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Research Article

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Solutions with Prescribed Local Blow-up Surface for the Nonlinear Wave Equation

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Abstract: We prove that any sufficiently differentiable space-like hypersurface of $\mathbb{R}^{1+N}$ coincides locally around any of its points with the blow-up surface of a finite-energy solution of the focusing nonlinear wave equation $\partial_{tt}u - \Delta u = |u|^{p-1}u$ on $\mathbb{R} \times \mathbb{R}^N$, for any $1 \le N \le 4$ and $1 < p \le \frac{N+2}{N-2}$. We follow the strategy developed in our previous work (2018) on the construction of solutions of the nonlinear wave equation blowing up at any prescribed compact set. Here to prove blow-up on a local space-like hypersurface, we first apply a change of variable to reduce the problem to blowup on a small ball at $t = 0$ for a transformed equation. The construction of an appropriate approximate solution is then combined with an energy method for the existence of a solution of the transformed problem that blows up at $t = 0$. To obtain a finite-energy solution of the original problem from trace arguments, we need to work with $H^2 \times H^1$ solutions for the transformed problem.

Keywords: Nonlinear Wave Equation, Finite-time Blowup, Blow-up Surface

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1 Introduction

1.1 Main Result

We consider the nonlinear energy-subcritical or -critical wave equation

$$\partial_{tt}u - \Delta u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

for $N \ge 1$ and $1 < p \le \frac{N+2}{N-2}$ ($1 < p < \infty$ if $N = 1, 2$). For simplicity, we restrict ourselves to space dimensions $1 \le N \le 4$. In this case, it is well known that the Cauchy problem for (1.1) is locally well posed in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. (See Remark B.1.)

When a solution u with initial data at $t = t_0$ is not globally defined ([1, 14, 23]), we introduce its *maximal influence domain* whose upper boundary is a 1-Lipschitz graph. See [1, Section III.2] and, for the present setting, Section 1.2.

We prove that any sufficiently differentiable space-like hypersurface of $\mathbb{R}^{1+N}$ coincides locally around any of its points with the blow-up surface of a finite-energy solution of the focusing nonlinear wave equation (1.1). More precisely, our main result is the following.

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Theorem 1.1. *Let $1 \le N \le 4$ and $1 < p \le \frac{N+2}{N-2}$. Let*

$$q_0 = 2\left\lfloor \frac{2p+2}{p-1} \right\rfloor + 3. \tag{1.2}$$

Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a function of class $\mathcal{C}^{q_0}$ such that

$$\varphi(0) = 0 \qquad \text{and} \qquad |\nabla\varphi(x)| < 1 \quad \text{for all } x \in \mathbb{R}^N. \tag{1.3}$$

There exist $\varepsilon > 0$, $\tau_0 > 0$ and $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that the upper boundary of the maximal influence domain of the solution u of (1.1) with initial data $(u, \partial_t u)(0) = (u_0, u_1)$ contains the local hypersurface $\{(t, x) : t = \tau_0 + \varphi(x)$ and $|x| < \varepsilon\}$. Moreover, u blows up on this local hypersurface in the sense that if $|x_0| < \varepsilon$ and $\sigma \in (|\nabla\varphi(0)|, 1)$, then

$$\liminf_{t \uparrow \tau_0+\varphi(x_0)} \frac{1}{\tau_0 + \varphi(x_0) - t} \int_t^{\tau_0+\varphi(x_0)} dt' \int_{\{|x-x_0|<\sigma(\tau_0+\varphi(x_0)-t')\}} |\partial_t u|^2 \, dx > 0. \tag{1.4}$$

It follows from (1.4) that $\partial_t u$ concentrates on the local hypersurface $\{(t, x) : t = \tau_0 + \varphi(x)$ and $|x| < \varepsilon\}$ in the sense of L^2. In particular, this local hypersurface is a *blow-up surface* for the solution u.

Compared to previous results (see Section 1.3), Theorem 1.1 applies to any space dimension $N \le 4$ and any subcritical or critical p. Moreover, our strategy is different. It mainly relies on the construction of an ansatz by elementary ODE arguments. (See Section 1.4.)

Remark 1.2. In the definition of q_0 above, we use the notation $y \mapsto \lfloor y \rfloor$ for the floor function which maps y to the greatest integer less than or equal to y. Note that $q_0 = 7$ for $p > 5$ and $q_0 \to \infty$ as $p \to 1^+$. See Remark 2.3 for comments on this condition.

1.2 Definition of the Maximal Influence Domain

We adapt the presentation of [1, Chapter III] (see also [24]) to the framework of $H^1 \times L^2$ solutions for the energy subcritical or critical wave equation in space dimension $N \ge 1$. Let

$$\mathbb{R}_+^{1+N} = [0, +\infty) \times \mathbb{R}^N.$$

For any $(t, x) \in \mathbb{R}_+^{1+N}$, we define the open (in $\mathbb{R}_+^{1+N}$) backward cone

$$C(t, x) = \{(s, y) \in \mathbb{R}_+^{1+N} : |x - y| < t - s\}. \tag{1.5}$$

Definition 1.3. An open set Ω of $\mathbb{R}_+^{1+N}$ is called an influence domain if $(t, x) \in \Omega$ implies $\overline{C}(t, x) \subset \Omega$.

For Ω an influence domain containing $\{0\} \times \mathbb{R}^N$, define for any $x \in \mathbb{R}^N$,

$$\phi(x) = \sup\{t \ge 0 : (t, x) \in \Omega\}.$$

From the above definition, either ϕ is identically ∞, or it is finite for all $x \in \mathbb{R}^N$. In the latter case, ϕ is a 1-Lipschitz continuous function.

Recall that by the Cauchy theory in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, for any $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ there exist $T > 0$ and a solution $(u, \partial_t u)$ of (1.1) belonging to $C([0, T], H^1(\mathbb{R}^N)) \cap C([0, T], L^2(\mathbb{R}^N))$. These solutions are unique in that class, except for the 3D critical case $p = 5$, where uniqueness is known in $C([0, T], H^1(\mathbb{R}^3)) \cap C([0, T], L^2(\mathbb{R}^3)) \cap L^8((0, T) \times \mathbb{R}^3)$. (See Remark B.1 for details.)

From the local Cauchy theory, it is standard to define the notion of maximal solution and maximal time of existence $T_{\max}(u_0, u_1) > 0$; if $T_{\max}(u_0, u_1) = \infty$, the solution is globally defined, otherwise it blows up as $t \uparrow T_{\max}(u_0, u_1)$ (in a suitable norm related to the resolution of the Cauchy problem).

To define the notion of maximal influence domain corresponding to an initial data, we first extend the Cauchy theory of $\mathbb{R}^N$ to truncated cones. For $x_0 \in \mathbb{R}^N$ and $0 \le \tau \le R$, we define

$$E(x_0, R, \tau) = \{(t, x) \in \mathbb{R}_+^{1+N} : 0 \le t < \tau \text{ and } |x - x_0| < R - t\}. \tag{1.6}$$

Suppose that $x_0 \in \mathbb{R}^N$ and $R > 0$, and let $(u_0, u_1) \in H^1(B(x_0, R)) \times L^2(B(x_0, R))$. Consider any extension $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ of (u_0, u_1), i.e. any function satisfying

$$\tilde{u}_0 = u_0 \quad \text{and} \quad \tilde{u}_1 = u_1 \quad \text{on } B(x_0, R).$$

Next, consider the solution $(\tilde{u}, \partial_t \tilde{u})$ of (1.1) corresponding to the initial data $(\tilde{u}_0, \tilde{u}_1)$ defined on a time interval $[0, \tilde{\tau}]$, where $\tilde{\tau} > 0$, given by the above Cauchy theory. Note that if $(\check{u}_0, \check{u}_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ is another extension of (u_0, u_1) and $(\check{u}, \partial_t \check{u})$ is the corresponding solution of (1.1) on a time interval $[0, \check{\tau}]$ ($\check{\tau} > 0$), then by finite speed of propagation (see Proposition B.2), the two solutions $(\tilde{u}, \partial_t \tilde{u})$ and $(\check{u}, \partial_t \check{u})$ are identically equal on the truncated cone $E(x_0, R, \min(\tilde{\tau}, \check{\tau}))$. In this way, we have defined a notion of solution of (1.1) on $E(x_0, R, \tau)$ for some $\tau > 0$ which is independent of the extension chosen and includes a uniqueness property. *From now on, for any* $(u_0, u_1) \in H^1(B(x_0, R)) \times L^2(B(x_0, R))$ *and any* $\tau > 0$, *we refer to the solution of* (1.1) *on* $E(x_0, R, \tau)$ *in this sense.*

By time-translation invariance of the equation and considering the map $(t, x) \in E(x_0, R, \tau) \mapsto u(t_0 + t, x)$, we extend this definition to any truncated cone in $\mathbb{R}_+^{1+N}$.

Now, we define the notion of solution in an influence domain.

Definition 1.4. Let $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Let Ω be an influence domain. We say that $(u, \partial_t u)$ is a solution of (1.1) on Ω with initial data (u_0, u_1) if the following hold:
(i) $u \in H^1_{\mathrm{loc}}(\Omega)$.
(ii) For any $t_0 \geq 0$, $x_0 \in \mathbb{R}^N$ and $R > 0$ such that $[0, t_0] \times B(x_0, R) \subset \Omega$, it holds
$$u_{|[0,t_0]\times B(x_0,R)} \in C([0, t_0], H^1(B(x_0, R))) \cap C^1([t_1, t_2], L^2(B(x_0, R)));$$
moreover, $u(0) \equiv u_0$ and $\partial_t u(0) \equiv u_1$ on $B(x_0, R)$.
(iii) For any $(t_0, x_0) \in \Omega$ and $R > 0$ such that $\{t_0\} \times B(x_0, R) \subset \Omega$, there exists τ with $0 < \tau < R$ such that $(t, x) \in E(x_0, R, \tau) \mapsto u(t_0 + t, x)$ is solution of (1.1) in the above sense.

Definition 1.5. For any $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, we denote $\Omega_{\max}(u_0, u_1)$ the union of all the influence domains Ω such that there exists a solution $(u, \partial_t u)$ with initial data (u_0, u_1) on Ω in the sense of Definition 1.4.

It follows that, for any initial data $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, $\Omega_{\max}(u_0, u_1)$ is the maximal influence domain on which a (unique) solution of (1.1) with initial data (u_0, u_1) exists. Finally, in the case $T_{\max}(u_0, u_1) < +\infty$ the upper boundary of the maximal influence domain is the graph of the 1-Lipschitz application

$$x \in \mathbb{R}^N \mapsto \phi(x) = \sup\{t \geq 0 : (t, x) \in \Omega_{\max}(u_0, u_1)\} \in (0, \infty).$$

1.3 Previous Results

Under certain assumptions, it is known that the upper boundary of the maximal influence domain is a blow-up surface in the sense that the solution blows up (at the same rate as the ODE) on the surface, and the blow-up surface is C^1. See [3, 4] and [1, Chapter III]. See also [9, 25, 26] and the references therein for further blow-up results.

Constructing solutions of the wave equation (1.1) with prescribed blow-up surface is a classical question. Results similar to Theorem 1.2 have been proved in several cases. For the wave equation with cubic nonlinearity, it is proved in [18, Theorem 10.14, p. 192] that there exist solutions (locally defined around the blow-up surface) blowing up exactly on a prescribed surface of class $H^r(\mathbb{R}^N)$ with $r > \frac{N}{2} + 7$. In [22, Theorem 1.1], an analogous result is proved in space dimension 1 for equation (1.1) for any $p > 1$. For previous results, see [1, 16, 17, 20, 21].

A related question is the study of the blow-up set, which is the intersection of the blow-up surface with the hyperplane $\{t = T_{\max}\}$. In [22, Corollary 1.2], it is proved for (1.1) in space dimension 1 that, given any compact subset K of $\mathbb{R}$, there exist smooth initial data for which the blow-up set is precisely K. This result is extended in [6, Theorem 1.1] to any space dimension and any energy-subcritical p. See [19] for a related result.

1.4 Strategy of the Proof of Theorem 1.1

We follow closely the strategy of [6] (see also [7]). It is based on the construction of an appropriate approximate solution which blows up at $t = 0$, combined with an energy method for the existence of an exact solution that also blows up at $t = 0$. Here, we wish to prove blowup on a local space-like hypersurface. In order to apply the previously recalled strategy, we therefore apply a change of variable to reduce the problem to blowup at $t = 0$ (Section 2.1). By doing so, we are led to study the transformed equation

$$(1 - |\nabla\psi|^2)\partial_{ss}v - 2\nabla\psi \cdot \nabla\partial_s v - (\Delta\psi)\partial_s v - \Delta v = f(v)$$

in the dual variables $(s, y) \in \mathbb{R} \times \mathbb{R}^N$. The construction of an appropriate ansatz for this equation (Sections 2.2 and 2.3) is similar to the construction made in [6]. In particular, it is based on elementary ODE arguments. The energy method for this transformed equation requires a smallness condition on $\|\nabla\psi\|_{L^\infty}$, and yields an existence time that depends on ψ. See Section 3. This smallness condition can be met through a localization argument (Section 4.1) and a Lorentz transform (Sections 4.2–4.4). Going back to the original variables, to obtain a solution in the framework of $H^1 \times L^2$, we are forced to apply a trace argument which requires higher regularity of the solution v (Section 4.5). This is why we use the energy method for v in the framework of $H^2 \times H^1$. The restriction $1 \le N \le 4$ implies that $H^2 \hookrightarrow L^q$ for every $2 \le q < \infty$, which simplifies the energy argument. The blow-up estimate (1.4) is a consequence of an ODE blow-up estimate for the solution of the transformed equation, and the change of variable (Section 4.6).

1.5 Notation

We fix a smooth, even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi \equiv 1 \text{ on } [0, 1], \quad \chi \equiv 0 \text{ on } [2, +\infty), \quad \chi' \le 0 \le \chi \le 1 \text{ on } [0, +\infty). \tag{1.7}$$

Let $f(u) = |u|^{p-1}u$ and $F(u) = \int_0^u f(v)\,dv$. For future reference, we state and justify two Taylor formulas involving the functions F and f (see Introduction of [7] for proofs). Let $\bar p = \min(2, p)$. For any $u > 0$ and any v, it holds

$$\Big|F(u + v) - F(v) - F'(u)v - \frac{1}{2}F''(u)v^2\Big| \lesssim |v|^{p+1} + u^{p-\bar p}|v|^{\bar p+1}, \tag{1.8}$$

$$|(f(u + v) - f(u) - f'(u)v)v| \lesssim |v|^{p+1} + u^{p-\bar p}|v|^{\bar p+1}, \tag{1.9}$$

$$|f'(u + v) - f'(u)| \lesssim |u|^{-1}|v|^p + |u|^{p-2}|v|, \tag{1.10}$$

$$\Big|f(u + v) - f(u) - f'(u)v - \frac{1}{2}f''(u)v^2\Big| \lesssim u^{-1}|v|^{p+1} + u^{p-\bar p-1}|v|^{\bar p+1}. \tag{1.11}$$

In the present article, we use multi-variate notation and results from [8]. For $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we set

$$|\beta| = \sum_{j=1}^N \beta_j, \qquad \beta! = \prod_{j=1}^N (\beta_j!),$$

$$x^\beta = \prod_{j=1}^N x_j^{\beta_j}, \quad \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1} \dots \partial_{x_N}^{\beta_N}} \quad \text{for } |\beta| > 0, \quad \partial_x^0 = \mathrm{Id}\,.$$

For $\beta, \beta' \in \mathbb{N}^N$, we write $\beta' \le \beta$ provided $\beta'_j \le \beta_j$, for all $j = 1, \dots, N$. Note that in this case $|\beta - \beta'| = |\beta| - |\beta'|$. For $\beta' \le \beta$, we denote

$$\binom{\beta}{\beta'} = \prod_{j=1}^N \binom{\beta_j}{\beta'_j} = \frac{\beta!}{(\beta'!)(\beta - \beta')!}.$$

Recall that for two functions $g, h : \mathbb{R}^{1+N} \to \mathbb{R}$, Leibniz's formula writes

$$\partial_x^\beta(gh) = \sum_{\beta' \le \beta} \binom{\beta}{\beta'}(\partial^{\beta'} g)(\partial^{\beta-\beta'} h). \tag{1.12}$$

We write $\beta' \prec \beta$ provided one of the following holds:

- $|\beta'| < |\beta|$,
- $|\beta'| = |\beta|$ and $\beta'_1 < \beta_1$,
- $|\beta'| = |\beta|$, $\beta'_1 = \beta_1, \dots, \beta'_\ell = \beta_\ell$ and $\beta'_{\ell+1} < \beta_{\ell+1}$ for some $1 \le \ell < N$.

We recall the Faa di Bruno formula (see in [8, Corollary 2.10]). Let $n = |\beta| \ge 1$. Then, for functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{1+N} \to \mathbb{R}$,

$$\partial_x^\beta (f \circ g) = \sum_{r=1}^n (f^{(r)} \circ g) \sum_{P(\beta,r)} (\beta!) \prod_{\ell=1}^n \frac{(\partial_x^{\beta_\ell} g)^{r_\ell}}{(r_\ell!)(\beta_\ell!)^{r_\ell}} \tag{1.13}$$

where

$$\begin{aligned} P(\beta, r) = \Big\{ (r_1, \dots, r_n; \beta_1, \dots, \beta_n) : {} & \text{there exists } 1 \le m \le n \text{ such that } r_\ell = 0 \text{ and } \beta_\ell = 0 \text{ for } 1 \le \ell \le n - m; \\ & r_\ell > 0 \text{ for } n - m + 1 \le \ell \le n; \\ & \text{and } 0 \prec \beta_{n-m+1} \prec \cdots \prec \beta_n \text{ are such that } \sum_{\ell=1}^n r_\ell = r,\ \sum_{\ell=1}^n r_\ell \beta_\ell = \beta \Big\}. \end{aligned}$$

We will also need to differentiate in space and time, so we define multi-index notation in space-time: $\boldsymbol{\nu} = (\alpha, \beta_1, \dots, \beta_N) \in \mathbb{N}^{1+N}$, $\beta = (\beta_1, \dots, \beta_N)$, and

$$|\boldsymbol{\nu}| = \alpha + |\beta|, \quad \boldsymbol{\nu}! = \alpha!\beta!, \quad \partial^{\boldsymbol{\nu}} = \partial_s^\alpha \partial_x^\beta.$$

For $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathbb{N}^{1+N}$, we write $\boldsymbol{\nu}' \le \boldsymbol{\nu}$ provided $\alpha' \le \alpha$ and $\beta'_j \le \beta_j$, for all $j = 1, \dots, N$. In such a case, we denote

$$\binom{\boldsymbol{\nu}}{\boldsymbol{\nu}'} = \binom{\alpha}{\alpha'}\binom{\beta}{\beta'}.$$

Then, for two functions $g, h : \mathbb{R}^{1+N} \to \mathbb{R}$,

$$\partial^{\boldsymbol{\nu}}(gh) = \sum_{\boldsymbol{\nu}' \le \boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\nu}'} (\partial^{\boldsymbol{\nu}'} g)(\partial^{\boldsymbol{\nu} - \boldsymbol{\nu}'} h). \tag{1.14}$$

We write $\boldsymbol{\nu}' \prec \boldsymbol{\nu}$ provided one of the following holds:

- $|\boldsymbol{\nu}'| < |\boldsymbol{\nu}|$,
- $|\boldsymbol{\nu}'| = |\boldsymbol{\nu}|$ and $\alpha' < \alpha$,
- $|\boldsymbol{\nu}'| = |\boldsymbol{\nu}|$, $\alpha' = \alpha$ and $\beta'_1 < \beta_1$,
- $|\boldsymbol{\nu}'| = |\boldsymbol{\nu}|$, $\alpha' = \alpha$, $\beta'_1 = \beta_1, \dots, \beta'_\ell = \beta_\ell$ and $\beta'_{\ell+1} < \beta_{\ell+1}$ for some $1 \le \ell < N$.

Last, we write in this context the Faa di Bruno formula. Let $n = |\boldsymbol{\nu}| \ge 1$. Then, for functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{1+N} \to \mathbb{R}$,

$$\partial^{\boldsymbol{\nu}} (f \circ g) = \sum_{r=1}^n (f^{(r)} \circ g) \sum_{P(\boldsymbol{\nu},r)} (\boldsymbol{\nu}!) \prod_{\ell=1}^n \frac{(\partial^{\boldsymbol{\nu}_\ell} g)^{r_\ell}}{(r_\ell!)(\boldsymbol{\nu}_\ell!)^{r_\ell}}, \tag{1.15}$$

where

$$\begin{aligned} P(\boldsymbol{\nu}, r) = \Big\{ (r_1, \dots, r_n; \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n) : {} & \text{there exists } 1 \le m \le n \text{ such that } r_\ell = 0 \text{ and } \boldsymbol{\nu}_\ell = 0 \text{ for } 1 \le \ell \le n - m; \\ & r_\ell > 0 \text{ for } n - m + 1 \le \ell \le n; \\ & \text{and } 0 \prec \boldsymbol{\nu}_{n-m+1} \prec \cdots \prec \boldsymbol{\nu}_n \text{ are such that } \sum_{\ell=1}^n r_\ell = r,\ \sum_{\ell=1}^n r_\ell \boldsymbol{\nu}_\ell = \boldsymbol{\nu} \Big\}. \end{aligned}$$

2 Blow-up Ansatz

2.1 Change of Variables

Let $\psi \in \mathcal{C}^{q_0}(\mathbb{R}^N, \mathbb{R})$, where q_0 is defined by (1.2), be such that for some $R \ge 2$,

$$\psi(x) = 0 \quad \text{for } |x| \ge R \qquad \text{and} \qquad \|\nabla \psi\|_{L^\infty} < 1. \tag{2.1}$$

We perform a change of variable related to ψ

$$u(t,x) = v(s,x), \quad s = \psi(x) - t$$

so that $s > 0$ is equivalent to $t < \psi(x)$. Then the following holds: for $j = 1, \dots, N$,

$$\begin{aligned}
\partial_{tt} u &= \partial_{ss} v, \\
\partial_{x_j} u &= (\partial_{x_j}\psi)\partial_s v + \partial_{x_j} v, \\
\partial_{x_j x_j} u &= (\partial_{x_j x_j}\psi)\partial_s v + (\partial_{x_j}\psi)^2 \partial_{ss} v + 2(\partial_{x_j}\psi)\partial_{x_j s} v + \partial_{x_j x_j} v, \\
\Delta u &= (\Delta\psi)\partial_s v + |\nabla\psi|^2 \partial_{ss} v + 2\nabla\psi \cdot \nabla\partial_s v + \Delta v.
\end{aligned}$$

Therefore, equation (1.1) on $u(t,x)$ rewrites

$$(1 - |\nabla\psi|^2)\partial_{ss} v - 2\nabla\psi \cdot \nabla\partial_s v - (\Delta\psi)\partial_s v - \Delta v = f(v). \tag{2.2}$$

In this section, we focus on finding ansatz for this equation under assumption (2.1).

2.2 First Blow-up Ansatz

Let

$$J = \left\lfloor \frac{2p+2}{p-1} \right\rfloor \quad \text{so that} \quad q_0 = 2J + 3, \tag{2.3}$$

where q_0 is defined by (1.2), and let

$$k \ge q_0 + 1 \tag{2.4}$$

be an integer.

We consider the function $A : \mathbb{R}^N \to [0, +\infty[$ given by

$$A(x) := \begin{cases} 0 & \text{if } |x| \le 1, \\ (|x| - \chi(x))^k & \text{if } 1 < |x| \le 2, \\ |x|^k & \text{if } |x| > 2. \end{cases} \tag{2.5}$$

It follows that A is of class $\mathcal{C}^{k-1}$ and that, for any $\beta \in \mathbb{N}^N$, with $|\beta| \le k - 1$,

$$\begin{cases} A \ge 0 \text{ and } |\partial_x^\beta A| \lesssim A^{1-\frac{|\beta|}{k}} & \text{on } \mathbb{R}^N, \\ A(x) = |x|^k & \text{for any } x \in \mathbb{R}^N \text{ such that } |x| \ge 2. \end{cases} \tag{2.6}$$

We define a basic blow-up ansatz V_0, for $s > 0$ and $x \in \mathbb{R}^N$,

$$V_0(s,x) = \kappa(x)(s + A(x))^{-\frac{2}{p-1}}, \tag{2.7}$$

where

$$\kappa(x) = \kappa_0 (1 - |\nabla\psi(x)|^2)^{\frac{1}{p-1}}, \quad \kappa_0 = \left[\frac{2(p+1)}{(p-1)^2} \right]^{\frac{1}{p-1}},$$

which satisfies

$$(1 - |\nabla\psi|^2)\partial_{ss} V_0 = V_0^p \quad \text{on } (0, +\infty) \times \mathbb{R}^N.$$

Since the functions ψ and A are of class $\mathcal{C}^{q_0}$, we remark that the function V_0 is of class $\mathcal{C}^\infty$ in the variable $s > 0$ and of class $\mathcal{C}^{q_0 - 1}$ in the variable $x \in \mathbb{R}^N$.

In view of (2.2), it is natural to set

$$\begin{aligned}
\mathcal{E}_0 &= -(1 - |\nabla\psi|^2)\partial_{ss} V_0 + 2\nabla\psi \cdot \nabla\partial_s V_0 + (\Delta\psi)\partial_s V_0 + \Delta V_0 + f(V_0) \\
&= 2\nabla\psi \cdot \nabla\partial_s V_0 + (\Delta\psi)\partial_s V_0 + \Delta V_0.
\end{aligned} \tag{2.8}$$

We gather in the next lemma the properties of V_0 and $\mathcal{E}_0$.

Lemma 2.1. *The function V_0 satisfies*

$$(1-|\nabla\psi|^2)^{\frac{1}{2}}\partial_s V_0 = -\Big(\frac{2}{p+1}V_0^{p+1}\Big)^{\frac{1}{2}}, \quad (1-|\nabla\psi|^2)\partial_{ss}V_0 = V_0^p. \tag{2.9}$$

Moreover, for any $\alpha\in\mathbb{N}$, $\beta\in\mathbb{N}^N$, $\rho\in\mathbb{R}$, $0<s<1$, $x\in\mathbb{R}^N$, the following hold:

(i) *If $0\le|\beta|\le q_0-1$ and $|x|\le R$, then*

$$|\partial_s^\alpha\partial_x^\beta(V_0^\rho)|\lesssim V_0^{\rho+(\alpha+\frac{|\beta|}{k})\frac{p-1}{2}}. \tag{2.10}$$

(ii) *If $|\beta|\le q_0-3$ and $|x|\le R$, then*

$$|\partial_s^\alpha\partial_x^\beta\mathcal{E}_0|\lesssim V_0^{\frac{p+1}{2}+(\alpha+\frac{1+|\beta|}{k})\frac{p-1}{2}}. \tag{2.11}$$

(iii) *If $|x|>R$, then*

$$|\partial_s^\alpha\partial_x^\beta V_0|\lesssim |x|^{-(\frac{2}{p-1}+\alpha)k-|\beta|}, \tag{2.12}$$

$$|\partial_s^\alpha\partial_x^\beta\mathcal{E}_0|\lesssim |x|^{-(\frac{2}{p-1}+\alpha)k-|\beta|-2}. \tag{2.13}$$

Furthermore, if $|x_0|<1$, then for any $\sigma>0$,

$$\liminf_{s\downarrow 0} s^{\frac{N+2-(N-2)p}{2(p-1)}}\|\partial_s V_0(s)\|_{L^2(|x-x_0|<\sigma s)}>0. \tag{2.14}$$

Proof. First, we observe that the function κ is constant for $|x|>R$ and satisfies $\kappa\gtrsim 1$, $|\partial_x^\beta\kappa|\lesssim 1$ on $\mathbb{R}^N$, for any $|\beta|\le q_0-1$.

Proof of (2.9). This follows from direct computations.

Proof of (2.10). For $0<s<1$ and $|x|\le R$, one has $0<s+A(x)\lesssim 1$ and thus, $V_0\gtrsim 1$. We introduce some notation:

$$h(z)=z^{-\frac{2}{p-1}} \quad \text{for } z>0, \qquad W(s,x)=s+A(x).$$

In particular, $V_0(s,x)=\kappa(x)h(W(s,x))$. Let $\alpha\ge 0$. Since $|h^{(\alpha)}(z)|\lesssim |z|^{-\frac{2}{p-1}-\alpha}$, we have

$$\partial_s^\alpha V_0=\kappa(x)h^{(\alpha)}(W(s,x)) \quad\text{and so}\quad |\partial_s^\alpha V_0|\lesssim |V_0|^{1+\alpha\frac{p-1}{2}}.$$

Let $\beta\in\mathbb{N}^N$ be such that $1\le|\beta|\le q_0-1$. Using (1.12), we have

$$\partial_s^\alpha\partial_x^\beta V_0=\sum_{\beta'\le\beta}\binom{\beta}{\beta'}(\partial_x^{\beta-\beta'}\kappa)(\partial_x^{\beta'}[h^{(\alpha)}(W)]).$$

For $0\le\beta'\le\beta$, it holds $|\partial_x^{\beta-\beta'}\kappa|\lesssim 1$. Thus, for $\beta'=0$ in the above sum, we have

$$|(\partial_x^\beta\kappa)h^{(\alpha)}(W)|\lesssim |V_0|^{1+\alpha\frac{p-1}{2}}.$$

For $1\le|\beta'|$, $\beta'\le\beta$, setting $n'=|\beta'|$ and using (1.13),

$$\partial_x^{\beta'}[h^{(\alpha)}(W)]=\sum_{r=1}^{n'}[h^{(\alpha+r)}(W)]\sum_{P(\beta',r)}(\beta'!)\prod_{l=1}^{n'}\frac{(\partial_x^{\beta_l}W)^{r_l}}{(r_l!)(\beta_l!)^{r_l}},$$

where

$$\begin{aligned}P(\beta',r)=\Big\{(r_1,\dots,r_n;\beta_1,\dots,\beta_{n'}) :{}& \text{there exists } 1\le m\le n' \text{ such that } r_\ell=0 \text{ and } \beta_\ell=0 \text{ for } 1\le\ell\le n'-m;\\ & r_\ell>0 \text{ for } n'-m+1\le\ell\le n';\\ & \text{and } 0\prec\beta_{n'-m+1}\prec\cdots\prec\beta_{n'} \text{ are such that } \sum_{\ell=1}^{n'}r_\ell=r,\ \sum_{\ell=1}^{n'}r_\ell\beta_\ell=\beta'\Big\}.\end{aligned}$$

As before, we use for $r\ge 1$, $|h^{(\alpha+r)}(W)|\lesssim W^{-\frac{2}{p-1}-r-\alpha}$. Moreover, using the assumption (2.6) on A, we have, for $1\le|\beta_\ell|\le q_0-1$,

$$|\partial_x^{\beta_\ell}W|\lesssim|\partial_x^{\beta_\ell}A|\lesssim A^{1-\frac{|\beta_\ell|}{k}}.$$

Since $\sum_{\ell=1}^{n'} r_\ell = r$, $\sum_{\ell=1}^{n'} r_\ell|\beta_\ell| = |\beta'|$ and $|\beta'| \le q_0 - 1 \le k - 1$, we obtain

$$\begin{aligned}|\partial_x^{\beta'}[h^{(\alpha)}(W)]| &\lesssim \sum_{r=1}^{n'} W^{-\frac{2}{p-1}-r-\alpha} \sum_{P(\beta',r)} \Big[A^{1-\frac{|\beta_\ell|}{k}}\Big]^{r_\ell} \\ &\lesssim \sum_{r=1}^{n'} W^{-\frac{2}{p-1}-r-\alpha} A^{r-\frac{|\beta'|}{k}} \lesssim W^{-\frac{2}{p-1}-\alpha-\frac{|\beta'|}{k}} \lesssim V_0^{1+(\alpha+\frac{|\beta'|}{k})\frac{p-1}{2}}.\end{aligned}$$

We obtain, for all $0 \le |\beta| \le q_0 - 1$ and $|x| \le R$,

$$|\partial_s^\alpha \partial_x^\beta V_0| \lesssim V_0^{1+(\alpha+\frac{|\beta|}{k})\frac{p-1}{2}}, \tag{2.15}$$

which proves (2.10) for $\rho = 1$.

We use the notation $\boldsymbol{v} = (\alpha, \beta_1, \dots, \beta_N)$ as in the context of formula (1.15). Let $n = |\boldsymbol{v}| \ge 1$. Then, by (1.15), for $\rho \in \mathbb{R}$,

$$\partial^{\boldsymbol{v}}(V_0^\rho) = \sum_{r=1}^{n} [\rho \cdots (\rho - r + 1)] V_0^{\rho-r} \sum_{P(\boldsymbol{v},r)} (\boldsymbol{v}!) \prod_{\ell=1}^{n} \frac{(\partial^{\boldsymbol{v}_\ell} V_0)^{r_\ell}}{(r_\ell!)(\boldsymbol{v}_\ell!)^{r_\ell}},$$

where

$$\begin{aligned}P(\boldsymbol{v}, r) = \Big\{(r_1, \dots, r_n; \boldsymbol{v}_1, \dots, \boldsymbol{v}_n) :\ &\text{there exists } 1 \le m \le n \text{ such that } r_\ell = 0 \text{ and } \boldsymbol{v}_\ell = 0 \text{ for } 1 \le \ell \le n - m; \\ &r_\ell > 0 \text{ for } n - m + 1 \le \ell \le n; \\ &\text{and } 0 \prec \boldsymbol{v}_{n-m+1} \prec \cdots \prec \boldsymbol{v}_n \text{ are such that } \sum_{\ell=1}^{n} r_\ell = r,\ \sum_{\ell=1}^{n} r_\ell \boldsymbol{v}_\ell = \boldsymbol{v}\Big\}.\end{aligned}$$

Using (2.15) and $\sum_{\ell=1}^{n} r_\ell = r$, $\sum_{\ell=1}^{n} r_\ell\alpha_\ell = \alpha$, $\sum_{\ell=1}^{n} r_\ell\beta_\ell = \beta$ in $P(\boldsymbol{v}, r)$, we estimate

$$\begin{aligned}|\partial^{\boldsymbol{v}}(V_0^\rho)| &\lesssim \sum_{r=1}^{n} V_0^{\rho-r} \sum_{P(\boldsymbol{v},r)} \prod_{\ell=1}^{n} V_0^{r_\ell[1+(\alpha_\ell+\frac{|\beta_\ell|}{k})\frac{p-1}{2}]} \\ &\lesssim \sum_{r=1}^{n} V_0^{\rho-r} V_0^{r+(\alpha+\frac{|\beta|}{k})\frac{p-1}{2}} \lesssim V_0^{\rho+(\alpha+\frac{|\beta|}{k})\frac{p-1}{2}}.\end{aligned}$$

Proof of (2.11). We estimate the three terms in (2.8). It follows from Leibniz's formula (1.14), the properties of ψ, $V_0 \gtrsim 1$, and estimate (2.10) that, for $|\beta| \le q_0 - 3$, and $|x| \le R$,

$$\begin{aligned}|\partial_s^\alpha \partial_x^\beta[\nabla\psi \cdot \nabla\partial_s V_0]| &\lesssim V_0^{1+(1+\alpha+\frac{1+|\beta|}{k})\frac{p-1}{2}}, \\ |\partial_s^\alpha \partial_x^\beta[(\Delta\psi)\partial_s V_0]| &\lesssim V_0^{1+(1+\alpha+\frac{|\beta|}{k})\frac{p-1}{2}}, \\ |\partial_s^\alpha \partial_x^\beta[\Delta V_0]| &\lesssim V_0^{1+(\alpha+\frac{2+|\beta|}{k})\frac{p-1}{2}}.\end{aligned}$$

Using once more that $V_0 \gtrsim 1$ for $|x| \le R$ and $k \ge 1$, these estimates imply (2.11).

Proof of (2.12). It follows from the properties of the functions ψ and A that

$$V_0(s, x) = \kappa_0(s + |x|^k)^{-\frac{2}{p-1}} \quad \text{for any } |x| \ge R.$$

Estimate (2.12) follows immediately. Then we have, for any $|x| \ge R$,

$$\begin{aligned}|\partial_s^\alpha \partial_x^\beta[\nabla\psi \cdot \nabla\partial_s V_0]| &= 0, \\ |\partial_s^\alpha \partial_x^\beta[(\Delta\psi)\partial_s V_0]| &= 0, \\ |\partial_s^\alpha \partial_x^\beta[\Delta V_0]| &\lesssim |x|^{-(\frac{2}{p-1}+\alpha)k-|\beta|-2},\end{aligned}$$

which implies (2.13).

Proof of (2.14). Since $|x_0| < 1$, we have for s small $|\partial_t V_0| \gtrsim s^{-\frac{p+1}{p-1}}$, and (2.14) follows. □

2.3 Refined Blow-up Ansatz

Starting from V_0, we define by induction a refined ansatz to the nonlinear wave equation (2.2).

Let V_0 be defined in (2.7) and let $\mathcal{E}_0$ be defined in (2.8). Let $s_0 = 1$. For $j \geq 1$, let

$$\begin{aligned} v_j &= -\frac{1}{3p+1}\Big[\frac{2(p+1)}{1-|\nabla\psi|^2}\Big]^{\frac{1}{2}}\Big(V_0^{\frac{p+1}{2}}\int_0^s V_0^{-p}\mathcal{E}_{j-1}\,ds' + V_0^{-p}\int_s^{s_{j-1}} V_0^{\frac{p+1}{2}}\mathcal{E}_{j-1}\,ds'\Big),\\ V_j &= V_0 + \sum_{\ell=1}^{j}\chi_\ell v_\ell,\\ \mathcal{E}_j &= -(1-|\nabla\psi|^2)\partial_{ss}V_j + 2\nabla\psi\cdot\nabla\partial_s V_j + (\Delta\psi)\partial_s V_j + \Delta V_j + f(V_j), \end{aligned}$$

where $\chi_j(x) = \chi(\frac{A(x)}{r_j})$ and $0 < r_j \leq 1$, $0 < s_j \leq 1$ are parameters to be defined for each $j = 1, \dots, J$. Since V_0 is of class $\mathcal{C}^\infty$ in s and of class $\mathcal{C}^{q_0-1}$ in x, the above expressions make sense as continuous functions for j such that $j \leq J$. This restriction is due to the spatial derivatives in V_j in the expression of $\mathcal{E}_j$.

Lemma 2.2. *There exist $0 < r_J \leq \cdots \leq r_1 \leq 1$ and $0 < s_J \leq \cdots \leq s_1 \leq 1$ such that for any $0 \leq j \leq J$, for any $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^N$, $0 < s \leq s_j$, $x \in \mathbb{R}^N$, the following hold:*

(i) *If $1 \leq j \leq J$, $|\beta| \leq q_0 - 2j - 1$ and $|x| \leq R$, then*

$$|\partial_s^\alpha \partial_x^\beta v_j| \lesssim V_0^{1+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}. \tag{2.16}$$

(ii) *If $1 \leq j \leq J$, then*

$$|V_j - V_0| \leq \frac{1}{4}(1-2^{-j})V_0, \quad |V_j - V_0| \leq (1-2^{-j})(1+V_0)^{-\frac{p-1}{4}}V_0, \tag{2.17}$$

$$|\partial_s V_j - \partial_s V_0| \lesssim V_0^{1+\frac{p-1}{2k}}. \tag{2.18}$$

(iii) *If $|\beta| \leq q_0 - 2(j+1) - 1$ and $|x| \leq R$, then*

$$|\partial_s^\alpha \partial_x^\beta \mathcal{E}_j| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}. \tag{2.19}$$

(iv) *If $|x| > R$, then*

$$|\partial_s^\alpha \partial_x^\beta V_j| \lesssim |x|^{-(\frac{2}{p-1}+\alpha)k-|\beta|}, \tag{2.20}$$

$$|\partial_s^\alpha \partial_x^\beta \mathcal{E}_j| \lesssim |x|^{-(\frac{2}{p-1}+\alpha)k-|\beta|-2}. \tag{2.21}$$

Remark 2.3. To complete the energy control in Section 3, we need an error estimate of the form

$$\|\mathcal{E}_J\|_{L^2} \lesssim s^{\frac{2}{p-1}+\delta},$$

as in [6] (see (3.28)), as well as an estimate of the form $\|\partial_s\mathcal{E}_J\|_{L^2} \lesssim s^{-1+\delta}$ (see the proof of (3.30)), with $\delta > 0$. This requires a sufficiently large J, see (2.3), and then a sufficiently large k. Compared with Lemma 2.3 (see also Remark 2.4) in [6], we need twice as many steps. This is due to the terms depending on $\partial_s V_j$ in the expression of the error term $\mathcal{E}_j$. These necessary restrictions have the important consequence that the minimal regularity of the hypersurface that we can consider in Theorem 1.1 depends on p, see (2.3).

Proof of Lemma 2.2. We observe that (2.19), (2.20) and (2.21) for $j = 0$ are exactly (2.11), (2.12) and (2.13) in Lemma 2.1. We proceed by induction on j: for any $1 \leq j \leq J$, we prove that estimate (2.19) for $\mathcal{E}_{j-1}$ implies (2.16)–(2.19) for v_j, V_j and $\mathcal{E}_j$. Let $s_0 = 1$.

Proof of (2.16). Let $1 \leq j \leq J$. First, assuming (2.19) for $\mathcal{E}_{j-1}$, we show the following estimates related to the two components of v_j: for $|\beta| \leq q_0 - 2j - 1$, $0 < s < s_{j-1}$ and $|x| \leq R$,

$$\Big|\partial_s^\alpha \partial_x^\beta\Big(\int_0^s V_0^{-p}\mathcal{E}_{j-1}\,ds'\Big)\Big| \lesssim V_0^{-\frac{p-1}{2}+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}, \tag{2.22}$$

$$\Big|\partial_s^\alpha \partial_x^\beta\Big(\int_s^{s_{j-1}} V_0^{\frac{p+1}{2}}\mathcal{E}_{j-1}\,ds'\Big)\Big| \lesssim V_0^{p+1+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}. \tag{2.23}$$

Indeed, we have by Leibniz's formula

$$\partial_s^\alpha \partial_x^\beta (V_0^{-p} \mathcal{E}_{j-1}) = \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (\partial_s^{\alpha'} \partial_x^{\beta'} (V_0^{-p}))(\partial_s^{\alpha-\alpha'} \partial_x^{\beta-\beta'} \mathcal{E}_{j-1}),$$

and thus using (2.10) and (2.19) for $\mathcal{E}_{j-1}$, we obtain

$$\begin{aligned}|\partial_s^\alpha \partial_x^\beta (V_0^{-p} \mathcal{E}_{j-1})| &\lesssim \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} V_0^{-p+(\alpha'+\frac{|\beta'|}{k})\frac{p-1}{2}} V_0^{\frac{p+1}{2}+(1-j+\alpha-\alpha'+\frac{j+|\beta-\beta'|}{k})\frac{p-1}{2}} \\ &\lesssim V_0^{(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}.\end{aligned}$$

For $\alpha = 0$, $|\beta| \le q_0 - 1 \le k - 1$ and $1 \le j \le J$, we note that

$$|\partial_x^\beta (V_0^{-p} \mathcal{E}_{j-1})| \lesssim V_0^{-a\frac{p-1}{2}} \lesssim (s + A)^a,$$

where

$$a = j - \frac{j + |\beta|}{k} = j\Big(1 - \frac{1}{k}\Big) - \frac{|\beta|}{k} \ge 0.$$

This means that we can integrate this term on $(0, s)$ for $0 < s \le s_{j-1}$. We obtain

$$\int_0^s |\partial_x^\beta (V_0^{-p} \mathcal{E}_{j-1})|\, ds' \lesssim (s + A)^{a+1} \lesssim V_0^{-\frac{p-1}{2}+(-j+\frac{j+|\beta|}{k})\frac{p-1}{2}}.$$

For $\alpha \ge 1$,

$$\Big|\partial_s^\alpha \partial_x^\beta \Big(\int_0^s V_0^{-p} \mathcal{E}_{j-1}\, ds' \Big)\Big| = |\partial_s^{\alpha-1} \partial_x^\beta (V_0^{-p} \mathcal{E}_{j-1})| \lesssim V_0^{-\frac{p-1}{2}+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}},$$

which proves (2.22). Similarly, using Leibniz's formula, we check the estimate

$$|\partial_s^\alpha \partial_x^\beta (V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1})| \lesssim V_0^{p+1+(1-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}.$$

In particular, for $\alpha = 0$,

$$|\partial_x^\beta (V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1})| \lesssim V_0^{b\frac{p-1}{2}} \lesssim (s + A)^{-b},$$

where, using $1 \le j \le J \le \frac{2p+2}{p-1}$,

$$b = \frac{2p + 2}{p - 1} + 1 - j + \frac{j + |\beta|}{k} \ge 1 + \frac{j + |\beta|}{k} > 1.$$

Thus, by integration on (s, s_{j-1}),

$$\Big|\partial_x^\beta \Big(\int_s^{s_{j-1}} V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1}\, ds' \Big)\Big| \lesssim (s + A)^{-b+1} \lesssim V_0^{p+1+(-j+\frac{j+|\beta|}{k})\frac{p-1}{2}}.$$

For $\alpha \ge 1$,

$$\Big|\partial_s^\alpha \partial_x^\beta \Big(\int_s^{s_{j-1}} V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1}\, ds' \Big)\Big| = |\partial_s^{\alpha-1} \partial_x^\beta (V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1})| \lesssim V_0^{p+1+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}},$$

which proves (2.23).

Using estimates (2.10), (2.22), (2.23) and again Leibniz's formula, we obtain, for all $s \in (0, s_{j-1}]$,

$$\Big|\partial_s^\alpha \partial_x^\beta \Big(V_0^{\frac{p+1}{2}} \int_0^s V_0^{-p} \mathcal{E}_{j-1}\, ds' \Big)\Big| \lesssim V_0^{1+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}},$$

$$\Big|\partial_s^\alpha \partial_x^\beta \Big(V_0^{-p} \int_s^{s_{j-1}} V_0^{\frac{p+1}{2}} \mathcal{E}_{j-1}\, ds' \Big)\Big| \lesssim V_0^{1+(-j+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}}.$$

These estimates imply (2.16) for v_j on $(0, s_{j-1}]$.

Proof of (2.17)–(2.18). For $j = 1$, we prove (2.17) as a consequence of (2.16). For $2 \le j \le J$, we prove (2.17) as a consequence of (2.16) for j and (2.17) for $j-1$.

For $|x| > R \ge 1$, (2.6) implies $A(x) \ge 2^k \ge 2r_1 \ge \cdots \ge 2r_j$, thus $\chi_j = 0$ and $V_j = V_0$.

For $0 < s \le s_{j-1}$ and $|x| < R$, by (2.16) with $\alpha = 0$ and $\beta = 0$, using the definition of χ_j and the bound $V_0 \gtrsim 1$, we have

$$\chi_j|v_j| \lesssim \chi_j V_0^{1-j(1-\frac{1}{k})\frac{p-1}{2}} \lesssim \chi_j V_0^{1-(1-\frac{1}{k})\frac{p-1}{2}} \lesssim \chi_j(s+A)^{1-\frac{1}{k}} V_0 \lesssim (s+r_j)^{1-\frac{1}{k}} V_0.$$

Choosing $0 < r_j \le 1$ and $0 < s_j \le s_{j-1}$ sufficiently small, we impose, for $s \in (0, s_j]$,

$$\chi_j|v_j| \le 2^{-j-2} V_0 \quad \text{and} \quad \chi_j|v_j| \le 2^{-j}(1+V_0)^{-\frac{p-1}{4}} V_0.$$

In the case $j = 1$, this proves (2.17). For $j \ge 2$, combining this estimate with (2.17) for $j-1$, we find, for all $s \in (0, s_j]$ and $x \in \mathbb{R}^N$,

$$\sum_{\ell=1}^{j} \chi_\ell|v_\ell| \le \frac{1}{4}(1-2^{-j})V_0 \quad \text{and} \quad \sum_{\ell=1}^{j} \chi_\ell|v_\ell| \le (1-2^{-j})(1+V_0)^{-\frac{p-1}{4}} V_0,$$

which is (2.17).

To prove (2.18), we note that by (2.16), and using $A \lesssim V_0^{-\frac{p-1}{2}}$,

$$\sum_{\ell=1}^{j} \chi_\ell|\partial_s v_\ell| \lesssim \sum_{\ell=1}^{j} \chi_\ell V_0^{1+(1-\ell(1-\frac{1}{k}))\frac{p-1}{2}} \lesssim V_0^{1+\frac{p-1}{2k}}.$$

Proof of (2.19). Note that (2.19) for $j = 0$ was already checked. Now, for $1 \le j \le J$, we prove (2.19) for $\mathcal{E}_j$ assuming (2.19) for $\mathcal{E}_{j-1}$, (2.16) for v_j and (2.17) for V_j. This suffices to complete the induction argument.

By direct computations, we briefly check that the function v_j satisfies

$$(1-|\nabla\psi|^2)\partial_{ss} v_j = f'(V_0)v_j + \mathcal{E}_{j-1}. \tag{2.24}$$

Indeed, we have

$$\partial_s v_j = -\frac{1}{3p+1}\left[\frac{2(p+1)}{1-|\nabla\psi|^2}\right]^{\frac{1}{2}}\left(\frac{p+1}{2}\partial_s V_0 V_0^{\frac{p-1}{2}} \int_0^s V_0^{-p}\mathcal{E}_{j-1}\,ds' - p\partial_s V_0 V_0^{-p-1} \int_s^{s_{j-1}} V_0^{\frac{p+1}{2}}\mathcal{E}_{j-1}\,ds'\right),$$

and thus, using (2.9),

$$(1-|\nabla\psi|^2)^{\frac{1}{2}}\partial_s v_j = \frac{1}{3p+1}\left[\frac{2(p+1)}{1-|\nabla\psi|^2}\right]^{\frac{1}{2}}\left(\left(\frac{p+1}{2}\right)^{\frac{1}{2}} V_0^p \int_0^s V_0^{-p}\mathcal{E}_{j-1}\,ds' - p\left(\frac{p+1}{2}\right)^{-\frac{1}{2}} V_0^{-\frac{p+1}{2}} \int_s^{s_{j-1}} V_0^{\frac{p+1}{2}}\mathcal{E}_{j-1}\,ds'\right).$$

Differentiating in s again, and using (2.9), we obtain

$$(1-|\nabla\psi|^2)\partial_{ss} v_j = pV_0^{p-1}v_j + \mathcal{E}_{j-1},$$

which is (2.24).

Using (2.24), $V_j = V_{j-1} + \chi_j v_j$ and the definition of $\mathcal{E}_{j-1}$, we have

$$\begin{aligned}\mathcal{E}_j &= \mathcal{E}_{j-1} - \chi_j(1-|\nabla\psi|^2)\partial_{ss}v_j + 2\nabla\psi\cdot\nabla\partial_s(\chi_j v_j) + (\Delta\psi)\partial_s(\chi_j v_j) + \Delta(\chi_j v_j) + f(V_j) - f(V_{j-1})\\ &= (1-\chi_j)\mathcal{E}_{j-1} + 2\nabla\psi\cdot\nabla\partial_s(\chi_j v_j) + (\Delta\psi)\partial_s(\chi_j v_j) + \Delta(\chi_j v_j) + f(V_j) - f(V_{j-1}) - f'(V_0)\chi_j v_j.\end{aligned}$$

We estimate each term of the right-hand side above for $|x| \le R$.

For the first term, recall that for $A \le r_j$, and any β', $1-\chi_j = 0$ and $\partial_x^{\beta'}\chi_j = 0$. Moreover, for $0 < s \le 1$, for x such that $A(x) > r_j$ and $|x| \le R$, one has $A \approx 1$ and $V_0 \approx 1$. Thus, using (2.19) for $\mathcal{E}_{j-1}$, we find

$$|\partial_s^\alpha \partial_x^\beta[(1-\chi_j)\mathcal{E}_{j-1}]| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}.$$

Now, we treat the next three terms in the expression of $\mathcal{E}_j$. By Leibniz's formula, the properties of ψ and χ_j, (2.16) and then $V_0 \gtrsim 1$, we have, for $0 < s \leq s_j$ and $|x| < R$,

$$\begin{aligned}\left|\partial_s^\alpha\partial_x^\beta[\nabla\psi\cdot\nabla\partial_s(\chi_j v_j)]\right| &\lesssim \sum_{\alpha'=1}^{\alpha+1}\sum_{|\beta'|\leq|\beta|+1}\left|\partial_s^{\alpha'}\partial_x^{\beta'}v_j\right| \lesssim V_0^{1+(-j+1+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}},\\ \left|\partial_s^\alpha\partial_x^\beta[(\Delta\psi)\partial_s(\chi_j v_j)]\right| &\lesssim \sum_{\alpha'=1}^{\alpha+1}\sum_{|\beta'|\leq|\beta|}\left|\partial_s^{\alpha'}\partial_x^{\beta'}v_j\right| \lesssim V_0^{1+(-j+1+\alpha+\frac{j+|\beta|}{k})\frac{p-1}{2}},\\ \left|\partial_s^\alpha\partial_x^\beta[\Delta(\chi_j v_j)]\right| &\lesssim \sum_{\alpha'=1}^{\alpha}\sum_{|\beta'|\leq|\beta|+2}\left|\partial_s^{\alpha'}\partial_x^{\beta'}v_j\right| \lesssim V_0^{1+(-j+\alpha+\frac{2+j+|\beta|}{k})\frac{p-1}{2}};\end{aligned}$$

we see that these three terms are estimated by $V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}$.

Finally, we estimate $\partial_s^\alpha\partial_x^\beta[f(V_j)-f(V_{j-1})-f'(V_0)\chi_j v_j]$ using Taylor expansions on f and its derivatives. We start with the case $\alpha=\beta=0$. Recall that by (2.17), we have $0<\frac{3}{4}V_0\leq V_j\leq\frac{5}{4}V_0$. The following Taylor expansions hold:

$$|f(V_j)-f(V_{j-1})-f'(V_{j-1})\chi_j v_j| \lesssim \chi_j^2 V_0^{p-2}v_j^2$$

and

$$|f'(V_{j-1})-f'(V_0)| \lesssim V_0^{p-2}\sum_{\ell=1}^{j-1}\chi_\ell|v_\ell|.$$

These estimates imply

$$|f(V_j)-f(V_{j-1})-f'(V_0)\chi_j v_j| \lesssim \chi_j V_0^{p-2}|v_j|\sum_{\ell=1}^{j}\chi_\ell|v_\ell|.$$

For $1\leq\ell\leq j$, using (2.16) and next $V_0\gtrsim 1$, we have

$$\begin{aligned}V_0^{p-2}|v_j||v_\ell| &\lesssim V_0^{p-2}V_0^{1-j(1-\frac{1}{k})\frac{p-1}{2}}V_0^{1-\ell(1-\frac{1}{k})\frac{p-1}{2}}\\ &\lesssim V_0^{p-(j+\ell)(1-\frac{1}{k})\frac{p-1}{2}} \lesssim V_0^{p-(j+1)(1-\frac{1}{k})\frac{p-1}{2}}.\end{aligned}$$

Thus, $|f(V_j)-f(V_{j-1})-f'(V_0)\chi_j v_j| \lesssim V_0^{\frac{p+1}{2}+(-j+\frac{1+j}{k})\frac{p-1}{2}}$ is proved.

Now, we consider the case $|\alpha|+|\beta|\geq 1$. By the Taylor formula with integral remainder we have for any V and w,

$$f(V+w)-f(V)-f'(V)w = w^2\int_0^1(1-\theta)f''(V+\theta w)\,d\theta.$$

Therefore, using the notation $\boldsymbol{\nu}=(\alpha,\beta_1,\ldots,\beta_N)$, by the Leibniz formula (1.14),

$$\partial^{\boldsymbol{\nu}}[f(V+w)-f(V)-f'(V)w] = \sum_{\boldsymbol{\nu}'\leq\boldsymbol{\nu}}\binom{\boldsymbol{\nu}}{\boldsymbol{\nu}'}(\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}(w^2))\int_0^1(1-\theta)\partial^{\boldsymbol{\nu}'}[f''(V+\theta w)]\,d\theta$$

and, by the Faa di Bruno formula (1.15), for $\boldsymbol{\nu}'\neq 0$, denoting $n'=|\boldsymbol{\nu}'|$,

$$\partial^{\boldsymbol{\nu}'}[f''(V+\theta w)] = \sum_{r=1}^{n'}f^{(r+2)}(V+\theta w)\sum_{P(\boldsymbol{\nu}',r)}(\boldsymbol{\nu}'!)\prod_{\ell=1}^{n'}\frac{(\partial^{\boldsymbol{\nu}_\ell}(V+\theta w))^{r_\ell}}{(r_\ell!)(\boldsymbol{\nu}_\ell!)^{r_j}}. \tag{2.25}$$

To estimate the term $\partial^{\boldsymbol{\nu}}[f(V_j)-f(V_{j-1})-f'(V_{j-1})\chi_j v_j]$, we apply these formulas to $V=V_{j-1}$ and $w=\chi_j v_j$. First, for $\boldsymbol{\nu}'\leq\boldsymbol{\nu}$, using (2.16) and the properties of χ, we obtain

$$\begin{aligned}\left|\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}[(\chi_j v_j)^2]\right| &\lesssim \sum_{\boldsymbol{\nu}''\leq\boldsymbol{\nu}-\boldsymbol{\nu}'}\left|\partial^{\boldsymbol{\nu}''}(\chi_j v_j)\right|\left|\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'-\boldsymbol{\nu}''}(\chi_j v_j)\right|\\ &\lesssim V_0^{2+(\alpha-\alpha'-2j+\frac{2j+|\beta-\beta'|}{k})\frac{p-1}{2}}.\end{aligned}$$

Thus, for $\boldsymbol{\nu}' = 0$ and $\theta \in [0,1]$, from (2.17), we obtain

$$\begin{aligned}\left|\partial^{\boldsymbol{\nu}}[(\chi_j v_j)^2] f''(V_{j-1}+\theta\chi_j v_j)\right| &\lesssim V_0^{2+(\alpha-2j+\frac{2j+|\beta|}{k})\frac{p-1}{2}} V_0^{p-2}\\ &\lesssim V_0^{\frac{p+1}{2}+(\alpha-j+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}.\end{aligned}$$

Second, for $\boldsymbol{\nu}' \neq 0$, $\boldsymbol{\nu}' \leq \boldsymbol{\nu}$ and $\theta \in [0,1]$, from formula (2.25), using (2.10) and (2.17), we have (the definition of $P(\boldsymbol{\nu}', r)$ implies $\sum_{\ell=1}^{n'} r_\ell = r$, $\sum_{\ell=1}^{n'} r_\ell \boldsymbol{\nu}_\ell = \boldsymbol{\nu}'$)

$$\begin{aligned}|\partial^{\boldsymbol{\nu}'}[f''(V_{j-1}+\theta\chi_j v_j)]| &\lesssim \sum_{r=1}^{n'} V_0^{p-r-2} \sum_{P(\boldsymbol{\nu}',r)} \prod_{\ell=1}^{n'} \Big(V_0^{1+(\alpha_\ell+\frac{|\beta_\ell|}{k})\frac{p-1}{2}}\Big)^{r_\ell}\\ &\lesssim \sum_{r=1}^{n'} V_0^{p-r-2} V_0^{r+(\alpha'+\frac{|\beta'|}{k})\frac{p-1}{2}} \lesssim V_0^{p-2+(\alpha'+\frac{|\beta'|}{k})\frac{p-1}{2}}.\end{aligned}$$

Thus, we have proved

$$\left|\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}[(\chi_j v_j)^2]\partial^{\boldsymbol{\nu}'}[f''(V_{j-1}+\theta\chi_j v_j)]\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}};$$

and so by integration in $\theta \in [0,1]$,

$$\left|\partial^{\boldsymbol{\nu}}[f(V_j)-f(V_{j-1})-f'(V_{j-1})\chi_j v_j]\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}. \tag{2.26}$$

We now estimate $\partial^{\boldsymbol{\nu}}[f(V_j)-f(V_{j-1})-f'(V_0)\chi_j v_j]$. For any V, W, w, we have

$$f'(V)-f'(W) = (V-W)\int_0^1 f''(W+\theta(V-W))\,d\theta,$$

and thus

$$\partial^{\boldsymbol{\nu}}[w(f'(V)-f'(W))] = \sum_{\boldsymbol{\nu}'\leq\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\nu}'} (\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}[w(V-W)]) \int_0^1 \partial^{\boldsymbol{\nu}'}[f''(W+\theta(V-W))]\,d\theta.$$

Moreover, for $\boldsymbol{\nu}' \neq 0$, formula (2.25) (with V replaced by W, and w by $V-W$) yields

$$\partial^{\boldsymbol{\nu}'}[f''(W+\theta(V-W))] = \sum_{r=1}^{n'} f^{(r+2)}(W+\theta(V-W)) \sum_{P(\boldsymbol{\nu}',r)} (\boldsymbol{\nu}'!) \prod_{\ell=1}^{n'} \frac{(\partial^{\boldsymbol{\nu}_\ell}(W+\theta(V-W)))^{r_\ell}}{(r_\ell!)(\boldsymbol{\nu}_\ell!)^{r_\ell}}.$$

To estimate the term $\partial^{\boldsymbol{\nu}}[\chi_j v_j(f'(V_{j-1})-f'(V_0))]$, we apply these formulas to $V = V_{j-1}$, $W = V_0$ and $w = \chi_j v_j$.

For $\boldsymbol{\nu}' \leq \boldsymbol{\nu}$, using (2.16) and Leibniz's formula, we have, for $1 \leq \ell \leq j-1$,

$$\begin{aligned}\left|\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}[\chi_j v_j \chi_\ell v_\ell]\right| &\lesssim V_0^{2+(-j-\ell+\alpha-\alpha'+\frac{j+\ell+|\beta-\beta'|}{k})\frac{p-1}{2}}\\ &\lesssim V_0^{-\frac{p-5}{2}+(-j+\alpha-\alpha'+\frac{1+j+|\beta-\beta'|}{k})\frac{p-1}{2}}.\end{aligned}$$

For $\boldsymbol{\nu}' = 0$ and $\theta \in [0,1]$, from (2.17), we obtain

$$\left|\partial^{\boldsymbol{\nu}}[\chi_j v_j(V_{j-1}-V_0)] f''(V_0+\theta(V_{j-1}-V_0))\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}.$$

Second, for $\boldsymbol{\nu}' \neq 0$, $\boldsymbol{\nu}' \leq \boldsymbol{\nu}$ and $\theta \in [0,1]$, by formula (2.25), using (2.10), (2.16) and (2.17), we have

$$\begin{aligned}|\partial^{\boldsymbol{\nu}'}[f''(V_0+\theta(V_{j-1}-V_0))]| &\lesssim \sum_{r=1}^{n'} V_0^{p-r-2} \sum_{P(\boldsymbol{\nu}',r)} \prod_{\ell=1}^{n'} \Big(V_0^{1+(\alpha_\ell+\frac{|\beta_\ell|}{k})\frac{p-1}{2}}\Big)^{r_\ell}\\ &\lesssim \sum_{r=1}^{n'} V_0^{p-r-2} V_0^{r+(\alpha'+\frac{|\beta'|}{k})\frac{p-1}{2}} \lesssim V_0^{p-2+(\alpha'+\frac{|\beta'|}{k})\frac{p-1}{2}}.\end{aligned}$$

Thus, we obtain

$$\left|\partial^{\boldsymbol{\nu}-\boldsymbol{\nu}'}[\chi_j v_j (V_{j-1} - V_0)]\partial^{\boldsymbol{\nu}'}[f''(V_0 + \theta(V_{j-1} - V_0))]\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}.$$

Integrating in $\theta \in [0, 1]$ and summing in $\boldsymbol{\nu}' \leq \boldsymbol{\nu}$, we obtain

$$\left|\partial^{\boldsymbol{\nu}}[\chi_j v_j (f'(V_{j-1}) - f'(V_0))]\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}. \tag{2.27}$$

Combining (2.26) and (2.27), we have proved for $s \in (0, s_j]$, $|x| \leq R$,

$$\left|\partial^{\boldsymbol{\nu}}[f(V_j) - f(V_{j-1}) - f'(V_0)\chi_j v_j]\right| \lesssim V_0^{\frac{p+1}{2}+(-j+\alpha+\frac{1+j+|\beta|}{k})\frac{p-1}{2}}.$$

In conclusion, we have estimated all terms in the expression of $\mathcal{E}_j$ and (2.19) for j is proved.

Proof of (2.20)–(2.21). For $|x| > R \geq 2$, (2.6) implies $A(x) \geq 2^k \geq 2r_1 \geq \cdots \geq 2r_j$, thus $\chi_j = 0$ and $V_j = V_0$, $\mathcal{E}_j = \mathcal{E}_0$. Thus, (2.20)–(2.21) follow from (2.12)–(2.13). □

3 Construction of a Solution of the Transformed Equation (2.2)

Let the function χ be given by (1.7), let $\psi \in \mathcal{C}^{q_0}(\mathbb{R}^N, \mathbb{R})$, where q_0 is defined by (1.2), satisfy (2.1), let J, q_0 and k be as in (2.3)–(2.4). Set

$$\lambda = \min\left\{\frac{1}{2}\left(J - \frac{p+3}{p-1}\right), \frac{1}{p}\right\} \in \left(0, \frac{1}{2}\right], \tag{3.1}$$

and impose the following additional condition on k:

$$k \geq \frac{2[p+1+\lambda(p-1)]}{\lambda(p-1)}. \tag{3.2}$$

Recall that $A : \mathbb{R}^N \to [0, +\infty[$ is defined by (2.5), and let V_J be defined as in Section 2.3.

Our main result of this section is the following.

Proposition 3.1. *Assume that*

$$\|\nabla\psi\|_{L^\infty} \leq \frac{\lambda}{8}\frac{p-1}{p+1}. \tag{3.3}$$

There exist $0 < \delta_0 < 1$ *and a function*

$$v \in C((0, \delta_0), H^2(\mathbb{R}^N)) \cap C^1((0, \delta_0), H^1(\mathbb{R}^N)) \cap C^2((0, \delta_0), L^2(\mathbb{R}^N)), \tag{3.4}$$

which is a solution of (2.2) *in* $C((0, \delta_0), L^2(\mathbb{R}^N))$, *and which satisfies*

$$\|(v - V_J)(s)\|_{H^2}^2 + \|\partial_s(v - V_J)(s)\|_{H^1}^2 \leq Cs^\lambda \quad \textit{for all } 0 < s < \delta_0, \tag{3.5}$$

with λ *given by* (3.1). *In addition, there exist a constant* C *and a function* $g \in L^\infty((0, \delta_0), H^1(\mathbb{R}^N))$ *such that*

$$|\partial_s v|^2 - |\nabla v|^2 \geq \frac{1}{4}|\partial_s V_0|^2 - C - g^2 \tag{3.6}$$

a.e. on $(0, \delta_0) \times \mathbb{R}^N$.

We construct the solution v of Proposition 3.1 by a compactness argument. For any n large, let $S_n = \frac{1}{n} < s_J$ and

$$B_n = \sup_{s\in[S_n, s_J]} \|V_J(s)\|_{L^\infty} \quad \text{so that} \quad \lim_{n\to\infty} B_n = \infty.$$

We let n be sufficiently large so that $B_n \geq 1$, and we define the function $f_n : \mathbb{R} \to [0, \infty)$ by

$$f_n(u) = f(u)\chi\left(\frac{|u|}{B_n}\right) \quad \text{so that} \quad f_n(u) = \begin{cases} f(u) & \text{for } |u| < B_n, \\ 0 & \text{for } |u| > 2B_n. \end{cases}$$

Let $F_n(u) = \int_0^u f_n(w)\,dw$. Note that Taylor's estimates such as (1.8)–(1.11) still hold for F_n and f_n with constants independent of n. We will refer to these inequalities for F_n and f_n with the same numbers (1.8), (1.9) and (1.11). In this proof, any implicit constant related the symbol $\lesssim$ is independent of n.

We define the sequence of solution v_n of

$$\begin{cases} (1-|\nabla\psi|^2)\partial_{ss}v_n - 2\nabla\psi\cdot\nabla\partial_s v_n - (\Delta\psi)\partial_s v_n - \Delta v_n = f_n(v_n), \\ v_n(S_n) = V_J(S_n), \quad \partial_s v_n(S_n) = \partial_s V_J(S_n). \end{cases} \tag{3.7}$$

The nonlinearity f_n being globally Lipschitz, the existence of a global solution $(v_n, \partial_s v_n)$ in $H^2\times H^1$ is a consequence of standard arguments from semigroups theory, see Appendix A, and in particular Section A.4.

We set, for all $s\in[S_n, s_J]$,

$$v_n(s) = V_J(s) + w_n(s),$$

thus $(w_n, \partial_s w_n)\in \mathcal{C}([S_n, s_J], H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N))\cap \mathcal{C}^1([S_n, s_J], H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N))$. The crucial step in the proof of Proposition 3.1 is the following estimate.

Proposition 3.2. *There exist $C>0$, $n_0>0$ and $0<\delta_0<1$ such that*

$$\|w_n(s)\|_{H^2}^2 + \|\partial_s w_n(s)\|_{H^1}^2 + \|\partial_{ss} w_n(s)\|_{L^2}^2 \le C(s-S_n)^\lambda \tag{3.8}$$

for all $n\ge n_0$ and $s\in[S_n, S_n+\delta_0]$.

Proof. We fix $n\ge n_0$ large, and we denote w_n simply by w in this proof. By (3.7) and the definition of $\mathcal{E}_J$, w satisfies the equation

$$\begin{cases} (1-|\nabla\psi|^2)\partial_{ss}w - 2\nabla\psi\cdot\nabla\partial_s w - (\Delta\psi)\partial_s w - \Delta w = f_n(V_J+w) - f_n(V_J) + \mathcal{E}_J, \\ w(S_n) = 0, \quad \partial_s w(S_n) = 0. \end{cases} \tag{3.9}$$

We define the auxiliary function Q as follows:

$$Q = (1-\chi+V_0)^{p+1},$$

where, by abuse of notation, we denote $\chi(x)=\chi(|x|)$. Note that $Q\gtrsim 1$. We make the following preliminary observation:

$$\begin{aligned} \partial_{ss}w &= \partial_{ss}[Q^{\frac12}(Q^{-\frac12}w)] = \partial_{ss}(Q^{\frac12})(Q^{-\frac12}w) + 2\partial_s(Q^{\frac12})\partial_s(Q^{-\frac12}w) + Q^{\frac12}\partial_{ss}(Q^{-\frac12}w) \\ &= \partial_{ss}(Q^{\frac12})(Q^{-\frac12}w) + Q^{-\frac12}\partial_s[Q\partial_s(Q^{-\frac12}w)]. \end{aligned}$$

Thus, setting

$$G = f'(V_0)Q^{\frac12} - (1-|\nabla\psi|^2)\partial_{ss}(Q^{\frac12})$$

(by the definition of Q and V_0, we expect G to be small in some sense), we rewrite the equation of w as follows:

$$\begin{aligned} (1-|\nabla\psi|^2)\partial_s[Q\partial_s(Q^{-\frac12}w)] &= Q^{\frac12}[2\nabla\psi\cdot\nabla\partial_s w + (\Delta\psi)\partial_s w + \Delta w] \\ &\quad + Q^{\frac12}[f_n(V_J+w) - f_n(V_J) - f_n'(V_0)w] + Gw + Q^{\frac12}\mathcal{E}_J. \end{aligned} \tag{3.10}$$

The nonlinear term $f_n(V_J+w)-f_n(V_J)-f_n'(V_0)w$ is mostly quadratic in w (some linear terms in w remain but they are also small in $V_J - V_0$), which is an important gain with respect to the previous formulation.

We define the following energy functional related to the above formulation of the equation of w:

$$\begin{aligned} \mathcal{H} = \int \Big\{ &(1-|\nabla\psi|^2)[Q\partial_s(Q^{-\frac12}w)]^2 + Q^2|\nabla(Q^{-\frac12}w)|^2 + \frac{\lambda}{16}s^{-2}Qw^2 \\ &- Q[2F_n(V_J+w) - 2F_n(V_J) - 2F_n'(V_J)w - F_n''(V_0)w^2]\Big\}. \end{aligned}$$

We also define a weighted norm related to the above functional

$$\mathcal{N} = \Big(\int [Q\partial_s(Q^{-\frac12}w)]^2 + Q^2|\nabla(Q^{-\frac12}w)|^2 + \frac{\lambda}{16}s^{-2}Qw^2\Big)^{\frac12}.$$

Since we may be dealing with $H^1 \times L^2$ supercritical nonlinearities (but $H^2 \times H^1$ subcritical by the condition $1 \le N \le 4$), we need higher order energy functionals. We set

$$\mathcal{K}_0 = \int \{(1 - |\nabla\psi|^2)(\partial_{ss} w)^2 + |\nabla \partial_s w|^2\},$$
$$\mathcal{K}_\ell = \int \{(1 - |\nabla\psi|^2)(\partial_s \partial_{x_\ell} w)^2 + |\nabla \partial_{x_\ell} w|^2\}, \quad 1 \le \ell \le N,$$

and

$$\mathcal{K} = \sum_{\ell=0}^{N} \mathcal{K}_\ell, \quad \mathcal{M} = (\|w\|_{H^2}^2 + \|\partial_s w\|_{H^1}^2 + \|\partial_{ss} w\|_{L^2}^2)^{\frac{1}{2}}.$$

For future reference, we establish two estimates on $\partial_s Q$ and ∇Q. By the expression of V_0 in (2.9), we have

$$\begin{aligned}\partial_s Q &= (p+1)\partial_s V_0 (1 - \chi + V_0)^p = (p+1)(\partial_s V_0) Q^{\frac{p}{p+1}}\\ &= -\sqrt{2(p+1)}(1 - |\nabla\psi|^2)^{-\frac{1}{2}} V_0^{\frac{p+1}{2}} Q^{\frac{p}{p+1}}.\end{aligned} \tag{3.11}$$

Thus, since $Q^{\frac{p-1}{2(p+1)}} \lesssim s^{-1}$,

$$|\partial_s Q| \lesssim V_0^{\frac{p+1}{2}} Q^{\frac{p}{p+1}} \lesssim Q^{1+\frac{p-1}{2(p+1)}} \lesssim s^{-1} Q. \tag{3.12}$$

Similarly, by (2.10),

$$|\nabla Q| = (p+1)|\nabla V_0|(1 - \chi + V_0)^p \lesssim V_0^{1+\frac{p-1}{2k}} Q^{\frac{p}{p+1}} \lesssim Q^{1+\frac{1}{k}\frac{p-1}{2(p+1)}} \lesssim s^{-\frac{1}{k}} Q \tag{3.13}$$

and

$$|\Delta Q| \lesssim s^{-\frac{2}{k}} Q, \quad |\nabla \partial_s Q| \lesssim s^{-1-\frac{1}{k}} Q. \tag{3.14}$$

Step 1: Coercivity. We claim the following estimates.

Lemma 3.3. *It holds*

$$\mathcal{M}^2 \lesssim \mathcal{N}^2 + \mathcal{K}. \tag{3.15}$$

For $0 < \delta \le s_J$ *and* $0 < \omega \le 1$ *sufficiently small, for n large, if* $\mathcal{N} \le \omega$ *and* $\mathcal{M} \le \omega$, *then*

$$2\mathcal{H} + \mathcal{K} \ge \mathcal{N}^2. \tag{3.16}$$

Proof. First, we prove the following estimates. For any $\rho \ge 0$, the following holds on $[S_n, \delta_0]$,

$$\int Q^\rho |\nabla w|^2 \lesssim \int Q^{\rho+1} |\nabla(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho + \frac{p-1}{k(p+1)}} w^2, \tag{3.17}$$

$$\int Q^{\rho+1} |\nabla(Q^{-\frac{1}{2}} w)|^2 \lesssim \int Q^\rho |\nabla w|^2 + \int Q^{\rho + \frac{p-1}{k(p+1)}} w^2, \tag{3.18}$$

$$\int Q^\rho |\partial_s w|^2 \lesssim \int Q^{\rho+1} |\partial_s(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho + \frac{p-1}{p+1}} w^2, \tag{3.19}$$

$$\int Q^{\rho+1} |\partial_s(Q^{-\frac{1}{2}} w)|^2 \lesssim \int Q^\rho |\partial_s w|^2 + \int Q^{\rho + \frac{p-1}{p+1}} w^2. \tag{3.20}$$

We have, using (3.13),

$$\begin{aligned}\int Q^\rho |\nabla w|^2 &= \int Q^\rho |Q^{\frac{1}{2}} \nabla(Q^{-\frac{1}{2}} w) + (\nabla Q^{\frac{1}{2}}) Q^{-\frac{1}{2}} w|^2\\ &\lesssim \int Q^{1+\rho} |\nabla(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho-1} |\nabla Q^{\frac{1}{2}}|^2 w^2\\ &\lesssim \int Q^{1+\rho} |\nabla(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho + \frac{p-1}{k(p+1)}} w^2.\end{aligned}$$

This proves (3.17) and the proof of (3.18) is similar. Moreover, using (3.12),

$$\begin{aligned}\int Q^\rho |\partial_s w|^2 &= \int Q^\rho |Q^{\frac{1}{2}} \partial_s(Q^{-\frac{1}{2}} w) + (\partial_s Q^{\frac{1}{2}}) Q^{-\frac{1}{2}} w|^2\\ &\lesssim \int Q^{\rho+1} \partial_s(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho-1} |\partial_s Q^{\frac{1}{2}}|^2 w^2\\ &\lesssim \int Q^{\rho+1} |\partial_s(Q^{-\frac{1}{2}} w)|^2 + \int Q^{\rho + \frac{p-1}{p+1}} w^2,\end{aligned}$$

which proves (3.19); the proof of (3.20) is similar.

We prove (3.15). The inequality $\|w\|_{L^2} \lesssim \mathcal{N}$ is obvious. Next, (3.19) with $\rho = 1$ and $Q^{\frac{p-1}{p+1}} \lesssim s^{-2}$ show that

$$\int |\partial_s w|^2 \lesssim \int Q|\partial_s w|^2 \lesssim \mathcal{N}^2.$$

Since $\|\nabla\psi\|_{L^\infty} \le \frac{1}{2}$ (from (3.3)), it follows (using $\|w\|_{H^2} \lesssim \|\Delta w\|_{L^2} + \|w\|_{L^2}$) that $\mathcal{M}^2 \lesssim \mathcal{N}^2 + \mathcal{K}$, which is (3.15).

Last, we prove (3.16). Let

$$A_1 = \Big|F_n(V_J + w) - F_n(V_J) - F_n'(V_J)w - \frac{1}{2}F_n''(V_0)w^2\Big|. \tag{3.21}$$

The triangle inequality and the Taylor inequality (1.8) yield

$$A_1 \lesssim \Big|F_n(V_J + w) - F_n(V_J) - F_n'(V_J)w - \frac{F_n''(V_J)}{2}w^2\Big| + |F_n''(V_J) - F_n''(V_0)|w^2 \lesssim \Lambda_1,$$

where

$$\Lambda_1 = |w|^{p+1} + V_0^{p-\bar{p}}|w|^{\bar{p}+1} + V_0^{p-2}|V_J - V_0|w^2. \tag{3.22}$$

From (2.17), $V_J \lesssim V_0$ and $|V_J - V_0| \lesssim (1 + V_0)^{-\frac{p-1}{4}} V_0 \lesssim Q^{-\frac{p-1}{4(p+1)}} V_0$. Moreover, $V_0^{p+1} \lesssim Q$. Thus,

$$\Lambda_1 \lesssim |w|^{p+1} + Q^{\frac{p-\bar{p}}{p+1}}|w|^{\bar{p}+1} + Q^{\frac{3}{4}\frac{p-1}{p+1}}w^2, \tag{3.23}$$

and so

$$\int Q\Lambda_1 \lesssim \int Q|w|^{p+1} + \int Q^{1+\frac{p-\bar{p}}{p+1}}|w|^{\bar{p}+1} + \int Q^{1+\frac{3}{4}\frac{p-1}{p+1}}w^2.$$

For the first term, we prove the following general estimate: for any $0 < \zeta \le 1$,

$$\int Q^{(1-\zeta)\frac{2p}{p+1}}|w|^{p+1} \lesssim \mathcal{N}^{p+1} + \mathcal{M}^{p+1}. \tag{3.24}$$

Indeed, using Hölder's inequality and the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $2 \le q < \infty$ (recall that $N \le 4$),

$$\begin{aligned}
\int Q^{(1-\zeta)\frac{2p}{p+1}}|w|^{p+1} &\lesssim s^{-2(1-\zeta)}\int Q^{1-\zeta}|w|^{p+1}\\
&\lesssim s^{-2(1-\zeta)}\Big(\int Qw^2\Big)^{1-\zeta}\Big(\int |w|^{\frac{p-1}{\zeta}+2}\Big)^{\zeta}\\
&\lesssim \mathcal{N}^{2(1-\zeta)}\mathcal{M}^{p-1+2\zeta} \lesssim \mathcal{N}^{p+1} + \mathcal{M}^{p+1}.
\end{aligned}$$

In particular, from (3.24), it holds

$$\int Q|w|^{p+1} \lesssim \mathcal{N}^{p+1} + \mathcal{M}^{p+1}.$$

In the case $1 < p \le 2$, one has $\bar{p} = p$ and the second term is identical to the first one. In the case $p > \bar{p} = 2$, the second term is estimated as follows. Using the inequality $|w|^3 \le aw^2 + a^{-(p-2)}|w|^{p+1}$ with $a = \varepsilon Q^{\frac{1}{p-1}}$, $\varepsilon > 0$ to be chosen later, and the estimate $Q^{\frac{p-1}{p+1}} \lesssim s^{-2}$, we see that

$$\begin{aligned}
Q^{\frac{2p-1}{p+1}}|w|^3 &\lesssim \varepsilon Q^{\frac{p-1}{p+1}}Qw^2 + \varepsilon^{-(p-2)}Q|w|^{p+1}\\
&\lesssim \varepsilon s^{-2}Qw^2 + \varepsilon^{-(p-2)}Q|w|^{p+1},
\end{aligned}$$

and so, using (3.24)

$$\int Q^{\frac{2p-1}{p+1}}|w|^{\bar{p}+1} \lesssim \varepsilon\mathcal{N}^2 + \varepsilon^{-(p-2)}(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}). \tag{3.25}$$

Last, since $Q^{\frac{p-1}{2(p+1)}} \lesssim s^{-1}$, we observe that

$$\int Q^{\frac{3}{4}\frac{p-1}{p+1}+1}w^2 \lesssim s^{-\frac{3}{2}}\int Qw^2 \lesssim s^{\frac{1}{2}}\mathcal{N}^2.$$

In conclusion, we have obtained, for $\mathcal{N} \le \omega$, $\mathcal{M} \le \omega$, $S_n \le s \le \delta$,

$$\begin{aligned}
\int QA_1 &\lesssim (\varepsilon + s^{\frac{1}{2}})\mathcal{N}^2 + (1 + \varepsilon^{-(p-2)})(\mathcal{N}^{p+1} + \mathcal{M}^{p+1})\\
&\lesssim (\varepsilon + \delta^{\frac{1}{2}})\mathcal{N}^2 + (1 + \varepsilon^{-(p-2)})\omega^{p-2}\mathcal{M}^2,
\end{aligned}$$

which, combined with (3.15), implies that for $\delta > 0$ and $\omega > 0$ small enough, it holds $2\mathcal{H}+\mathcal{K} \ge \mathcal{N}^2$ on $[S_n, \delta_0]$. (Recall that $1 - |\nabla\psi|^2 \ge \frac{3}{4}$ by (3.3) and $\lambda \le \frac{1}{2}$.) □

Step 2: Energy Control. We claim that there exist $C>0$ such that

$$\frac{d\mathcal{H}}{ds} \le Cs^{-1+\lambda}\mathcal{N} + \frac{\lambda}{4}s^{-1}\mathcal{N}^2 + Cs^{-\frac{1}{2}}\mathcal{N}^2 + Cs^{-1}(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}) \tag{3.26}$$

provided $\mathcal{N} \le \omega$ and $\mathcal{M} \le \omega$ with ω sufficiently small.

Proof of (3.26). We compute $\frac{d\mathcal{H}}{ds}$:

$$\begin{aligned}
\frac{1}{2}\frac{d\mathcal{H}}{ds} = \int \Big\{&(1-|\nabla\psi|^2)Q\partial_s(Q^{-\frac{1}{2}}w)\partial_s[Q\partial_s(Q^{-\frac{1}{2}}w)] \\
&+ Q^2\nabla(Q^{-\frac{1}{2}}w)\cdot\partial_s[\nabla(Q^{-\frac{1}{2}}w)] + \frac{\lambda}{16}s^{-2}Q^{\frac{3}{2}}w\partial_s(Q^{-\frac{1}{2}}w) \\
&- Q^{\frac{3}{2}}[f_n(V_J+w) - f_n(V_J) - f_n'(V_0)w]\partial_s(Q^{-\frac{1}{2}}w)\Big\} \\
&+ \int (\partial_s Q)Q|\nabla(Q^{-\frac{1}{2}}w)|^2 + \frac{\lambda}{32}s^{-2}(\partial_s Q)w^2 - \frac{\lambda}{16}s^{-3}Qw^2 \\
&- \frac{1}{2}\int \partial_s Q[2F_n(V_J+w) - 2F_n(V_J) - 2F_n'(V_J)w - F_n''(V_0)w^2] \\
&- \frac{1}{2}\int \partial_s Q[f_n(V_J+w) - f_n(V_J) - f_n'(V_0)w]w \\
&- \frac{1}{2}\int Q\partial_s V_0[2f_n(V_J+w) - 2f_n(V_J) - 2f_n'(V_J)w - f_n''(V_0)w^2] \\
&- \frac{1}{2}\int Q\partial_s(V_J - V_0)[2f_n(V_J+w) - 2f_n(V_J) - 2f_n'(V_J)w] \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.&
\end{aligned}$$

First, we remark the negative contribution of I_2. Since $\partial_s Q \le 0$ by (3.11), we have

$$I_2 \le -\frac{\lambda}{16}s^{-3}\int Qw^2. \tag{3.27}$$

Second, we compute I_1 using equation (3.10) of w:

$$\begin{aligned}
I_1 = &\int Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)[2\nabla\psi\cdot\nabla\partial_s w + (\Delta\psi)\partial_s w] \\
&+ \int \{Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)(\Delta w) + Q^2\nabla(Q^{-\frac{1}{2}}w)\cdot\nabla[\partial_s(Q^{-\frac{1}{2}}w)]\} \\
&+ \int Q\partial_s(Q^{-\frac{1}{2}}w)Gw + \int Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)\mathcal{E}_J + \frac{\lambda}{16}s^{-2}\int Q^{\frac{3}{2}}w\partial_s(Q^{-\frac{1}{2}}w) \\
= I_7 &+ I_8 + I_9 + I_{10} + I_{11}.
\end{aligned}$$

For I_7, we first observe that

$$\begin{aligned}
2\int Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla\partial_s w) = 2\int Q^2\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla\partial_s(Q^{-\frac{1}{2}}w)) + \int Q(\partial_s(Q^{-\frac{1}{2}}w))^2(\nabla\psi\cdot\nabla Q) \\
+ \int Q\partial_s Q\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla(Q^{-\frac{1}{2}}w)) + \int Q^{\frac{1}{2}}w\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla\partial_s Q) \\
- \frac{1}{2}\int Q^{-\frac{1}{2}}\partial_s Qw\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla Q).
\end{aligned}$$

Second, by integration by parts,

$$\begin{aligned}
2\int Q^2\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla\partial_s(Q^{-\frac{1}{2}}w)) &= \int Q^2\nabla\psi\cdot\nabla[(\partial_s(Q^{-\frac{1}{2}}w))^2] \\
&= -\int Q^2\Delta\psi[\partial_s(Q^{-\frac{1}{2}}w)]^2 - 2\int Q[\partial_s(Q^{-\frac{1}{2}}w)]^2(\nabla\psi\cdot\nabla Q).
\end{aligned}$$

By the definition of $\mathcal{N}$, we estimate

$$\left|\int Q^2\Delta\psi[\partial_s(Q^{-\frac{1}{2}}w)]^2\right| \lesssim \mathcal{N}^2.$$

Using (3.13), we also have

$$\left|\iint Q(\partial_s(Q^{-\frac{1}{2}}w))^2(\nabla\psi\cdot\nabla Q)\right| \lesssim s^{-\frac{1}{k}}\mathcal{N}^2.$$

Now, by the expressions of Q and V_0, we have

$$|\partial_s Q| = (p+1)|\partial_s V_0|(1-\chi+V_0)^p = 2\frac{p+1}{p-1}(s+A(x))^{-1}V_0(1-\chi+V_0)^p \le 2\frac{p+1}{p-1}s^{-1}Q,$$

and thus

$$\left|\iint Q\partial_s Q\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla(Q^{-\frac{1}{2}}w))\right| \le 2\frac{p+1}{p-1}s^{-1}\int|\nabla\psi|Q^2|\partial_s(Q^{-\frac{1}{2}}w)||\nabla(Q^{-\frac{1}{2}}w)| \le \frac{p+1}{p-1}s^{-1}\|\nabla\psi\|_{L^\infty}\mathcal{N}^2.$$

Similarly, using (3.12), (3.13), (3.14)

$$\left|\iint Q^{\frac{1}{2}}w\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla\partial_s Q)\right| \lesssim s^{-\frac{1}{k}}\left(\int Q^2|\partial_s(Q^{-\frac{1}{2}}w)|^2+s^{-2}\int Qw^2\right) \lesssim s^{-\frac{1}{k}}\mathcal{N}^2$$

and

$$\left|\iint Q^{-\frac{1}{2}}\partial_s Qw\partial_s(Q^{-\frac{1}{2}}w)(\nabla\psi\cdot\nabla Q)\right| \lesssim s^{-\frac{1}{k}}\mathcal{N}^2.$$

Using the same estimates and then (3.19), we finish estimating I_7 as follows:

$$\left|\iint Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)(\Delta\psi)\partial_s w\right| \lesssim \int Q^2|\partial_s(Q^{-\frac{1}{2}}w)|^2+\int Q(\partial_s w)^2 \lesssim \int Q^2|\partial_s(Q^{-\frac{1}{2}}w)|^2+s^{-2}\int Qw^2 \lesssim \mathcal{N}^2.$$

Thus, for some constant $C>0$, using (3.3),

$$|I_7| \le \frac{p+1}{p-1}s^{-1}\|\nabla\psi\|_{L^\infty}\mathcal{N}^2+Cs^{-\frac{1}{k}}\mathcal{N}^2 \le \frac{\lambda}{8}s^{-1}\mathcal{N}^2+Cs^{-\frac{1}{k}}\mathcal{N}^2.$$

Next, integrating by parts, using the identities

$$\begin{aligned}Q^2\nabla[\partial_s(Q^{-\frac{1}{2}}w)] &= Q^{\frac{1}{2}}\nabla[Q^{\frac{3}{2}}(\partial_s(Q^{-\frac{1}{2}}w)]-\frac{3}{2}\partial_s(Q^{-\frac{1}{2}}w)\nabla Q,\\ -\nabla w+Q^{\frac{1}{2}}\nabla(Q^{-\frac{1}{2}}w) &= -Q^{-\frac{1}{2}}\nabla(Q^{\frac{1}{2}}),\end{aligned}$$

and integrating again by parts, we find

$$\begin{aligned}I_8 &= -\int\nabla(Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w))\cdot\nabla w+\int Q^2\nabla(Q^{-\frac{1}{2}}w)\cdot\nabla[\partial_s(Q^{-\frac{1}{2}}w)]\\ &= -\int[\nabla(Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w))\cdot\nabla(Q^{\frac{1}{2}})]\,Q^{-\frac{1}{2}}w-\frac{3}{2}\int Q\partial_s(Q^{-\frac{1}{2}}w)[\nabla Q\cdot\nabla(Q^{-\frac{1}{2}}w)]\\ &= -\int Q\partial_s(Q^{-\frac{1}{2}}w)[\nabla(Q^{-\frac{1}{2}}w)\cdot\nabla Q]+\int\Delta(Q^{\frac{1}{2}})Qw\partial_s(Q^{-\frac{1}{2}}w).\end{aligned}$$

By (3.13) and the definition of $\mathcal{N}$,

$$\left|\iint Q\partial_s(Q^{-\frac{1}{2}}w)[\nabla(Q^{-\frac{1}{2}}w)\cdot\nabla Q]\right| \lesssim s^{-\frac{1}{k}}\mathcal{N}^2.$$

Similarly, using (3.13) and (3.14), we have $|\Delta(Q^{\frac{1}{2}})Q^{\frac{3}{2}}| \lesssim |\nabla Q|^2+|\Delta Q|Q \lesssim s^{-\frac{2}{k}}Q^2$, and thus

$$\left|\iint\Delta(Q^{\frac{1}{2}})Qw\partial_s(Q^{-\frac{1}{2}}w)\right| \lesssim \int Q^2[\partial_s(Q^{-\frac{1}{2}}w)]^2+s^{-\frac{4}{k}}\int Qw^2 \lesssim \mathcal{N}^2.$$

For I_9, we start by an estimate of $G=f_n'(V_0)Q^{\frac{1}{2}}-(1-|\nabla\psi|^2)\partial_{ss}(Q^{\frac{1}{2}})$. By the definition of $Q=(1-\chi+V_0)^{p+1}$ and (2.9), we observe

$$\begin{aligned}\partial_{ss}(Q^{\frac{1}{2}}) &= \partial_{ss}[(1-\chi+V_0)^{\frac{p+1}{2}}]\\ &= \frac{p+1}{2}\frac{p-1}{2}(\partial_s V_0)^2(1-\chi+V_0)^{\frac{p-3}{2}}+\frac{p+1}{2}\partial_{ss}V_0(1-\chi+V_0)^{\frac{p-1}{2}}\\ &= (1-|\nabla\psi|^2)^{-1}\left[\frac{p-1}{2}V_0^{p+1}Q^{\frac{p-3}{2(p+1)}}+\frac{p+1}{2}V_0^pQ^{\frac{p-1}{2(p+1)}}\right].\end{aligned}$$

Thus,

$$\begin{aligned} G &= pV_0^{p-1}Q^{\frac{1}{2}} - \frac{p-1}{2}V_0^{p+1}Q^{\frac{p-3}{2(p+1)}} - \frac{p+1}{2}V_0^pQ^{\frac{p-1}{2(p+1)}} \\ &= V_0^{p-1}Q^{\frac{p-3}{2(p+1)}}[p(1-\chi+V_0) + \frac{p-1}{2}V_0](1-\chi). \end{aligned}$$

For $|x|>1$, we have $V_0 \lesssim 1$ and $Q \lesssim 1$; since also $Q \gtrsim 1$, we see that $\|G\|_{L^\infty} \lesssim 1$. Therefore,

$$|I_9| \lesssim \|G\|_{L^\infty}\mathcal{N}^2 \lesssim \mathcal{N}^2.$$

For I_{10}, by the Cauchy–Schwarz inequality

$$|I_{10}| = \left|\int Q^{\frac{3}{2}}\partial_s(Q^{-\frac{1}{2}}w)\mathcal{E}_J\right| \lesssim \|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2}\mathcal{N},$$

and we need only estimate $\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2}$. From (2.21), for $|x| \geq R$, we have

$$Q^{\frac{1}{2}}|\mathcal{E}_J| \lesssim |\mathcal{E}_J| \lesssim |x|^{-\frac{2k}{p-1}-2}.$$

Since $1 \leq N \leq 4$, this implies $\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2(|x|>R)} \lesssim 1$. Next, from (2.19), for $|x| \leq R$, we have

$$Q^{\frac{1}{2}}|\mathcal{E}_J| \lesssim Q^{\frac{1}{2}}V_0^{\frac{p+1}{2}+(-J+\frac{1+J}{k})\frac{p-1}{2}} \lesssim V_0^{p+1+(-J+\frac{1+J}{k})\frac{p-1}{2}} \lesssim V_0^{p+1+\frac{1}{k}\frac{p-1}{2}-J(1-\frac{1}{k})\frac{p-1}{2}}.$$

Recall that by (3.1),

$$-J\frac{p-1}{2} \leq -\frac{p+3}{2} - \lambda(p-1)$$

and that (3.2) is equivalent to

$$\frac{p+1}{k} - \lambda(p-1)\Big(1-\frac{1}{k}\Big) \leq -\frac{\lambda(p-1)}{2}.$$

Thus, for $|x| \leq R$,

$$\begin{aligned} Q^{\frac{1}{2}}|\mathcal{E}_J| &\lesssim V_0^{p+1+\frac{1}{k}\frac{p-1}{2}-\frac{p+3}{2}(1-\frac{1}{k})-\lambda(p-1)(1-\frac{1}{k})} \lesssim V_0^{\frac{p-1}{2}+\frac{p+1}{k}-\lambda(p-1)(1-\frac{1}{k})} \\ &\lesssim V_0^{\frac{p-1}{2}-\lambda\frac{p-1}{2}} \lesssim (s+A(x))^{-1+\lambda} \lesssim s^{-1+\lambda}. \end{aligned}$$

It follows that

$$\|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2} \lesssim s^{-1+\lambda}. \tag{3.28}$$

For this term, we have obtained

$$|I_{10}| \lesssim \|Q^{\frac{1}{2}}\mathcal{E}_J\|_{L^2}\mathcal{N} \lesssim s^{-1+\lambda}\mathcal{N}.$$

Finally, by the Cauchy–Schwarz inequality,

$$|I_{11}| \leq s^{-1}\frac{\lambda}{16}\int Q^2|\partial_s(Q^{-\frac{1}{2}}w)|^2 + s^{-3}\frac{\lambda}{16}\int Qw^2.$$

Using (3.27), we obtain

$$|I_{11}| \leq \frac{\lambda}{16}s^{-1}\mathcal{N}^2 - I_2.$$

In conclusion for $I_1 + I_2$, we find

$$I_1 + I_2 \leq Cs^{-1+\lambda}\mathcal{N} + \frac{3\lambda}{16}s^{-1}\mathcal{N}^2 + Cs^{-\frac{1}{k}}\mathcal{N}^2.$$

To continue with the proof of (3.26), we estimate the term I_3. To that end, recall that $\bar{p} = \min(2,p)$. First, by (3.21)–(3.23) and (3.12)

$$\begin{aligned} |\partial_s Q|A_1 &\lesssim |\partial_s Q|\Lambda_1 \lesssim V_0^{\frac{p+1}{2}}Q^{\frac{p}{p+1}}\Lambda_1 \lesssim Q^{\frac{3p+1}{2(p+1)}}\Lambda_1 \\ &\lesssim Q^{\frac{3p+1}{2(p+1)}}|w|^{p+1} + Q^{\frac{3p+1}{2(p+1)}}Q^{\frac{p-\bar{p}}{p+1}}|w|^{\bar{p}+1} + Q^{\frac{9p-1}{4(p+1)}}w^2. \end{aligned}$$

Using (3.24), the first term is controlled as follows:

$$\int Q^{\frac{3p+1}{2(p+1)}}|w|^{p+1} \lesssim \mathcal{N}^{p+1} + \mathcal{M}^{p+1}.$$

In the case $1 < p \le 2$, one has $\bar{p} = p$ and the second term is identical to the first one. In the case $p > \bar{p} = 2$, using $Q^{\frac{p-1}{2(p+1)}} \lesssim s^{-1}$ and (3.25),

$$\int Q^{\frac{3p+1}{2(p+1)}} Q^{\frac{p-\bar{p}}{p+1}} |w|^{\bar{p}+1} = \int Q^{\frac{5p-3}{2(p+1)}} |w|^{\bar{p}+1} \lesssim s^{-1} \int Q^{\frac{2p-1}{p+1}} |w|^{\bar{p}+1}$$
$$\lesssim \varepsilon s^{-1} \mathcal{N}^2 + \varepsilon^{-(p-2)} s^{-1} (\mathcal{N}^{p+1} + \mathcal{M}^{p+1}),$$

where $\varepsilon > 0$ is to be chosen. Last, we observe that $Q^{\frac{9p-1}{4(p+1)}} w^2 \lesssim s^{-\frac{5}{2}} Q w^2$, and thus

$$\int Q^{\frac{9p-1}{4(p+1)}} w^2 \lesssim s^{-\frac{1}{2}} \mathcal{N}^2.$$

In conclusion, we have proved

$$|I_3| \lesssim \int |\partial_s Q| \Lambda_1 \lesssim \int Q^{\frac{3p+1}{2(p+1)}} \Lambda_1 \lesssim (s^{-\frac{1}{2}} + \varepsilon s^{-1}) \mathcal{N}^2 + (1 + \varepsilon^{-(p-2)} s^{-1})(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}). \tag{3.29}$$

We proceed similarly for I_4. Indeed, setting

$$A_2 = \big|f_n(V_J + w) - f_n(V_J) - f_n'(V_J) w\big| |w|$$
$$\lesssim \big|f_n(V_J + w) - f_n(V_J) - f_n'(V_J) w\big| |w| + |f_n'(V_0) - f_n'(V_J)| w^2,$$

by (1.9) and Taylor's inequality,

$$A_2 \lesssim |w|^{p+1} + V_0^{p-\bar{p}} |w|^{\bar{p}+1} + V_0^{p-2} |V_J - V_0| w^2 = \Lambda_1.$$

Using (3.29), we conclude that

$$|I_4| \lesssim \int |\partial_s Q| \Lambda_1 \lesssim (s^{-\frac{1}{2}} + \varepsilon s^{-1}) \mathcal{N}^2 + (1 + \varepsilon^{-(p-2)} s^{-1})(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}).$$

Now, we estimate I_5 and we set

$$A_3 = \big|2 f_n(V_J + w) - 2 f_n(V_J) - 2 f_n'(V_J) w - f_n''(V_0) w^2\big|.$$

By the triangle inequality, Taylor inequality (1.11), $|\partial_s V_0| \lesssim V_0^{\frac{p+1}{2}}$ (see (2.9)), we have

$$Q |\partial_s V_0| A_3 \lesssim Q V_0^{\frac{p+1}{2}} \big|2 f_n(V_J + w) - 2 f_n(V_J) - 2 f_n'(V_J) w - f_n''(V_J) w^2\big| + Q V_0^{\frac{p+1}{2}} \big|f_n''(V_J) - f_n''(V_0)\big| w^2$$
$$\lesssim Q V_0^{\frac{p-1}{2}} \Lambda_1 \lesssim Q^{\frac{3p+1}{2(p+1)}} \Lambda_1.$$

Using (3.29), we conclude that

$$|I_5| \lesssim (s^{-\frac{1}{2}} + \varepsilon s^{-1}) \mathcal{N}^2 + (1 + \varepsilon^{-(p-2)} s^{-1})(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}).$$

Finally, we estimate I_6 and we set

$$A_4 = \big|2 f_n(V_J + w) - 2 f_n(V_J) - 2 f_n'(V_J) w\big|.$$

By the triangle inequality and Taylor's inequality (1.11)

$$A_4 \lesssim \big|2 f_n(V_J + w) - 2 f_n(V_J) - 2 f_n'(V_J) w - f_n''(V_J) w^2\big| + |f_n''(V_J)| w^2$$
$$\lesssim V_0^{-1} |w|^{p+1} + V_0^{p-\bar{p}-1} |w|^{\bar{p}+1} + V_0^{p-2} w^2.$$

Using (2.18), $V_0 \lesssim Q^{\frac{1}{p+1}}$, $Q \gtrsim 1$, $Q \lesssim s^{-\frac{2(p+1)}{p-1}}$ and $k \ge 1$, we obtain

$$Q |\partial_s (V_J - V_0)| A_4 \lesssim Q V_0^{\frac{p-1}{2k}} \big(|w|^{p+1} + V_0^{p-\bar{p}} |w|^{\bar{p}+1} + V_0^{p-1} w^2\big)$$
$$\lesssim Q^{1 + \frac{p-1}{2k(p+1)}} \big(|w|^{p+1} + V_0^{p-\bar{p}} |w|^{\bar{p}+1}\big) + Q^{1 + \frac{p-1}{p+1}(1 + \frac{1}{2k})} w^2$$
$$\lesssim Q^{\frac{3p+1}{2(p+1)}} \big(|w|^{p+1} + V_0^{p-\bar{p}} |w|^{\bar{p}+1}\big) + s^{-\frac{2k+1}{k}} Q w^2$$
$$\lesssim Q^{\frac{3p+1}{2(p+1)}} \Lambda_1 + s^{-\frac{1}{k}} s^{-2} Q w^2.$$

Using (3.29) and $k \ge 2$, we conclude that

$$|I_6| \lesssim (s^{-\frac{1}{2}} + \varepsilon s^{-1}) \mathcal{N}^2 + (1 + \varepsilon^{-(p-2)} s^{-1})(\mathcal{N}^{p+1} + \mathcal{M}^{p+1}).$$

Choosing $\varepsilon \le \frac{\lambda}{16}$, then ω sufficiently small, and collecting the above estimates, we have proved (3.26).

Step 3: Higher-Order Energy Terms. We claim that for any $\ell = 0, 1, \dots, N$,

$$\left|\frac{d\mathcal{K}_\ell}{ds}\right| \lesssim s^{-1+\lambda}(\mathcal{N} + \mathcal{N}^2 + \mathcal{M}^2) + \mathcal{N}^{p+1} + \mathcal{M}^{p+1}. \tag{3.30}$$

Differentiating (3.9) with respect to s, setting $z_0 = \partial_s w$, we have

$$(1 - |\nabla\psi|^2)\partial_{ss} z_0 - 2\nabla\psi \cdot \nabla\partial_s z_0 - (\Delta\psi)\partial_s z_0 - \Delta z_0 = f_n'(V_J + w)z_0 + (f_n'(V_J + w) - f_n'(V_J))\partial_s V_J + \partial_s \mathcal{E}_J. \tag{3.31}$$

Differentiating $\mathcal{K}_0 = \int (1 - |\nabla\psi|^2)(\partial_s z_0)^2 + |\nabla z_0|^2$, we find from (3.31) and integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d\mathcal{K}_0}{ds} &= \int \{f_n'(V_J + w)z_0 + (f_n'(V_J + w) - f_n'(V_J))\partial_s V_J + \partial_s \mathcal{E}_J\}\partial_s z_0 \\ &= I_{12} + I_{13} + I_{14}.\end{aligned}$$

First, by the Cauchy–Schwarz inequality

$$|I_{12}| \lesssim \|f_n'(V_J + w)z_0\|_{L^2}\mathcal{K}_0^{\frac{1}{2}} \lesssim \|f_n'(V_J + w)z_0\|_{L^2}\mathcal{M}.$$

From

$$|f_n'(V_J + w)| \lesssim |V_0|^{p-1} + |w|^{p-1} \lesssim s^{-\frac{p-1}{p}} Q^{\frac{p-1}{2p}} + |w|^{p-1} \lesssim s^{-\frac{p-1}{p}} Q^{\frac{1}{2}} + |w|^{p-1}$$

and then (3.19) with $\rho = 1$, we have (recall that $1 \le N \le 4$ and thus $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \ge 2$, and $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $2 \le q \le 4$)

$$\begin{aligned}\|f'(V_J + w)z_0\|_{L^2}^2 &\lesssim s^{-\frac{2(p-1)}{p}} \int Q(\partial_s w)^2 + \int |w|^{2(p-1)}(\partial_s w)^2 \\ &\lesssim s^{-\frac{2(p-1)}{p}}\mathcal{N}^2 + \|w\|_{L^{4p}}^{2(p-1)}\|\partial_s w\|_{L^{\frac{4p}{p+1}}}^2 \\ &\lesssim s^{-\frac{2(p-1)}{p}}\mathcal{N}^2 + \|w\|_{H^2}^{2(p-1)}\|\partial_s w\|_{H^1}^2.\end{aligned}$$

Thus, using also $-\frac{p-1}{p} \ge -1 + \lambda$ (since $\lambda \le \frac{1}{p}$) we have

$$|I_{12}| \lesssim s^{-\frac{p-1}{p}}\mathcal{N}\mathcal{M} + \mathcal{M}^{p+1} \lesssim s^{-1+\lambda}(\mathcal{N}^2 + \mathcal{M}^2) + \mathcal{M}^{p+1}.$$

Second, using (2.10) and (2.18),

$$|\partial_s V_J| \lesssim V_0^{1+\frac{p-1}{2}} + V_0^{1+\frac{p-1}{2k}} \lesssim V_0 Q^{\frac{p-1}{2(p+1)}},$$

so that Taylor's inequality (1.10) yields

$$\begin{aligned}|(f_n'(V_J + w) - f_n'(V_J))\partial_s V_J| &\lesssim Q^{\frac{p-1}{2(p+1)}}|w|^p + Q^{\frac{p-1}{2(p+1)}} V_0^{p-1}|w| \\ &\lesssim Q^{\frac{p-1}{2(p+1)}}|w|^p + Q^{\frac{3(p-1)}{2(p+1)}}|w|.\end{aligned}$$

We have

$$\begin{aligned}\int Q^{\frac{p-1}{p+1}}|w|^{2p} &\lesssim \int (Qw^2)^{\frac{p-1}{p+1}}|w|^{2\frac{p^2+1}{p+1}} \lesssim \Big(\int Qw^2\Big)^{\frac{p-1}{p+1}}\Big(\int w^{p^2+1}\Big)^{\frac{2}{p+1}} \\ &\lesssim \mathcal{N}^{2\frac{p-1}{p+1}}\mathcal{M}^{2\frac{p^2+1}{p+1}} \lesssim (\mathcal{N} + \mathcal{M})^{2p}.\end{aligned}$$

Moreover, since $\frac{2(p-2)}{p+1} \le \frac{(2p-1)(p-1)}{p(p+1)}$, $Q \gtrsim 1$, $Q \lesssim s^{-\frac{2(p+1)}{p-1}}$, and $\lambda \le \frac{1}{p}$, we have

$$\begin{aligned}\int Q^{\frac{3(p-1)}{p+1}} w^2 &\lesssim \int Q^{\frac{2(p-2)}{p+1}} Qw^2 \lesssim \int Q^{\frac{(2p-1)(p-1)}{p(p+1)}} Qw^2 \\ &\lesssim s^{-\frac{2(2p-1)}{p}} \int Qw^2 \lesssim s^{-\frac{2(p-1)}{p}}\mathcal{N}^2 \lesssim s^{-2(1-\lambda)}\mathcal{N}^2.\end{aligned}$$

Thus,

$$\begin{aligned}|I_{13}| &\lesssim \Big(\|Q^{\frac{p-1}{2(p+1)}}|w|^p\|_{L^2} + \|Q^{\frac{3(p-1)}{2(p+1)}}\|_{L^2}\Big)\mathcal{M} \\ &\lesssim (s^{-1+\lambda}\mathcal{N} + (\mathcal{N} + \mathcal{M})^p)\mathcal{M} \lesssim s^{-1+\lambda}(\mathcal{N}^2 + \mathcal{M}^2) + \mathcal{N}^{p+1} + \mathcal{M}^{p+1}.\end{aligned}$$

Third, from (2.21), for $|x| \geq R$, $|\partial_s \mathcal{E}_J| \lesssim |x|^{-\frac{k(p+1)}{p-1}-2}$ and thus, since $1 \leq N \leq 4$, $\|\partial_s \mathcal{E}_J\|_{L^2(|x|\geq R)} \lesssim 1$. Now, from (2.19), for $|x| \leq R$,

$$|\partial_s \mathcal{E}_J| \lesssim V_0^{\frac{p+1}{2}+(1-J+\frac{1+J}{k})\frac{p-1}{2}} \lesssim V_0^{p+\frac{1}{k}\frac{p-1}{2}-J(1-\frac{1}{k})\frac{p-1}{2}},$$

and thus, following the proof of (3.28), we have $\|\partial_s \mathcal{E}_J\|_{L^2(|x|\leq R)} \lesssim s^{-1+\lambda}$. Thus,

$$|I_{14}| \lesssim \|\partial_s \mathcal{E}_J\|_{L^2}\mathcal{M} \lesssim s^{-1+\lambda}\mathcal{M}.$$

The above estimates prove (3.30) for $\mathcal{K}_0$.

We now prove (3.30) for $\ell \in \{1, \dots, N\}$. Differentiating (3.9) with respect to x_ℓ, setting $z_\ell = \partial_{x_\ell} w$, we have

$$\begin{aligned}(1-|\nabla\psi|^2)\partial_{ss}z_\ell - 2\nabla\psi\cdot\nabla\partial_s z_\ell - (\Delta\psi)\partial_s z_\ell - \Delta z_\ell = f_n'(V_J+w)z_\ell &+ (f_n'(V_J+w)-f_n'(V_J))\partial_{x_\ell}V_J \\ &+ \partial_{x_\ell}\mathcal{E}_J + 2(\nabla\psi\cdot\nabla\partial_{x_\ell}\psi)\partial_{ss}w \\ &+ 2\nabla\partial_{x_\ell}\psi\cdot\nabla\partial_s w + (\Delta\partial_{x_\ell}\psi)\partial_s w.\end{aligned} \tag{3.32}$$

Differentiating $\mathcal{K}_\ell = \int (1-|\nabla\psi|^2)(\partial_s z_\ell)^2 + |\nabla z_\ell|^2$, we find from (3.32) and integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d\mathcal{K}_\ell}{ds} &= \int \{f_n'(V_J+w)z_\ell + (f_n'(V_J+w)-f_n'(V_J))\partial_{x_\ell}V_J + \partial_{x_\ell}\mathcal{E}_J\}\partial_s z_\ell \\ &\quad + \int \{2(\nabla\psi\cdot\nabla\partial_{x_\ell}\psi)\partial_{ss}w + 2\nabla\partial_{x_\ell}\psi\cdot\nabla\partial_s w + (\Delta\partial_{x_\ell}\psi)\partial_s w\}\partial_s z_\ell \\ &= I_{15}+I_{16}+I_{17}+I_{18}.\end{aligned}$$

The term I_{15} is estimated exactly like I_{12}. Next, it follows from (2.10), (2.16), (2.20) and the properties of χ that

$$|\partial_{x_\ell} V_J| \lesssim V_0^{1+\frac{p-1}{2k}},$$

so that I_{16} is estimated like I_{13}.

Moreover, from (2.21), for $|x| \geq R$, $|\partial_{x_\ell}\mathcal{E}_J| \lesssim |x|^{-3}$ and thus, since $1 \leq N \leq 4$, $\|\partial_s \mathcal{E}_J\|_{L^2(|x|\geq R)} \lesssim 1$. Now, from (2.19), for $|x| \leq R$,

$$|\partial_{x_\ell}\mathcal{E}_J| \lesssim V_0^{\frac{p+1}{2}+(-J+\frac{2+J}{k})\frac{p-1}{2}} \lesssim V_0^{\frac{p-1}{2}+\frac{1}{k}\frac{p-1}{2}-J(1-\frac{1}{k})\frac{p-1}{2}},$$

and thus, following the proof of (3.28), we have $\|\partial_s \mathcal{E}_J\|_{L^2(|x|\leq R)} \lesssim s^{-1+\lambda}$. Thus we see that I_{17} is estimated like I_{14}.

Finally,

$$\left|2(\nabla\psi\cdot\nabla\partial_{x_\ell}\psi)\partial_{ss}w + 2\nabla\partial_{x_\ell}\psi\cdot\nabla\partial_s w + (\Delta\partial_{x_\ell}\psi)\partial_s w\right| \lesssim |\partial_{ss}w| + |\nabla\partial_s w| + |\partial_s w|,$$

so that $I_{18} \lesssim \mathcal{M}^2$. Therefore, estimate (3.30) holds for $\ell \in \{1, \dots, N\}$.

Step 4: Conclusion. Since $\mathcal{H}(S_n) = \mathcal{K}(S_n) = 0$, the following is well defined:

$$S_n^\star = \sup\{s \in [S_n, \delta] : \text{for all } s' \in [S_n, s],\ \mathcal{N}^2 + \mathcal{K} \leq \min(s^\lambda, \omega)\},$$

and by continuity, $S_n^\star \in (S_n, \delta]$. It follows from (3.26), (3.30), (3.15), and $\lambda \leq \frac{1}{2}$ that

$$\frac{d}{dt}(\mathcal{H}+\mathcal{K}) \leq s^{-1+\lambda}\Big(C\mathcal{N} + \frac{\lambda}{4}s^{-\lambda}\mathcal{N}^2 + C(\mathcal{N}^2+\mathcal{K}) + Cs^{-\lambda}(\mathcal{N}^2+\mathcal{K})^{\frac{p+1}{2}}\Big)$$

for some constant $C > 0$ independent of δ. By the definition of $S_n^\star$, we deduce that

$$\frac{d}{dt}(\mathcal{H}+\mathcal{K}) \leq \lambda s^{-1+\lambda}\Big(Cs^{\frac{1}{2}\lambda} + Cs^{\frac{p-1}{2}\lambda} + \frac{1}{4}\Big)$$

for some constant $C > 0$ independent of δ. We fix $0 < \delta_0 \leq \delta$ such that

$$C\delta_0^{\frac{1}{2}\lambda} + C\delta_0^{\frac{p-1}{2}\lambda} + \frac{1}{4} \leq \frac{1}{3}, \qquad \delta_0^\lambda \leq \omega.$$

This gives, for all $S_n \leq s \leq \min(S_n^\star, \delta_0)$, $\frac{d}{dt}(\mathcal{H}+\mathcal{K}) \leq \frac{\lambda}{3}s^{-1+\lambda}$.

By integration, using $\mathcal{H}(s_n) = \mathcal{K}(s_n) = 0$, we find for $S_n \le s \le \min(S_n^\star, \delta_0)$,

$$\mathcal{H}(s) + \mathcal{K}(s) \le \frac{1}{3}(s^\lambda - S_n^\lambda) \le \frac{1}{3}(s - S_n)^\lambda.$$

Thus, from (3.16), it holds, for $S_n \le s \le \min(S_n^\star, \delta_0)$,

$$\mathcal{N}^2(s) + \mathcal{K}(s) \le \frac{2}{3}(s - S_n)^\lambda.$$

It follows from (3.15) and the definition of $S_n^\star$ that $S_n^\star \ge \delta_0$ and so, for all $s \in [S_n, \delta_0]$,

$$\mathcal{M}(s) \lesssim (s - S_n)^{\frac{\lambda}{2}}.$$

This completes the proof of the proposition. □

Proof of Proposition 3.1. We set

$$Z_n(s) = V_J(S_n + s), \quad \eta_n(s, y) = w_n(S_n + s), \quad \mathcal{F}_n(s) = \mathcal{E}_J(S_n + s).$$

From Proposition 3.2, there exist $C > 0$, $n_0 > 0$ and $0 < \delta_0 < 1$ such that

$$\|\eta_n(s)\|_{H^2} + \|\partial_s \eta_n(s)\|_{H^1} + \|\partial_{ss}\eta_n(s)\|_{L^2} \le C s^{\frac{\lambda}{2}} \tag{3.33}$$

for all $n \ge n_0$ and $s \in [0, \delta_0]$. Moreover, from (3.9),

$$(1 - |\nabla\psi|^2)\partial_{ss}\eta_n - 2\nabla\psi \cdot \nabla\partial_s\eta_n - (\Delta\psi)\partial_s\eta_n - \Delta\eta_n = f_n(Z_n + \eta_n) - f_n(Z_n) + \mathcal{F}_n. \tag{3.34}$$

It follows from estimate (3.33) that there exist a subsequence of (η_n) (still denoted by (η_n)) and a map $\eta \in L^\infty((0, \delta_0), H^2(\mathbb{R}^N)) \cap W^{1,\infty}((0, \delta_0), H^1(\mathbb{R}^N)) \cap W^{2,\infty}((0, \delta_0), L^2(\mathbb{R}^N))$ such that

$$\begin{aligned}
\eta_n &\xrightarrow[n\to\infty]{} \eta && \text{in } L^\infty((0, \delta_0), H^2(\mathbb{R}^N)) \text{ weak}^\star, && (3.35)\\
\partial_s\eta_n &\xrightarrow[n\to\infty]{} \partial_s\eta && \text{in } L^\infty((0, \delta_0), H^1(\mathbb{R}^N)) \text{ weak}^\star, && (3.36)\\
\partial_{ss}\eta_n &\xrightarrow[n\to\infty]{} \partial_{ss}\eta && \text{in } L^\infty((0, \delta_0), L^2(\mathbb{R}^N)) \text{ weak}^\star, && (3.37)\\
\eta_n(s) &\xrightarrow[n\to\infty]{} \eta(s) && \text{weakly in } H^2(\mathbb{R}^N), \text{ for all } s \in [0, \delta_0], && (3.38)\\
\partial_s\eta_n(s) &\xrightarrow[n\to\infty]{} \partial_s\eta(s) && \text{weakly in } H^1(\mathbb{R}^N)), \text{ for all } s \in [0, \delta_0]. && (3.39)
\end{aligned}$$

It is then easy to pass to the limit in (3.34), and it follows that

$$(1 - |\nabla\psi|^2)\partial_{ss}\eta - 2\nabla\psi \cdot \nabla\partial_s\eta - (\Delta\psi)\partial_s\eta - \Delta\eta = f(V_J + \eta) - f_n(V_J) + \mathcal{E}_J$$

in $L^\infty((0, \delta_0), L^2(\mathbb{R}^N))$. Therefore, setting

$$v(s) = V_J(s) + \eta(s), \quad s \in (0, \delta_0),$$

it holds

$$v \in L^\infty_{\mathrm{loc}}((0, \delta_0), H^2(\mathbb{R}^N)) \cap W^{1,\infty}_{\mathrm{loc}}((0, \delta_0), H^1(\mathbb{R}^N)) \cap W^{2,\infty}_{\mathrm{loc}}((0, \delta_0), L^2(\mathbb{R}^N)) \tag{3.40}$$

and, using the definition of $\mathcal{E}_J$, we see that v is a solution of equation (2.2) in $L^\infty_{\mathrm{loc}}((0, \delta_0), L^2(\mathbb{R}^N))$. Estimate (3.5) follows by letting $n \to \infty$ in (3.8) and using (3.38) and (3.39). We now prove that v satisfies (3.4). By standard semigroup theory (see Section A.3) it suffices to prove that $|v|^{p-1}v \in C((0, \delta_0), H^1(\mathbb{R}^N))$. Since by (3.40) $v \in C((0, \delta_0), H^{2-\eta}(\mathbb{R}^N))$ for every $\eta > 0$, and $N \le 4$, we have by Sobolev's embeddings $v \in C((0, \delta_0), W^{1,q}(\mathbb{R}^N))$ for all $2 \le q < 4$ and $|v|^{p-1} \in C((0, \delta_0), L^r(\mathbb{R}^N))$ for $\max\{1, \frac{2}{p-1}\} \le r < \infty$. Choosing for instance $q = \frac{4(p+1)}{p+3}$ and $r = \frac{4(p+1)}{p-1}$ yields $|v|^{p-1}v \in C((0, \delta_0), H^1(\mathbb{R}^N))$.

Finally, we prove (3.6). We write

$$|\partial_s v| \ge |\partial_s V_0| - |\partial_s(V_J - V_0)| - |\partial_s(v - V_J)|.$$

On the other hand, $V_0^{1+\frac{p-1}{2}} \lesssim |\partial_s V_0|$, so that $V_0^{1+\frac{p-1}{2k}} \lesssim |\partial_s V_0|^{1-\frac{(p-1)(k-1)}{k(p+1)}}$. Therefore, given any $\eta > 0$, there exists a constant C_η such that

$$V_0^{1+\frac{p-1}{2k}} \le \eta|\partial_s V_0| + C_\eta. \tag{3.41}$$

Since $|\partial_s(V_j - V_0)| \lesssim V_0^{1+\frac{p-1}{2k}}$ by (2.18), we see that there exists a constant C such that

$$|\partial_s v|^2 \geq \frac{1}{2}|\partial_s V_0|^2 - C|\partial_s(v - V_J)|^2 - C. \tag{3.42}$$

Next, we write

$$|\nabla v| \leq |\nabla(v - V_J)| + |\nabla V_J| \leq |\nabla(v - V_J)| + |\nabla V_0| + \sum_{j=1}^{J} |\nabla v_j|.$$

It follows from (2.20) that $|\nabla V_J| \lesssim 1$ for $|x| \geq R$. For $|x| < R$, by (2.16) and $k \geq 2$, $|\nabla v_j| \lesssim V_0$ for $j \geq 1$; and

$$|\nabla V_0| \lesssim V_0^{1+\frac{p-1}{2k}}$$

by (2.10). Using again (3.41), we conclude that

$$|\nabla v|^2 \leq \frac{1}{4}|\partial_s V_0|^2 - C|\nabla(v - V_J)|^2 - C. \tag{3.43}$$

Since $|\partial_s(v - V_J)| + |\nabla(v - V_J)| \in L^\infty((0, \delta_0), H^1(\mathbb{R}^N))$ by (3.5), the lower estimate (3.6) follows from (3.42) and (3.43). □

4 Proof of Theorem 1.1

In this section, we use the following notation. We let $\{\mathbf{e}_k : k = 1, \dots, N\}$ be the canonical basis of $\mathbb{R}^N$. If $N \geq 2$, then for $x \in \mathbb{R}^N$, we denote $x = (x_1, x_2, \dots, x_N)$ and $\bar{x} = (x_2, \dots, x_N)$. We set $\bar{\Delta} u = \sum_{k=2}^N \partial_{x_k x_k} u$. If $N = 1$, we ignore $\bar{x}$ and $\bar{\Delta}$.

4.1 Cut-Off of the Local Hypersurface

Let φ be a function satisfying (1.3) (see statement of Theorem 1.1). Without loss of generality, by the invariance by rotation of equation (1.1), we assume that

$$\nabla\varphi(0) = \ell \mathbf{e}_1 \quad \text{where } 0 \leq \ell < 1.$$

(For dimension 1, the reduction is done by possibly changing $x \mapsto -x$.) For a positive real $r < 1$ small to be defined later, set

$$\widetilde{\varphi}(x) = (\varphi(x) - \ell x_1)\chi\Big(\frac{|x|}{r}\Big) + \ell x_1.$$

On the one hand, from this definition and the properties of χ, it holds

$$\widetilde{\varphi}(x) = \varphi(x) \text{ for } |x| < r, \quad \widetilde{\varphi}(x) = \ell x_1 \text{ for } |x| > 2r, \quad \nabla\widetilde{\varphi}(0) = \ell \mathbf{e}_1. \tag{4.1}$$

On the other hand, from $\varphi(0) = 0$ and $\nabla\varphi(0) = \ell\mathbf{e}_1$, there exists a constant $C > 1$ such that for $|x| < 1$, it holds $|\varphi(x) - \ell x_1| \leq C|x|^2$ and $|\nabla\varphi(x) - \ell\mathbf{e}_1| \leq C|x|$. In particular, since

$$\nabla\widetilde{\varphi}(x) = (\nabla\varphi(x) - \ell\mathbf{e}_1)\chi\Big(\frac{|x|}{r}\Big) + \ell\mathbf{e}_1 + \frac{1}{r}(\varphi(x) - \ell x_1)\chi'\Big(\frac{|x|}{r}\Big)\frac{x}{|x|},$$

it holds on $\mathbb{R}^N$,

$$|\nabla\widetilde{\varphi}(x) - \ell\mathbf{e}_1| \leq Cr.$$

We fix $r > 0$ small enough so that

$$\|\nabla\widetilde{\varphi} - \ell\mathbf{e}_1\|_{L^\infty} \leq (1 - \ell)\min\Big\{\frac{\lambda}{8}\frac{p-1}{p+1}, \frac{1}{2}\Big\}. \tag{4.2}$$

The first constraint on $\widetilde{\varphi}$ is related to assumption (3.3) in Proposition 3.2, and the second implies

$$\|\nabla\widetilde{\varphi}\|_{L^\infty} \leq \frac{\ell + 1}{2} < 1. \tag{4.3}$$

4.2 Construction of the Function ψ

We claim that for any $y \in \mathbb{R}^N$, there exists $X_1(y) \in \mathbb{R}$ such that

$$y_1 = \frac{X_1(y) - \ell\widetilde{\varphi}(X_1(y), \bar{y})}{(1-\ell^2)^{\frac{1}{2}}}. \tag{4.4}$$

(As observed before, we ignore $\bar{y}$ in dimension 1.) To prove the claim, we define

$$\Phi(x_1, \bar{y}) = \frac{x_1 - \ell\widetilde{\varphi}(x_1, \bar{y})}{(1-\ell^2)^{\frac{1}{2}}}, \tag{4.5}$$

and we compute, using (4.3),

$$\partial_{x_1}\Phi(x_1, \bar{y}) = \frac{1 - \ell\partial_{x_1}\widetilde{\varphi}(x_1, \bar{y})}{(1-\ell^2)^{\frac{1}{2}}} \ge \frac{1-\ell}{(1-\ell^2)^{\frac{1}{2}}} = \Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}} > 0 \tag{4.6}$$

and

$$\partial_{x_1}\Phi(x_1, \bar{y}) \le \frac{1+\ell}{(1-\ell^2)^{\frac{1}{2}}} \le \Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}}. \tag{4.7}$$

Thus, for fixed $\bar{y} \in \mathbb{R}^{N-1}$, the function $x_1 \in \mathbb{R} \mapsto \widetilde{\Phi}_{\bar{y}}(x_1) =: \Phi(x_1, \bar{y}) \in \mathbb{R}$ is increasing and surjective. It has an inverse function $\widetilde{\Phi}_{\bar{y}}^{-1}$ on $\mathbb{R}$, which is also (strictly) increasing, and we set $X_1(y_1, \bar{y}) = \widetilde{\Phi}_{\bar{y}}^{-1}(y_1)$ for $y_1 \in \mathbb{R}$. Setting $X_1(y) = X_1(y_1, \bar{y})$, we have proved the claim. Note that

$$X_1(\Phi(x_1, \bar{y}), \bar{y}) = \widetilde{\Phi}_{\bar{y}}^{-1}(\Phi(x_1, \bar{y})) = \widetilde{\Phi}_{\bar{y}}^{-1}(\widetilde{\Phi}_{\bar{y}}(x_1)) = x_1,$$

so that by (4.5)

$$x_1 = X_1\Big(\frac{x_1 - \ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}}, \bar{x}\Big) \tag{4.8}$$

for all $x \in \mathbb{R}^N$. Moreover, it follows from (4.6)–(4.7) that

$$\Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}} \le \frac{\partial X_1}{\partial y_1} \le \Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}} \tag{4.9}$$

on $\mathbb{R}^N$. Setting $X(y) = (X_1(y), \bar{y})$, it holds

$$y_1 = \frac{X_1(y) - \ell\widetilde{\varphi}(X(y))}{(1-\ell^2)^{\frac{1}{2}}}. \tag{4.10}$$

Moreover, using (4.9), we see that

$$|X(y)| \ge \max\{|X_1(y)|, |\bar{y}|\} \xrightarrow[|y|\to\infty]{} \infty. \tag{4.11}$$

For all $y \in \mathbb{R}^N$, we define the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(y) = \frac{\widetilde{\varphi}(X(y)) - \ell X_1(y)}{(1-\ell^2)^{\frac{1}{2}}}. \tag{4.12}$$

Equivalently, the functions ψ and $\widetilde{\varphi}$ are uniquely related by the following relation on $\mathbb{R}^N$:

$$\widetilde{\varphi}(x) = (1-\ell^2)^{\frac{1}{2}}\psi\Big(\frac{x_1 - \ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}}, \bar{x}\Big) + \ell x_1. \tag{4.13}$$

We check that ψ is of class $\mathcal{C}^{q_0}$, where q_0 is defined in (1.2), and satisfies assumptions (2.1) and (3.3).

First, since φ is of class $\mathcal{C}^{q_0}$ and χ is of class $\mathcal{C}^\infty$, it follows from their definitions that $\widetilde{\varphi}$ and then the functions X and ψ are of class $\mathcal{C}^{q_0}$ in $\mathbb{R}^N$. Since $\varphi(0) = \widetilde{\varphi}(0) = 0$, from (4.13), we also have $\psi(0) = 0$.

Second, from (4.1), it follows that $\widetilde{\varphi}(x) = \ell x_1$ for any $|x| > 2r$. From (4.11) and (4.12), we see that $\psi(y) = 0$ for $|y|$ large.

Last, we estimate $|\nabla\psi|$. From (4.13)

$$(1-\ell\partial_{x_1}\widetilde{\varphi}(x))\partial_{y_1}\psi\Big(\frac{x_1-\ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}},\bar{x}\Big)=\partial_{x_1}\widetilde{\varphi}(x)-\ell, \tag{4.14}$$

and for $j\neq 1$,

$$\partial_{y_j}\psi\Big(\frac{x_1-\ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}},\bar{x}\Big)=(1-\ell^2)^{-\frac{1}{2}}\partial_{x_j}\widetilde{\varphi}(x)\Big(1+\ell\partial_{y_1}\psi\Big(\frac{x_1-\ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}},\bar{x}\Big)\Big). \tag{4.15}$$

It follows from (4.3) that $|1-\ell\partial_{x_1}\widetilde{\varphi}(x)|\geq 1-\ell$, so that (4.14) and (4.2) yield

$$\|\partial_{y_1}\psi\|_{L^\infty}\leq\frac{1}{1-\ell}\|\partial_{x_1}\widetilde{\varphi}-\ell\|_{L^\infty}\leq\frac{\lambda}{8}\frac{p-1}{p+1}.$$

In particular, we see that $\|\partial_{y_1}\psi\|_{L^\infty}\leq 1$. Since $\|\partial_{x_j}\widetilde{\varphi}\|_{L^\infty}\leq(1-\ell)\frac{\lambda}{8}\frac{p-1}{p+1}$ by (4.2), we deduce from (4.15) that

$$\|\partial_{y_j}\psi\|_{L^\infty}\leq(1-\ell^2)^{-\frac{1}{2}}(1-\ell)\frac{\lambda}{8}\frac{p-1}{p+1}(1+\ell)=(1-\ell^2)^{\frac{1}{2}}\frac{\lambda}{8}\frac{p-1}{p+1}\leq\frac{\lambda}{8}\frac{p-1}{p+1}$$

so that (3.3) is proved.

4.3 Definition of an Appropriate Solution of the Transformed Equation

We assume (2.3), (2.4), (3.1), (3.2) and we consider the function ψ defined in (4.12)–(4.13). Note that ψ is of class $\mathcal{C}^{q_0}$ where q_0 is defined in (1.2), and satisfies assumptions (2.1) and (3.3). Let the function A be given by (2.5). We consider the solution $v\in C((0,\delta_0),H^2(\mathbb{R}^N))\cap C^1((0,\delta_0),H^1(\mathbb{R}^N))\cap C^2((0,\delta_0),L^2(\mathbb{R}^N))$ of (2.2) given by Proposition 3.1.

4.4 Returning to the Original Variable

Let

$$\tau_0=\Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}}\frac{\delta_0}{6},\qquad\varepsilon_0=\frac{1-\ell}{2+\ell}\tau_0. \tag{4.16}$$

Recall (see (4.1) and (4.3)) that $\widetilde{\varphi}(0)=0$ and $|\nabla\widetilde{\varphi}|\leq\frac{\ell+1}{2}$, so that $|\widetilde{\varphi}(x)|\leq\frac{\ell+1}{2}|x|$. Thus we see that

$$\tau_0+\inf_{|x|\leq\tau_0+\varepsilon_0}\widetilde{\varphi}(x)>0. \tag{4.17}$$

It follows that the space-time region

$$\mathcal{T}=\{(t,x)\in\mathbb{R}^{1+N}_+:0\leq t<\tau_0+\widetilde{\varphi}(x),\ |x|<\tau_0+\varepsilon_0-t\}$$

is an influence domain in the sense of Section 1.2. (See Figure 1.) Moreover, let $|x|\leq\frac{\varepsilon_0}{2}$. We have $\widetilde{\varphi}(x)\leq\frac{\varepsilon_0}{2}$. Therefore, if $0\leq t<\tau_0+\widetilde{\varphi}(x)$, then $t<\tau_0+\frac{\varepsilon_0}{2}$ so that $|x|<\tau_0+\varepsilon_0-t$. It follows that

$$|x|\leq\frac{\varepsilon_0}{2}\implies\max\{t>0:(t,x)\in\mathcal{T}\}=\tau_0+\widetilde{\varphi}(x). \tag{4.18}$$

Given $0\leq\ell<1$ and $\tau_0\in\mathbb{R}$, we define the Lorentz transform $\Lambda_{\ell,\tau_0}:\mathbb{R}^{1+N}\to\mathbb{R}^{1+N}$ by

$$\Lambda_{\ell,\tau_0}(t,x)=(s,y)=(s,y_1,\bar{y}),\qquad\text{where } s=\frac{t-\tau_0-\ell x_1}{(1-\ell^2)^{\frac{1}{2}}},\quad y_1=\frac{x_1-\ell(t-\tau_0)}{(1-\ell^2)^{\frac{1}{2}}},\quad\bar{y}=\bar{x}.$$

It is well known that Λ_{ℓ,τ_0} is a $\mathcal{C}^\infty$ diffeomorphism with Jacobian determinant $|\det J_{\Lambda_{\ell,\tau_0}}|=1$. We also define the transformation $\Lambda_\psi:\mathbb{R}^{1+N}\to\mathbb{R}^{1+N}$ by

$$\Lambda_\psi(t',x')=(s',y'),\qquad\text{where } s'=\psi(x')-t',\quad y'=x'.$$

Since ψ is of class $\mathcal{C}^{q_0}$ where q_0 is defined in (1.2) (see Section 4.2), it follows easily that Λ_ψ is a diffeomorphism of class $\mathcal{C}^{q_0}$. Moreover, $|\det J_{\Lambda_\psi}|=1$. We define the map $\Lambda:\mathbb{R}^{1+N}\to\mathbb{R}^{1+N}$ as the composition of the above two maps, i.e.

$$\Lambda=\Lambda_\psi\circ\Lambda_{\ell,\tau_0}.$$

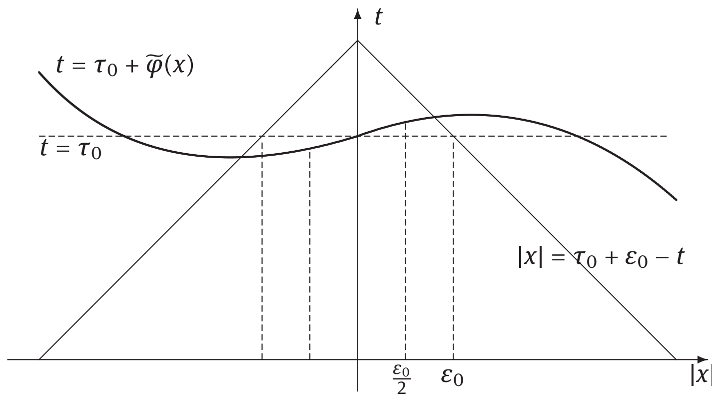

Figure 1: The set $\mathcal{T}$ is the part of the cone $|x| < \tau_0 + \varepsilon_0 - t$ below the surface $t = \tau_0 + \widetilde{\varphi}(x)$.

The map Λ has the expression

$$\Lambda(t,x) = (s,y) = (s,y_1,\bar{y}), \qquad \text{where } s = \psi(y) - \frac{t-\tau_0-\ell x_1}{(1-\ell^2)^{\frac{1}{2}}}, \quad y_1 = \frac{x_1-\ell(t-\tau_0)}{(1-\ell^2)^{\frac{1}{2}}}, \quad \bar{y}=\bar{x} \tag{4.19}$$

and it follows that $\Lambda : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}$ is a diffeomorphism of class $\mathcal{C}^{q_0}$ and that $|\det J_\Lambda| = 1$.

We prove that

$$s > 0 \iff t < \tau_0 + \widetilde{\varphi}(x) \tag{4.20}$$

and that

$$\Lambda(\mathcal{T}) \subset \Big(0, \frac{\delta_0}{2}\Big) \times \mathbb{R}^N. \tag{4.21}$$

In the case where $\ell = 0$, by (4.4), we have $X(y) = y$ and thus by (4.12), $\psi(y) = \widetilde{\varphi}(y)$. Thus in this case,

$$\Lambda(t,x) = (\widetilde{\varphi}(x) - t + \tau_0, x). \tag{4.22}$$

Property (4.20) follows. Moreover,

$$s \le \widetilde{\varphi}(x) + \tau_0 \le |x| + \tau_0 \le 2\tau_0 + \varepsilon_0 \le 3\tau_0 < \frac{\delta_0}{2}$$

by (4.16). Thus (4.21) is proved in this case.

In the case where $\ell \ne 0$, we observe that from (4.12),

$$s = \frac{\widetilde{\varphi}(X(y)) - \ell X_1(y)}{(1-\ell^2)^{\frac{1}{2}}} - \frac{t-\tau_0-\ell x_1}{(1-\ell^2)^{\frac{1}{2}}}.$$

Using (4.10), we replace $\widetilde{\varphi}(X(y)) = \frac{1}{\ell}(X_1(y) - (1-\ell^2)^{\frac{1}{2}} y_1)$ so that

$$\begin{aligned} \ell(1-\ell^2)^{-\frac{1}{2}} s &= \frac{\ell\widetilde{\varphi}(X(y)) - \ell^2 X_1(y)}{(1-\ell^2)} - \frac{\ell(t-\tau_0) - \ell^2 x_1}{(1-\ell^2)} \\ &= X_1(y) - \frac{y_1}{(1-\ell^2)^{\frac{1}{2}}} + \frac{x_1 - \ell(t-\tau_0)}{(1-\ell^2)} - x_1 \\ &= X_1(y) - x_1. \end{aligned}$$

Recall that by (4.8), we have

$$x_1 = X_1\Big(\frac{x_1 - \ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}}, \bar{x}\Big),$$

which means that

$$\ell(1-\ell^2)^{-\frac{1}{2}} s = X_1\Big(\frac{x_1-\ell(t-\tau_0)}{(1-\ell^2)^{\frac{1}{2}}}, \bar{x}\Big) - X_1\Big(\frac{x_1-\ell\widetilde{\varphi}(x)}{(1-\ell^2)^{\frac{1}{2}}}, \bar{x}\Big), \tag{4.23}$$

hence, using (4.9),

$$-\Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}} \le \frac{\partial s}{\partial t} \le -\Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}} \tag{4.24}$$

on $\mathbb{R}^{1+N}$. Thus we see that $s > 0$ is equivalent to $t < \tau_0 + \widetilde{\varphi}(x)$, i.e. (4.20) holds. Moreover, by (4.24), we have on $\mathcal{T}$

$$s \le \Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}} |t - \tau_0 - \widetilde{\varphi}(x)| = \Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}} (\tau_0 + \widetilde{\varphi}(x) - t).$$

Using (4.16), we see that $\tau_0 + \widetilde{\varphi}(x) - t < \widetilde{\varphi}(x) + \tau_0 \le 3\tau_0$, so that $s < \frac{\delta_0}{2}$. Thus (4.21) is proved in all cases.

We now set

$$u(t,x) = v(\Lambda(t,x)), \quad (t,x) \in \mathcal{T}. \tag{4.25}$$

We refer to [18, Exercise 10.7.c] for a similar use of the Lorentz transform. Note that by (4.21), u is well defined.

Let ω be an open subset of $\mathbb{R}^N$ and let $0 \le a < b$. Suppose that $[a,b] \times \overline{\omega} \subset \mathcal{T}$. We claim that

$$u \in H^2((a,b)\times\omega), \tag{4.26}$$

$$u \in L^q((a,b)\times\omega) \quad \text{for all } 1 \le q < \infty, \tag{4.27}$$

$$\partial_{tt} u = \Delta u + |u|^{p-1} u \quad \text{in } L^2((a,b)\times\omega). \tag{4.28}$$

Since $[a,b]\times\overline{\omega}$ is a compact subset of $\mathbb{R}^{1+N}$, it follows that $\Lambda([a,b]\times\overline{\omega})$ is a compact subset of $\mathbb{R}^{1+N}$. Moreover, it follows from (4.20)–(4.21) that $\Lambda([a,b]\times\overline{\omega})$ is a compact subset of $(0,\delta_0)\times\mathbb{R}^N$. Let $1 \le q < \infty$. Since $v \in C((0,\delta_0), H^2(\mathbb{R}^N))$ and $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for every $r < \infty$ (because $N \le 4$), we have $v \in L^q(\Lambda((a,b)\times\omega))$; and so (4.27) follows from (4.25) and the change of variable formula. Next, let $\theta \in C_c^\infty((0,\delta_0)\times\mathbb{R}^N)$ such that $\theta(x) \equiv 1$ on $\Lambda([a,b]\times\overline{\omega})$. Thus we may replace v by θv in formula (4.25), this does not change the values of u on $(a,b)\times\omega$. Since $\theta v \in H^2((0,\delta_0)\times\mathbb{R}^N)$, we can approximate θv in $H^2((0,\delta_0)\times\mathbb{R}^N)$ by a sequence $(w_n)_{n\ge1} \subset C_c^\infty((0,\delta_0)\times\mathbb{R}^N)$ supported in a fixed compact of $(0,\delta_0)\times\mathbb{R}^N$. Setting $u_n = w_n \circ \Lambda$, we have

$$\begin{aligned}\int_{(a,b)\times\omega} |u_n - u|^2 &= \int_{\Lambda((a,b)\times\omega)} |w_n - v|^2 |\det J_\Lambda|^{-1}\\ &= \int_{\Lambda((a,b)\times\omega)} |w_n - \theta v|^2 \xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{4.29}$$

Next, it follows from (4.25) that

$$\begin{aligned}(1-\ell^2)\partial_{tt} u_n = {}& [\ell^2(\partial_{y_1}\psi(\cdot))^2 + 2\ell\partial_{y_1}\psi(\cdot) + 1]\partial_{ss} w_n(\cdot) + \ell^2 \partial_{y_1y_1}\psi(\cdot)\partial_s w_n(\cdot)\\ &+ 2\ell(\ell\partial_{y_1}\psi(\cdot) + 1)\partial_{sy_1} w_n(\cdot) + \ell^2 \partial_{y_1y_1} w_n(\cdot)\end{aligned}$$

and

$$\begin{aligned}(1-\ell^2)\Delta u_n = {}& \Big[(\partial_{y_1}\psi(\cdot))^2 + (1-\ell^2)\sum_{k\ne1}(\partial_{y_k}\psi(\cdot))^2 + 2\ell\partial_{y_1}\psi(\cdot) + \ell^2\Big]\partial_{ss} w_n(\cdot) + \partial_{y_1y_1}\psi(\cdot)\partial_s w_n(\cdot)\\ &+ 2(\partial_{y_1}\psi(\cdot) + \ell)\partial_{sy_1} w_n(\cdot) + 2(1-\ell^2)\sum_{k\ne1}\partial_{y_k}\psi(\cdot)\partial_{sy_k} w_n(\cdot)\\ &+ \partial_{y_1y_1} w_n(\cdot) + (1-\ell^2)\bar{\Delta} w_n(\cdot) + (1-\ell^2)(\bar{\Delta}\psi(\cdot))\partial_s w_n(\cdot),\end{aligned}$$

where the argument of ψ is y and the argument of w_n is Λ.

Similar formulas hold for all first and second space-time derivatives of u_n, so arguing as in (4.29) we conclude that u_n is a Cauchy sequence in $H^2((a,b)\times\omega)$, from which (4.26) follows. In addition, the above two formulas imply that

$$\partial_{tt} u_n - \Delta u_n = [1 - |\nabla\psi(\cdot)|^2]\partial_{ss} w_n(\cdot) - 2\nabla\psi(\cdot)\cdot\nabla\partial_s w_n(\cdot) - \Delta\psi(\cdot)\partial_s w_n(\cdot) - \Delta w_n(\cdot).$$

Since $u_n \to u$ in $H^2((a,b)\times\omega)$ and $w_n \to \theta v$ in $H^2((0,\delta_0)\times\mathbb{R}^N)$, we may pass to the limit in the above equation. Since $\theta v = v$ in $\Lambda((a,b)\times\omega)$, we obtain using (2.2)

$$\begin{aligned}\partial_{tt} u - \Delta u &= [1 - |\nabla\psi(\cdot)|^2]\partial_{ss} v(\cdot) - 2\nabla\psi(\cdot)\cdot\nabla\partial_s v(\cdot) - \Delta\psi(\cdot)\partial_s v(\cdot) - \Delta v(\cdot)\\ &= (|v|^{p-1}v)(\cdot) = |u|^{p-1}u\end{aligned}$$

in $L^2((a,b)\times\omega)$. This proves (4.28).

Set

$$\widetilde{\rho} = \tau_0 + \frac{\varepsilon_0}{2}$$

and

$$\widetilde{\tau} = \min\Big\{\frac{\varepsilon_0}{2}, \tau_0 + \inf_{|x|\le\tau_0+\varepsilon_0}\{\widetilde{\varphi}(x)\}\Big\}$$

so that $\widetilde{\tau} > 0$ by (4.17). We see that $(0,\widetilde{\tau}) \times B_{\widetilde{\rho}} \subset \mathcal{T}$ so that $u \in H^2((0,\widetilde{\tau}) \times B_{\widetilde{\rho}}) \cap L^q((0,\widetilde{\tau}) \times B_{\widetilde{\rho}})$ for all $q < \infty$. In particular, $u \in C([0,\widetilde{\tau}], H^1(B_{\widetilde{\rho}})) \cap C^1([0,\widetilde{\tau}], L^2(B_{\widetilde{\rho}}))$, so that $u(0) \in H^1(B_{\widetilde{\rho}})$ and $\partial_t u(0) \in L^2(B_{\widetilde{\rho}})$ are well defined.

4.5 Choice of a Solution of the Nonlinear Wave Equation

We apply Section 1.2 to extend u, which is a solution of (1.1) on $\mathcal{T}$, to a solution of (1.1) on a maximal domain of influence that contains $\mathcal{T}$. For this, we consider any pair $(\widetilde{u}_0, \widetilde{u}_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that $\widetilde{u}_0$ and $\widetilde{u}_1$ coincide with $u(0)$ and $\partial_t u(0)$, respectively, on $B_{\widetilde{\rho}}$. The initial data $(\widetilde{u}_0, \widetilde{u}_1)$ give rise to a solution $\widetilde{u}$ of (1.1) defined on the maximal influence domain $\Omega_{\max}(\widetilde{u}_0, \widetilde{u}_1)$ in the sense of Section 1.2. We claim that this maximal influence domain contains

$$\widetilde{\mathcal{T}} = \mathcal{T} \cap \{(t,x) \in [0,\widetilde{\rho}) \times \mathbb{R}^N : |x| < \widetilde{\rho} - t\}$$

and that $\widetilde{u}$ coincides with u on $\widetilde{\mathcal{T}}$. Indeed, let $(t,x) \in \widetilde{\mathcal{T}}$ and consider the corresponding open backward cone $C(t,x)$. The cone $C(t,x)$ is an influence domain, and it follows easily, using Proposition B.2 and (4.27), that u is a solution of (1.1) in $C(t,x)$ with initial data (u_0, u_1), so that $C(t,x) \subset \Omega_{\max}(\widetilde{u}_0, \widetilde{u}_1)$. Since $(t,x) \in \widetilde{\mathcal{T}}$ is arbitrary, this proves the claim. From now on, we denote by u this solution.

4.6 Blowup on the Local Hypersurface and End of the Proof

We show blowup on the local hypersurface by proving (1.4). For this, we further restrict the size of the hypersurface. Arguing as in the proof of (4.18), we see that

$$|x| \le \frac{\varepsilon_0}{4} \implies \max\{t > 0 : (t,x) \in \widetilde{\mathcal{T}}\} = \tau_0 + \widetilde{\varphi}(x).$$

Thus we see that if $|x_0| \le \frac{\varepsilon_0}{4}$, then the open backward cone $C(\tau_0 + \widetilde{\varphi}(x_0), x_0)$ is a subset of $\widetilde{\mathcal{T}}$.

We fix $\ell < \sigma \le 1$ and $|x_0| \le \frac{\varepsilon_0}{4}$, and we prove (1.4). We use the geometric property that the image by the map Λ of a cone of slope σ contains at least a small cone (estimate (4.31)), and the lower estimate (3.6) for v on this small cone.

Let $s_0 \ge 0$ and $y_0 \in \mathbb{R}^N$ be given by $\Lambda(\tau_0 + \widetilde{\varphi}(x_0), x_0) = (s_0, y_0)$. We first note that $s_0 = 0$ by (4.12) and (4.19). Moreover, it follows from (4.19), (4.3) and (4.16) that

$$|y_0| \le \frac{\varepsilon_0}{4}\Big(1 + \frac{2}{(1-\ell^2)^{\frac{1}{2}}}\Big) \le 1. \tag{4.30}$$

Given $0 \le t < \tau_0 + \widetilde{\varphi}(x_0)$, we set

$$K(t) = \{(t', x) \in \mathbb{R}^{1+N};\ t < t' < \tau_0 + \widetilde{\varphi}(x_0), |x - x_0| < \sigma(\tau_0 + \widetilde{\varphi}(x_0) - t')\}$$

and, given $s > 0$ and $\sigma' > 0$ we set

$$L(s,\sigma') = \{(s', y) \in \mathbb{R}^{1+N};\ 0 < s' < s, |y - y_0| < \sigma' s'\}.$$

We claim that there exist $\sigma' > 0$ and $\eta > 0$ such that

$$L(s(t), \sigma') \subset \Lambda(K(t)), \tag{4.31}$$

where

$$s(t) = \eta(\tau_0 + \widetilde{\varphi}(x_0) - t). \tag{4.32}$$

Assuming (4.31)–(4.32), we conclude the proof of (1.4). Given $(t,x)\in\widetilde{\mathcal{T}}$, it follows from (4.25) and (4.19) that

$$\partial_t u(t,x)=\frac{-1}{(1-\ell^2)^{\frac{1}{2}}}[\partial_s v(\Lambda(t,x))+\ell\partial_{y_1}v(\Lambda(t,x))]$$

so that, using $2\ell xy\le \ell^2x^2+y^2$,

$$|\partial_t u(t,x)|^2\ge|\partial_s v(\Lambda(t,x))|^2-|\partial_{y_1}v(\Lambda(t,x))|^2.$$

Therefore,

$$\int_{K(t)}|\partial_t u|^2\ge\int_{\Lambda(K(t))}(|\partial_s v|^2-|\partial_{y_1}v|^2)$$

Applying (3.6) and (4.31), we deduce that

$$\int_{K(t)}|\partial_t u|^2\ge\frac14\int_{L(s(t),\sigma')}|\partial_s V_0|^2-C|\Lambda(K(t))|-\int_{\Lambda(K(t))}g^2. \tag{4.33}$$

It follows from (4.32), (2.14) and (4.30) that

$$\liminf_{t\uparrow\tau_0+\widetilde{\varphi}(x_0)}\frac{1}{\tau_0+\widetilde{\varphi}(x_0)-t}\int_{L(s(t),\sigma')}|\partial_s V_0|^2\ge\liminf_{s\downarrow 0}\frac1s\int_{L(s,\sigma')}|\partial_s V_0|^2>0. \tag{4.34}$$

Furthermore,

$$|\Lambda(K(t))|\lesssim|K(t)|\lesssim(\tau_0+\widetilde{\varphi}(x_0)-t)^{1+N}. \tag{4.35}$$

Next, $H^1(\mathbb{R}^N)\hookrightarrow L^{\frac{2(N+2)}{N}}(\mathbb{R}^N)$, so that $g^2\in L^{\frac{N+2}{N}}((0,\delta_0)\times\mathbb{R}^N)$; and so by (4.35),

$$\int_{\Lambda(K(t))}g^2\lesssim|\Lambda(K(t))|^{\frac{2}{N+2}}\lesssim(\tau_0+\widetilde{\varphi}(x_0)-t)^{1+\frac{N}{N+2}}. \tag{4.36}$$

Estimate (1.4) follows from (4.33)–(4.36).

It remains to prove the claim (4.31)–(4.32). Let $(s',y)\in\mathbb{R}^{1+N}_+$ and $(t',x)\in\mathbb{R}^{1+N}$ such that $(s',y)=\Lambda(t',x)$. In particular, $t'\le\tau_0+\widetilde{\varphi}(x)$ by (4.20). We prove that

$$s'\le\Big(\frac{1+\ell}{1-\ell}\Big)^{\frac12}(\tau_0+\widetilde{\varphi}(x_0)-t'+|x-x_0|). \tag{4.37}$$

In the case $\ell=0$, this follows from (4.22) and the inequality $|\widetilde{\varphi}(x)-\widetilde{\varphi}(x_0)|\le|x-x_0|$ (see (4.3)). In the case $\ell\ne0$, then by (4.19) and (4.23),

$$\ell(1-\ell^2)^{-\frac12}s'=X_1(y_1,\bar y)-X_1\Big(y_1-\frac{\ell(\tau_0+\widetilde{\varphi}(x)-t')}{(1-\ell^2)^{\frac12}},\bar y\Big).$$

Using the right-hand side inequality in (4.9), and then (4.3), we deduce

$$s'\le\Big(\frac{1+\ell}{1-\ell}\Big)^{\frac12}(\tau_0+\widetilde{\varphi}(x)-t')$$

and (4.37) by using again (4.3).

Next we claim that

$$|x-x_0|\le|y-y_0|+\ell(\tau_0+\widetilde{\varphi}(x_0)-t'). \tag{4.38}$$

Indeed, by (4.19) for (t',x) and for $(\tau_0+\widetilde{\varphi}(x_0),x_0)$,

$$\begin{aligned} y_1-(y_0)_1&=\frac{x_1-(x_0)_1}{(1-\ell^2)^{\frac12}}+\frac{\ell(\tau_0+\widetilde{\varphi}(x_0)-t')}{(1-\ell^2)^{\frac12}},\\ \bar y-\bar y_0&=\bar x-\bar x_0\end{aligned}$$

so that

$$\begin{aligned}|x_1-(x_0)_1|&\le|y_1-(y_0)_1|+\ell(\tau_0+\widetilde{\varphi}(x_0)-t'),\\ |\bar x-\bar x_0|&=|\bar y-\bar y_0|.\end{aligned}$$

Estimate (4.38) follows by using the triangle inequality $\sqrt{(a+b)^2+c^2} \le \sqrt{a^2+c^2} + |b|$. Assuming now $(s', y) \in L(s, \sigma')$ for some $s > 0$ and $\sigma' > 0$, we deduce from (4.38) that

$$|x - x_0| \le \sigma' s' + \ell(\tau_0 + \widetilde{\varphi}(x_0) - t').$$

Estimating s' by (4.37), we obtain

$$\Big(1 - \sigma'\Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}}\Big)|x - x_0| \le \Big(\ell + \sigma'\Big(\frac{1+\ell}{1-\ell}\Big)^{\frac{1}{2}}\Big)(\tau_0 + \widetilde{\varphi}(x_0) - t').$$

Since $\sigma > \ell$, we see that if $\sigma' > 0$ and $\delta > 0$ are sufficiently small, then

$$|x - x_0| \le (\sigma - \delta)(\tau_0 + \widetilde{\varphi}(x_0) - t'). \tag{4.39}$$

It now remains to prove that if $s' \le \eta(\tau_0 + \widetilde{\varphi}(x_0) - t)$ for some sufficiently small $\eta > 0$, then $t' \ge t$. By (4.24), and then (4.3), we deduce

$$s' \ge \Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}}(\tau_0 + \widetilde{\varphi}(x) - t') \ge \Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}}(\tau_0 + \widetilde{\varphi}(x_0) - t' - |x - x_0|).$$

Using (4.39), we obtain

$$s' \ge (1 - \sigma + \delta)\Big(\frac{1-\ell}{1+\ell}\Big)^{\frac{1}{2}}(\tau_0 + \widetilde{\varphi}(x_0) - t'),$$

which proves the claim for $\eta = (1 - \sigma + \delta)(\frac{1-\ell}{1+\ell})^{\frac{1}{2}}$.

Finally, we prove that the hypersurface $\{(t, x) \in \mathbb{R}_+^{1+N} : |x_0| < \frac{\varepsilon_0}{4}, t = \tau_0 + \widetilde{\varphi}(x_0)\}$ is contained in the upper boundary of the maximal influence domain $\Omega_{\max}$ of the solution u. Indeed, otherwise there would exist $|x_0| < \frac{\varepsilon_0}{4}$ and $t > \tau_0 + \widetilde{\varphi}(x_0)$ such that $C(t, x_0) \subset \Omega_{\max}$ with the notation (1.5). In particular,

$$\partial_t u \in C\Big([0, \tau_0 + \widetilde{\varphi}(x_0)), L^2\Big(\Big\{|x - x_0| < \frac{t - \tau_0 - \widetilde{\varphi}(x_0)}{2}\Big\}\Big)\Big).$$

This is absurd, since by (1.4), given $\ell < \sigma \le 1$, there exist a sequence $t_n \uparrow \tau_0 + \widetilde{\varphi}(x_0)$ and $\delta > 0$ such that

$$\int\limits_{\{|x - x_0| < \sigma(\tau_0 + \widetilde{\varphi}(x_0) - t_n)\}} |\partial_t u(t_n)|^2 \ge \delta.$$

This completes the proof of the theorem, where τ_0 and ε_0 are given by (4.16), and $\varepsilon = \min\{\frac{\varepsilon_0}{4}, r\}$ with r defined in Section 1.1 (recall that $\varphi = \widetilde{\varphi}$ on $\{|x| < r\}$).

A The Wave Equation (2.2)

Let $\psi \in \mathcal{C}^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfy $\|\nabla\psi\|_{L^\infty} < 1$. It follows in particular that $(1 - |\nabla\psi|^2)^{-1} \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$.

A.1 The Associated Semigroup

Let X be the Hilbert space $H^1 \times L^2$, equipped with the (equivalent) scalar product

$$\langle (a, b), (\tilde{a}, \tilde{b}) \rangle_X = \int (\nabla a \cdot \nabla \tilde{a} + a\tilde{a}) + \int b\tilde{b}(1 - |\nabla\psi|^2),$$

and consider the linear operator $\mathcal{A}$ on X defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \frac{\Delta - 1}{1 - |\nabla\psi|^2} & \frac{2\nabla\psi\cdot\nabla + \Delta\psi}{1 - |\nabla\psi|^2} \end{pmatrix},$$

with domain $D(\mathcal{A}) = H^2 \times H^1$. We compute

$$\langle \mathcal{A}(a, b), (a, b) \rangle_X = \int (\nabla a \cdot \nabla b + ab) + \int (\Delta a - a)b + (2\nabla\psi \cdot \nabla b + (\Delta\psi)b)b = 0,$$

which proves that $\mathcal{A}$ is dissipative in X. Moreover, for any $(c, d) \in X$, there exist $(a, b) \in D(\mathcal{A})$ such that $(a, b) - \mathcal{A}(a, b) = (c, d)$. Indeed, this system reduces to

$$\begin{cases} b = a - c, \\ 2a - \Delta a - 2\nabla\psi \cdot \nabla a - (\Delta\psi)a = -2\nabla\psi \cdot \nabla c - (\Delta\psi)c + c + (1 - |\nabla\psi|^2)d. \end{cases}$$

It is easy to solve the second equation by the Lax–Milgram theorem, and we obtain a solution $a \in H^1(\mathbb{R}^N)$. Since, by the equation, $\Delta a \in L^2(\mathbb{R}^N)$, we see that $a \in H^2(\mathbb{R}^N)$. The first equation then yields $b \in H^1(\mathbb{R}^N)$. In particular, $\mathcal{A}$ is maximal dissipative, hence is the generator of a C_0 semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on X. (See, e.g., [27, Chapter 1, Theorem 4.3, p. 14].)

A.2 The Nonlinear Equation

Using the notation $U = \binom{v}{\partial_s v}$, we rewrite equation (2.2) as

$$\partial_s U = \mathcal{A}U + \mathcal{F}(U), \tag{A.1}$$

where

$$\mathcal{F}\begin{pmatrix} a \\ b \end{pmatrix} = (1 - |\nabla\psi|^2)^{-1} \begin{pmatrix} 0 \\ f(a) + a \end{pmatrix}. \tag{A.2}$$

A.3 Regularity

Suppose $T > 0$ and $U \in L^\infty((0, T), D(\mathcal{A})) \cap W^{1,\infty}((0, T), X)$ is such that $\mathcal{F}(U) \in L^\infty((0, T), X)$ and U satisfies equation (A.1) for a.a. $0 < t < T$. If $\mathcal{F}(U) \in C((0, T), D(\mathcal{A}))$, then $U \in C((0, T), D(\mathcal{A})) \cap C^1((0, T), X)$. Indeed, U is weakly continuous $(0, T) \to D(\mathcal{A})$. In particular, $U(t) \in D(\mathcal{A})$ for all $0 < t < T$ and the result follows easily, see, e.g., [27, Chapter 4, Corollary 2.6, p. 108].

A.4 The Case of Equation (3.7)

Equation (3.7) is equation (A.1), where f is replaced by f_n in (A.2). Since $f_n(0) = 0$ and f_n is globally Lipschitz $\mathbb{R} \to \mathbb{R}$, we see that the map $u \mapsto f_n(u)$ is globally Lipschitz $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. In particular, $\mathcal{F} : X \to X$ is globally Lipschitz, and the existence and uniqueness of a global, mild solution $U \in C([0, \infty), X)$ of (A.1) with the initial condition $U(0) = U_0 \in X$ is a direct consequence of standard semigroup theory. (See, e.g., [27, Chapter 6, Theorem 1.2, p. 184].) Moreover, since f_n is globally Lipschitz and C^1, it follows easily that the map $u \mapsto f_n(u)$ is continuous $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$. Therefore $\mathcal{F}$ is continuous $X \to D(\mathcal{A})$, so that $\mathcal{F}(U) \in C([0, \infty), D(\mathcal{A}))$. It follows, again by the semigroup theory, that if the initial value is in $D(\mathcal{A})$, then $U \in C([0, \infty), D(\mathcal{A})) \cap C^1([0, \infty), X)$ is a solution of (A.1). (See, e.g., [27, Chapter 4, Corollary 2.6, p. 108].)

B Uniqueness on Light Cones

We state and prove a uniqueness property for solutions of the nonlinear wave equation on light cones (Proposition B.2), for which we could not find a reference. We first recall in the following remark the relevant results concerning the local well-posedness of the Cauchy problem.

Remark B.1 (Local Well-Posedness). Let $N \geq 1$, let p such that $1 < p \leq \frac{N+2}{N-2}$ ($1 < p < \infty$ if $N = 1, 2$) and let $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. We summarize some results on the existence of $T > 0$ and a local solution

$$u \in C([0, T], H^1(\mathbb{R}^N)) \cap C^1([0, T], L^2(\mathbb{R}^N)) \tag{B.1}$$

of the wave equation

$$\begin{cases} \partial_{tt}u - \Delta u = |u|^{p-1}u, \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \tag{B.2}$$

We also discuss the property

$$u \in L^{\frac{2(N+1)}{N-2}}((0,T)\times\mathbb{R}^N) \tag{B.3}$$

in the case $N \geq 3$.

(i) Case $N = 1, 2$. There exist $T > 0$ and a unique solution u of (B.2) in the class (B.1). See, e.g., [5, Theorem 6.2.2].

(ii) Case $N \geq 3$, $p < \frac{N+2}{N-2}$. There exist $T > 0$ and a unique solution u of (B.2) in the class (B.1), and this solution satisfies (B.3) by possibly choosing T smaller. Indeed, existence follows from [12, Proposition 2.3] and uniqueness from [11, Proposition 3.1]. Moreover, applying [11, Lemma 3.3] with $\rho = \frac{N}{2(N-1)}$, $r = \frac{2(N^2-1)}{N^2-2N+3}$ and $q = \frac{2(N+1)}{N-1}$, we see that $u \in L^q((0,T), \dot{B}^{\rho}_{r,2}(\mathbb{R}^N))$, hence (B.3) by Sobolev's embedding.

(iii) Case $N = 3$, $p = 5$. There exist $T > 0$ and a solution u of (B.2) in the class (B.1)–(B.3). See, e.g., [15, Theorem 2.7]. Moreover, solutions of (B.2) in the class (B.1)–(B.3) are unique. This last property is not explicitly stated in [15], but it easily follows from the proof. (It also follows from Proposition B.2.)

(iv) Case $N \geq 4$, $p = \frac{N+2}{N-2}$. There exist $T > 0$ and a unique solution u of (B.2) in the class (B.1), and this solution satisfies property (B.3) by possibly choosing T smaller. Indeed, existence is established in [10] (see also [15, Theorem 2.7] for the case $N = 4, 5$ and [2, Theorem 3.3] for the case $N \geq 6$). Uniqueness is proved in [28, Theorem 2] for $N = 4$, in [28, Theorem 3] for $N = 5$ and in [2, Theorem 3.4] for $N \geq 6$. Property (B.3) follows from [15, Theorem 2.7] in the case $N = 4, 5$. In the case $N \geq 6$, it follows from [2, Theorem 3.3] that $u \in L^q((0,T), \dot{B}^{\rho}_{r,2}(\mathbb{R}^N))$ with $\rho = \frac{N}{2(N-1)}$, $r = \frac{2(N^2-1)}{N^2-2N+3}$ and $q = \frac{2(N+1)}{N-1}$, hence (B.3) by Sobolev's embedding.

Proposition B.2 (Uniqueness on Light Cones). *Let $N \geq 1$ and let p satisfy $1 < p \leq \frac{N+2}{N-2}$ ($1 < p < \infty$ if $N = 1, 2$). Let $R > 0$, $0 < \tau < R$, and let B_R be the open ball of center 0 and radius R in $\mathbb{R}^N$. Let*

$$u, v \in C([0,\tau], H^1(B_R)) \cap C^1([0,\tau], L^2(B_R)) \cap C^2([0,\tau], H^{-1}(B_R)))$$

be two solutions of the wave equation $\partial_{tt}u = \Delta u + |u|^{p-1}u$ in $H^{-1}(B_R)$) for $\leq t \leq \tau$. If $N \geq 3$ and $p > \frac{N}{N-2}$, suppose in addition that $u, v \in L^{\frac{2(N+1)}{N-2}}((0,T)\times B_R)$. If $u(0) = v(0)$ and $\partial_t u(0) = \partial_t v(0)$, then

$$u = v \quad \text{on } \{(t,x) \in (0,\tau)\times B_R : |x| < R - t\}.$$

The proof of Proposition B.2 relies on the following local estimates.

Lemma B.3. *Let $R > 0$, $0 < \tau < R$, $h \in C([0,\tau], L^q(B_R))$ for some $q \geq 1$, $q \geq \frac{2N}{N+2}$ (so that $h \in C([0,\tau], H^{-1}(B_R))$), and let*

$$z \in C([0,\tau], L^2(B_R)) \cap C^1([0,\tau], H^{-1}(B_R)) \cap C^2([0,\tau], H^{-2}(B_R))$$

satisfy $\partial_{tt}z = \Delta z + h$ in $H^{-2}(B_R)$ for all $0 \leq t \leq \tau$ and if $z(0) = \partial_t z(0) = 0$. If $h_{|E(0,R,\tau)} \in L^2(E(0,R,\tau))$ with the notation (1.6), then $z(t) \in H^1(B_{R-t})$ for all $0 < t < \tau$, and

$$\|z(t)\|_{H^1(B_{R-t})} \leq Ce^{Ct}\|h\|_{L^2(E(0,R,t))} \tag{B.4}$$

for all $0 < t < \tau$. If $N \geq 2$ and $h \in L^{\frac{2(N+1)}{N+3}}(E(0,R,\tau))$, then $z_{|E(0,R,\tau)} \in L^{\frac{2(N+1)}{N-1}}(E(0,R,\tau))$ and

$$\|z\|_{L^{\frac{2(N+1)}{N-1}}(E(0,R,\tau))} \leq C\|h\|_{L^{\frac{2(N+1)}{N+3}}(E(0,R,\tau))}. \tag{B.5}$$

In (B.4) and (B.5), the constant C independent of h, R, τ and t.

Proof. We define $\widetilde{h} \in C([0,\tau], L^q(\mathbb{R}^N))$ by

$$\widetilde{h} = \begin{cases} h & \text{on } (0,\tau)\times B_R, \\ 0 & \text{elsewhere.} \end{cases}$$

We let $\tilde{z} \in C([0,\tau], L^2(\mathbb{R}^N)) \cap C^1([0,\tau], H^{-1}(\mathbb{R}^N)) \cap C^2([0,\tau], H^{-2}(\mathbb{R}^N))$ be the solution of the wave equation $\partial_{tt}\tilde{z} - \Delta\tilde{z} = \tilde{h}$ on $\mathbb{R}^N$ with the initial conditions $\tilde{z}(0) = \partial_t\tilde{z}(0) = 0$. Note that, given any $0 < t \le \tau < R$ and $1 \le r \le \infty$,

$$\|\tilde{h}\|_{L^r(E(0,R,t))} = \|h\|_{L^r(E(0,R,t))}.$$

Therefore, estimate (B.4) with z replaced by $\tilde{z}$ follows from the standard energy inequality for $\tilde{z}$; and (B.5) with z replaced by $\tilde{z}$ follows from the Strichartz estimates (see [13, Corollary 1.3]).

To conclude the proof, we show that z and $\tilde{z}$ coincide on $E(0,R,\tau)$. We let $w(t) = (z(t) - \tilde{z}(t))_{|B_R}$ for all $0 \le t \le \tau$, so that

$$w \in C([0,\tau], L^2(B_R)) \cap C^1([0,\tau], H^{-1}(B_R)) \cap C^2([0,\tau], H^{-2}(B_R)) \tag{B.6}$$

satisfies $\partial_{tt}w = \Delta w$ in $H^{-2}(B_R)$ for all $0 \le t \le \tau$ and $w(0) = \partial_t w(0) = 0$. Thus we need to show that $w = 0$ a.e. on $E(0,R,\tau)$. Let $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \ge 0$, be radially symmetric, supported in B_1, and satisfy $\int \rho = 1$. Given $\varepsilon > 0$, let $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(\frac{x}{\varepsilon})$. Let $0 < \eta < R$ and $0 < \varepsilon < \frac{\eta}{2}$. Since ρ_ε is supported in B_ε, it follows that $\rho_\varepsilon \star w$ is well defined in $B_{R-\eta}$, and we set $w_\varepsilon = (\rho_\varepsilon \star w)_{|B_{R-\eta}}$. We claim that

$$w_\varepsilon \in C^2([0,\tau] \times \overline{B_{R-\eta}}), \tag{B.7}$$

$$\partial_{tt}w_\varepsilon = \Delta w_\varepsilon \quad \text{on } [0,\tau] \times \overline{B_{R-\eta}}, \tag{B.8}$$

$$w_\varepsilon(0) = \partial_t w_\varepsilon(0) = 0 \quad \text{on } B_{R-\eta}. \tag{B.9}$$

By finite speed of propagation, it follows that w_ε identically vanishes on $E(0, R-\eta, \tau)$. Letting $\varepsilon \to 0$, we deduce that w vanishes a.e. on $E(0, R-\eta, \tau)$; and letting $\eta \to 0$, we see that w vanishes a.e. on $E(0,R,\tau)$. It remains to prove the claims (B.7)–(B.9). Given $m \in \mathbb{N}$ and $\theta \in H^{-m}(B_R))$, recall that $\rho_\varepsilon \star \theta \in H^{-m}(B_{R-\eta}))$ is given by

$$\langle \rho_\varepsilon \star \theta, \varphi\rangle_{H^{-m}(B_{R-\eta}), H_0^m(B_{R-\eta})} = \langle \theta, \rho_\varepsilon \star \varphi\rangle_{H^{-m}(B_R), H_0^m(B_R)}$$

for all $\varphi \in C_c^\infty(B_{R-\eta})$. It is well known that

$$\rho_\varepsilon \star \theta \in C^\infty(\overline{B_{R-\eta}})),$$

$$D^\alpha(\rho_\varepsilon \star \theta) = \rho_\varepsilon \star (D^\alpha\theta) \quad \text{for all } \alpha \in \mathbb{N}^N,$$

$$\|\rho_\varepsilon \star \theta\|_{C(\overline{B_{R-\eta}})} \lesssim \|\theta\|_{H^{-m}(B_R)}.$$

On the other hand, it follows from (B.6) that $D^\alpha\partial_t^\beta w \in C([0,\tau], H^{-2}(B_R))$ for all $\alpha \in \mathbb{N}^N$ and $\beta \in \mathbb{N}$ such that $|\alpha| + \beta \le 2$. Thus we see that $D^\alpha\partial_t^\beta w_\varepsilon \in C([0,\tau] \times \overline{B_{R-\eta}})$ and that

$$D^\alpha\partial_t^\beta w_\varepsilon = \rho_\varepsilon \star D^\alpha\partial_t^\beta w.$$

Properties (B.7)–(B.9) easily follow. □

Proof of Proposition B.2. We need only prove the result for τ small, the general case follows by iteration.

The Case $p \le \frac{N}{N-2}$ (any $1 < p < \infty$ if $N = 1, 2$). We note that $r = 2p$ satisfies $2 < r \le \frac{2N}{N-2}$ ($2 < r < \infty$ if $N = 1, 2$), so that by (B.4) and Sobolev's embedding

$$\|(u-v)(t)\|_{L^r(B_{R-t})}^2 \le Ce^{Ct} \int_0^t \||u|^{p-1}u - |v|^{p-1}v\|_{L^2(B_{R-s})}^2 \, ds.$$

Since $\||u|^{p-1}u - |v|^{p-1}v\|_{L^2} \le C(\|u\|_{L^r} + \|v\|_{L^r})^{p-1}\|u-v\|_{L^r}$ and u and v are bounded in H^1, hence in L^r, the result follows by Gronwall's inequality.

The Case $N \ge 3$ and $p = \frac{N+2}{N-2}$. We note that, since $p \le \frac{N+2}{N-2}$,

$$\big||u|^{p-1}u - |v|^{p-1}v\big| \lesssim (|u|^{p-1} + |v|^{p-1})|u-v| \lesssim (1 + |u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}})|u-v|.$$

By Hölder's inequality, it follows that

$$\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{2(N+1)}{N+3}}} \lesssim \Big(\tau^{\frac{2}{N+1}} + \|u\|_{L^{\frac{2(N+1)}{N-2}}}^{\frac{4}{N-2}} + \|u\|_{L^{\frac{2(N+1)}{N-2}}}^{\frac{4}{N-2}}\Big)\|u-v\|_{L^{\frac{2(N+1)}{N-1}}},$$

where all the integrals are on $E(0, R, \tau)$, with the notation (1.6). Applying the Strichartz inequality (B.5), we deduce that

$$\|u - v\|_{L^{\frac{2(N+1)}{N-1}}} \leq C\Big(\tau^{\frac{2}{N+1}} + \|u\|^{\frac{4}{N-2}}_{L^{\frac{2(N+1)}{N-2}}} + \|u\|^{\frac{4}{N-2}}_{L^{\frac{2(N+1)}{N-2}}}\Big)\|u - v\|_{L^{\frac{2(N+1)}{N-1}}},$$

where all the integrals are on $E(0, R, \tau)$. Since

$$\|u\|_{L^{\frac{2(N+1)}{N-2}}(E(0,R,\tau))} + \|v\|_{L^{\frac{2(N+1)}{N-2}}(E(0,R,\tau))} \xrightarrow[\tau\downarrow 0]{} 0,$$

the conclusion follows by choosing τ sufficiently small. □

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