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Research Article

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Existence of a Heteroclinic Solution for a Double Well Potential Equation in an Infinite Cylinder of $\mathbb{R}^N$

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Abstract: This paper is concerned with the existence of a heteroclinic solution for the following class of elliptic equations:

$$-\Delta u + A(\epsilon x, y)V'(u) = 0 \quad \text{in } \Omega,$$

where $\epsilon > 0$, $\Omega = \mathbb{R} \times \mathcal{D}$ is an infinite cylinder of $\mathbb{R}^N$ with $N \geq 2$. Here, we consider a large class of potentials V that includes the Ginzburg–Landau potential $V(t) = (t^2 - 1)^2$ and two geometric conditions on the function A. In the first condition we assume that A is asymptotic at infinity to a periodic function, while in the second one A satisfies

$$0 < A_0 = A(0, y) = \inf_{(x,y)\in\Omega} A(x, y) < \liminf_{|(x,y)|\to+\infty} A(x, y) = A_\infty < \infty \quad \text{for all } y \in \mathcal{D}.$$

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1 Introduction

This paper is concerned with the existence of a heteroclinic solution for the following class of elliptic equations:

$$-\Delta u + A(\epsilon x, y)V'(u) = 0 \quad \text{in } \Omega, \tag{PDE}$$

together with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x, y) = 0, \quad x \in \mathbb{R},\ y \in \partial\mathcal{D}, \tag{NC}$$

where $N \geq 2$, $\epsilon > 0$, Ω is an infinite cylinder of the type $\Omega = \mathbb{R} \times \mathcal{D}$ with $\mathcal{D} \subset \mathbb{R}^{N-1}$ being a smooth bounded domain and $\nu = \nu(y)$ is the normal vector outward pointing to $\partial\mathcal{D}$. Related to the functions $A : \overline{\Omega} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$, we are assuming the following conditions.

Conditions on V:
(V_1) $V \in C^1(\mathbb{R}, \mathbb{R})$,
(V_2) $V(-1) = V(1) = 0$ and $V(t) \geq 0$ for all $t \in \mathbb{R}$,
(V_3) $V(t) > 0$ for all $t \neq -1, 1$.
An example of V satisfying (V_1)–(V_3) is the Ginzburg–Landau potential $V(t) = (t^2 - 1)^2$.

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Conditions on A: Throughout this paper, A is a C^1-function that belongs to one of the following classes:

Class 1. The function A is asymptotic at infinity to a periodic function.

In this class, we assume that there exists a C^1-function $A_p : \overline{\Omega} \to \mathbb{R}$, which is 1-periodic in x, such that
(A_1) $|A(x,y) - A_p(x,y)| \to 0$ as $|(x,y)| \to +\infty$,
(A_2) $0 < A_0 = \inf_{(x,y)\in\Omega} A(x,y) \le A(x,y) < A_p(x,y)$ for all $(x,y) \in \Omega$.
This type of condition is well known when we are working with periodic asymptotically problem of the type

$$-\Delta u + A(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

see for example Alves, Carrião and Miyagaki [7], Jianfu and Xiping [16] and their references.

Class 2. The function A satisfies the Rabinowitz condition.

In this class of functions, we suppose that
(A_3) $0 < \inf_{(x,y)\in\overline{\Omega}} A(x,y) \le \max_{y\in\overline{\mathcal{D}}} A(0,y) < \liminf_{|(x,y)|\to+\infty} A(x,y) = A_\infty < \infty$.

A condition like the above has been introduced by Rabinowitz [10, Theorem 4.33] to study the existence of solution for a PDE of the type

$$-\epsilon^2\Delta u + A(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $\epsilon > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $A : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$0 < \inf_{x\in\mathbb{R}^N} A(x) < \liminf_{|x|\to\infty} A(x).$$

By using variational methods, more precisely the mountain pass theorem, Rabinowitz has established the existence of solution for ϵ small enough. For this reason, throughout this article, we will call (A_3) of Rabinowitz's condition.

By (V_1)–(V_3), V is a double well potential and we are interested in the existence of solutions for (PDE) and (NC) that are heteroclinic in x from 1 to -1. A heteroclinic solution from 1 to -1 is a function $u \in C^2(\overline{\Omega}, \mathbb{R})$ verifying (PDE)–(NC) with

$$u(x,y) \to 1 \quad \text{as } x \to -\infty \qquad \text{and} \qquad u(x,y) \to -1 \quad \text{as } x \to +\infty, \qquad \text{uniformly in } y \in \mathcal{D}.$$

In [11], Rabinowitz has proved the existence of a heteroclinic solution for elliptic equations of the type

$$-\Delta u = g(x,y,u) \quad \text{in } \Omega,$$

together with the boundary condition (NC) and also with the Dirichlet boundary condition, that is,

$$u(x,y) = 0, \quad x \in \mathbb{R},\ y \in \partial\mathcal{D}.$$

In order to prove the existence of heteroclinic solution, in [11, Section 2], Rabinowitz has used variational methods by supposing on g the conditions below:
(g_1) $g \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
(g_2) $g(x,y,t)$ is even and 1-periodic in x.
In [11, Section 3], Rabinowitz has considered some conditions on g that permit to study other classes of nonlinearity. From these comments, we see that if
(g) $g(x,y,t) = A(x,y)V'(t)$,
Rabinowitz has studied the case when $A(x,y)$ is 1-periodic in x, see [11, Section 2]. Here, we continue this study, because we will work with two new classes of function A that were not considered in that paper, more precisely Classes 1 and 2.

After Byeon, Montecchiari and Rabinowitz [8] have established the existence of heteroclinic solution $u : \Omega \to \mathbb{R}^m$ for a large class of elliptic system like

$$-\Delta u + V_u(x,u) = 0 \quad \text{in } \Omega,$$

together with the boundary condition (NC) by supposing the following conditions on potential V:

(H_1) $V \in C^1(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R})$ and $V(x_1 + 1, x_2, \dots, x_N, y) = V(x, y)$, i.e., V is 1-periodic in x_1.

(H_2) There are points $a^- \neq a^+$ such that $V(x, a^\pm) = 0$ for all $x \in \Omega$ and $V(x, y) > 0$ otherwise.

(H_3) There is a constant $\underline{V} > 0$ such that $\liminf_{|t|\to\infty} V(x, t) \geq \underline{V}$ uniformly in $x \in \Omega$.

(H_4) For $N \geq 2$, there exist constants $c_1, C_1 > 0$ such that

$$|V_u(x, t)| \leq c_1 + C_1|t|^p,$$

where $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and there is no upper growth restriction on p if $N = 2$.

In the present paper, we are working with the potential $V(x, y, u) = A(x, y)V(u)$, with A belonging to Classes 1 or 2 and V satisfying (H_1)–(H_4). Our paper also continues the study made in [8] for $m = 1$, because we are working with other classes of function A. Here, it is very important to mention that the study of elliptic system as above is very subtle because some arguments used for the scalar case $m = 1$ cannot be used for general case $m > 1$ as for example maximum principle.

In the literature we also find interesting papers that study the existence of heteroclinic solution for elliptic equations in whole $\mathbb{R}^N$ like

$$-\Delta u(x, y) + A(x, y)V'(u(x, y)) = 0, \quad (x, y) \in \mathbb{R}^N,$$

by supposing different conditions on A and V, see for example Alessio and Montecchiari [5, 6], Alessio, Jeanjean and Montecchiari [3, 4], Alessio, Gui and Montecchiari [2], Rabinowitz [12], Rabinowitz and Stredulinsky [13–15] and their references. The reader can find versions for elliptic systems of the above equation in Alama, Bronsard and Gui [1], Alessio, Jeanjean and Montecchiari [4], Montecchiari and Rabinowitz [9] and the references therein.

Motivated by papers [8] and [11], we intend to establish the existence of a heteroclinic solution for equation (PDE) under the Neumann boundary conditions by working with Classes 1 and 2. As in the above papers, we have used variational method, more precisely minimization technical on a special set, however new ideas have been introduced in the study of the problem, see for example Proposition 3.1. The regularity and behavior of the heteroclinic are obtained by using the same arguments found in [8].

Our main results are the following:

Theorem 1.1. *Assume* (V_1)–(V_3), $\epsilon = 1$ *and that A belongs to Class 1. Then problem* (PDE)–(NC) *has a heteroclinic solution from* 1 *to* -1.

Theorem 1.2. *Assume* (V_1)–(V_3) *and that A belongs to Class 2. Then there is a constant* $\epsilon_0 > 0$ *such that problem* (PDE)–(NC) *possesses a heteroclinic solution from* 1 *to* -1 *for all* $\epsilon \in (0, \epsilon_0)$.

The plan of the paper is as follows: In Section 2, we prove some technical results, which will be useful to prove the above theorems. In Section 3 we prove Theorem 1.1, while in Section 4 we prove Theorem 1.2.

2 Preliminary Results

Consider problem (PDE)–(NC) with $\epsilon = 1$, more precisely,

$$\begin{cases} -\Delta u + A(x, y)V'(u) = 0, & (x, y) \in \Omega = \mathbb{R} \times \mathcal{D}, \\ \dfrac{\partial u}{\partial \nu}(x, y) = 0, & x \in \mathbb{R},\ y \in \partial\mathcal{D}. \end{cases}$$

In the sequel, we define the set

$$\Gamma = \{U \in W^{1,2}_{\mathrm{loc}}(\Omega) : |\nabla U| \in L^2(\Omega),\ \|P_k U - 1\|_{L^2(\Omega_1)} \to 0 \text{ as } k \to -\infty,\ \|P_k U + 1\|_{L^2(\Omega_1)} \to 0 \text{ as } k \to +\infty\},$$

where $\Omega_1 = (0, 1) \times \mathcal{D}$ and

$$P_k U(x, y) = U(x + k, y) \quad \text{for } (x, y) \in \Omega \text{ and } k \in \mathbb{Z}.$$

It is very important to observe that $\Gamma \neq \emptyset$, because the function Φ given by

$$\Phi(x,y) = \begin{cases} 1, & \text{if } x \leq j,\ y \in \mathcal{D}, \\ 2j+1-2x, & \text{if } j < x \leq j+1,\ y \in \mathcal{D}, \\ -1, & \text{if } j+1 < x,\ y \in \mathcal{D}, \end{cases} \tag{2.1}$$

belongs to Γ. Furthermore, we also fix

$$\mathcal{L}(u) = \frac{1}{2}|\nabla u|^2 + A(x,y)V(u),$$

and the functionals $J : \Gamma \to \mathbb{R} \cup \{+\infty\}$ given by

$$J(U) = \sum_{k \in \mathbb{Z}} I_k(U) \tag{2.2}$$

and $I_k : W^{1,2}((k,k+1) \times \mathcal{D}) \to \mathbb{R}$ defined by

$$I_k(U) = \int_k^{k+1} \int_{\mathcal{D}} \mathcal{L}(U)\, dx\, dy.$$

Associated with functional J we have the number

$$\Theta^* = \inf\{J(U) : U \in \Gamma\}. \tag{2.3}$$

By (2.1), $\Phi \in \Gamma$, then $\Theta^* < +\infty$. By the definition of Θ^*, there exists a minimizing sequence $(U_n) \subset \Gamma$ for J, that is,

$$J(U_n) \to \Theta^* \quad \text{as } n \to \infty. \tag{2.4}$$

Without loss of generality, we can assume that (U_n) verifies

$$-1 \leq U_n(x,y) \leq 1 \quad \text{for all } (x,y) \in \Omega. \tag{2.5}$$

Indeed, for each $n \in \mathbb{N}$ let us consider

$$\tilde{U}_n(x,y) = \begin{cases} -1, & \text{if } U_n(x,y) \leq -1, \\ U_n(x,y), & \text{if } -1 \leq U_n(x,y) \leq 1, \\ 1, & \text{if } U_n(x,y) \geq 1. \end{cases}$$

It is easy to check that $\tilde{U}_n \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with

$$|\tilde{U}_n(x,y) - 1| \leq |U_n(x,y) - 1| \quad \text{for all } (x,y) \in \Omega,$$

and

$$|\tilde{U}_n(x,y) + 1| \leq |U_n(x,y) + 1| \quad \text{for all } (x,y) \in \Omega.$$

Hence $(\tilde{U}_n) \subset \Gamma$, and so

$$\Theta^* \leq J(\tilde{U}_n) \quad \text{for all } n \in \mathbb{N}.$$

Since

$$J(\tilde{U}_n) \leq J(U_n) \quad \text{for all } n \in \mathbb{N},$$

it follows that

$$\Theta^* \leq J(\tilde{U}_n) \leq J(U_n) = \Theta^* + o_n(1),$$

thereby showing that $(\tilde{U}_n)$ is also a minimizing sequence for J on Γ with

$$-1 \leq \tilde{U}_n(x,y) \leq 1 \quad \text{for all } (x,y) \in \Omega.$$

From (2.4)–(2.5), there is $M > 0$ independent of k and m such that

$$\||\nabla U_m|\|_{L^2((k,k+1)\times\mathcal{D})} + \|U_m\|_{L^2((k,k+1)\times\mathcal{D})} \leq M \quad \text{for all } m \in \mathbb{N} \text{ and } k \in \mathbb{Z}.$$

Consequently, (U_n) is bounded in $E_k = W^{1,2}((k, k+1) \times \mathcal{D})$, endowed with the usual norm, for all $k \in \mathbb{Z}$. Then for some subsequence, there is $U \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$U_n \rightharpoonup U \quad \text{in } E_k \text{ for all } k \in \mathbb{Z}, \tag{2.6}$$

$$U_n \to U \quad \text{in } L^2((k, k+1) \times \mathcal{D}) \text{ for all } k \in \mathbb{Z}, \tag{2.7}$$

and

$$U_n(x, y) \to U(x, y) \quad \text{a.e. in } \Omega. \tag{2.8}$$

Therefore, from (2.4)–(2.8),

$$J(U) \leq \Theta^* \quad \text{and} \quad -1 \leq U(x, y) \leq 1 \quad \text{a.e. in } \Omega. \tag{2.9}$$

In the next section, our main goal is to prove that U is the desired heteroclinic solution, and in this point, the conditions on the function A play their role. However, before doing that, we need to say that if A is 1-periodic in x, the same arguments explored in [8] guarantee the existence of a heteroclinic solution W^* from 1 to -1.

3 Proof of Theorem 1.1: *A* is Asymptotic at Infinity to a Periodic Function

By hypothesis,

$$A(x, y) < A_p(x, y) \quad \text{for all } (x, y) \in \Omega.$$

If $W^* \in \Gamma$ is a heteroclinic solution for the periodic case, we must have

$$\Theta^* \leq J(W^*) < J_p(W^*) = \Theta^*_p,$$

that is,

$$\Theta^* < \Theta^*_p. \tag{3.1}$$

Here $J_p : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is given by

$$J_p(U) = \sum_{k\in\mathbb{Z}} I_{k,p}(U)$$

and $I_{k,p} : W^{1,2}((k, k+1) \times \mathcal{D}) \to \mathbb{R}$ is defined by

$$I_{k,p}(U) = \int_k^{k+1} \int_{\mathcal{D}} \mathcal{L}_p(U)\, dx\, dy,$$

where

$$\mathcal{L}_p(u) = \frac{1}{2}|\nabla u|^2 + A_p(x, y)V(u).$$

Moreover, Θ^*_p is the real number given by

$$\Theta^*_p = \inf\{J_p(U) : U \in \Gamma\}.$$

In what follows, $(U_m) \subset \Gamma$ is a minimizing sequence for J with

$$-1 \leq U_m(x, y) \leq 1 \quad \text{for all } (x, y) \in \Omega,$$

and $U \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies (2.6)–(2.9).

The next proposition is a key point in our approach.

Proposition 3.1 (Main Proposition). *For fixed $\tau \in (0, \sqrt{|\Omega_1|})$, there is $j_0 \in \mathbb{N}$ such that*

$$\|U-1\|_{L^2((-j,-j+1)\times\mathcal{D})} \le \tau \quad \textit{and} \quad \|U+1\|_{L^2((j,j+1)\times\mathcal{D})} \le \tau \quad \textit{for all } j \ge j_0. \tag{3.2}$$

We will assume for a moment that Proposition 3.1 is proved and show Theorem 1.1.

Proof of Theorem 1.1. From the limit $J(U_n) \to \Theta^*$, we get

$$\sum_{j\in\mathbb{Z}} I_j(U) = J(U) \le \Theta^*, \tag{3.3}$$

from where it follows that

$$I_j(U) \to 0 \quad \text{as } j \to \pm\infty,$$

or equivalently,

$$\int_0^1\int_{\mathcal{D}} |\nabla P_j U|^2 \,dx\,dy + \int_0^1\int_{\mathcal{D}} A(x,y)V(P_jU)\,dx\,dy \to 0 \quad \text{as } j \to \pm\infty.$$

As $U \in L^\infty(\Omega)$, we have that $(P_{-j}U)$ is a bounded sequence in $W^{1,2}(\Omega_1)$. Thus, there are a subsequence $(P_{-j_k}U)$ of $(P_{-j}U)$ and $\hat{U} \in W^{1,2}(\Omega_1)$ such that

$$\begin{aligned} P_{-j_k}U &\rightharpoonup \hat{U} \quad \text{in } W^{1,2}(\Omega_1) \quad \text{as } j_k \to +\infty \\ P_{-j_k}U &\to \hat{U} \quad \text{in } L^2(\Omega_1) \qquad \text{as } j_k \to +\infty \end{aligned}$$

and

$$P_{-j_k}U(x,y) \to \hat{U}(x,y) \quad \text{a.e. in } \Omega_1 \quad \text{as } j_k \to +\infty.$$

From this,

$$\int_0^1\int_{\mathcal{D}} V(\hat{U})\,dx\,dy = 0,$$

then

$$\hat{U} = 1 \quad \text{or} \quad \hat{U} = -1,$$

and so

$$P_{-j_k}U \to 1 \quad \text{or} \quad P_{-j_k}U \to -1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j_k \to +\infty.$$

Since $\tau \in (0, \sqrt{|\Omega_1|})$, these limits combine with (3.2) to give

$$P_{-j_k}U \to 1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j_k \to +\infty.$$

The above argument also yields

$$P_{-j}U \to 1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j \to +\infty.$$

Similar reasoning proves

$$P_jU \to -1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j \to +\infty.$$

Consequently, $U \in \Gamma$ and $-1 \le U(x,y) \le 1$ for all $(x,y) \in \overline{\Omega}$. Moreover, by (3.3),

$$J(U) = \Theta^*.$$

Now, we claim that for each $\phi \in C_0^\infty(\overline{\Omega})$, we have $\frac{\partial J}{\partial\phi}(U) = 0$, where $\frac{\partial J}{\partial\phi}(U)$ denotes the directional derivative of J at U in the direction of $\phi \in C_0^\infty(\overline{\Omega})$, where

$$C_0^\infty(\overline{\Omega}) = \{\phi : \overline{\Omega} \to \mathbb{R} : \text{there exists } \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \text{ such that } \psi(x) = \phi(x) \text{ for all } x \in \overline{\Omega}\}.$$

Indeed, taking $w = U + t\phi$ with $\phi \in C_0^\infty(\overline{\Omega})$ and $t \in \mathbb{R}$, we derive that for k large enough, let us say, $|k| > \ell_0$, we have

$$I_k(U + t\phi) = I_k(U) \quad \text{for all } |k| > \ell_0.$$

Thereby

$$\frac{J(U+t\phi)-J(U)}{t}=\frac{1}{t}\sum_{k\in\mathbb{Z}}(I_k(U+t\phi)-I_k(U))=\sum_{k=-\ell_0}^{\ell_0}\Big(\frac{I_k(U+t\phi)-I_k(U)}{t}\Big),$$

and so

$$\frac{\partial J}{\partial\phi}(U)=\lim_{t\to0}\frac{J(U+t\phi)-J(U)}{t}=\sum_{k=-\ell_0}^{\ell_0}I_k'(U)\phi.$$

As $w\in\Gamma$ and $J(U)\le J(w)$, a standard argument ensures that $\frac{\partial J}{\partial\phi}(U)=0$ for all $\phi\in C_0^\infty(\overline{\Omega})$. Therefore,

$$\int_\Omega\nabla U\nabla\phi\,dx+\int_\Omega A(x,y)V(U)\phi\,dx=0\quad\text{for all }\phi\in C_0^\infty(\overline{\Omega}).$$

From this, U is a weak solution of (PDE). A regularity argument from [8, Section 6] implies that $U\in C^2(\overline{\Omega},\mathbb{R})$, and that U is a classical solution of

$$-\Delta U+A(x,y)V'(U)=0\quad\text{in }\Omega\qquad\text{and}\qquad\frac{\partial U}{\partial\nu}=0,\quad x\in\mathbb{R},\ y\in\partial\mathcal{D},$$

with

$$U(x,y)\to1\quad\text{as }x\to-\infty\qquad\text{and}\qquad U(x,y)\to-1\quad\text{as }x\to+\infty,\qquad\text{uniformly in }y\in\mathcal{D}.$$

From this, U is a heteroclinic solution from 1 to −1, which finishes the proof of Theorem 1.1. □

Proof of Proposition 3.1. Arguing as in the proof of Theorem 1.1, there is a sequence $(P_{j_k}U)$ such that

$$P_{j_k}U\to1\quad\text{in }L^2(\Omega_1)\quad\text{or}\quad P_{j_k}U\to-1\quad\text{in }L^2(\Omega_1)\quad\text{as }j_k\to+\infty.$$

Claim 1. *The limit $P_{j_n}U\to-1$ in $L^2(\Omega_1)$ as $j_n\to+\infty$ holds.*

If the claim is not true, we must have

$$P_{j_n}U\to1\quad\text{in }L^2(\Omega_1)\quad\text{as }j_n\to+\infty.\tag{3.4}$$

Hence, as $U_m\to U$ in $L^2([j,k]\times\mathcal{D})$ for all $j,k\in\mathbb{Z}$ with $j<k$ and $U_m\in\Gamma$, there are a subsequence of (U_m), still denoted by itself, $i_*,k_m\in\mathbb{N}$ with $i_*<k_m$ and $k_m\to+\infty$ such that

$$\|U_m(\cdot+j,y)-1\|_{L^2(\Omega_1)}<\tau\quad\text{and}\quad\|U_m(\cdot+k_m,y)-1\|_{L^2(\Omega_1)}\ge\tau\quad\text{for all }j\in[i_*,k_m-1]\cap\mathbb{N}.\tag{3.5}$$

Indeed, by (3.4), there is $i_*\in\mathbb{N}$ such that

$$\|P_{j_n}U-1\|_{L^2(\Omega_1)}<\tau\quad\text{for all }j_n\ge i_*,\tag{3.6}$$

in particular

$$\|U-1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}<\tau.$$

From this, there exists $m_1\in\mathbb{N}$ such that

$$\|U_{m_1}-1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}<\tau.$$

Since $U_{m_1}\in\Gamma$, let us fix $k_1\in\mathbb{N}$ and $k_1\ge i_*+1$ as the first number satisfying

$$\|U_{m_1}-1\|_{L^2([j,j+1]\times\mathcal{D})}<\tau\quad\text{and}\quad\|U_{m_1}-1\|_{L^2([k_1,k_1+1]\times\mathcal{D})}\ge\tau\quad\text{for all }j\in[i_*,k_1]\cap\mathbb{N}.$$

By (3.6), we also have

$$\|U-1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}\quad\text{and}\quad\|U-1\|_{L^2([i_*+1,i_*+2]\times\mathcal{D})}<\tau.$$

Hence, there exists $m_2\in\mathbb{N}$ such that

$$\|U_{m_2}-1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}\quad\text{and}\quad\|U_{m_2}-1\|_{L^2([i_*+1,i_*+2]\times\mathcal{D})}<\tau.$$

Using the fact that $U_{m_2} \in \Gamma$, we fix $k_2 \in \mathbb{N}$ and $k_2 \geq i_* + 2$ as the first number such that

$$\|U_{m_1} - 1\|_{L^2([j,j+1]\times\mathcal{D})} < \tau \quad \text{and} \quad \|U_{m_2} - 1\|_{L^2([k_2,k_2+1]\times\mathcal{D})} \geq \tau \quad \text{for all } j \in [i_*, k_2 - 1].$$

Now, using the fact the inequalities below hold

$$\|U - 1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}, \quad \|U - 1\|_{L^2([i_*+1,i_*+2]\times\mathcal{D})} < \tau \quad \text{and} \quad \|U - 1\|_{L^2([i_*+2,i_*+3]\times\mathcal{D})} < \tau,$$

there exists $m_3 \in \mathbb{N}$ such that

$$\|U_{m_3} - 1\|_{L^2([i_*,i_*+1]\times\mathcal{D})}, \quad \|U_{m_3} - 1\|_{L^2([i_*+1,i_*+2]\times\mathcal{D})} < \tau \quad \text{and} \quad \|U_{m_3} - 1\|_{L^2([i_*+2,i_*+3]\times\mathcal{D})} < \tau.$$

The fact that $U_{m_3} \in \Gamma$ yields that there is $k_3 \in \mathbb{N}$ and $k_3 \geq i_* + 3$ such that

$$\|U_{m_1} - 1\|_{L^2([j,j+1]\times\mathcal{D})} < \tau \quad \text{and} \quad \|U_{m_3} - 1\|_{L^2([k_3,k_3+1]\times\mathcal{D})} \geq \tau \quad \text{for all } j \in [i_*, k_3 - 1].$$

Repeating the above argument, we will find sequences (U_{m_s}) and (k_s) verifying (3.5). We would like point out that $k_s \to +\infty$ as $s \to +\infty$, because $k_s \geq i_* + s$ for all $s \in \mathbb{N}$.

Since $\|U_m\|_\infty \leq 1$ and $(\||\nabla U_m|\|_{L^2((k,k+1)\times\mathcal{D})})$ is a bounded sequence, the sequence $Q_m(x, y) = U_m(x+k_m, y)$ is bounded in E_k, for all $k \in \mathbb{Z}$. Thus, for some subsequence, there is $W \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$\begin{aligned} Q_m &\rightharpoonup W && \text{in } E_k \text{ for all } k \in \mathbb{Z}, && (3.7)\\ Q_m &\to W && \text{in } L^2((k, k+1)\times\mathcal{D}) \text{ for all } k \in \mathbb{Z}, && (3.8)\\ Q_m(x, y) &\to W(x, y) && \text{a.e. in } \Omega, && (3.9) \end{aligned}$$

and

$$-1 \leq W(x, y) \leq 1 \quad \text{a.e. in } \Omega. \tag{3.10}$$

A simples change of variables gives us

$$\sum_{k\in\mathbb{Z}} \tilde{I}_k(Q_m) = J(U_m) = \Theta^* + o_m(1) \leq \Theta^* + 1, \tag{3.11}$$

where

$$\tilde{I}_k(U) = \int_k^{k+1} \int_{\mathcal{D}} \Big(\frac{1}{2}|\nabla u|^2 + A(x + k_m, y)V(u)\Big)\, dx\, dy.$$

Consequently, Fatou's Lemma together with (A_1) and (3.7)–(3.11) provide

$$J_p(W) \leq \Theta^*, \tag{3.12}$$

which gives

$$I_{j,p}(W) \to 0 \quad \text{as } j \to \pm\infty. \tag{3.13}$$

Setting for each $j \in \mathbb{N}$ the function $\widetilde{W}_j = P_{-j}W$, the fact that $W \in L^\infty(\Omega)$ together with the Sobolev embeddings guarantee the existence of $W_0 \in L^2(\Omega_1)$, and a subsequence of $(\widetilde{W}_j)$, still denoted by itself, such that

$$\widetilde{W}_j \to W_0 \quad \text{in } L^2(\Omega_1),$$

that is,

$$\|\widetilde{W}_j - W_0\|_{L^2(\Omega_1)} \to 0.$$

By (3.5), for each $j \in \mathbb{N}$, there is $m_* = m_*(j) \in \mathbb{N}$ such that

$$\|P_{-j}Q_m - 1\| \leq \tau \quad \text{for all } m \geq m_*.$$

Therefore, taking the limit of $m \to +\infty$, we get

$$\|\widetilde{W}_j - 1\| \leq \tau \quad \text{for all } j \in \mathbb{N}.$$

Now, taking the limit of $j \to +\infty$, it follows that

$$\|W_0 - 1\|_{L^2(\Omega_1)} \leq \tau.$$

On the other hand, by (3.13),

$$I_{0,p}(W_0) = 0,$$

from where it follows that $W_0 = 1$ or $W_0 = -1$. As $\tau \in (0, \sqrt{|\Omega_1|})$, we must have $W_0 = 1$. Thereby,

$$\|\widetilde{W}_j - 1\|_{L^2(\Omega_1)} \to 0 \quad \text{as } j \to +\infty.$$

Now, fixing $W_j = P_j W$ for $j \in \mathbb{N}$, the same reasoning works to show that there exist $\widehat{W}_0 \in L^2(\Omega_1)$ and a subsequence of (W_j), still denoted by itself, such that

$$W_j \to \widehat{W}_0 \quad \text{in } L^2(\Omega_1) \quad \text{as } j \to \infty,$$

or equivalently,

$$\|W_j - \widehat{W}_0\|_{L^2(\Omega_1)} \to 0.$$

This information gathering with (3.13) leads to $\widehat{W}_0 = 1$ or $\widehat{W}_0 = -1$. Next we are going to show that $\widehat{W}_0 = -1$. To see why, assume by contradiction that $\widehat{W}_0 = 1$. From (3.5), there is $j_1 \in \mathbb{N}$ such that

$$\|W - 1\|_{L^2((j_1-1,j_1)\times\mathcal{D})} \geq \tau \quad \text{and} \quad \|W - 1\|_{L^2((j_1,j_1+1)\times\mathcal{D})} \leq \tau.$$

As $Q_m \to W$ in $L^2((j_1 - 1, j_1 + 1) \times \mathcal{D})$, there is $m_0 \in \mathbb{N}$ satisfying

$$\|Q_m - 1\|_{L^2((j_1-1,j_1)\times\mathcal{D})} \geq \frac{\tau}{2} \quad \text{and} \quad \|Q_m - 1\|_{L^2((j_1,j_1+1)\times\mathcal{D})} \leq 2\tau \quad \text{for all } m \geq m_0.$$

In what follows, we denote by $\beta = \beta(\tau)$ the real number given by

$$\frac{\beta}{\tilde{A}_0} = \inf_{u\in\mathcal{N}_\tau} I_{*,\tau}(u),$$

where $\tilde{A}_0 = \min\{1, A_0\}$,

$$\mathcal{N}_\tau = \Big\{u \in W^{1,2}((-1,1)\times\mathcal{D}) : \|u\|_{L^\infty((-1,1)\times\mathcal{D})} \leq 1,\ \|u - 1\|_{L^2((-1,0)\times\mathcal{D})} \geq \frac{\tau}{2} \text{ and } \|u - 1\|_{L^2((0,1)\times\mathcal{D})} \leq 2\tau\Big\}$$

and $I_{*,\tau} : W^{1,2}((-1,1)\times\mathcal{D}) \to \mathbb{R}$ is defined by

$$I_{*,\tau}(u) = \int_{-1}^{1}\int_{\mathcal{D}} (|\nabla u|^2 + V(u))\,dx\,dy.$$

Hence, by a simple change of variable

$$\int_{j_1-1}^{j_1+1}\int_{\mathcal{D}} (|\nabla Q_m|^2 + V(Q_m))\,dx\,dy \geq \frac{\beta}{\tilde{A}_0} \quad \text{for all } m \geq m_0. \tag{3.14}$$

Here we would like point out that the same arguments found in [8, Proposition 2.14] work to show that $\beta > 0$. Having this in mind, we can assume without loss of generality that

$$J(U_m) \leq \Theta^* + \frac{\beta}{4} \quad \text{for all } m \geq m_0. \tag{3.15}$$

In the sequel, for each $j \geq j_1 + 2$ and $m \geq m_0$, let us consider the function

$$Z_{j,m}(x,y) = \begin{cases} 1, & \text{if } x \leq j,\ y \in \mathcal{D}, \\ ((j+1) - x) + (x - j)Q_m(x,y), & \text{if } j < x \leq j+1,\ y \in \mathcal{D}, \\ Q_m(x,y), & \text{if } j+1 < x,\ y \in \mathcal{D}. \end{cases}$$

By a direct computation, we see that $Z_{j,m} \in \Gamma$ and

$$J_p(Z_{j,m}) = I_{p,j}(Z_{j,m}) + \sum_{k=j+1}^{\infty} I_{p,k}(Q_m) = I_{p,j}(Z_{j,m}) + \sum_{k=j+1+k_m}^{\infty} I_{p,k}(U_m),$$

and so

$$\Theta_p^* \le J_p(Z_{j,m}) = I_{p,j}(Z_{j,m}) + \sum_{k=j+1+k_m}^{\infty} I_{p,k}(U_m).$$

As A verifies (A_1)–(A_2) and $(J(U_n))$ is bounded, increasing m_0 if necessary, we have

$$\sum_{k=j+1+k_m}^{\infty} I_{p,k}(U_m) \le \sum_{k=j+1+k_m}^{\infty} I_k(U_m) + \frac{\beta}{4} \quad \text{for all } m \ge m_0. \tag{3.16}$$

Here, we have used the fact $k_m \to +\infty$. Now, as $j \ge j_1 + 2$, (3.16) implies in the inequality

$$\Theta_p^* \le I_{p,j}(Z_{j,m}) + J(U_m) - \tilde{A}_0 \int_{j_1-1}^{j_1+1} \int_{\mathcal{D}} (|\nabla Q_m|^2 + V(Q_m))\, dx\, dy + \frac{\beta}{4},$$

which combined with (3.14)–(3.15) gives

$$\Theta_p^* \le I_{p,j}(Z_{j,m}) + \Theta^* + \frac{\beta}{4} - \tilde{A}_0 \frac{\beta}{\tilde{A}_0} + \frac{\beta}{4} = I_{p,j}(Z_{j,m}) + \Theta^* - \frac{\beta}{2}. \tag{3.17}$$

Since

$$-1 \le W_j(x, y) \le 1 \quad \text{and} \quad W_j \to 1 \quad \text{in } W^{1,2}(\Omega_1) \quad \text{as } j \to +\infty,$$

it is easy to check that

$$\lim_{j\to+\infty} \int_0^1 \int_{\mathcal{D}} A(x + j, y)V((-x + 1 + xW_j)\, dx\, dy = 0$$

and

$$\lim_{j\to+\infty} \int_0^1 \int_{\mathcal{D}} |1 - W_j|^2\, dx\, dy = 0.$$

Thus, given $\delta > 0$, there is $j_0 = j_0(\delta) > j_1 + 2$, which is independent of m, such that

$$\int_0^1 \int_{\mathcal{D}} A(x + j, y)V(-x + 1 + xW_j)\, dx\, dy < \delta \quad \text{for all } j \ge j_0$$

and

$$\int_0^1 \int_{\mathcal{D}} |1 - W_j|^2\, dx\, dy < \delta \quad \text{for all } j \ge j_0. \tag{3.18}$$

To continue, we further claim that there are $j = j(m) \ge j_0$ and $m \ge m_0$ such that

$$I_{p,j}(Z_{j,m}) = \int_j^{j+1} \int_{\mathcal{D}} \mathcal{L}_p(Z_{j,m})\, dx\, dy < \frac{\beta}{2}. \tag{3.19}$$

If the claim does not hold, for each $j \ge j_0$, there exists $m_1 = m_1(j) \ge m_0$ verifying

$$\int_j^{j+1} \int_{\mathcal{D}} \mathcal{L}_p(Z_{j,m})\, dx\, dy \ge \frac{\beta}{2} \quad \text{for all } m \ge m_1.$$

From the definition of $Z_{j,m}$ and condition (A_2),

$$\int_j^{j+1} \int_{\mathcal{D}} |\nabla Z_{j,m}|^2\, dx\, dy \ge \frac{\beta}{2} - \int_j^{j+1} \int_{\mathcal{D}} A(x, y)V((j + 1) - x + (x - j)Q_m)\, dx\, dy.$$

Recalling that

$$\lim_{m\to+\infty}\int_j^{j+1}\int_{\mathcal{D}} A(x,y)V((j+1)-x+(x-j)Q_m)\,dx\,dy=\int_0^1\int_{\mathcal{D}} A(x+j,y)V(-x+1+xW_j)\,dx\,dy<\delta,$$

for $j\geq j_0$ and $\delta<\frac{\beta}{4}$, there exists $m_2=m_2(j)\geq m_1(j)$ such that

$$\int_j^{j+1}\int_{\mathcal{D}} |\nabla Z_{j,m}|^2\,dx\,dy\geq\frac{\beta}{4}\quad\text{for all } m\geq m_2.$$

By using again the definition of $Z_{j,m}$, there is a constant $C>0$ such that

$$\int_j^{j+1}\int_{\mathcal{D}} |\nabla Z_{j,m}|^2\,dx\,dy\leq C\bigg(\int_0^1\int_{\mathcal{D}} |1-P_j(Q_m)|^2\,dx\,dy+\int_j^{j+1}\int_{\mathcal{D}} |\nabla Q_m|^2\,dx\,dy\bigg).$$

Now, fixing $\delta<\frac{\beta}{8C}$ in (3.18), we obtain

$$\int_j^{j+1}\int_{\mathcal{D}} |\nabla Q_m|^2\,dx\,dy\geq\frac{\beta}{8}\quad\text{for all } m\geq m_2(j).$$

Let $l\in\mathbb{N}$ such that

$$(l+1)\frac{\beta}{8}>\Theta^*+1$$

and fix $m>\max\{m_2(j):j_0\leq j\leq j_0+l\}$. Then

$$\sum_{k\in\mathbb{Z}}\tilde{I}_k(Q_m)\geq\Theta^*+1,$$

which contradicts (3.11), thereby showing (3.19). Thus, by (3.17) and (3.19),

$$\Theta_p^*<\Theta^*,$$

contrary to (3.1). This ensures that $\widehat{W}_0=-1$. From the above study, we deduce that $W\in\Gamma$, then by (3.12),

$$\Theta_p^*\leq J(W)\leq\Theta^*,$$

which is absurd. This proves Claim 1. As a byproduct of Claim 1, there is $j_*\in\mathbb{N}$ such that

$$\|P_jU+1\|_{L^2(\Omega_1)}<\tau\quad\text{for all } j\geq j_*.$$

A similar argument works to prove that

$$\|P_jU-1\|_{L^2(\Omega_1)}\leq\tau\quad\text{for all } j\leq -k_*.$$

Therefore, Proposition 3.1 follows with $j_0=\max\{j_*,k_*\}$. □

4 Proof of Theorem 1.2: *A* Verifies Rabinowitz's Condition

In this section we establish the existence of a heteroclinic solution for Class 2. In what follows, we are considering the equation

$$-\Delta u+A(\epsilon x,y)V'(u)=0\quad\text{in }\Omega,$$

together with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x,y)=0,\quad x\in\mathbb{R},\ y\in\partial\mathcal{D},$$

where ϵ is a positive parameter and A satisfies

(A_3) $0<\inf_{(x,y)\in\overline{\Omega}}A(x,y)\leq\max_{y\in\overline{\mathcal{D}}}A(0,y)<\liminf_{|(x,y)|\to+\infty}A(x,y)=A_\infty<\infty$.

From now on, we are denoting by $J_\epsilon, J_\infty : \Gamma \to \mathbb{R} \cup \{+\infty\}$ the functionals

$$J_\epsilon(U) = \sum_{k\in\mathbb{Z}} I_{\epsilon,k}(U) \quad \text{and} \quad J_\infty(U) = \sum_{k\in\mathbb{Z}} I_{\infty,k}(U),$$

where $I_{\epsilon,k}, I_{\infty,k} : E_k \to \mathbb{R}$ are given by

$$I_{\epsilon,k}(U) = \int_k^{k+1}\int_{\mathcal{D}} (|\nabla U|^2 + A(\epsilon x, y)V(U))\,dx\,dy \quad \text{and} \quad I_{\infty,k}(U) = \int_k^{k+1}\int_{\mathcal{D}} (|\nabla U|^2 + A_\infty V(U))\,dx\,dy.$$

Moreover, we denote by Θ_ϵ and Θ_∞ the following numbers:

$$\Theta_\epsilon = \inf\{J_\epsilon(U) : U \in \Gamma\} \quad \text{and} \quad \Theta_\infty = \inf\{J_\infty(U) : U \in \Gamma\}.$$

From Section 2, we know that there are $W_{\max}, W_\infty \in \Gamma$ verifying $J_{\max}(W_{\max}) = \Theta_{\max}$ and $J_\infty(W_\infty) = \Theta_\infty$. Here

$$J_{\max}(U) = \sum_{k\in\mathbb{Z}} I_{\max,k}(U) \quad \text{and} \quad I_{\max,k}(U) = \int_k^{k+1}\int_{\mathcal{D}} (|\nabla U|^2 + A_0 V(U))\,dx\,dy$$

with $A_0 = \max_{y\in\overline{\mathcal{D}}} A(0, y)$. Moreover,

$$\Theta_{\max} = \inf\{J_{\max}(U) : U \in \Gamma\}.$$

This fact permits us to prove the following lemma:

Lemma 4.1. *We have* $\limsup_{\epsilon\to 0} \Theta_\epsilon \le \Theta_{\max}$ *and* $\Theta_{\max} < \Theta_\infty$.

Proof. For each $\epsilon > 0$,

$$\Theta_\epsilon \le J_\epsilon(W_{\max}).$$

Since

$$\lim_{\epsilon\to 0} J_\epsilon(W_{\max}) \le J_{\max}(W_{\max}) = \Theta_{\max},$$

it follows that

$$\limsup_{\epsilon\to 0} \Theta_\epsilon \le \Theta_{\max}.$$

On the other hand, by (A_3),

$$\Theta_{\max} \le J_{\max}(W_\infty) < J_\infty(W_\infty) = \Theta_\infty,$$

which shows the lemma. □

In the sequel, we fix $\epsilon_0 > 0$ small enough such that

$$\Theta_\epsilon < \Theta_\infty \quad \text{for all } \epsilon \in (0, \epsilon_0). \tag{4.1}$$

4.1 Proof of Theorem 1.2

Arguing as in Section 2, for each $\epsilon > 0$ there is a minimizing sequence $(U_n) \subset \Gamma$ with $-1 \le U_n(x, y) \le 1$ for all $(x, y) \in \Omega$ and $U \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$\begin{aligned}
&J_\epsilon(U_n) \to \Theta_\epsilon, \\
&U_n \rightharpoonup U && \text{in } E_k \text{ for all } k \in \mathbb{Z}, \\
&U_n \to U && \text{in } L^2((k, k+1) \times \mathcal{D}) \text{ for all } k \in \mathbb{Z}, \\
&U_n(x, y) \to U(x, y) && \text{a.e. in } \Omega, \\
&-1 \le U(x, y) \le 1 && \text{for all } (x, y) \in \Omega,
\end{aligned}$$

and

$$J_\epsilon(U) \le \Theta_\epsilon.$$

Proposition 4.1. *For fixed $\tau \in (0, \sqrt{|\Omega_1|})$ and $\epsilon \in (0, \epsilon_0)$, there is $j_0 \in \mathbb{N}$ such that*

$$\|U - 1\|_{L^2((-j,-j+1)\times\mathcal{D})} \leq \tau \quad \textit{and} \quad \|U + 1\|_{L^2((j,j+1)\times\mathcal{D})} \leq \tau \quad \textit{for all } j \geq j_0.$$

Proof. For fixed $\epsilon \in (0, \epsilon_0)$, arguing as in the proof of Proposition 3.1, there is a sequence $(P_{j_k} U)$ such that

$$P_{j_k} U \to 1 \quad \text{in } L^2(\Omega_1) \quad \text{or} \quad P_{j_k} U \to -1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j_k \to +\infty.$$

Claim 2. *The limit $P_{j_k} U \to -1$ in $L^2(\Omega_1)$ as $j_k \to +\infty$ holds.*

If the claim is not true, we must have

$$P_{j_k} U \to 1 \quad \text{in } L^2(\Omega_1) \quad \text{as } j_k \to +\infty.$$

Hence, as $U_m \to U$ in $L^2([j, k] \times \mathcal{D})$ for all $j, k \in \mathbb{Z}$ with $j < k$, there are a subsequence of (U_m), still denoted by itself, $i_*, k_m \in \mathbb{N}$ with $i_* < k_m$ and $k_m \to +\infty$ such that

$$\|U_m(\cdot + j, y) - 1\|_{L^2(\Omega_1)} < \tau \quad \text{and} \quad \|U_m(\cdot + k_m, y) - 1\|_{L^2(\Omega_1)} \geq \tau \quad \text{for all } j \in [i_*, k_m - 1] \cap \mathbb{N}. \tag{4.2}$$

From the definition of (k_m), we have that $k_m \to +\infty$ as $m \to +\infty$. Since $\|U_m\|_\infty \leq 1$ and $(\||\nabla U_m|\|_{L^2((k,k+1)\times\mathcal{D})})$ is a bounded sequence, the sequence $Q_m(x, y) = U_m(x + k_m, y)$ is bounded in E_k for all $k \in \mathbb{Z}$. Thus, for some subsequence, there is $W \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$\begin{aligned} Q_m &\rightharpoonup W && \text{in } E_k \text{ for all } k \in \mathbb{N}, \\ Q_m(x, y) &\to W(x, y) && \text{a.e. in } \Omega, \end{aligned}$$

and

$$-1 \leq W(x, y) \leq 1 \quad \text{a.e. in } \Omega.$$

By a simple change variable,

$$\sum_{k\in\mathbb{Z}} \tilde{I}_k(W_m) \leq J_\epsilon(U_m) = \Theta_\epsilon + o_m(1), \tag{4.3}$$

where

$$\tilde{I}_{\epsilon,k}(U) = \int_k^{k+1} \int_{\mathcal{D}} \tilde{\mathcal{L}}_{\epsilon,m}(U)\, dx\, dy$$

with

$$\tilde{\mathcal{L}}_{\epsilon,m}(u) = \frac{1}{2}|\nabla u|^2 + A(\epsilon x + \epsilon k_m, y)V(u).$$

Now, Fatou's Lemma combined with (4.3) lead to

$$J_\infty(W) \leq \Theta_\epsilon.$$

Then

$$I_{\infty,j}(W) \to 0 \quad \text{as } j \to \pm\infty. \tag{4.4}$$

Setting for each $j \in \mathbb{N}$ the function $\widetilde{W}_j = P_{-j} W$, the fact that $W \in L^\infty(\Omega)$ implies that there are $W_0 \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R})$ and a subsequence of $(\widetilde{W}_j)$, still denoted by itself, such that

$$\widetilde{W}_j \to W_0 \quad \text{in } W^{1,2}(\Omega_1) \quad \text{as } j \to +\infty,$$

and so

$$\|\widetilde{W}_j - W_0\|_{L^2(\Omega_1)} \to 0.$$

Arguing as in the proof of Proposition 3.1, the first inequality in (4.2) leads to

$$\|W_0 - 1\|_{L^2(\Omega_1)} \leq \tau.$$

On the other hand, by (4.4),

$$I_{\infty,0}(W_0) = 0,$$

which gives $W_0 = 1$ or $W_0 = -1$. As $\tau \in (0, \sqrt{|\Omega_1|})$, we must have $W_0 = 1$. Then

$$\|W_j - 1\|_{L^2(\Omega_1)} \to 0 \quad \text{as } j \to +\infty.$$

By using the same type of argument, fixing $W_j = P_j W$ for $j \in \mathbb{N}$, it is possible to prove that there exist $\widehat{W}_0 \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ and a subsequence of (W_j), still denoted by itself, such that

$$W_j \to \widehat{W}_0 \quad \text{in } W^{1,2}(\Omega_1) \quad \text{as } j \to -\infty,$$

and so

$$\|W_j - \widehat{W}_0\|_{L^2(\Omega_1)} \to 0.$$

Thereby, $\widehat{W}_0 = 1$ or $\widehat{W}_0 = -1$. Here, as in the previous section, we have that $\widehat{W}_0 = -1$. Indeed, assuming by contradiction that $\widehat{W}_0 = 1$, we set the function

$$H_j(x,y) = \begin{cases} 1, & x \leq j,\ y \in \mathcal{D}, \\ ((j+1) - x) + (x - j)Q_m(x,y), & j < x \leq j+1,\ y \in \mathcal{D}, \\ Q_m(x,y), & j+1 < x,\ y \in \mathcal{D}. \end{cases}$$

Arguing again as in the proof of Proposition 3.1, we will find

$$\Theta_\infty \leq \Theta_\epsilon,$$

which contradicts (4.1), and then $\widehat{W}_0 = -1$. Now we follow the same idea explored in Proposition 3.1 to conclude the proof. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. As an immediate consequence of the last proposition, for each $\epsilon \in (0, \epsilon_0)$, there is $j_0 \in \mathbb{N}$ such that

$$\|U - 1\|_{L^2((-j,-j+1)\times\mathcal{D})} < \tau \quad \text{and} \quad \|U + 1\|_{L^2((j,j+1)\times\mathcal{D})} < \tau \quad \text{for all } j \geq j_0.$$

Now, arguing as in the proof of Theorem 1.1, it follows that $U \in C^2(\overline{\Omega}, \mathbb{R})$. Moreover, U is a classical solution of

$$-\Delta U + A(\epsilon x, y)V'(U) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial U}{\partial \nu} = 0, \quad x \in \mathbb{R},\ y \in \partial\mathcal{D}$$

with

$$U(x,y) \to 1 \quad \text{as } x \to -\infty \qquad \text{and} \qquad U(x,y) \to -1 \quad \text{as } x \to +\infty, \qquad \text{uniformly in } y \in \mathcal{D}.$$

From this, U is a heteroclinic solution from 1 to -1, which finishes the proof of Theorem 1.2. □

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