a reflection group, it takes the form $H_m = \{e, h\}$ for a certain matrix $h \in O(2)$ satisfying $\det h = -1$. Then $k^{-1}H_m k = \{e, k^{-1}hk\}$ and $\det(k^{-1}hk) = -1$. Hence, $k^{-1}H_m k$ is also a reflection group. Since k^{-1} leaves P_{4n} invariant, it moves the line L_m to another line passing through a midpoint. Denote it by $L_{m'}$, that is, $k^{-1}(L_m) = L_{m'}$. We shall show that each point in $L_{m'}$ is a fixed point of $k^{-1}H_m k$. Note that $h(x) = x$ for $x \in L_m$ because $H_m = \{e, h\}$ is a reflection group with respect to L_m. For $x \in L_{m'}$, we can write $x = k^{-1}y$ with $y \in L_m$, and so $k^{-1}hk(x) = k^{-1}h(y) = k^{-1}(y) = x$. Thus, $k^{-1}H_m k$ does not move any point in $L_{m'}$, and therefore it is a reflection with respect to $L_{m'}$, that is, $k^{-1}H_m k = H_{m'}$. Consequently, u_v is invariant under the action of both H_v and $H_{m'}$. Let G be a group generated by the union of H_v and $H_{m'}$. Then u is G-invariant. Moreover, $H_v \subsetneq G$ and $H_v(x) \subset G(x)$. Since L_v and $L_{m'}$ are not perpendicular to each other by Lemma 3.4, we have that $|G(x)| \geq 3$ for any $x \neq 0$ by Lemma 3.5. On the other hand, $|H_v(x)| \leq 2$ for any $x \neq 0$ because H_v is a reflection. Thus, $H_v(x) \subsetneq G(x)$ for $x \neq 0$. Theorem 2.4 proves that $I_{H_v} < I_G$. However, u_v is an H_v-minimum solution and it is G-invariant. A contradiction occurs. Therefore, no L_v-solution is equivalent to any L_m-solution. The proof is complete. □

In Theorem 3.1, we considered the hollow polygon. Let us generalize it to regular polytopes in $\mathbb{R}^N$. It is known that all the regular polytopes are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron in $\mathbb{R}^3$; the 5-cell, the 8-cell, the 16-cell, the 24-cell, the 120-cell and the 600-cell in $\mathbb{R}^4$; and the $(N+1)$-cell, the $2N$-cell and the 2^N-cell in $\mathbb{R}^N$ for $N \geq 5$ (see [7]). Let P be an interior of any regular polytope in $\mathbb{R}^N$. Put

$$\mathrm{HP}(\varepsilon) := (1+\varepsilon)P \setminus \overline{P}.$$

This is a *hollow regular polytope*. Define

$$\mathrm{SO}(P) := \{g \in \mathrm{SO}(N) : g(P) = P\},$$

which is called a *regular polytope group*. Clearly, the orbit $\mathrm{SO}(P)(x)$ has at least two points for any $x \neq 0$. Hence, Theorem 2.6 ensures the next result.

Corollary 3.6. *For $\varepsilon > 0$ small enough, no global minimum solution for $\mathrm{HP}(\varepsilon)$ is $\mathrm{SO}(P)$-invariant.*

Define

$$O(P) := \{g \in O(N) : g(P) = P\}. \tag{3.1}$$

In the literature, the definition above is often referred to a regular polytope group. Since $\mathrm{SO}(P) \subset O(P)$, the $\mathrm{SO}(P)$-non-invariance implies the $O(P)$-non-invariance. So, Corollary 3.6 is valid for $O(P)$ as well.

Corollary 3.6 remains valid for semiregular polytopes, with $\mathrm{SO}(P)$ replaced by $O(P)$. A polytope P is called *semiregular* if it is a convex polytope whose faces are $(N-1)$-dimensional regular polytopes and its symmetric group is transitive on the set of vertices. More precisely, for any vertices x and y of P, there exists an isometric transformation which leaves P invariant and maps x to y (see [20, p. 15]). Therefore, all vertices of a semiregular polytope lie on a certain sphere. Choose the center of this sphere as the coordinate origin. Then the symmetric group of P is defined by (3.1). Accordingly, for any vertices x, y, there exists a $g \in O(P)$ such that $gx = y$.

Let E be the union of all edges of a semiregular polytope P. Let $\mathrm{SP}(\varepsilon)$ be an ε-neighborhood of E. This is a *skeleton semiregular polytope*.

Theorem 3.7. *Let P be an interior of a semiregular polytope centered at the origin in $\mathbb{R}^N$. Let Ω be either the hollow semiregular polytope $\mathrm{HP}(\varepsilon)$ or the skeleton semiregular polytope $\mathrm{SP}(\varepsilon)$. For $\varepsilon > 0$ small enough, no global minimum solution is $O(P)$-invariant.*

For example, consider a soccer ball (a truncated icosahedron) in $\mathbb{R}^3$. Let P be the interior of a soccer ball centered at the origin, and define $\Omega := (1+\varepsilon)P \setminus \overline{P}$. This is a hollow soccer ball. For $\varepsilon > 0$ small enough, no global minimum solution is invariant under the soccer ball group. The soccer ball is constructed by truncating vertices from the icosahedron. Therefore, the soccer ball group is equal to the icosahedral group.

Proof of Theorem 3.7. Put $G := O(P)$. If we could prove that $G(x)$ has at least two points for $x \neq 0$, then Theorem 2.6 proves Theorem 3.7. Although this claim seems clear, we give a proof to make the paper rigorous. Suppose on the contrary that $G(x_0) = \{x_0\}$ at some $x_0 \neq 0$. Let M be the orthogonal complement of x_0, which is defined by

$$M := \{y \in \mathbb{R}^N : (x_0, y) = 0\},$$

where (x_0, y) is a standard inner product in $\mathbb{R}^N$. Since $gx_0 = x_0$, we have that $g^{-1}x_0 = x_0$. Then

$$(gy, x_0) = (y, {}^t g x_0) = (y, g^{-1}x_0) = (y, x_0) = 0$$

for $g \in G$ and $y \in M$, where ${}^t g$ is the transpose of g. Therefore, $g(M) \subset M$ for $g \in G$. This shows that $g(M) = M$ for $g \in G$. Let x_1 be a vertex of P. Choose $\lambda \in \mathbb{R}$ so that $x_1 \in \lambda x_0 + M$, where $\lambda x_0 + M$ is a hyperplane defined by

$$\lambda x_0 + M := \{\lambda x_0 + y : y \in M\}.$$

Since $g(M) = M$ and $gx_0 = x_0$ for $g \in G$, it follows that $g(\lambda x_0 + M) = \lambda x_0 + M$. Denote the set of all vertices by V. Since G is transitive on V (this is a definition of a semiregular polytope), it is written as

$$V = \{gx_1 : g \in G\}.$$

Since $x_1 \in \lambda x_0 + M$ and $g(\lambda x_0 + M) = \lambda x_0 + M$, we have that $V \subset \lambda x_0 + M$. Taking the convex hull of both sides, we have $P \subset \lambda x_0 + M$. This is impossible because $\dim M = N - 1$. The proof is complete. □

Example 3.8. We give an easy example of G and Ω. Let $G := \{e, -e\}$, with e being the unit matrix. In this case, a G-invariant function is even, i.e., $u(-x) = u(x)$. Let D be a bounded domain whose closure does not contain the origin and which is point symmetric with respect to the origin, that is, $x \in D$ implies $-x \in D$. Let Ω be a point symmetric subdomain of D for which $\lambda_1(\Omega)$ is large enough. Then no global minimum solution is even. This assertion follows from Theorem 2.6 because $G(x) = \{x, -x\}$ for any $x \in \overline{D}$.

We shall give two examples of H, G and Ω and in these examples, we shall solve Problems 1.5 and 1.6.

Example 3.9. Let us consider Problem 1.5. We generalize it to a hollow regular n-gon. Let $\Omega := \mathrm{HP}_n(\varepsilon)$ be a hollow regular n-gon with $\varepsilon > 0$ small enough and let L be an axis of reflection symmetry of Ω. We choose the center of the n-gon as the coordinate origin. Let H be the reflection group with respect to L. Let G be a group generated by the union of H and the rotations of $2\pi j/n$, with $j = 0, 1, \dots, n-1$, that is, G is a dihedral group. It is clear that $H \subsetneq G$ and $H(x) \subset G(x)$ for $x \in \overline{D}$, where $D := \mathrm{HP}_n(1)$. We shall show that $H(x) \subsetneq G(x)$ for $x \in \overline{D}$. For $x \neq 0$, $|H(x)| \leq 2$ because H is a reflection. However, $|G(x)| \geq n \geq 3$ because G includes rotations of $2\pi j/n$, with $j = 0, 1, \dots, n-1$. Therefore, $H(x) \subsetneq G(x)$ for $x \in \overline{D}$. By Theorem 2.4, no H-minimum solution is G-invariant. Therefore, Problem 1.5 is solved in the affirmative.

Example 3.10. We consider Problem 1.6. Define $\mathrm{HC}_N(\varepsilon)$ and $\mathrm{HCP}_N(\varepsilon)$ by (1.8) and (1.9), respectively, and $\mathrm{SC}_N(\varepsilon)$ and $\mathrm{SCP}_N(\varepsilon)$ by the ε-neighborhood of the union of all the edges of C_N and CP_N, respectively. Let H and G be as defined after (1.8), i.e., H is the set of $g \in O(N)$, each element of which is equal to 0 or 1, and G is the set of $g \in O(N)$ each element of which is equal to 0, 1 or -1. We shall show that $H(x) \subsetneq G(x)$ for $x \neq 0$. Since $H \subset G$, we have that $H(x) \subset G(x)$. Let $x = (x_1, \dots, x_N) \neq 0$ be any point. Since $x \neq 0$, at least one element x_i is not 0. Suppose that it is positive. For any $y \in H(x)$, at least one element y_j is positive because $(y_1, \dots, y_N)$ is a permutation of $(x_1, \dots, x_N)$. However, there exists a point in $G(x)$, all elements of which are non-positive because we can define it by $(\tau_1 x_1, \dots, \tau_N x_N)$, where $\tau_i := -1$ if $x_i > 0$ and $\tau_i = 1$ if $x_i \leq 0$. Thus, $H(x) \subsetneq G(x)$. By Theorem 2.4, no H-minimum solution is G-invariant. Choose an H-minimum solution. Then it satisfies (1.10) but does not satisfy (1.11). Consequently, Problem 1.6 is solved in the affirmative. Moreover, in the annulus $A_N(a, \varepsilon)$, an H-minimum solution and a G-minimum solution are not radially symmetric. Indeed, since they are finite groups, Corollary 2.11 ensures the assertion above.

4 Existence of G-Minimum Solutions

In this section, we shall prove the existence of G-minimum solutions. We denote the $L^q(\Omega)$ norm of u by $\|u\|_q$. We employ the norm $\|\nabla u\|_p$ in the space $W_0^{1,p}(\Omega)$ because of the Poincaré inequality. Throughout this section, we suppose Assumption 2.2. Recall that $f(x, u)$ is extended as an odd function of u. We shall investigate the sign of $J(tu)$ as t varies in $(0, \infty)$.

Lemma 4.1. *For each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a unique $\mu > 0$ such that $J(tu) > 0$ for $t \in (0, \mu)$, $J(\mu u) = 0$, and $J(tu) < 0$ for $t \in (\mu, \infty)$. Therefore, μu belongs to the Nehari manifold $\mathcal{N}$.*

Proof. Since $f(x, u)/u^{q-1}$ is increasing with respect to u by (2.2), we have

$$0 < \frac{f(x, u)}{u^{q-1}} \leq f(x, 1) \leq C \quad \text{for } 0 < u < 1,\ x \in \overline{\Omega},$$

for a constant $C > 0$. Accordingly, $|f(x, u)| \leq C|u|^{q-1}$ for $|u| \leq 1$ because $f(x, u)$ is odd with respect to u. This inequality with Assumption 2.2 (iii) shows that

$$|f(x, u)| \leq C(|u|^{q-1} + |u|^{r-1}) \quad \text{for } u \in \mathbb{R},\ x \in \Omega, \tag{4.1}$$

for some $C > 0$. Recall that $f(x, u)$ is odd with respect to u, and $uf(x, u) > 0$ for $u \neq 0$. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then $J(tu)$, for $t > 0$, is written as

$$J(tu) = t^p \int_\Omega \Big(|\nabla u|^p - \frac{f(x, t|u|)}{|tu|^{p-1}} |u|^p \Big)\, dx.$$

By (4.1), we have

$$\int_\Omega \frac{f(x, t|u|)}{|tu|^{p-1}} |u|^p \, dx \leq C \int_\Omega (t^{q-p}|u|^q + t^{r-p}|u|^r)\, dx \to 0 \quad \text{as } t \to +0.$$

Accordingly, $J(tu) > 0$ for small $t > 0$.

We shall show that $J(tu) < 0$ for $t > 0$ large enough. By Assumption 2.2 (i) and (ii), we have a constant $c_0 > 0$ such that

$$\frac{f(x,u)}{u^{q-1}} \ge f(x,1) \ge c_0 \quad \text{for } u \ge 1. \tag{4.2}$$

This shows that

$$\min_{x\in\overline{\Omega}} f(x,u)u^{-(p-1)} \to \infty \quad \text{as } u \to \infty.$$

Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. For $\varepsilon > 0$, we put

$$E := \{x \in \Omega : |u(x)| > \varepsilon\}.$$

Since $u(x) \not\equiv 0$, we can choose $\varepsilon > 0$ so small that the Lebesgue measure of E (denoted by $|E|$) is positive. Then

$$\int_\Omega \frac{f(x,t|u|)}{|tu|^{p-1}}|u|^p\,dx \ge \int_E \frac{f(x,t\varepsilon)}{(t\varepsilon)^{p-1}}\varepsilon^p\,dx \ge \varepsilon^p|E| \min_{x\in\overline{\Omega}} f(x,t\varepsilon)(t\varepsilon)^{-(p-1)} \to \infty \quad \text{as } t \to \infty.$$

Thus, $J(tu) < 0$ for $t > 0$ large. Since $f(x,u)/u^{p-1}$ is strictly increasing with respect to u, $J(tu)$ has a unique zero $\mu > 0$. Accordingly, $J(tu) > 0$ in $(0,\mu)$, and $J(\mu u) = 0$ and $J(tu) < 0$ in (μ,∞). The proof is complete. □

In view of Lemma 4.1, for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we define $\mu(u)$ by $\mu > 0$ satisfying $J(\mu u) = 0$.

Lemma 4.2. *$\mu(\cdot)$ is continuous.*

Proof. Let u_n converge to $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$. Define $\underline{\mu} := \liminf_{n\to\infty} \mu(u_n)$. If $\underline{\mu} > 0$, for $\mu \in (0,\underline{\mu})$, we have that $J(\mu u_n) > 0$ for all large n, by Lemma 4.1. As $n \to \infty$, we have $J(\mu u_0) \ge 0$. If $\mu > \underline{\mu}$, then $J(\mu u_n) < 0$ along a subsequence. Hence, $J(\mu u_0) \le 0$. Consequently, $J(\mu u_0) \ge 0$ when $\mu < \underline{\mu}$ and $J(\mu u_0) \le 0$ when $\mu > \underline{\mu}$. Thus, $\underline{\mu} = \mu(u_0)$. The same argument shows that $\limsup_{n\to\infty} \mu(u_n) = \mu(u_0)$. Therefore, $\mu(\cdot)$ is continuous. □

Let $L^p(\Omega, G)$ and $C_0^\infty(\Omega, G)$ denote the set of G-invariant functions in $L^p(\Omega)$ and $C_0^\infty(\Omega)$, respectively. The space $W_0^{1,p}(\Omega, G)$ has already been defined in Section 1. Then the following lemma holds.

Lemma 4.3. *Let G be a closed subgroup of $O(N)$ and let Ω be a G-invariant open set in $\mathbb{R}^N$. Then $C_0^\infty(\Omega, G)$ is dense in $L^p(\Omega, G)$ and $W_0^{1,p}(\Omega, G)$.*

Proof. In [12, Lemma 3.1], it is proved that $C_0^\infty(\Omega, G)$ is dense in $L^p(\Omega, G)$. The same method, as in that lemma, is still valid for proving the density in $W_0^{1,p}(\Omega, G)$. □

By Lemmas 4.1 and 4.3, $\mathcal{N}$ and $\mathcal{N}(G)$ are not empty. Therefore, I_0 and I_G are well defined. We shall show that $\mathcal{N}$ is bounded away from the origin.

Lemma 4.4. *There exists a constant $c > 0$ such that $\|\nabla u\|_p \ge c$ for $u \in \mathcal{N}$. Moreover, I_0, given by (1.2), is positive.*

Proof. Let q and r be the exponents given in Assumption 2.2. From (4.2) and Assumption 2.2 (iii), it follows that $q \le r$, and hence $p < q \le r$. By this inequality and (4.1), for any $\varepsilon > 0$, there exists a constant $C > 0$ such that $|f(x,u)| \le \varepsilon|u|^{p-1} + C|u|^{r-1}$ for $u \in \mathbb{R}$ and $x \in \overline{\Omega}$. Using the Sobolev embedding, we estimate $J(u)$:

$$J(u) = \|\nabla u\|_p^p - \int_\Omega f(x,u)u\,dx \ge \|\nabla u\|_p^p - \varepsilon\|u\|_p^p - C\|u\|_r^r \ge \|\nabla u\|_p^p - \varepsilon C'\|\nabla u\|_p^p - C'\|\nabla u\|_p^r$$

for some $C' > 0$. Let $u \in \mathcal{N}$. Since $J(u) = 0$, we have

$$C'\|\nabla u\|_p^{r-p} \ge 1 - \varepsilon C'.$$

Choose $\varepsilon > 0$ so small that the right-hand side is positive. Hence, we have the first assertion of the lemma.

It follows from (2.2) that $f_u(x,u)u > (q-1)f(x,u)$. Integrating it over $[0,u]$, we have $F(x,u) \le (1/q)uf(x,u)$ for $u \ge 0$. This inequality is still valid for $u \in \mathbb{R}$ because both sides are even with respect to u. Let $u \in \mathcal{N}$. Since $J(u) = 0$ and $\|\nabla u\|_p \ge c$, we get

$$I(u) \ge \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega uf(x,u)\,dx = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|\nabla u\|_p^p \ge \Big(\frac{q-p}{pq}\Big)c^p.$$

Therefore, $I_0 \ge ((q-p)/pq)c^p$. The proof is complete. □

In the next lemma, we shall show the existence of a global minimum solution only. However, the same method is applicable to a G-minimum solution with the help of the principle of symmetric criticality by Palais [27].

Lemma 4.5. *There exists a global minimum solution. Any global minimum solution u is a critical point of I, and it becomes a positive solution of* (1.1) *after replacing u by $-u$ if necessary.*

Proof. The lemma can be proved in a standard method (we refer the readers to [31, Theorems 4.2 and 4.3]). By the strong maximum principle, a global minimum solution u must be positive after replacing u by $-u$ if necessary. □

5 Proof of the Main Theorem

In this section, we shall prove Theorem 2.4 only, which readily yields all the other theorems in Section 2. We always suppose Assumptions 2.1 and 2.2. For a G-invariant bounded domain Ω, we have already defined $L^p(\Omega, G)$ and $H_0^1(\Omega, G)$, which are G-invariant function spaces in $L^p(\Omega)$ and $H_0^1(\Omega)$, respectively. Here we denote the orthogonal complement of $L^2(\Omega, G)$ in $L^2(\Omega)$ by $L^2(\Omega, G)^\perp$, and the orthogonal complement of $H_0^1(\Omega, G)$ in $H_0^1(\Omega)$ by $H_0^1(\Omega, G)^\perp$, that is,

$$L^2(\Omega, G)^\perp := \{u \in L^2(\Omega) : (u, v)_{L^2} = 0 \text{ for } v \in L^2(\Omega, G)\},$$
$$H_0^1(\Omega, G)^\perp := \{u \in H_0^1(\Omega) : (u, v)_{H_0^1} = 0 \text{ for } v \in H_0^1(\Omega, G)\}.$$

Here the inner products are defined by

$$(u, v)_{L^2} := \int_\Omega u(x)v(x)\,dx, \quad (u, v)_{H_0^1} := \int_\Omega \nabla u(x)\nabla v(x)\,dx.$$

We present four lemmas below, which have been proved in [14].

Lemma 5.1 ([14, Lemma 5.2]). *Let Ω be a G-invariant bounded domain. Then the following assertions hold:*
(i) *If $u \in L^p(\Omega, G)$ and $v \in L^q(\Omega) \cap L^2(\Omega, G)^\perp$, with $1/p + 1/q = 1$ and $1 \le p \le \infty$, then $\int_\Omega u(x)v(x)\,dx = 0$.*
(ii) $H_0^1(\Omega, G)^\perp \subset L^2(\Omega, G)^\perp$.

We here introduce a Haar measure. Since a closed subgroup G of $O(N)$ is a compact Lie group, it has a unique Haar measure dg which satisfies

$$\int_G f(g)\,dg = \int_G f(g'g)\,dg = \int_G f(gg')\,dg = \int_G f(g^{-1})\,dg, \qquad \int_G f(g)\,dg > 0 \quad \text{if } f \ge 0, f \not\equiv 0, \qquad \int_G 1\,dg = 1,$$

for any $g' \in G$ and any real valued integrable function f on G. For the Haar measure, we refer to [28] or [24, Chapter 2].

Let $M(N)$ be the set of all $N \times N$ real matrices. We define the norm by

$$\|g\| := \max_{|x|\le 1} |gx| \quad \text{for } g \in M(N),$$

where $|x|$ is the standard Euclidean norm. Let G be a closed subgroup of $O(N)$. For $h \in G$ and $r > 0$, we define a ball $B(h, r; G)$ in G by

$$B(h, r; G) := \{g \in G : \|g - h\| < r\}.$$

We define the volume of $B(h, r; G)$ (denoted by $|B(h, r; G)|$) by

$$|B(h, r; G)| := \int_{B(h,r;G)} 1\,dg,$$

where dg is the Haar measure of G.

Lemma 5.2 ([14, Lemma 5.6]). *The volume $|B(h, r; G)|$ is independent of h.*

Let H and G be closed subgroups of $O(N)$ satisfying $H \subsetneq G$. Since H and G are compact Lie groups, we can define

$$Q(x, g) := \min_{h \in H} |gx - hx|, \quad P(x) := \max_{g \in G} Q(x, g).$$

Lemma 5.3 ([14, Lemma 5.5]). *We have that*

$$|P(x) - P(y)| \le 2|x - y| \quad \text{for } x, y \in \mathbb{R}^N.$$

By Assumption 2.1, for any $x \in \overline{D}$, there exists a $g \in G$ such that $hx \neq gx$ for any $h \in H$. This assertion is equivalent to a condition that $P(x) > 0$ for $x \in \overline{D}$. By Lemma 5.3, $P(x)$ is continuous. Hence, the minimum of $P(x)$ on $\overline{D}$ is positive and we define

$$\varepsilon := \frac{1}{4} \min_{\overline{D}} P(x) > 0. \tag{5.1}$$

This definition implies that for any $x \in \overline{D}$, there exists a $g \in G$ such that

$$|gx - hx| \ge 4\varepsilon \quad \text{for all } h \in H. \tag{5.2}$$

Denote a ball in $\mathbb{R}^N$ centered at x with radius $r > 0$ by $B(x, r)$. Let $\varepsilon > 0$ be defined by (5.1). Since $\overline{D}$ is compact, we can take a finite covering $B(x_i, \varepsilon/4)$ with some $x_1, \dots, x_k \in \mathbb{R}^N$ such that

$$\overline{D} \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon/4), \tag{5.3}$$

where k is the smallest integer satisfying the inclusion above, and hence it depends only on D and ε. Let Ω be a G-invariant subdomain of D satisfying $\lambda_1(\Omega) > \Lambda$, where $\Lambda > 0$ will be determined later on. Let u be a G-minimum solution. We put $u(x) = 0$ outside Ω. Choose a point $y_0 \in \mathbb{R}^N$ satisfying

$$\sup_{y \in \mathbb{R}^N} \int_{B(y, \varepsilon/4)} f(x, u) u \, dx = \int_{B(y_0, \varepsilon/4)} f(x, u) u \, dx.$$

If $B(y_0, \varepsilon/4) \cap \Omega = \emptyset$, then the right-hand side is zero and a contradiction occurs. Thus, we can choose a point $x_0 \in B(y_0, \varepsilon/4) \cap \Omega$, which ensures that $B(y_0, \varepsilon/4) \subset B(x_0, \varepsilon/2)$. Then

$$\sup_{y \in \mathbb{R}^N} \int_{B(y, \varepsilon/4)} f(x, u) u \, dx \le \int_{B(x_0, \varepsilon/2)} f(x, u) u \, dx. \tag{5.4}$$

Combining (5.3) and (5.4), we have

$$\int_{\Omega} f(x, u) u \, dx = \int_{D} f(x, u) u \, dx \le k \int_{B(x_0, \varepsilon/2)} f(x, u) u \, dx. \tag{5.5}$$

Note that k is independent of Ω and $u(x)$. We define a radially symmetric function $\Phi(x) \in C_0^\infty(\mathbb{R}^N)$ which satisfies $0 \le \Phi(x) \le 1$ in $\mathbb{R}^N$,

$$\Phi(x) = 1 \quad \text{for } |x| \le \varepsilon, \qquad \Phi(x) = 0 \quad \text{for } |x| \ge 2\varepsilon, \tag{5.6}$$

$$|\nabla \Phi(x)| \le 2/\varepsilon \quad \text{for } \varepsilon < |x| < 2\varepsilon, \tag{5.7}$$

where ε has been defined by (5.1). Let dg and dh be the Haar measures of G and H, respectively. We define

$$\phi(x) := \int_G \Phi(x - gx_0) \, dg - \int_H \Phi(x - hx_0) \, dh, \tag{5.8}$$

where x_0 has been defined before (5.4). Then $\phi \in C_0^\infty(\mathbb{R}^N)$. Define

$$\Omega' := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 2\varepsilon\}, \quad \operatorname{dist}(x, \Omega) := \inf\{|x - y| : y \in \Omega\}.$$

Then $\Omega \subset \Omega'$. Since Ω is G-invariant, so is Ω'. Since $x_0 \in \Omega$, we have that $hx_0, gx_0 \in \Omega$ for $h \in H$ and $g \in G$. Therefore, the support of $\phi(x)$ is in Ω'.

Lemma 5.4. *We have* $\phi \in C_0^\infty(\Omega') \cap H_0^1(\Omega', G)^\perp \cap H_0^1(\Omega', H)$.

Proof. This lemma is proved in [14, Theorem 2.2]. □

Let u be a G-minimum solution and let $x_0 \in \Omega$ satisfy (5.4). Let ϕ be given by (5.8). We define

$$g(t,s) := I((1+t)(1+s\phi)u) \quad \text{for } t, s \in \mathbb{R}. \tag{5.9}$$

Since $u \in W_0^{1,p}(\Omega)$ and $\phi \in C_0^\infty(\Omega')$, the function $(1+s\phi)u$ belongs to $W_0^{1,p}(\Omega)$, and therefore $g(t,s)$ is well defined.

We shall explain our idea to prove Theorem 2.4. For a small $s > 0$, we shall show that there exists a number t such that

$$(1+t)(1+s\phi)u \in \mathcal{N}(H), \quad I((1+t)(1+s\phi)u) < I(u).$$

Then it follows that

$$I_H \le I((1+t)(1+s\phi)u) < I(u) = I_G.$$

Thus, we get Theorem 2.4. To accomplish the method above, we investigate the behavior of $g(t,s)$ near $(t,s) = (0,0)$.

Lemma 5.5. *Let u be a G-minimum solution and let ϕ be defined by* (5.8). *Define $g(t,s)$ by* (5.9). *Then we have that*

$$g(t,s) = g(0,0) + \frac{t^2}{2} g_{tt}(0,0) + \frac{s^2}{2} g_{ss}(0,0) + o(t^2+s^2) \tag{5.10}$$

as $t, s \to 0$, where $o(t^2+s^2)/(t^2+s^2) \to 0$ as $t, s \to 0$, and $g_{tt}(0,0)$ denotes the second partial derivatives with respect to t. Moreover, we have the representations

$$g_{tt}(0,0) = \int_\Omega ((p-1)f(x,u)u - f_u(x,u)u^2)\,dx, \tag{5.11}$$

$$g_{ss}(0,0) = (p-1)\int_\Omega (f(x,u)u\phi^2 + |\nabla u|^{p-2}|\nabla\phi|^2u^2)\,dx - \int_\Omega f_u(x,u)u^2\phi^2\,dx. \tag{5.12}$$

Proof. Since $I'(u) = 0$, we have

$$g_t(0,0) = I'(u)u = 0, \quad g_s(0,0) = I'(u)\phi u = 0.$$

The Taylor theorem shows that

$$g(t,s) = g(0,0) + \frac{t^2}{2} g_{tt}(0,0) + ts g_{ts}(0,0) + \frac{s^2}{2} g_{ss}(0,0) + o(t^2+s^2) \tag{5.13}$$

as $t, s \to 0$. By Assumption 2.2 (iii) and since $p \ge 2$, $I(u)$ has a second derivative which is a bilinear form, i.e.,

$$I''(u)[v,w] = \int_\Omega ((p-1)|\nabla u|^{p-2}\nabla v\nabla w - f_u(x,u)vw)\,dx$$

for $u, v, w \in W_0^{1,p}(\Omega)$. Noting that $I'(u) = 0$, we compute the second derivatives of $g(t,s)$:

$$g_{tt}(0,0) = I''(u)[u,u] = \int_\Omega ((p-1)|\nabla u|^p - f_u(x,u)u^2)\,dx, \tag{5.14}$$

$$g_{ts}(0,0) = I''(u)[u,\phi u] + I'(u)\phi u = (p-1)\int_\Omega (|\nabla u|^p\phi + |\nabla u|^{p-2}(\nabla u\nabla\phi)u)\,dx - \int_\Omega f_u(x,u)u^2\phi\,dx, \tag{5.15}$$

where $\nabla u\nabla\phi = \sum_{i=1}^N (\partial u/\partial x_i)(\partial\phi/\partial x_i)$, and

$$\begin{aligned} g_{ss}(0,0) &= I''(u)[\phi u, \phi u] \\ &= (p-1)\int_\Omega (|\nabla u|^p\phi^2 + 2|\nabla u|^{p-2}(\nabla u\nabla\phi)u\phi)\,dx \\ &\quad + \int_\Omega ((p-1)|\nabla u|^{p-2}|\nabla\phi|^2u^2 - f_u(x,u)u^2\phi^2)\,dx. \end{aligned} \tag{5.16}$$

We shall show that $g_{ts}(0,0) = 0$. Put $u(x) = 0$ outside Ω. Then u belongs to $W_0^{1,p}(\Omega', G)$. Accordingly, $|\nabla u|^p \in L^1(\Omega', G)$. Since $\phi \in H_0^1(\Omega', G)^\perp$ by Lemma 5.4, it belongs to $L^2(\Omega', G)^\perp$ by Lemma 5.1. We use Lemma 5.1 (i) to obtain

$$\int_\Omega |\nabla u|^p \phi \, dx = \int_{\Omega'} |\nabla u|^p \phi \, dx = 0,$$

which is the first term on the right-hand side of (5.15). In the same way, we have

$$\int_\Omega f_u(x,u) u^2 \phi \, dx = 0.$$

Therefore, the third term of (5.15) vanishes. If $u \in C_0^\infty(\Omega, G)$, then we have

$$\int_\Omega |\nabla u|^{p-2} (\nabla u \nabla \phi) u \, dx = -\int_{\Omega'} \operatorname{div}(|\nabla u|^{p-2} \nabla u u) \phi \, dx = 0$$

because $\operatorname{div}(|\nabla u|^{p-2}\nabla u u)$ is G-invariant. The space $C_0^\infty(\Omega, G)$ is dense in $W_0^{1,p}(\Omega, G)$ by Lemma 4.3. For $u \in W_0^{1,p}(\Omega, G)$, we choose a sequence u_n in $C_0^\infty(\Omega, G)$ converging to u in $W_0^{1,p}(\Omega, G)$. Substituting u_n in the equation above and letting $n \to \infty$, we obtain

$$\int_\Omega |\nabla u|^{p-2} (\nabla u \nabla \phi) u \, dx = 0,$$

which is the second term on the right-hand side of (5.15). Consequently, $g_{ts}(0,0) = 0$. Hence, (5.13) is reduced to (5.10).

We shall prove the expressions (5.11) and (5.12). Multiplying (1.1) by u and integrating it over Ω, we get

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega f(x,u) u \, dx. \tag{5.17}$$

Substituting this relation into (5.14), we obtain (5.11). Multiplying (1.1) by $u\phi^2$ and integrating it, we have

$$\int_\Omega \left(|\nabla u|^p \phi^2 + 2|\nabla u|^{p-2} (\nabla u \nabla \phi) u \phi \right) dx = \int_\Omega f(x,u) u \phi^2 \, dx.$$

Substituting this relation into (5.16), we get (5.12). The proof is complete. □

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Let u be a G-minimum solution and let $x_0 \in \Omega$ satisfy (5.4). For this x_0, by (5.2), there exists a $g_0 \in G$ such that

$$|g_0 x_0 - h x_0| \geq 4\varepsilon \quad \text{for } h \in H.$$

Hence, $|x - hx_0| \geq 2\varepsilon$ for $x \in B(g_0 x_0, 2\varepsilon)$ and $h \in H$. This inequality and (5.6) show that

$$\Phi(x - hx_0) = 0 \quad \text{for } x \in B(g_0 x_0, 2\varepsilon),\ h \in H,$$

which, by (5.8), implies that

$$\phi(x) = \int_G \Phi(x - gx_0) dg \quad \text{for } x \in B(g_0 x_0, 2\varepsilon). \tag{5.18}$$

Since D is bounded, there exists a constant $R > 0$ such that $|x| \leq R$ for $x \in D$. We define $\nu := \varepsilon/(2R)$ and $d := |B(g, \nu; G)|$. By Lemma 5.2, d is independent of $g \in G$. For $x \in B(g_0 x_0, \varepsilon/2)$ and $g \in B(g_0, \nu; G)$, we have that

$$|x - gx_0| \leq |x - g_0 x_0| + |g_0 x_0 - g x_0| < \varepsilon/2 + \|g_0 - g\| |x_0| < \varepsilon.$$

This inequality and (5.6) prove that

$$\Phi(x - gx_0) = 1 \quad \text{for } x \in B(g_0x_0, \varepsilon/2),\ g \in B(g_0, \nu; G),$$

which, by (5.18), implies that

$$\phi(x) \geq \int_{B(g_0,\nu;G)} \Phi(x - gx_0)\, dg \geq |B(g_0, \nu; G)| = d > 0 \tag{5.19}$$

for $x \in B(g_0x_0, \varepsilon/2)$. Thus, $\phi \not\equiv 0$ in Ω. We define $g(t, s)$ by (5.9). We shall show that $g_{tt}(0, 0)$ and $g_{ss}(0, 0)$ are negative. We use (5.11) and (2.2) to obtain

$$g_{tt}(0, 0) = \int_\Omega ((p - 1)f(x, u)u - f_u(x, u)u^2)\, dx \leq \int_\Omega ((p - 1)f(x, u)u - (q - 1)f(x, u)u)\, dx < 0,$$

because u is a positive solution. We again write (5.12) as

$$g_{ss}(0, 0) = (p - 1) \int_\Omega (f(x, u)u\phi^2 + |\nabla u|^{p-2}|\nabla\phi|^2u^2)\, dx - \int_\Omega f_u(x, u)u^2\phi^2\, dx. \tag{5.20}$$

Since $|\nabla\Phi(x)| \leq 2/\varepsilon$ in $\mathbb{R}^N$, by (5.6) and (5.7), we have

$$|\nabla\phi(x)| \leq \int_G |\nabla\Phi(x - gx_0)|\, dg + \int_H |\nabla\Phi(x - hx_0)|\, dh \leq 4/\varepsilon.$$

Using the inequality above and employing the Hölder inequality, we estimate the second term in (5.20):

$$\int_\Omega |\nabla u|^{p-2}|\nabla\phi|^2u^2\, dx \leq 16\varepsilon^{-2} \int_\Omega |\nabla u|^{p-2}u^2\, dx \leq 16\varepsilon^{-2}\|\nabla u\|_p^{p-2}\|u\|_p^2.$$

Since $\lambda_1(\Omega)$ is the first eigenvalue, we have that $\lambda_1(\Omega)\|u\|_p^p \leq \|\nabla u\|_p^p$. Using this inequality with the assumption $\lambda_1(\Omega) > \Lambda$ and employing (5.17) and (5.5), we get

$$\int_\Omega |\nabla u|^{p-2}|\nabla\phi|^2u^2\, dx \leq 16\varepsilon^{-2}\Lambda^{-2/p}\|\nabla u\|_p^p = 16\varepsilon^{-2}\Lambda^{-2/p} \int_\Omega f(x, u)u\, dx \leq 16k\varepsilon^{-2}\Lambda^{-2/p} \int_{B(x_0,\varepsilon/2)} f(x, u)u\, dx.$$

Since u and f are G-invariant, the right-hand side is equal to

$$16k\varepsilon^{-2}\Lambda^{-2/p} \int_{B(g_0x_0,\varepsilon/2)} f(x, u)u\, dx.$$

Since $\phi(x) \geq d$ in $B(g_0x_0, \varepsilon/2)$ by (5.19), we have

$$\int_\Omega |\nabla u|^{p-2}|\nabla\phi|^2u^2\, dx \leq 16kd^{-2}\varepsilon^{-2}\Lambda^{-2/p} \int_{B(g_0x_0,\varepsilon/2)} f(x, u)u\phi^2\, dx \leq 16kd^{-2}\varepsilon^{-2}\Lambda^{-2/p} \int_\Omega f(x, u)u\phi^2\, dx.$$

Denoting the coefficient of the last integral by a, we get

$$\int_\Omega |\nabla u|^{p-2}|\nabla\phi|^2u^2\, dx \leq a \int_\Omega f(x, u)u\phi^2\, dx.$$

Using this inequality, we estimate (5.20) as

$$\begin{aligned} g_{ss}(0, 0) &\leq \int_\Omega ((p - 1)(a + 1)f(x, u)u\phi^2 - f_u(x, u)u^2\phi^2)\, dx \\ &\leq -[q - 1 - (p - 1)(a + 1)] \int_\Omega f(x, u)u\phi^2\, dx, \end{aligned} \tag{5.21}$$

where we have used (2.2). Recall that $a = 16kd^{-2}\varepsilon^{-2}\Lambda^{-2/p}$. Here we choose $\Lambda > 0$ so large that

$$(p-1)[16kd^{-2}\varepsilon^{-2}\Lambda^{-2/p} + 1] < q - 1. \tag{5.22}$$

Then the right-hand side of (5.21) is negative. We note that the constants k, d and ε in (5.22) depend only on G, H and D and are independent of Ω and $u(x)$.

We have proved that $g_{tt}(0,0)$ and $g_{ss}(0,0)$ are negative. It then follows from (5.10) that

$$g(t,s) = I((1+t)(1+s\phi)u) < g(0,0) = I(u),$$

when $|t|$, $|s|$ are small enough and $(t,s) \neq (0,0)$. For $v \in W_0^{1,p}(\Omega) \setminus \{0\}$, we recall that $\mu(v)$ is a unique positive number satisfying $\mu(v)v \in \mathcal{N}$. Since u is a G-minimum solution, it belongs to $\mathcal{N}$. Therefore, $\mu(u) = 1$. Since $\mu(\cdot)$ is continuous by Lemma 4.2, $\mu((1+s\phi)u)$ converges to $\mu(u) = 1$ as $s \to 0$. Define $t(s) := \mu((1+s\phi)u) - 1$. Then $t(s) \to 0$ as $s \to 0$. Since u is G-invariant and ϕ is H-invariant, $(1+s\phi)u$ is H-invariant. Thus, we find that $(1+t(s))(1+s\phi)u$ belongs to $\mathcal{N}(H)$. When $s > 0$ is small enough, so is $|t(s)|$. Therefore, for $s > 0$ small, we have

$$I_H \leq I((1+t(s))(1+s\phi)u) < I(u) = I_G.$$

The proof is complete. □

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