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Research Article

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Large Time Behavior of Solutions to the Nonlinear Heat Equation with Absorption with Highly Singular Antisymmetric Initial Values

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Abstract: In this paper, we study global well-posedness and long-time asymptotic behavior of solutions to the nonlinear heat equation with absorption, $u_t - \Delta u + |u|^\alpha u = 0$, where $u = u(t,x) \in \mathbb{R}$, $(t,x) \in (0,\infty) \times \mathbb{R}^N$ and $\alpha > 0$. We focus particularly on highly singular initial values which are antisymmetric with respect to the variables $x_1, x_2, \dots, x_m$ for some $m \in \{1,2,\dots,N\}$, such as $u_0 = (-1)^m \partial_1 \partial_2 \cdots \partial_m |\cdot|^{-\gamma} \in \mathcal{S}'(\mathbb{R}^N)$, $0 < \gamma < N$. In fact, we show global well-posedness for initial data bounded in an appropriate sense by u_0 for any $\alpha > 0$. Our approach is to study well-posedness and large time behavior on sectorial domains of the form $\Omega_m = \{x \in \mathbb{R}^N : x_1, \dots, x_m > 0\}$, and then to extend the results by reflection to solutions on $\mathbb{R}^N$ which are antisymmetric. We show that the large time behavior depends on the relationship between α and $\frac{2}{\gamma+m}$, and we consider all three cases, α equal to, greater than, and less than $\frac{2}{\gamma+m}$. Our results include, among others, new examples of self-similar and asymptotically self-similar solutions.

Keywords: Heat Equation, Absorption, Complexity, Sectorial Domains, Asymptotic Behavior, Antisymmetric Solutions, Scaling, Self-similar Solutions

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1 Introduction

In this paper, we study the long-time behavior of solutions to the nonlinear heat equation with absorption,

$$u_t - \Delta u + |u|^\alpha u = 0, \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}$, $(t,x) \in (0,\infty) \times \mathbb{R}^N$ and $\alpha > 0$, which are antisymmetric with respect to the variables $x_1, x_2, \dots, x_m$ for some $m \in \{1,2,\dots,N\}$. Our goal is to see how some well-known results [1, 3, 5, 6] for the long-time behavior of solutions to (1.1) carry over with the additional hypothesis of antisymmetry. For example, some of the results in the cited works concern positive solutions. We will see that these results have analogues for antisymmetric solutions which are positive on an appropriate sector in $\mathbb{R}^N$. In particular, these solutions are not positive on $\mathbb{R}^N$. Moreover, in many cases, the range of allowable powers $\alpha > 0$ will be larger with the additional hypothesis of antisymmetry than without. Also, the condition of antisymmetry allows consideration of a class of highly singular initial values.

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Our previous paper [9] considered the *linear* heat equation on $\mathbb{R}^N$ with antisymmetric solutions. The results and the theoretical framework from [9] were applied to the nonlinear heat equation with source term

$$u_t - \Delta u - |u|^\alpha u = 0 \tag{1.2}$$

in [12]. In the current paper, these ideas are applied to (1.1). We mention that this approach was earlier developed in [11], where solutions to (1.2) with antisymmetric initial values of the form $u_0 = (-1)^m \partial_1 \partial_2 \cdots \partial_m \delta$ were studied. In the current paper, as in [9, 12], initial values of the form $u_0 = (-1)^m \partial_1 \partial_2 \cdots \partial_m |\cdot|^{-\gamma}$ for some $0 < \gamma < N$ are considered.

In order to state our results precisely, we begin by recalling the definition of an antisymmetric function.

Definition 1.1. Let $m \in \{1, 2, \dots, N\}$. A function $f\colon \mathbb{R}^N \to \mathbb{R}$ is antisymmetric with respect to $x_1, \dots, x_m$ if it satisfies

$$T_1 f = T_2 f = \cdots = T_m f = -f, \tag{1.3}$$

where T_i, $i \in \{1, 2, \dots, N\}$, denotes the operator

$$[T_i f](x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N) = f(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N).$$

We denote the set of functions antisymmetric with respect to $x_1, \dots, x_m$ by

$$\mathcal{A} = \mathcal{A}_m = \{f\colon \mathbb{R}^N \to \mathbb{R}; f \text{ satisfies (1.3)}\}.$$

A function on $\mathbb{R}^N$ which is antisymmetric with respect to $x_1, x_2, \dots, x_m$ for some $m \in \{1, 2, \dots, N\}$ is determined by its values on Ω_m, the sector of $\mathbb{R}^N$ defined by

$$\Omega_m = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N;\ x_1 > 0,\ x_2 > 0,\ \dots,\ x_m > 0\}.$$

Note that, by definition, an antisymmetric function must take the value 0 on the boundary $\partial\Omega_m$. Since the operators T_i defined above commute with the operations in equation (1.1), the study of antisymmetric solutions to (1.1) reduces to the study of solutions on Ω_m with Dirichlet boundary conditions. This point is discussed in detail in [12, Section 3], and that discussion applies as well to the heat equation with absorption. Moreover, as in [12], we will construct certain classes of antisymmetric solutions to (1.1) on $\mathbb{R}^N$ by constructing solutions on Ω_m and extending them to $\mathbb{R}^N$ by antisymmetry.

Since both the present paper and [12] are based on the framework developed in [9], we need to recall some definitions and notation used in [9]. Let ρ_m be the weight function defined on Ω_m by

$$\rho_m(x) = \frac{|x|^{\gamma+2m}}{x_1 \cdots x_m} \quad \text{for all } x \in \Omega_m,$$

where $0 < \gamma < N$. We consider the Banach space

$$\mathcal{X}_{m,\gamma} = \{\psi\colon \Omega_m \to \mathbb{R},\ \rho_m \psi \in L^\infty(\Omega_m)\},$$

endowed with the norm $\|\psi\|_{\mathcal{X}_{m,\gamma}} = \|\rho_m \psi\|_{L^\infty(\Omega_m)}$ for all $\psi \in \mathcal{X}_{m,\gamma}$. The closed ball of radius M on $\mathcal{X}_{m,\gamma}$ is denoted by

$$\mathcal{B}_{m,\gamma,M} = \{\psi \in \mathcal{X}_{m,\gamma} \text{ such that } \|\psi\|_{\mathcal{X}_{m,\gamma}} \le M\}.$$

As observed in [9, p. 344], $\mathcal{B}^\star_{m,\gamma,M}$ the closed ball $\mathcal{B}_{m,\gamma,M}$ endowed with the weak * topology of $\mathcal{X}_{m,\gamma}$ is a compact metric space (hence complete and separable).

Let $\sigma > 0$. For each $\lambda > 0$, we let D^σ_λ denote the dilation operator defined by

$$D^\sigma_\lambda u(x) = \lambda^\sigma u(\lambda x), \tag{1.4}$$

where u is a function defined on Ω_m, or on $\mathbb{R}^N$. A function $\psi\colon \Omega_m \to \mathbb{R}$ is homogeneous of degree $-\sigma$ if $D^\sigma_\lambda \psi = \psi$ for all $\lambda > 0$. The operators D^σ_λ, $\lambda > 0$, act on the spaces $\mathcal{X}_{m,\gamma}$, but leave the norm invariant, i.e. leave the ball $\mathcal{B}_{m,\gamma,M}$ invariant, if and only if $\sigma = \gamma + m$. In fact, we have

$$\|D^\sigma_\lambda \psi\|_{\mathcal{X}_{m,\gamma}} = \lambda^\sigma \|\rho_m \psi(\lambda\,\cdot)\|_{L^\infty(\Omega_m)} = \lambda^{\sigma-(\gamma+m)} \|\rho_m(\lambda\,\cdot)\psi(\lambda\,\cdot)\|_{L^\infty(\Omega_m)} = \lambda^{\sigma-(\gamma+m)} \|\psi\|_{\mathcal{X}_{m,\gamma}} \tag{1.5}$$

for all $\lambda > 0$ and $\psi \in \mathcal{X}_{m,\gamma}$.

The function ψ_0 defined on Ω_m by

$$\psi_0(x) = c_{m,\gamma}(\rho_m(x))^{-1} = c_{m,\gamma}x_1\cdots x_m|x|^{-\gamma-2m}, \quad x \in \Omega_m, \tag{1.6}$$

where $c_{m,\gamma} = \gamma(\gamma+2)\cdots(\gamma+2m-2)$, will play a central role. It is homogeneous of degree $-(\gamma+m)$, belongs to $\mathcal{X}_{m,\gamma}$, and satisfies $\|\psi_0\|_{\mathcal{X}_{m,\gamma}} = c_{m,\gamma}$. Moreover, we have $D_\lambda^\sigma\psi_0(x) = \lambda^{\sigma-(\gamma+m)}\psi_0(x)$ for all $\sigma, \lambda > 0$, and so $\|D_\lambda^\sigma\psi_0\|_{\mathcal{X}_{m,\gamma}} = \lambda^{\sigma-(\gamma+m)}c_{m,\gamma}$. Its interest lies in the fact that

$$\psi_0(x) = (-1)^m\partial_1\partial_2\cdots\partial_m(|x|^{-\gamma}), \quad x \in \Omega_m.$$

The heat semigroup on Ω_m, denoted $e^{t\Delta_m}$, is given by

$$e^{t\Delta_m}\psi(x) = \int_{\Omega_m} K_t(x,y)\psi(y)\,dy \tag{1.7}$$

for all $t > 0$, where

$$K_t(x,y) = (4\pi t)^{-\frac{N}{2}}\prod_{j=m+1}^{N} e^{-\frac{|x_j-y_j|^2}{4t}}\prod_{i=1}^{m}[e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}]. \tag{1.8}$$

See, for example, [11, Proposition 3.1, p. 514]. It is well known that $e^{t\Delta_m}$ is a C_0 semigroup on $C_0(\Omega_m)$, the space of continuous functions $f\colon \overline{\Omega}_m \to \mathbb{R}$ such that $f \equiv 0$ on the boundary $\partial\Omega_m$ and $f(x) \to 0$ as $|x| \to \infty$ in Ω_m. It is also well defined on $\mathcal{X}_{m,\gamma}$, and $e^{t\Delta_m}\colon \mathcal{X}_{m,\gamma} \to C_0(\Omega_m) \cap \mathcal{X}_{m,\gamma}$ is continuous for all $t > 0$. See [9, Theorem 1.1, p. 343]. We recall the commutation relation between $e^{t\Delta_m}$ and the operators D_λ^σ,

$$D_\lambda^\sigma e^{\lambda^2 t\Delta_m} = e^{t\Delta_m}D_\lambda^\sigma \tag{1.9}$$

for all $\lambda > 0$ and $\sigma > 0$, and for future use, we note the following identity, which is immediate to verify:

$$\int_{\Omega_m} K_t(x,y)y_1\cdots y_m\,dy = x_1\cdots x_m \tag{1.10}$$

for all $t > 0$ and all $x \in \Omega_m$.

In terms of behavior on the sectors Ω_m, our goal is to study the well-posedness of equation (1.1) on the space $\mathcal{X}_{m,\gamma}$ and to obtain results on the large time behavior of solutions in the three cases $\alpha = \frac{2}{\gamma+m}$, $\alpha > \frac{2}{\gamma+m}$, and $\alpha < \frac{2}{\gamma+m}$. By interpreting these results for antisymmetric solutions on $\mathbb{R}^N$, we will extend some known results, [3, Theorem 1.3, Theorem 1.4] and [6], in the case $m = 0$. We now describe these results in detail.

In Section 2, we consider the Cauchy problem

$$\begin{cases} u_t - \Delta u + |u|^\alpha u = 0, \\ \qquad\qquad u(0) = u_0 \in \mathcal{X}_{m,\gamma}. \end{cases} \tag{1.11}$$

It is well known that, given any $u_0 \in C_0(\mathbb{R}^N)$, there exists a unique function $u \in C([0,\infty), C_0(\mathbb{R}^N))$ which is a classical solution of (1.1) on $\mathbb{R}^N$ for $t > 0$ and such that $u(0) = u_0$, which we denote by

$$u(t) = \mathcal{S}(t)u_0, \tag{1.12}$$

where $u(t) = u(t,\cdot)$. Likewise, for any $u_0 \in C_0(\Omega_m)$, there exists a unique function $u \in C([0,\infty), C_0(\Omega_m))$ which is a classical solution of (1.1) for $t > 0$ and such that $u(0) = u_0$. This defines a global semiflow $\mathcal{S}_m(t)$ on $C_0(\Omega_m)$. In other words,

$$\mathcal{S}_m(t)u_0 = u(t), \tag{1.13}$$

where $u(t) = u(t,\cdot)$ is the solution of (1.1) with initial value $u_0 \in C_0(\Omega_m)$. In fact, existence and uniqueness of solutions in $C_0(\Omega_m)$ follows from the existence and uniqueness of solutions in $u_0 \in C_0(\mathbb{R}^N)$ since $\mathcal{S}(t)$ preserves antisymmetry; it suffices to consider the antisymmetric extension of $u_0 \in C_0(\Omega_m)$ to an element of $C_0(\mathbb{R}^N) \cap \mathcal{A}$.

Similarly, given any $u_0 \in L^q(\Omega_m)$, $1 \le q < \infty$, we deduce by Kato's parabolic inequality (see Lemma A.1 and Corollary A.2 in the appendix) and the fact that $\mathcal{D}(\Omega_m)$ is dense in $L^q(\Omega_m)$ that there exists a unique

$u \in C([0,\infty), L^q(\Omega_m))$ which is a classical solution of (1.1) for $t > 0$ and such that $u(0) = u_0$. Alternatively, see [6, Proposition 1.1, p. 261] for a proof using accretive operators. Again by preservation of antisymmetry, the result of [6], valid for $\mathbb{R}^N$, holds also on Ω_m. Thus, the semiflow $\mathcal{S}_m(t)$ extends to $L^q(\Omega_m)$, and formula (1.13) is valid also for $u_0 \in L^q(\Omega_m)$.

Here we consider initial data $u_0 \in \mathcal{X}_{m,\gamma}$. Our first main result is the following.

Theorem 1.2. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$. If $u_0 \in \mathcal{X}_{m,\gamma}$, then there exists a unique solution $u \in C((0,\infty), C_0(\Omega_m))$ of equation* (1.1) *such that*
(i) *$u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$,*
(ii) *there exists $C > 0$, independent of u_0, such that $\|u(t)\|_{\mathcal{X}_{m,\gamma}} \le C\|u_0\|_{\mathcal{X}_{m,\gamma}}$ for all $t > 0$.*
In addition, the following properties hold.
(iii) *For all $v_0 \in \mathcal{X}_{m,\gamma}$, $|u(t) - v(t)| \le e^{t\Delta_m}|u_0 - v_0|$, where v is the solution of* (1.1) *with initial value v_0 satisfying* (i) *and* (ii).
(iv) *There exists $C > 0$ such that $|u(t,x)| \le Cx_1 \cdots x_m(t + |x|^2)^{-\frac{\gamma+2m}{2}}\|u_0\|_{\mathcal{X}_{m,\gamma}}$ for all $t > 0$ and for all $u_0 \in \mathcal{X}_{m,\gamma}$.*
(v) *The solution $u(t)$ satisfies the integral equation*

$$u(t) = e^{t\Delta_m}u_0 - \int_0^t e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s))\,ds \tag{1.14}$$

for all $t > 0$, where the integrand is in $L^1((0,t); C_0(\Omega_m))$.
(vi) *If $v_0 \in \mathcal{X}_{m,\gamma}$, $v_0 \ge 0$, and $|u_0| \le v_0$, then $|u(t)| \le v(t)$, where v is the solution of* (1.1) *with initial value v_0 satisfying* (i) *and* (ii).

In other words, the nonlinear operators $\mathcal{S}_m(t)$, $t > 0$, extend in a natural way to $\mathcal{X}_{m,\gamma}$. We remark that, in the case $\alpha < \frac{2}{\gamma+m}$, this well-posedness result was established in [12, Theorems 2.3 and 2.6] by a different method and with plus and minus sign in the term of the nonlinearity. Furthermore, the analogous results on the whole space $\mathbb{R}^N$ follow from [1, Theorem 8.8, p. 536].

Definition 1.3. Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$. Given $u_0 \in \mathcal{X}_{m,\gamma}$ we set $\mathcal{S}_m(t)u_0 = u(t)$ for all $t > 0$, where $u \in C((0,\infty), C_0(\Omega_m))$ is the unique solution of (1.1) satisfying (i) and (ii) of Theorem 1.2.

We also establish the continuous dependence properties of solutions of equation (1.1) with initial values in $\mathcal{X}_{m,\gamma}$.

Theorem 1.4. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. It follows that $\mathcal{S}_m(t)$ is continuous $\mathcal{B}^\star_{m,\gamma,M} \to C_0(\Omega_m)$ for all $t > 0$, where $\mathcal{B}^\star_{m,\gamma,M}$ denotes the compact metric space topology induced by the weak* topology on $\mathcal{B}_{m,\gamma,M}$.*

It is well known that any solution $u(t)$ of (1.1), for example, as constructed in Theorem 1.2, is always bounded by the spatially independent solution; more precisely,

$$|u(t,x)| \le \left(\frac{1}{\alpha t}\right)^{1/\alpha} \tag{1.15}$$

for all $t > 0$, throughout the spatial domain of existence. See, for example, [6, p. 261]. In addition, it is clear from Theorem 1.2 that if u is the solution of (1.1) with positive initial data $u_0 \ge 0$, then

$$u(t) \le e^{t\Delta_m}u_0 \tag{1.16}$$

for any $t > 0$. We have the following upper estimate for solutions of (1.1) that combines (1.16) and (1.15) into one estimate which implies them both. Its proof is given in Section 3.

Proposition 1.5. *Let $N \ge 1$, $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$. Let $u_0 \in \mathcal{X}_{m,\gamma}$, $u_0 \ge 0$. Then the solution u of* (1.1) *with initial data $u(0) = u_0$ satisfies the upper estimate*

$$u(t,x) \le \frac{e^{t\Delta_m}u_0(x)}{(1 + \alpha t(e^{t\Delta_m}u_0(x))^\alpha)^{1/\alpha}}$$

for all $t > 0$ and all $x \in \Omega_m$.

After proving global well-posedness of Cauchy problem (1.11), i.e. Theorems 1.2 and 1.4, we seek to describe the large time behavior of solutions of (1.1) on Ω_m with initial values in $\mathcal{X}_{m,\gamma}$. Our basic approach is to study the effect of certain space-time dilations on such a solution, and to relate the resulting behavior to the effect of related spatial dilations on the initial value. In particular, we consider the space-time dilation operators Γ_λ^σ, $\lambda > 0$, defined by

$$\Gamma_\lambda^\sigma u(t,x) = \lambda^\sigma u(\lambda^2 t, \lambda x) = D_\lambda^\sigma[u(\lambda^2 t)](x) \tag{1.17}$$

for all $\lambda, \sigma > 0$, where the spatial dilation operator D_λ^σ is given by (1.4). If $u \in C((0,\infty), C_0(\Omega_m))$ is solution of equation (1.1), then $\Gamma_\lambda^\sigma u$ is a solution of (1.1) if and only if $\sigma = \frac{2}{\alpha}$. Moreover, if a solution u has initial value u_0, either in the sense of $C_0(\Omega_m)$ or in some more general sense, then $\Gamma_\lambda^{2/\alpha} u$ has initial value $D_\lambda^{2/\alpha} u_0$. If $u_0 \in \mathcal{X}_{m,\gamma}$, the function $D_\lambda^{2/\alpha} u_0$ belongs to $\mathcal{X}_{m,\gamma}$ for all $\lambda > 0$, and the uniqueness of solutions of (1.1) implies that $\Gamma_\lambda^{2/\alpha} u$ coincides with $\mathcal{S}_m(\cdot) D_\lambda^{2/\alpha} u_0$. Thus, we have the relation

$$\Gamma_\lambda^{2/\alpha}[\mathcal{S}_m(\cdot)u_0] = \mathcal{S}_m(\cdot)[D_\lambda^{2/\alpha} u_0] \tag{1.18}$$

for all $u_0 \in \mathcal{X}_{m,\gamma}$. We emphasize that, at this point, there is no assumed relationship between α and m. Formula (1.18) holds for any semiflow generated by (1.1) in place of $\mathcal{S}_m(\cdot)$, as long as the space of initial values is invariant under the dilations $D_\lambda^{2/\alpha}$ and initial values give rise to unique solutions.

A solution u of (1.1) is self-similar if $\Gamma_\lambda^{2/\alpha} u = u$ for all $\lambda > 0$ or, equivalently, if

$$u(t,x) = t^{-1/\alpha} f\Big(\frac{x}{\sqrt{t}}\Big) = D_{\frac{1}{\sqrt{t}}}^{2/\alpha} f(x),$$

where $f(x) = u(1,x)$ is called the profile of u. It follows that if a self-similar solution u of (1.1) has initial value u_0, then $D_\lambda^{2/\alpha} u_0 = u_0$ for all $\lambda > 0$, i.e. u_0 is homogeneous of degree $-\frac{2}{\alpha}$. Conversely, if u_0 is homogeneous of degree $-\frac{2}{\alpha}$ and $u(t)$ is a solution with initial value u_0 in some appropriate sense, then $\Gamma_\lambda^{2/\alpha} u$ has the same initial value for all $\lambda > 0$. Assuming that uniqueness of solutions having a given initial value has been proved in the appropriate class of functions, one then concludes that $u = \Gamma_\lambda^{2/\alpha} u$ for all $\lambda > 0$, i.e. that u is a self-similar solution.

More generally, we say that a solution u of (1.1) is asymptotically self-similar if

$$\lim_{\lambda\to\infty} \Gamma_\lambda^{2/\alpha} u = U, \tag{1.19}$$

in some appropriate sense, and that U is also a solution to (1.1). If so, the limit is necessarily a self-similar solution. See [5, Section 3] for a discussion of several equivalent definitions of asymptotically self-similar solutions. Formally, if we put $t = 0$ in (1.19), we obtain

$$\lim_{\lambda\to\infty} D_\lambda^{2/\alpha} u_0 = \varphi, \tag{1.20}$$

where $\varphi = U(0)$ is homogeneous of degree $-\frac{2}{\alpha}$. In Section 4, we study the long-time asymptotic behavior of solutions to (1.1) with initial values in $\mathcal{X}_{m,\gamma}$ in the case $\alpha = \frac{2}{\gamma+m}$. The first result shows that (1.20) implies (1.19).

Theorem 1.6. *Let $m \in \{1,2,\dots,N\}$, $0 < \gamma < N$ and $\psi \in \mathcal{B}_{m,\gamma,M}$. Let $\alpha > 0$ be such that $\alpha = \frac{2}{\gamma+m}$. Suppose that there exists $\varphi \in \mathcal{B}_{m,\gamma,M}$ such that $\lim_{\lambda\to\infty} D_\lambda^{\gamma+m}\psi = \varphi$ in $\mathcal{B}^\star_{m,\gamma,M}$. It follows that φ is homogeneous of degree $-(\gamma+m)$ and that the solution $u(t) = \mathcal{S}_m(t)\psi$ is asymptotically self-similar to the self-similar solution $U(t) = \mathcal{S}_m(t)\varphi$.*

As is by now well established [3–5], the notion of asymptotically self-similar solution can be naturally extended by allowing different limits in (1.20) and (1.19) along different sequences $(\lambda_n)_{n\geq 0}$ with $\lambda_n \to \infty$. The next step in our analysis is to generalize Theorem 1.6 in this fashion. To accomplish this, for $u_0 \in \mathcal{X}_{m,\gamma}$ and $M \geq \|u_0\|_{\mathcal{X}_{m,\gamma}}$, we consider the set of all accumulation points of $D_\lambda^{\gamma+m} u_0$ as $\lambda \to \infty$, given by

$$\mathcal{Z}^\gamma(u_0) = \{z \in \mathcal{B}_{m,\gamma,M};\ \text{there exists } \lambda_n \to \infty \text{ such that } \lim_{n\to\infty} D_{\lambda_n}^{\gamma+m} u_0 = z \text{ in } \mathcal{B}^\star_{m,\gamma,M}\}. \tag{1.21}$$

Since $\mathcal{B}^\star_{m,\gamma,M}$ is a compact metric space, $\mathcal{Z}^\gamma(u_0)$ is a nonempty compact subset for all $u_0 \in \mathcal{X}_{m,\gamma}$, and independent of $M \geq \|u_0\|_{\mathcal{X}_{m,\gamma}}$ by [9, Proposition 3.1, p. 356]. In particular, if u_0 is homogeneous of degree $-(\gamma+m)$, then $\mathcal{Z}^\gamma(u_0) = \{u_0\}$. We set $u(t) = \mathcal{S}_m(t)u_0$, and we also define the omega-limit set of all accumulation points of $\Gamma^{\gamma+m}_{\sqrt{t}} u(1,\cdot) = t^{\frac{\gamma+m}{2}} u(t, \sqrt{t}\,\cdot)$ as $t \to \infty$ by

$$\mathcal{Q}^\gamma(u_0) = \{f \in C_0(\Omega_m);\ \text{there exists } t_n \to \infty \text{ such that } \lim_{n\to\infty} \|\Gamma^{\gamma+m}_{\sqrt{t_n}} \mathcal{S}_m(1)u_0 - f\|_{L^\infty(\Omega_m)} = 0\}. \tag{1.22}$$

Relation (1.18) and Theorem 1.4 are the essential elements needed to investigate the relationship between $\mathcal{Q}^\gamma(u_0)$ and $\mathcal{Z}^\gamma(u_0)$, which is given by our next main result.

Theorem 1.7. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$ such that $\alpha = \frac{2}{\gamma+m}$. If $u_0 \in \mathcal{X}_{m,\gamma}$, then*

$$\mathcal{Q}^\gamma(u_0) = \mathcal{S}_m(1)\mathcal{Z}^\gamma(u_0).$$

In particular, $\mathcal{Q}^\gamma(u_0) \subset \mathcal{S}_m(1)\mathcal{B}^\star_{m,\gamma,M}$ and is therefore a compact subset of $C_0(\Omega_m)$.

The last relation shows that, in the case $\alpha = \frac{2}{\gamma+m}$, the complexity in the large time behavior of a solution, as expressed in $\mathcal{Q}^\gamma(u_0)$, is determined by the complexity in the spatial asymptotic behavior of its initial value as expressed in $\mathcal{Z}^\gamma(u_0)$. Furthermore, Theorem 1.7 above is inspired from [3, Theorem 1.3, p. 83] which requires $\alpha \geq \frac{2}{N}$, and we observe that, in Theorem 1.7, if $\gamma + m > N$, then $\alpha < \frac{2}{N}$. Since $\mathcal{B}^\star_{m,\gamma,M}$ is separable and $\mathcal{Z}^\gamma(u_0)$ can contain any countable subset of $\mathcal{B}^\star_{m,\gamma,M}$, we show that $\mathcal{Z}^\gamma(U_0) = \mathcal{B}^\star_{m,\gamma,M}$ for some choice of $U_0 \in \mathcal{B}^\star_{m,\gamma,M}$.

Using [9, Theorem 1.4, p. 345], we obtain the following result.

Corollary 1.8. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let $\alpha > 0$ be such that $\alpha = \frac{2}{\gamma+m}$. Then there exists*

$$U_0 \in \mathcal{B}_{m,\gamma,M} \cap C^\infty(\Omega_m) \cap C_0(\Omega_m)$$

such that $\mathcal{Q}^\gamma(U_0) = \mathcal{S}_m(1)\mathcal{B}_{m,\gamma,M}$.

Remark 1.9. If u_0 belongs to $\mathcal{X}_{m,\gamma} \cap \mathcal{X}_{m,\gamma'}$ with $\gamma < \gamma' < N$, then $\mathcal{Z}^\gamma(u_0) = \{0\}$. In fact, for all $\lambda > 0$,

$$|D^{\gamma+m}_\lambda u_0(x)| = \lambda^{\gamma+m}|u_0(\lambda x)| \leq C\lambda^{\gamma-\gamma'}|x|^{-(\gamma'+m)} \to 0$$

as $\lambda \to \infty$ uniformly on $\{x \in \Omega_m;\ |x| \geq \varepsilon\}$ for all $\varepsilon > 0$. Thus, $\mathcal{Z}^\gamma(u_0) = \{0\}$. For example, if $\alpha = \frac{2}{\gamma+m} > \frac{2}{\gamma'+m}$, the function $\varphi(x) = x_1 \cdots x_m|x|^{-\gamma'-2m}\,\mathbb{1}_{\{|x|>1\}} \in \mathcal{X}_{m,\gamma'} \cap \mathcal{X}_{m,\gamma}$. It follows from Theorem 1.7 that $\mathcal{Q}^\gamma(\varphi) = \{0\}$. However, we might have $\mathcal{Q}^{\gamma'}(\varphi) \neq \{0\}$.

In Section 5 of this paper, we consider the case $\alpha > \frac{2}{\gamma+m}$. Since $\alpha \neq \frac{2}{\gamma+m}$, there is a disconnect between the transformations which preserve the set of solutions to (1.1), i.e. $\Gamma^{2/\alpha}_\lambda$, and those which leave invariant the norm of the space $\mathcal{X}_{m,\gamma}$ where the solutions live, i.e. $\Gamma^{\gamma+m}_\lambda$. Indeed, by (1.17) and (1.5), it follows that, for $u_0 \in \mathcal{X}_{m,\gamma}$,

$$\|\Gamma^\sigma_\lambda \mathcal{S}_m(t)u_0\|_{\mathcal{X}_{m,\gamma}} = \|D^\sigma_\lambda \mathcal{S}_m(\lambda^2 t)u_0\|_{\mathcal{X}_{m,\gamma}} = \lambda^{\sigma-(\gamma+m)}\|\mathcal{S}_m(\lambda^2 t)u_0\|_{\mathcal{X}_{m,\gamma}}. \tag{1.23}$$

Since $\|\mathcal{S}_m(\lambda^2 t)u_0\|_{\mathcal{X}_{m,\gamma}} \leq C\|u_0\|_{\mathcal{X}_{m,\gamma}}$ for some $C > 0$, by Theorem 1.2, it follows, setting $\sigma = \frac{2}{\alpha}$ in (1.23), that if $\frac{2}{\alpha} < \gamma + m$, then $\|\Gamma^{2/\alpha}_\lambda \mathcal{S}_m(t)u_0\|_{\mathcal{X}_{m,\gamma}} \to 0$ as $\lambda \to \infty$, uniformly for all u_0 in a bounded set of $\mathcal{X}_{m,\gamma}$ and all $t > 0$.

It is clear from (1.23) that, for $u_0 \in \mathcal{X}_{m,\gamma}$, the transformations most likely to yield some nontrivial asymptotic behavior are $\Gamma^{\gamma+m}_\lambda$. In other words, we still need to study $\mathcal{Q}^\gamma(u_0)$ as given by (1.22), and likewise $\mathcal{Z}^\gamma(u_0)$ as given by (1.21). However, we cannot expect the relationship between these two objects to be given as in Theorem 1.7 since the transformations do not preserve solutions of (1.1).

If u is a solution of (1.1), then $v = \Gamma^{\gamma+m}_\lambda u$ is the solution of $v_t - \Delta v + \lambda^{2-(\gamma+m)\alpha}|v|^\alpha v = 0$. If $\alpha > \frac{2}{\gamma+m}$, it follows that, as $\lambda \to \infty$, the function v satisfies an equation which approaches the *linear* heat equation. Hence we should not be surprised if, in this case, $\mathcal{Q}^\gamma(u_0)$ and $\mathcal{Z}^\gamma(u_0)$ are related by the linear heat equation. The next theorem makes this idea precise, both in the asymptotically self-similar case, and the more general case of arbitrary $u_0 \in \mathcal{X}_{m,\gamma}$. It is analogous to [3, Lemma 5.1, p. 110].

Theorem 1.10. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let α be such that $\alpha > \frac{2}{\gamma+m}$. We then have the following conclusions.*

(i) *If $u_0, \varphi \in \mathcal{B}_{m,\gamma,M}$ is such that $\lim_{\lambda\to\infty} D^{\gamma+m}_\lambda u_0 = \varphi$ in $\mathcal{B}^\star_{m,\gamma,M}$, then φ is homogeneous of degree $-(\gamma + m)$, and $u(t) = \mathcal{S}_m(t)u_0$ is asymptotically self-similar to $U(t) = e^{t\Delta_m}\varphi$.*

(ii) *$\mathcal{Q}^\gamma(u_0) = e^{\Delta_m}\mathcal{Z}^\gamma(u_0)$ for all $u_0 \in \mathcal{X}_{m,\gamma}$.*

(iii) *There exists $U_0 \in \mathcal{B}_{m,\gamma,M} \cap C^\infty(\Omega_m) \cap C_0(\Omega_m)$ such that $\mathcal{Q}^\gamma(U_0) = e^{\Delta_m}\mathcal{B}_{m,\gamma,M}$.*

In Section 6 of this paper, we consider the case $\alpha < \frac{2}{\gamma+m}$. As in the case of Theorem 1.10, the transformations which leave solutions invariant, i.e. $\Gamma_\lambda^{2/\alpha}$, do not leave invariant the norm of $\mathcal{X}_{m,\gamma}$, which is the space where the solution lives. Nonetheless, unlike in the case $\alpha > \frac{2}{\gamma+m}$, the transformations $\Gamma_\lambda^{2/\alpha}$ reveal nontrivial asymptotic behavior. Because of (1.23), to study this asymptotic behavior, we need to leave the context of the space $\mathcal{X}_{m,\gamma}$.

This is best illustrated by the result of Gmira and Véron [6] in the case of $\mathbb{R}^N$. If we express the upper bound (1.15) in terms more suggestive of the long-time asymptotic behavior of the solution, we see that, considering only positive solutions,

$$(\Gamma_{\sqrt{t}}^{2/\alpha}u)(1,x) = t^{1/\alpha}u(t, x\sqrt{t}) \le \left(\frac{1}{\alpha}\right)^{1/\alpha}. \tag{1.24}$$

The main result of [6] can be stated as follows. Suppose $\alpha < \frac{2}{N}$. Let $u_0 \in L^q(\mathbb{R}^N)$ for some $1 \le q < \infty$, or $C_0(\mathbb{R}^N)$, with $u_0 \ge 0$ be such that, for every $k > 0$, there exists $R_0 > 0$ such that $u_0(x) \ge k|x|^{-2/\alpha}$, $|x| \ge R_0$, i.e. $\liminf_{|x|\to\infty}|x|^{2/\alpha}u_0(x) = \infty$. It follows that if $u(t,x)$ is the resulting solution of (1.1), then

$$t^{1/\alpha}u(t, x\sqrt{t}) \to \left(\frac{1}{\alpha}\right)^{1/\alpha} \tag{1.25}$$

uniformly on compact subsets of $\mathbb{R}^N$. In light of the upper bound (1.24), the result (1.25) is rather sharp.

In the case of the sector Ω_m, we have the following result, where $C_0^{b,u}(\Omega_m)$ denotes the space of bounded uniformly continuous functions on Ω_m which are zero on $\partial\Omega_m$.

Theorem 1.11. *Let $m \in \{1,\dots,N\}$, $0 < \gamma < N$ and $\alpha > 0$ such that $\alpha < \frac{2}{\gamma+m}$. Let $u_0 \in \mathcal{X}_{m,\gamma}$ with $u_0 \ge 0$, and let $u(t) = \mathcal{S}_m(t)u_0$ be the resulting solution of* (1.1) *as given by Theorem 1.2. Suppose that there exist $R_0 > 0$ and $c_0 > 0$ such that*

$$u_0(x) \ge c_0\psi_0(x), \quad x \in \Omega_m,\ |x| \ge R_0, \tag{1.26}$$

where ψ_0 is given by (1.6). *Then*

$$\lim_{t\to\infty} t^{1/\alpha}u(t, x\sqrt{t}) = g(x) \tag{1.27}$$

uniformly on compact subsets of $\overline{\Omega}_m$, where $g \in C_0^{b,u}(\Omega_m)$ is the profile of the self-similar solution of (1.1) *given by Proposition 6.3.*

Remark 1.12. Condition (1.26) implies that, for any $c > 0$,

$$\lim_{|x|\to\infty,\, x_1\cdots x_m|x|^{-m}\ge c} |x|^{2/\alpha}u_0(x) = \infty$$

since $\frac{2}{\alpha} > \gamma + m$.

Remark 1.13. Using (6.9) below and (1.8), we have that g in (1.27) satisfies the explicit bound

$$\alpha^{-1/\alpha}I_m(1,x) \le g(x) \le (\alpha\varepsilon)^{-1/\alpha}I_m((1-\varepsilon),x), \quad x \in \Omega_m,$$

for all $0 < \varepsilon < 1$, where

$$I_m(\delta,x) = \prod_{i=1}^m\left(\frac{1}{\sqrt{\pi}}\int_{-\frac{x_i}{2\sqrt{\delta}}}^{\frac{x_i}{2\sqrt{\delta}}} e^{-y^2}\,dy\right).$$

In Section 7 of this paper, we reinterpret the results of the previous sections on the global well-posedness and the asymptotic behavior of $\mathcal{S}_m(t)u_0$, $u_0 \in \mathcal{X}_{m,\gamma}$, in the case of antisymmetric functions defined on the whole space $\mathbb{R}^N$. Recall that the heat semigroup on $\mathbb{R}^N$ is given by $e^{t\Delta}\varphi = G_t \star \varphi$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, where G_t is the Gauss kernel on $\mathbb{R}^N$,

$$G_t(x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$$

for all $t > 0$ and $x \in \mathbb{R}^N$. The heat semigroup $e^{t\Delta}$ was studied in [4] on the space

$$\mathcal{W}^\sigma = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N\backslash\{0\});\ |x|^\sigma u(x) \in L^\infty(\mathbb{R}^N)\} \quad \text{with } 0 < \sigma < N. \tag{1.28}$$

It was observed in [9] that we can consider the case $N \le \sigma < 2N$ for some class of antisymmetric initial values in $\mathcal{W}^\sigma$. See [9, Corollary 1.7, p. 346] and the discussion just after.

If $\psi\colon \Omega_m \to \mathbb{R}$, we denote by $\widetilde{\psi}$ its pointwise extension to $\mathbb{R}^N$ which is antisymmetric with respect to $x_1, x_2, \dots, x_m$. If $\psi \in \mathcal{X}_{m,\gamma}$, $\widetilde{\psi}$ has a natural interpretation as an element of $\mathcal{S}'(\mathbb{R}^N)$. See [9, Definition 1.6, p. 346]. We also define the space

$$\widetilde{\mathcal{X}_{m,\gamma}} = \{\widetilde{\psi};\ \psi \in \mathcal{X}_{m,\gamma}\} \subset \mathcal{S}'(\mathbb{R}^N)$$

with the norm $\|\varphi\|_{\widetilde{\mathcal{X}_{m,\gamma}}} = \|\varphi_{|\Omega_m}\|_{\mathcal{X}_{m,\gamma}}$ for all $\varphi \in \widetilde{\mathcal{X}_{m,\gamma}}$. We also consider

$$\widetilde{\mathcal{B}_{m,\gamma,M}} = \{\widetilde{\psi};\ \psi \in \mathcal{B}_{m,\gamma,M}\}.$$

We denote by $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$ the ball $\widetilde{\mathcal{B}_{m,\gamma,M}}$ endowed with the weak $\star$ topology. $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$ inherits the metric space structure from $\mathcal{B}^\star_{m,\gamma,M}$. In addition, we observe that $\widetilde{\mathcal{X}_{m,\gamma}} \subset \mathcal{W}^{\gamma+m}$ with continuous injection. However the two norms are not equivalent. On the other hand, $\widetilde{\mathcal{B}^\star_{m,\gamma,M}} \subset (\mathcal{B}^{\gamma+m}_M)^\star$ where $(\mathcal{B}^{\gamma+m}_M)^\star$ denote the closed ball of radius M on $\mathcal{W}^{\gamma+m}$ endowed with the weak $\star$ topology, but here the metric on $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$ is equivalent to the one it inherits from the metric space $(\mathcal{B}^{\gamma+m}_M)^\star$. See Proposition 7.1 below.

The heat semigroup $e^{t\Delta}$ is well-defined on $\widetilde{\mathcal{X}_{m,\gamma}}$, and $\widetilde{e^{t\Delta_m}\psi} = e^{t\Delta}\widetilde{\psi}$. See [9, Proposition 5.1, p. 361]. The last formula is the key to the study equation (1.1) in the space $\widetilde{\mathcal{X}_{m,\gamma}}$. The following result is essentially a reformulation of Theorem 1.2 for antisymmetric functions on $\mathbb{R}^N$.

Theorem 1.14. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$. If $v_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$, then there exists a unique solution $v \in C((0, \infty), C_0(\mathbb{R}^N) \cap \mathcal{A})$ of equation* (1.1) *such that*
(i) *$v(t) \to v_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\})$ as $t \to 0$,*
(ii) *there exists $C > 0$, independent of v_0, such that $\|v(t)\|_{\widetilde{\mathcal{X}_{m,\gamma}}} \le C\|v_0\|_{\widetilde{\mathcal{X}_{m,\gamma}}}$ for all $t > 0$.*
In addition, the following properties hold.
(iii) *For all $w_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$, $|v(t) - w(t)| \le e^{t\Delta}|v_0 - w_0|$, where w is the solution of* (1.1) *with initial value w_0 satisfying* (i) *and* (ii).
(iv) *$v(t)$ satisfies the integral equation*

$$v(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}(|v(s)|^\alpha v(s))\,ds \quad \text{for all } t > 0.$$

Since $\widetilde{\mathcal{X}_{m,\gamma}} \subset \mathcal{W}^{\gamma+m}$, where $\mathcal{W}^{\gamma+m}$ is given by (1.28), the last result gives a new class of initial values for which we have global well-posedness of solutions in the case $\alpha < \frac{2}{N}$ (when $\gamma + m > N$). See [1] and [3, Section 4] for information about non-uniqueness of solutions in the case $\alpha < \frac{2}{N}$.

The semiflow $\mathcal{S}(t)$ defined by (1.12) extends to $\widetilde{\mathcal{X}_{m,\gamma}}$ as the following.

Definition 1.15. Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$. Given $v_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$, we set $\mathcal{S}(t)v_0 = v(t)$ for all $t > 0$, where $v \in C((0, \infty), C_0(\mathbb{R}^N) \cap \mathcal{A})$ is the unique solution of (1.1) given by Theorem 1.14.

From the construction in Theorem 1.14 and the uniqueness part, we have $\mathcal{S}(t)\widetilde{u_0} = \widetilde{\mathcal{S}_m(t)u_0}$ for all $t > 0$ and $u_0 \in \mathcal{X}_{m,\gamma}$. As in the case of the sectors Ω_m, the flow $\mathcal{S}(t)$ depends continuously on the initial values. The following is an adaptation of Theorem 1.4.

Theorem 1.16. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Then $\mathcal{S}(t)$ is continuous, $\widetilde{\mathcal{B}^\star_{m,\gamma,M}} \to C_0(\mathbb{R}^N)$, for all $t > 0$.*

We now consider the long-time asymptotic behavior of the solutions described in Theorem 1.14. In analogy with (1.21) and (1.22) above, and using a notation consistent with [4, formulas (1.17) and (1.18)] and [3, Definition 1.2], we make the following definitions. For $v_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$, we define the ω-limit set of possible asymptotic forms of v_0 by

$$\Omega^{\gamma+m}(v_0) = \{z \in \widetilde{\mathcal{B}^\star_{m,\gamma,M}};\ \text{there exists } \lambda_n \to \infty \text{ such that } \lim_{n\to\infty} D^{\gamma+m}_{\lambda_n} v_0 = z \text{ in } \widetilde{\mathcal{B}^\star_{m,\gamma,M}}\},$$

and the ω-limit set of all limits of $\Gamma^{\gamma+m}_{\sqrt{t}}\mathcal{S}(1)v_0$ as $t \to \infty$ by

$$\omega^{\gamma+m}(v_0) = \{f \in C_0(\mathbb{R}^N);\ \text{there exists } t_n \to \infty \text{ such that } \lim_{n\to\infty} \|\Gamma^{\gamma+m}_{\sqrt{t_n}}\mathcal{S}(1)v_0 - f\|_{L^\infty(\mathbb{R}^N)} = 0\}.$$

The following three theorems are reformulations of the results above on the asymptotic behavior of solutions, adapted from the case of the sectors Ω_m to the case of antisymmetric functions on $\mathbb{R}^N$, in the three cases: α equals, is greater than, and is less than $\frac{2}{\gamma+m}$.

Theorem 1.17. *Let $m \in \{1, 2, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let $\alpha > 0$ be such that $\alpha = \frac{2}{\gamma+m}$. It follows that*
(i) *if $v_0, \varphi \in \widetilde{\mathcal{B}_{m,\gamma,M}}$ are such that $\lim_{\lambda\to\infty} D_\lambda^{\gamma+m} v_0 = \varphi$ in $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$, then φ is homogeneous of degree $-(\gamma+m)$, and the solution $v(t) = \mathcal{S}(t)v_0$ of (1.1) is asymptotically self-similar to $U(t) = \mathcal{S}(t)\varphi$,*
(ii) *$\omega^{\gamma+m}(v_0) = \mathcal{S}(1)\Omega^{\gamma+m}(v_0)$ for all $v_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$,*
(iii) *there exists $V_0 \in \widetilde{\mathcal{B}_{m,\gamma,M}} \cap C^\infty(\mathbb{R}^N)$ such that $\omega^{\gamma+m}(V_0) = \mathcal{S}(1)\widetilde{\mathcal{B}_{m,\gamma,M}}$.*

Theorem 1.18. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let $\alpha > 0$ be such that $\alpha > \frac{2}{\gamma+m}$. It follows that*
(i) *if $v_0, \varphi \in \widetilde{\mathcal{B}_{m,\gamma,M}}$ are such that $\lim_{\lambda\to\infty} D_\lambda^{\gamma+m} v_0 = \varphi$ in $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$, then φ is homogeneous of degree $-(\gamma+m)$, and the solution $v(t) = \mathcal{S}(t)v_0$ of (1.1) is asymptotic to the self-similar solution of the linear heat equation $U(t) = e^{t\Delta}\varphi$,*
(ii) *$\omega^{\gamma+m}(v_0) = e^{\Delta}\Omega^{\gamma+m}(v_0)$ for all $v_0 \in \widetilde{\mathcal{B}_{m,\gamma,M}}$,*
(iii) *there exists $V_0 \in \widetilde{\mathcal{B}_{m,\gamma,M}} \cap C^\infty(\mathbb{R}^N)$ such that $\omega^{\gamma+m}(V_0) = e^{\Delta}\widetilde{\mathcal{B}_{m,\gamma,M}}$.*

Theorem 1.19. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$ such that $\alpha < \frac{2}{\gamma+m}$. Let $v_0 \in \widetilde{\mathcal{X}_{m,\gamma}}$ with $v_0|_{\Omega_m} \geq 0$, and let $v(t) = \mathcal{S}(t)v_0$ be the resulting solution of (1.1) as given by Definition 1.15. Suppose that there exist $R_0 > 0$ and $c_0 > 0$ such that $v_0(x) \geq c_0\psi_0(x)$, $x \in \Omega_m$, $|x| \geq R_0$, where ψ_0 is given by (1.6). Then*

$$\lim_{t\to\infty} t^{1/\alpha} v(t, x\sqrt{t}) = g(x)$$

uniformly on compact subsets of $\mathbb{R}^N$, where $g \in C^{b,u}(\mathbb{R}^N)$ is the antisymmetric (bounded, uniformly continuous) profile of the self-similar solution of (1.1) given by Proposition 7.3.

Finally, in the appendix, for completeness, we give a proof of Kato's parabolic inequality and the main application for which we use it. Also, we present some results which we found during the course of research for this article, which we feel have some independent interest, but which ultimately were not needed for the proofs of the main results. One of them concerns the lowest eigenvalue and corresponding eigenfunction for $-\Delta$ on $B_1 = \{x \in \Omega_m : |x| < 1\}$ with Dirichlet boundary conditions.

2 Existence and Continuity Properties of Solutions

The purpose of this section is to study well-posedness of equation (1.1) with initial values in $\mathcal{X}_{m,\gamma}$ and to give the proofs of Theorem 1.2 and Theorem 1.4. For this purpose, we need several results from [9], sometimes in a slightly stronger version. The first result below is a slight improvement of [9, Proposition 2.5, p. 353].

Proposition 2.1. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $\psi \in \mathcal{X}_{m,\gamma}$. Then $e^{t\Delta_m}\psi \to \psi$ as $t \to 0$ in $L^1_{\mathrm{loc}}(\Omega_m)$. In particular, the convergence is also in $\mathcal{D}'(\Omega_m)$.*

Proof. Let $\psi \in \mathcal{X}_{m,\gamma}$, and let K be a fixed compact in Ω_m. Let $\varepsilon = d(0, K) > 0$, and let $\eta \in C^\infty(\mathbb{R}^N)$ denote a radial cut-off function satisfying
(i) $0 \leq \eta \leq 1$ for all $x \in \mathbb{R}^N$,
(ii) $\eta(x) = 1$ for all $x \in \mathbb{R}^N$ with $|x| \leq \frac{\varepsilon}{4}$,
(iii) $\eta(x) = 0$ for all $x \in \mathbb{R}^N$ with $|x| \geq \frac{\varepsilon}{2}$.
We write

$$e^{t\Delta_m}\psi = e^{t\Delta_m}[\eta\psi] + e^{t\Delta_m}[(1-\eta)\psi]. \tag{2.1}$$

Using the inequality

$$e^{-\frac{(x_i-y_i)^2}{4t}} - e^{-\frac{(x_i+y_i)^2}{4t}} = e^{-\frac{x_i^2}{4t}} e^{-\frac{y_i^2}{4t}} \int_{-\frac{x_iy_i}{2t}}^{\frac{x_iy_i}{2t}} e^s\,ds \leq \frac{x_iy_i}{t} e^{-\frac{(x_i-y_i)^2}{4t}} \quad \text{for all } i \in \{1, \dots, m\},$$

we deduce from (1.8) that, for all $x, y \in \Omega_m$,

$$K_t(x,y) \le t^{-m}\Big(\prod_{i=1}^{m} x_i y_i\Big) G_t(x-y). \tag{2.2}$$

Therefore,

$$|e^{t\Delta_m}[\eta\psi](x)| \le C \int_{\Omega_m} K_t(x,y) y_1 \cdots y_m \eta(y)|y|^{-\gamma-2m}\,dy \le Ct^{-m} x_1 \cdots x_m \int_{|y|\le\frac{\varepsilon}{2}} G_t(x-y)|y|^{-\gamma}\,dy.$$

Since, for $x \in K$ (hence $|x| \ge \varepsilon$) and $|y| \le \frac{\varepsilon}{2}$, we have $|x-y| \ge |x| - |y| \ge \frac{\varepsilon}{2}$, it follows that

$$|e^{t\Delta_m}[\eta\psi](x)| \le Cx_1 \cdots x_m t^{-(m+N/2)} e^{-\frac{\varepsilon^2}{16t}} \int_{|y|\le\frac{\varepsilon}{2}} |y|^{-\gamma}\,dy \quad \text{for all } x \in K.$$

This implies that $e^{t\Delta_m}[\eta\psi] \to 0$ a.e. pointwise on K as $t \to 0$. Moreover, by Proposition [9, Theorem 1.1 (i), p. 343], we have $|e^{t\Delta_m}[\eta\psi]| \le C\psi_0$ for all $t > 0$. Thus, by the dominated convergence theorem, $e^{t\Delta_m}[\eta\psi] \to 0$ in $L^1(K)$ as $t \to 0$.

On the other hand, since $(1-\eta)\psi \in L^p(\Omega_m)$ for $p > \max\{1, \frac{N}{\gamma+m}\}$, it follows that $e^{t\Delta_m}[(1-\eta)\psi] \to (1-\eta)\psi$ in $L^p(\Omega_m)$ as $t \to 0$. In particular, since $K \subset \Omega_m$ is compact, $e^{t\Delta_m}[(1-\eta)\psi] \to \psi$ in $L^1(K)$ as $t \to 0$. Using (2.1), we obtain $e^{t\Delta_m}\psi \to \psi$ in $L^1_{\text{loc}}(\Omega_m)$ as $t \to 0$. This completes the proof. ☐

We also need to use a stronger version of [9, Lemma 2.6, p. 355], as follows.

Lemma 2.2. *Let $m \in \{1, \dots, N\}$ and $0 < \gamma < N$. There exists $C > 0$ such that*

$$|e^{t\Delta_m}\psi(x)| \le Cx_1 \cdots x_m (t + |x|^2)^{-\frac{\gamma+2m}{2}} \|\psi\|_{\mathcal{X}_{m,\gamma}}$$

for all $t > 0$, $x \in \Omega_m$ and $\psi \in \mathcal{X}_{m,\gamma}$.

Proof. It suffices to prove the lemma for $\psi = \psi_0$. Since ψ_0 is homogeneous, we know that $e^{t\Delta_m}\psi_0$ is self-similar, and so

$$e^{t\Delta_m}\psi_0(x) = t^{-\frac{\gamma+m}{2}} f\Big(\frac{x}{\sqrt{t}}\Big), \tag{2.3}$$

where $f := e^{\Delta_m}\psi_0$. By [9, Proposition 2.2, p. 349], we have

$$f(x) = e^{\Delta_m}\psi_0(x) \le C\psi_0(x) \le Cx_1 \cdots x_m |x|^{-\gamma-2m}$$

for all $x \in \Omega_m$. Therefore, there exists $C > 0$ such that $f(x) \le Cx_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}$ for $|x| \ge 1$. On the other hand, for all $x \in \Omega_m$, we have

$$f(x) = \int_{\Omega_m} K_1(x,y)\psi_0(y)\,dy.$$

Using inequality (2.2), we obtain

$$\begin{aligned} f(x) &\le Cx_1 \cdots x_m \int_{\Omega_m} G_1(x-y) y_1^2 \cdots y_m^2 |y|^{-\gamma-2m}\,dy \le Cx_1 \cdots x_m \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4}} |y|^{-\gamma}\,dy \\ &\le Cx_1 \cdots x_m (e^{\Delta}|\cdot|^{-\gamma})(x) \le Cx_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma}{2}} \end{aligned}$$

by [1, Corollary 8.3, p. 531]. Hence, for $|x| \le 1$, we have

$$f(x) \le Cx_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

Therefore, there exists $C > 0$ such that

$$f(x) \le Cx_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}$$

for all $x \in \Omega_m$. Using relation (2.3), we deduce that

$$e^{t\Delta_m}\psi_0(x) \le Cx_1 \cdots x_m (t + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

This proves the result. ☐

The following is a version of [1, Corollary 8.3, p. 531] adapted from $\mathbb{R}^N$ to Ω_m.

Corollary 2.3. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $A > 0$. There exists $C > 0$ such that if $\tau \geq 0$ and $u_0 \in \mathcal{X}_{m,\gamma}$ is such that $|u_0(x)| \leq A x_1 \cdots x_m (\tau + |x|^2)^{-\frac{\gamma+2m}{2}}$ for $x \in \Omega_m$, then*

$$|e^{t\Delta_m} u_0(x)| \leq C x_1 \cdots x_m (\tau + t + |x|^2)^{-\frac{\gamma+2m}{2}}$$

for all $t > 0$ and all $x \in \Omega_m$.

Proof. From Lemma 2.2, the result is true for $\tau = 0$. Next, we consider the case $\tau = 1$. We put

$$g(x) = x_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

Using (2.2), we obtain

$$\begin{aligned} e^{t\Delta_m} g(x) &\leq t^{-m} x_1 \cdots x_m \int_{\mathbb{R}^N} G_t(x-y) y_1^2 \cdots y_m^2 (1 + |y|^2)^{-\frac{\gamma+2m}{2}} \, dy \\ &\leq t^{-m} x_1 \cdots x_m \int_{\mathbb{R}^N} G_t(x-y)(1 + |y|^2)^{-\frac{\gamma}{2}} \, dy. \end{aligned}$$

By [1, Corollary 8.3, p. 531], we have

$$e^{t\Delta_m} g(x) \leq C x_1 \cdots x_m t^{-m} (1 + t + |x|^2)^{-\frac{\gamma}{2}} \leq C x_1 \cdots x_m (1 + t + |x|^2)^{-\frac{\gamma+2m}{2}}$$

for $t \geq 1 + |x|^2$, so that $(2t)^{-m} \leq (1 + t + |x|^2)^{-m}$.

If $t \leq 1 + |x|^2$, we have $(1 + |x|^2)^{-\frac{\gamma+2m}{2}} \leq C(1 + t + |x|^2)^{-\frac{\gamma+2m}{2}}$, so it suffices to prove that

$$e^{t\Delta_m} g(x) \leq C x_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

Using (1.10), we obtain

$$e^{t\Delta_m} g(x) = \int_{\Omega_m} K_t(x, y) y_1 \cdots y_m (1 + |y|^2)^{-\frac{\gamma+2m}{2}} \, dy \leq x_1 \cdots x_m.$$

Hence, for $|x| \leq 1$,

$$e^{t\Delta_m} g(x) \leq C x_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

In addition, $g \in \mathcal{X}_{m,\gamma}$, so, by [9, Theorem 1.1 (i), p. 343],

$$e^{t\Delta_m} g(x) \leq C\psi_0(x) \leq C x_1 \cdots x_m |x|^{-\gamma-2m}.$$

Therefore, if $|x| > 1$ so that $(1 + |x|^2)^{\frac{\gamma+2m}{2}} \leq (2|x|)^{\gamma+2m}$, we have

$$e^{t\Delta_m} g(x) \leq C x_1 \cdots x_m (1 + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

It follows that

$$|e^{t\Delta_m} u_0(x)| \leq e^{t\Delta_m} g(x) \leq C x_1 \cdots x_m (1 + t + |x|^2)^{-\frac{\gamma+2m}{2}} \tag{2.4}$$

for all $x \in \Omega_m$ and all $t > 0$. This proves the result for $\tau = 1$.

For the general case, we proceed by scaling and observe that

$$D^{\gamma+m}_{\frac{1}{\sqrt{\tau}}} g(x) = x_1 \cdots x_m (\tau + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

Using formula (1.9) and inequality (2.4), we obtain

$$|e^{t\Delta_m} u_0(x)| \leq e^{t\Delta_m} [D^{\gamma+m}_{\frac{1}{\sqrt{\tau}}} g](x) = D^{\gamma+m}_{\frac{1}{\sqrt{\tau}}} [e^{\frac{t}{\tau}\Delta_m} g](x) \leq C x_1 \cdots x_m (\tau + t + |x|^2)^{-\frac{\gamma+2m}{2}}.$$

This completes the proof. □

We will also use the following lemma, which gives a property of convergence in $L^1_{\mathrm{loc}}(\Omega_m)$ which is not shared by convergence in $\mathcal{D}'(\Omega_m)$.

Lemma 2.4. *Let $(w_k)_{k\geq 1} \subset \mathcal{B}_{m,\gamma,M}$ and $w \in \mathcal{B}_{m,\gamma,M}$ be such that $w_k \xrightarrow[k\to\infty]{} w$ in $L^1_{\mathrm{loc}}(\Omega_m)$. Then*

$$e^{t\Delta_m}|w_k| \xrightarrow[k\to\infty]{} e^{t\Delta_m}|w| \quad \text{in } C_0(\Omega_m).$$

Proof. Since $w_k \to w$ in $L^1_{\mathrm{loc}}(\Omega_m)$, then $|w_k| \to |w|$ in $L^1_{\mathrm{loc}}(\Omega_m)$; hence $|w_k| \to |w|$ in $\mathcal{D}'(\Omega_m)$. From [9, Proposition 3.1 (i), p. 356], and since $(|w_k|)_{k\geq 1}, |w| \subset \mathcal{B}_{m,\gamma,M}$, we deduce that $|w_k| \to |w|$ in $\mathcal{B}^\star_{m,\gamma,M}$. Since by [9, Proposition 4.1 (ii), p. 359], $e^{t\Delta_m} : \mathcal{B}^\star_{m,\gamma,M} \to C_0(\Omega_m)$ is continuous, it follows that $e^{t\Delta_m}|w_k| \to e^{t\Delta_m}|w|$ in $C_0(\Omega_m)$ as $k \to \infty$. ☐

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $u_0 \in \mathcal{X}_{m,\gamma}$ and let $(\mathcal{K}_n)_{n\geq 1}$ be the sequence of nondecreasing compacts in Ω_m defined by

$$\mathcal{K}_n = \Big\{x \in \Omega_m \text{ such that } d(x, \partial\Omega_m) \geq \frac{1}{n} \text{ and } |x| \leq n\Big\}.$$

We consider the function

$$u_{0,n} = \xi_n u_0, \tag{2.5}$$

where ξ_n is a cut-off function satisfying
(i) $\xi_n \in C^\infty(\Omega_m)$,
(ii) $0 \leq \xi_n \leq 1$ for all $x \in \Omega_m$,
(iii) $\xi_n(x) = 1$ for all $x \in \mathcal{K}_n$,
(iv) $\xi_n(x) = 0$ for all $x \in \Omega_m \backslash \mathcal{K}_{n+1}$.
Note that $u_{0,n} \in \mathcal{X}_{m,\gamma}$ for all $n \geq 1$, and
- $\|u_{0,n}\|_{\mathcal{X}_{m,\gamma}} \leq \|u_{0,n+1}\|_{\mathcal{X}_{m,\gamma}} \leq \|u_0\|_{\mathcal{X}_{m,\gamma}}$;
- $u_{0,n} \to u_0$ pointwise and in $L^1_{\mathrm{loc}}(\Omega_m)$ (hence in $\mathcal{D}'(\Omega_m)$) as $n \to \infty$;
- for a fixed compact K on Ω_m, there exists n_0 such that $u_{0,n} = u_0$ on K for all $n \geq n_0$.

Existence: The proof is motivated by the proof of [1, Theorem 8.8, p. 536]. Since $u_{0,n} \in L^p(\Omega_m)$, $1 \leq p < \infty$, we consider the unique solution $u_n \in C([0,\infty), L^p(\Omega_m)) \cap C((0,\infty), C_0(\Omega_m))$ of (1.1) with initial value $u_{0,n} \in L^p(\Omega_m)$. It follows from Kato's parabolic inequality (see Corollary A.2 in the appendix) that, for all $n, \ell \in \mathbb{N}^\star$,

$$|u_n(t) - u_\ell(t)| \leq e^{t\Delta_m}|u_{0,n} - u_{0,\ell}| \quad \text{for all } t > 0. \tag{2.6}$$

Since $|u_{0,n} - u_{0,\ell}| \leq |u_{0,n} - u_0|$ for all $\ell > n$, we have

$$|u_n(t) - u_\ell(t)| \leq e^{t\Delta_m}|u_{0,n} - u_0| \tag{2.7}$$

for all $t > 0$ and $\ell > n$. In addition, $\|u_{0,n} - u_0\|_{\mathcal{X}_{m,\gamma}} \leq 2\|u_0\|_{\mathcal{X}_{m,\gamma}}$ for all $n \geq 1$, and $(u_{0,n} - u_0) \to 0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $n \to \infty$, and so it follows from Lemma 2.4 that $e^{t\Delta_m}|u_{0,n} - u_0| \to 0$ in $C_0(\Omega_m)$ as $n \to \infty$. Therefore, from (2.7), $u_n(t)$ is a Cauchy sequence in $C_0(\Omega_m)$ for all $t > 0$, and so there exists a function $u(t)$ such that $u_n(t)$ converge to $u(t)$ in $C_0(\Omega_m)$. Furthermore, by letting $\ell \to \infty$ in (2.6), we obtain

$$|u_n(t) - u(t)| \leq e^{(t-\varepsilon)\Delta_m}[e^{\varepsilon\Delta_m}|u_{0,n} - u_0|]$$

for all $t > \varepsilon > 0$. Since $e^{\varepsilon\Delta_m}|u_{0,n} - u_0| \to 0$ in $C_0(\Omega_m)$ as $n \to \infty$, and $e^{(t-\varepsilon)\Delta_m}$ is C_0 contraction on $C_0(\Omega_m)$, we deduce that u_n converges to u on $L^\infty([\varepsilon,\infty), C_0(\Omega_m))$ for all $\varepsilon > 0$. The limit function $u \in C((0,\infty), C_0(\Omega_m))$ is clearly a solution of (1.1).

Again by Corollary A.2, we have

$$|u_n(t)| \leq e^{t\Delta_m}|u_{0,n}| \leq e^{t\Delta_m}|u_0| \in \mathcal{X}_{m,\gamma}$$

for all $n \geq 1$. In addition, by letting $n \to \infty$, we obtain

$$|u(t)| \leq e^{t\Delta_m}|u_0|, \tag{2.8}$$

and so, by [9, Theorem 1.1 (i), p. 343], we deduce that $\|u(t)\|_{\mathcal{X}_{m,\gamma}} \leq C\|u_0\|_{\mathcal{X}_{m,\gamma}}$ for all $t > 0$. This proves (ii).

It remains now to show that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$. We fix a compact subset $K \subset \Omega_m$ and n such that $u_{0,n} = u_0$ on K. Thus,

$$\int_K |u(t) - u_0| = \int_K |u(t) - u_{0,n}| \le \int_K |u(t) - u_n(t)| + \int_K |u_n(t) - u_{0,n}|.$$

By letting $\ell \to \infty$ in (2.6), we have

$$|u_n(t) - u(t)| \le e^{t\Delta_m}|u_{0,n} - u_0|.$$

Proposition 2.1 shows that $e^{t\Delta_m}|u_{0,n} - u_0| \to |u_{0,n} - u_0|$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$. Therefore,

$$\int_K e^{t\Delta_m}|u_{0,n} - u_0| \xrightarrow[t\to 0]{} \int_K |u_{0,n} - u_0| = 0,$$

and so $\int_K |u(t) - u_n(t)| \to 0$ as $t \to 0$. Since $u_n \in C([0,\infty), L^p(\Omega_m))$, we have $u_n(t) \to u_{0,n}$ in $L^p(\Omega_m)$ as $t \to 0$ so that

$$\int_K |u_n(t) - u_{0,n}| \to 0 \quad \text{as } t \to 0.$$

This proves that $u(t)$ converges to u_0 in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$, and so (i) is proved.

Uniqueness: Let $s > 0$, and let u, v be two solutions of (1.1) satisfying (i) and (ii). We have

$$|u(t+s) - v(t+s)| \le e^{t\Delta_m}|u(s) - v(s)| \le e^{t\Delta_m}|u(s) - u_0| + e^{t\Delta_m}|v(s) - u_0|$$

for all $t > s > 0$. Let $M \ge C\|u_0\|_{\mathcal{X}_{m,\gamma}}$. Since $u(s), v(s) \in \mathcal{B}^{\star}_{m,\gamma,M}$ for all $s > 0$ and $u(s), v(s) \to u_0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $s \to 0$, it follows from Lemma 2.4 that the right-hand side of the last inequality tends to 0 in $C_0(\Omega_m)$ as $s \to 0$. This gives $|u(t+s) - v(t+s)| \to 0$ as $s \to 0$. But, since $u, v \in C((0,\infty), C_0(\Omega_m))$, we deduce that $|u(t+s) - v(t+s)| \to |u(t) - v(t)|$ as $s \to 0$ for every fixed $t > 0$. By uniqueness of the limit, we have $u(t) = v(t)$ for all $t > 0$.

Additional properties: We next give the proof of statements (iii), (iv), and (vi). In fact, by (2.8), we have $|u(t)| \le e^{t\Delta_m}|u_0|$, and so, from Lemma 2.2, we obtain

$$|u(t,x)| \le C x_1 \cdots x_m (t + |x|^2)^{-\frac{\gamma+2m}{2}} \|u_0\|_{\mathcal{X}_{m,\gamma}}$$

for all $t > 0$ and $x \in \Omega_m$. In addition, if $u_0, v_0 \in \mathcal{X}_{m,\gamma}$, we denote $u(t)$ and $v(t)$ the corresponding solutions. For all $n \ge 1$, we let $u_{0,n} = u_0\xi_n$ and $v_{0,n} = v_0\xi_n$, where ξ_n is the cut-off function defined by (2.5). Then, for all $n \ge 1$,

$$|u_n(t) - v_n(t)| \le e^{t\Delta_m}|u_{0,n} - v_{0,n}|.$$

Letting $n \to \infty$ and using Lemma 2.4, we deduce that

$$|u(t) - v(t)| \le e^{t\Delta_m}|u_0 - v_0|.$$

Finally, assertion (vi) is true since, under the same conditions, $|u_n(t)| \le v_n(t)$ by well-known comparison results.

Integral equation: Since $u_{0,n} \in L^p(\Omega_m)$ for all $p > \max[1, \frac{N\alpha}{2}]$, the corresponding solution $u_n(t)$ satisfies the integral equation

$$u_n(t) = e^{t\Delta_m}u_{0,n} - \int_0^t e^{(t-s)\Delta_m}(|u_n(s)|^\alpha u_n(s))\,ds$$

for all $t > 0$, where each term is in $C([0,\infty); L^p(\Omega_m))$.

Since $u_{0,n} \to u_0$ in $\mathcal{B}^{\star}_{m,\gamma,M}$ as $n \to \infty$, we know, for example by Lemma 2.4, that $e^{t\Delta_m}u_{0,n} \to e^{t\Delta_m}u_0$ in $C_0(\Omega_m)$ as $n \to \infty$ for all $t > 0$. On the other hand, for all $0 < s < t$,

$$e^{(t-s)\Delta_m}(|u_n(s)|^\alpha u_n(s)) \to e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s)) \quad \text{on } C_0(\Omega_m) \quad \text{as } n \to \infty.$$

From property (iii) above and [6, inequality (1.8), p. 261], we have

$$|u_n(s)| \le Cx_1 \cdots x_m(s+|x|^2)^{-\frac{\gamma+2m}{2}} \quad \text{and} \quad |u_n(s)| \le (\alpha s)^{-1/\alpha}$$

for all $s > 0$. Therefore, for all $0 < \varepsilon < 1$,

$$\begin{aligned} |u_n(s)|^{\alpha+1} &= |u_n(s)|^{\alpha(1-\varepsilon)}|u_n(s)|^{1+\alpha\varepsilon} \\ &\le C|u_n(s)|^{\alpha(1-\varepsilon)}(\omega(x))^{1+\alpha\varepsilon}(s+|x|^2)^{-\frac{(\gamma+m)}{2}(1+\alpha\varepsilon)} \\ &\le \frac{C}{s^{1-\varepsilon}}\omega(x)(s+|x|^2)^{-\frac{(\gamma+m)}{2}(1+\alpha\varepsilon)}, \end{aligned}$$

where $\omega(x) = x_1 \cdots x_m(s+|x|^2)^{-\frac{m}{2}} \le 1$. We then take $\varepsilon < \frac{N-\gamma}{\alpha(\gamma+m)}$ so that $(\gamma+m)(1+\alpha\varepsilon) = \gamma'+m$, $0 < \gamma' < N$ and $0 < \gamma < \gamma'$. It follows that

$$|u_n(s)|^{\alpha+1} \le \frac{C}{s^{1-\varepsilon}}x_1 \cdots x_m(s+|x|^2)^{-\frac{\gamma'+2m}{2}}. \tag{2.9}$$

We deduce by Corollary 2.3 that

$$e^{(t-s)\Delta_m}|u_n(s)|^{\alpha+1} \le \frac{C}{s^{1-\varepsilon}}x_1 \cdots x_m(t+|x|^2)^{-\frac{\gamma'+2m}{2}}$$

for all $t > s > 0$. Likewise, since $u_n(s) \to u(s)$ in $C_0(\Omega_m)$,

$$e^{(t-s)\Delta_m}|u(s)|^{\alpha+1} \le \frac{C}{s^{1-\varepsilon}}x_1 \cdots x_m(t+|x|^2)^{-\frac{\gamma'+2m}{2}} \tag{2.10}$$

for all $t > s > 0$ so that $s \to e^{(t-s)\Delta_m}|u(s)|^{\alpha+1}$ is in $L^1((0,t);C_0(\Omega_m))$.

We deduce, using the dominated convergence theorem,

$$\int_0^t e^{(t-s)\Delta_m}(|u_n(s)|^\alpha u_n(s))\,ds \to \int_0^t e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s))\,ds \quad \text{as } n \to \infty,$$

and so that the solution $u(t)$ satisfies

$$u(t) = e^{t\Delta_m}u_0 - \int_0^t e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s))\,ds.$$

This proves (v).

Note that (2.10) implies that

$$\left|\int_0^t e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s))\,ds\right| \le Ct^\varepsilon x_1x_2 \cdots x_m(t+|x|^2)^{-\frac{\gamma'+2m}{2}},$$

with ε and γ' as above. □

The following lemma is needed to establish Theorem 1.4.

Lemma 2.5. *Let $(u_n)_{n\ge 0}$ be a sequence of solutions of (1.1), $u_n \in C((0,\infty), C_0(\Omega_m))$, satisfying*

$$|u_n(t,x)| \le Cx_1 \cdots x_m(t+|x|^2)^{-\frac{\gamma+2m}{2}} \quad \textit{for all } x \in \Omega_m \textit{ and all } t > 0. \tag{2.11}$$

There exists a subsequence $(u_{n_k})_{k\ge 0}$ and a solution $g \in C((0,\infty), C_0(\Omega_m))$ of (1.1) such that $u_{n_k} \to g$ as $k \to \infty$, in $C([\tau,\infty), C_0(\Omega_m))$ for every $\tau > 0$.

Proof. Fix $\tau > 0$. Using (2.11) with $t = \frac{\tau}{2}$, we deduce that the set $\{u_n(\frac{\tau}{2}),\ n \ge 1\}$ is bounded in $L^p(\Omega_m)$ for all p satisfying $\max(1, \frac{N}{\gamma+m}) < p \le \infty$. By standard smoothing effects, we see that the $u_n(\tau) = \mathcal{S}_m(\frac{\tau}{2})u_n(\frac{\tau}{2})$ are uniformly bounded in $W^{1,\infty}(\Omega_m)$. Thus, $\{u_n(\tau)\}$ is relatively compact in $C(\overline{\Omega_m^R})$ for all $R > 1$, where

$\Omega_m^R = \{x \in \Omega_m;\ |x| \le R\}$. Using the decay estimate (2.11), $\{u_n(\tau)\}$ is also relatively compact on $C_0(\Omega_m)$. By continuous dependence in $C_0(\Omega_m)$ of (1.1), it follows that $\{u_n(\cdot), n \ge 1\}$ is relatively compact in $C([\tau, T], C_0(\Omega_m))$ for all $T > \tau$, the limit points being solutions of (1.1). Since, by (2.11), $\|u_n(t)\|_{L^\infty} \to 0$ as $t \to \infty$ uniformly in $n \ge 1$, we may let $T = \infty$ in the previous property. By letting $\tau \to 0$ and using a diagonal procedure, we see that there exists a solution $g \in C((0, \infty), C_0(\Omega_m))$ of (1.1) and a subsequence $(u_{n_k})_{k\ge0}$ such that $u_{n_k} \to g$ as $k \to \infty$ in $C([\tau, \infty), C_0(\Omega_m))$ for every $\tau > 0$. □

Proof of Theorem 1.4. Let $(u_{0,n})_{n\ge0} \subset \mathcal{B}_{m,\gamma,M}$ and $u_0 \in \mathcal{B}_{m,\gamma,M}$ such that $u_{0,n} \to u_0$ in $\mathcal{B}^\star_{m,\gamma,M}$ as $n \to \infty$. Let $u(t) = \mathcal{S}_m(t)u_0$ and $u_n(t) = \mathcal{S}_m(t)u_{0,n}$ for all $t > 0$ be the corresponding solutions of (1.1) as in Definition 1.3. By Theorem 1.2 (iv), we have

$$|u_n(t, x)| \le Cx_1 \cdots x_m(t + |x|^2)^{-\frac{\gamma+2m}{2}} \tag{2.12}$$

for all $x \in \Omega_m$ and $t > 0$. It follows from Lemma 2.5 that there exists a solution $g \in C((0, \infty), C_0(\Omega_m))$ of (1.1) and a subsequence $(u_{n_k})_{k\ge0}$ such that $u_{n_k} \to g$ as $k \to \infty$ in $C([\tau, \infty), C_0(\Omega_m))$ for every $\tau > 0$. To see that $g(t) \to u_0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$, we consider a compact $K \subset \Omega_m$ and let $\mathcal{O}$ be an open, bounded, and regular subset of Ω_m with $K \subset \mathcal{O}$. By (2.12), we have $|u_n(t, x)| \le C$ for all $x \in \mathcal{O}$, $t > 0$, and $n \ge 0$. Since $u_0, (u_{0,n})_{n\ge0} \subset \mathcal{B}^\star_{m,\gamma,M}$ and $u_{0,n} \to u_0$ in $\mathcal{B}^\star_{m,\gamma,M}$ as $n \to \infty$, we have by [9, Proposition 3.1 (i), p. 356] that $u_{0,n} \to u_0$ in $\mathcal{D}'(\Omega_m)$; we conclude using [3, Lemma 2.6, p. 89] that $g(t) \to u_0$ in $L^1(\mathcal{O})$ as $t \to 0$. Therefore, $g(t) \to u_0$ in $L^1_{\mathrm{loc}}(\Omega_m)$ as $t \to 0$, and from uniqueness of solutions of (1.1), we have $g \equiv u$ so that the limit g is determined by u_0. In particular, it does not depend on the subsequence $(u_{n_k})_{k\ge0}$ so that the whole sequence $(u_n)_{n\ge0}$ converges to u in $C([\tau, \infty), C_0(\Omega_m))$ for every $\tau > 0$. This completes the proof. □

3 An Upper Bound on Solutions

In this section, we prove Proposition 1.5. This proposition is stated for solutions on the domain Ω_m, but in fact is valid for solutions of (1.1), or rather the associated integral equation, on any domain Ω. Accordingly, we state here the more general version. Both the statement and proof are inspired by the statement and proof of [13, Theorem 1]. Moreover, we introduce some notation which will be used solely in this section.

Let $\Omega \subset \mathbb{R}^N$ be a domain, not necessarily bounded, and let $C_0(\Omega)$ be the space of continuous functions $f\colon \overline{\Omega} \to \mathbb{R}$ such that $f \equiv 0$ on the boundary $\partial\Omega$ and $f(x) \to 0$ as $|x| \to \infty$ in Ω. Let $e^{t\Delta}$ be the heat semigroup on $C_0(\Omega)$, given by a kernel $k_t = k_t^\Omega$ as follows:

$$e^{t\Delta}f(x) = \int_\Omega k_t(x, y)f(y)\,dy. \tag{3.1}$$

In particular, if $f \in L^1_{\mathrm{loc}}(\Omega)$, $f \ge 0$, then $e^{t\Delta}f$ is likewise defined by formula (3.1).

Theorem 3.1. *Fix $\alpha > 0$. Let $u_0 \in L^1_{\mathrm{loc}}(\Omega)$, $u_0 \ge 0$, and suppose that the continuous function $u\colon (0, T) \to C_0(\Omega)$ is a nonnegative solution of the integral equation*

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}(u(s)^{\alpha+1})\,ds.$$

It follows that

$$u(t, x) \le ((e^{t\Delta}u_0(x))^{-\alpha} + \alpha t)^{-1/\alpha} = \frac{e^{t\Delta}u_0(x)}{(1 + \alpha t(e^{t\Delta}u_0(x))^\alpha)^{1/\alpha}}$$

for all $0 < t < T$ and all $x \in \Omega$.

Proof. Fix $0 < \tau < T$, and set

$$G(t) = e^{(\tau-t)\Delta}u(t) = e^{\tau\Delta}u_0 - \int_0^t e^{(\tau-s)\Delta}(u(s)^{\alpha+1})\,ds \quad \text{for all } 0 \le t \le \tau. \tag{3.2}$$

It is clear from the integral expression in (3.2) that $G\colon [0,\tau] \to C_0(\Omega)$ is a continuous, decreasing function with $G(0) = e^{\tau\Delta}u_0$ and $G(\tau) = u(\tau)$. Furthermore, $G\colon (0,\tau] \to C_0(\Omega)$ is continuously differentiable, and

$$G'(t) = -e^{(\tau-t)\Delta}(u(t)^{\alpha+1}) = -\int_\Omega k_{\tau-t}(\cdot\,,y)u(t,y)^{\alpha+1}\,dy.$$

Since, for all $x \in \Omega$, the measure $k_{\tau-t}(x,y)\,dy$ on Ω has total mass less than or equal to 1, Jensen's inequality implies that

$$G'(t) = -\int_\Omega k_{\tau-t}(\cdot\,,y)u(t,y)^{\alpha+1}\,dy \le -\Big(\int_\Omega k_{\tau-t}(\cdot\,,y)u(t,y)\,dy\Big)^{\alpha+1} = -(e^{(\tau-t)\Delta}u(t))^{\alpha+1} = -G(t)^{\alpha+1}.$$

Integrating this last differential inequality on $[0,t]$, we obtain

$$G(t) \le \frac{1}{(G(0)^{-\alpha} + \alpha t)^{1/\alpha}},$$

which is the same as

$$e^{(\tau-t)\Delta}u(t) \le \frac{1}{((e^{\tau\Delta}u_0)^{-\alpha} + \alpha t)^{1/\alpha}}.$$

This is true for $0 < \tau < T$ and $0 \le t \le \tau$. The result follows by setting $t = \tau > 0$. □

Remark 3.2. Using an argument similar to the above, one can obtain an analogous estimate for positive solutions of the more general equation

$$u_t = \Delta u - f(u),$$

where f is a positive, convex, increasing C^2 function in $(0,\infty)$ such that $F(s) = \int_s^\infty \frac{1}{f(\sigma)}\,d\sigma < \infty$ for all $s > 0$. Precisely, we have

$$u(t) \le F^{-1}(F(e^{t\Delta}u_0) + t),$$

where F^{-1} is the inverse function of F.

4 Self-similar Asymptotic Behavior on Sectors

In this section, we consider equation (1.1) in the case $\frac{2}{\alpha} = \gamma + m$. Let $u_0 \in \mathcal{X}_{m,\gamma}$, and set $u(t) = \mathcal{S}_m(t)u_0$. Using (1.18), we can re-write definition (1.22) of the ω-limits set $\mathcal{Q}^\gamma(u_0)$ in the equivalent form

$$\mathcal{Q}^\gamma(u_0) = \{f \in C_0(\Omega_m);\ \text{there exists } \lambda_n \to \infty \text{ such that } \lim_{n\to\infty} \|\mathcal{S}_m(1)D_{\lambda_n}^{2/\alpha}u_0 - f\|_{L^\infty(\Omega_m)} = 0\}.$$

We begin by proving Theorem 1.6 which corresponds to the particular case when $\mathcal{Q}^\gamma(u_0)$ contains one nontrivial element.

Proof of Theorem 1.6. Using limits in the sense of $\mathcal{D}'(\Omega_m)$, we have

$$D_\mu^{\gamma+m}\varphi = D_\mu^{\gamma+m}\big(\lim_{\lambda\to\infty} D_\lambda^{\gamma+m}\psi\big) = \lim_{\lambda\to\infty} D_\mu^{\gamma+m}D_\lambda^{\gamma+m}\psi = \lim_{\lambda\to\infty} D_{\mu\lambda}^{\gamma+m}\psi = \varphi$$

for all $\mu > 0$. It follows that φ is homogeneous of degree $-(\gamma+m)$. By uniqueness of solutions of (1.1), we deduce that the corresponding solution $U(t) = \mathcal{S}_m(t)\varphi$ is self-similar. By Theorem 1.4, we have

$$\lim_{\lambda\to\infty} \mathcal{S}_m(t)D_\lambda^{\gamma+m}\psi = \mathcal{S}_m(t)\varphi$$

in $C_0(\Omega_m)$ for all $t > 0$. From (1.18) and Theorem 1.2 (iii), we obtain

$$\lim_{\lambda\to\infty} \Gamma_\lambda^{\gamma+m}\mathcal{S}_m(\cdot)\psi = U(\cdot) \quad \text{in } C([\tau,t]; C_0(\Omega_m))$$

for all $0 < \tau < t$ so that $\mathcal{S}_m(t)\psi$ is asymptotically self-similar to the self-similar solution $U(t)$. □

We give now the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $u_0 \in \mathcal{X}_{m,\gamma}$ and $M > 0$ be such that $M > \|u_0\|_{\mathcal{X}_{m,\gamma}}$. If $f \in \mathcal{S}_m(1)\mathcal{Z}^\gamma(u_0)$, then there exists $z \in \mathcal{Z}^\gamma(u_0)$ such that $f = \mathcal{S}_m(1)z$. Since $z \in \mathcal{Z}^\gamma(u_0)$, there exists $\lambda_n \to \infty$ such that $D^{\gamma+m}_{\lambda_n} u_0 \to z$ in $\mathcal{B}^\star_{m,\gamma,M}$. We deduce by Theorem 1.4 that $\mathcal{S}_m(1)D^{\gamma+m}_{\lambda_n} u_0 \to \mathcal{S}_m(1)z = f$ in $C_0(\Omega_m)$. Then $f \in \mathcal{Q}^\gamma(u_0)$, and so $\mathcal{S}_m(1)\mathcal{Z}^\gamma(u_0) \subset \mathcal{Q}^\gamma(u_0)$.

Conversely, if $f \in \mathcal{Q}^\gamma(u_0)$, then there exists $\lambda_n \to \infty$ such that $\mathcal{S}_m(1)D^{\gamma+m}_{\lambda_n} u_0 \to f$ in $C_0(\Omega_m)$. Since $\mathcal{B}^\star_{m,\gamma,M}$ is compact, there exist a subsequence $(\lambda_{n_k})_{k\geq 1}$ such that $D^{\gamma+m}_{\lambda_{n_k}} u_0 \to w$ in $\mathcal{B}^\star_{m,\gamma,M}$. Again by Theorem 1.4, $\mathcal{S}_m(1)D^{\gamma+m}_{\lambda_{n_k}} u_0 \to \mathcal{S}_m(1)w$ in $C_0(\Omega_m)$ as $k \to \infty$. Therefore, we have $f = \mathcal{S}_m(1)w \in \mathcal{S}_m(1)\mathcal{Z}^\gamma(u_0)$. This proves the result. □

Proof of Corollary 1.8. This follows immediately from [9, Theorem 1.4, p. 345] and Theorem 1.7. □

5 Linear Asymptotic Behavior on Sectors

In this section, we study the long-time asymptotic behavior of solutions to (1.1) in the case $\frac{2}{\alpha} < \gamma + m$. The key point is that, under the dilations $D^{\gamma+m}_{\sqrt{t}}$, which preserve the norm of $\mathcal{X}_{m,\gamma}$, the integral term in (1.14) decays faster than the difference between the two other terms. This is the content of the next proposition.

Proposition 5.1. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $\alpha > 0$ such that $\alpha > \frac{2}{\gamma+m}$. Let $u_0 \in \mathcal{X}_{m,\gamma}$ and $u(t) = \mathcal{S}_m(t)u_0$. It follows that $D^{\gamma+m}_{\sqrt{t}}(u(t) - e^{t\Delta_m}u_0) \to 0$ in $C_0(\Omega_m)$ as $t \to \infty$.*

Proof. We know that, for all $t > 0$,

$$u(t) - e^{t\Delta_m}u_0 = -\int_0^t e^{(t-s)\Delta_m}(|u(s)|^\alpha u(s))\,ds = -t\int_0^1 e^{t(1-\sigma)\Delta_m}(|u(\sigma t)|^\alpha u(\sigma t))\,d\sigma.$$

Therefore, using (1.9), we have

$$D^{\gamma+m}_{\sqrt{t}}(u(t) - e^{t\Delta_m}u_0) = -t\int_0^1 D^{\gamma+m}_{\sqrt{t}} e^{t(1-\sigma)\Delta_m}(|u(\sigma t)|^\alpha u(\sigma t))\,d\sigma = -t\int_0^1 e^{(1-\sigma)\Delta_m}(D^{\gamma+m}_{\sqrt{t}}|u(\sigma t)|^\alpha u(\sigma t))\,d\sigma.$$

On the other hand, estimating as in (2.9), we see that

$$|u(\sigma t)|^{\alpha+1} \leq \frac{C}{(\sigma t)^{1-\varepsilon}} x_1 \cdots x_m(\sigma t + |x|^2)^{-\frac{\gamma'+2m}{2}}$$

for all $0 < \varepsilon < \min(1, \frac{N-\gamma}{\alpha(\gamma+m)})$, where $\gamma < \gamma' < N$ satisfies $\gamma - \gamma' = -\alpha\varepsilon(\gamma + m)$. Therefore,

$$D^{\gamma+m}_{\sqrt{t}}|u(\sigma t)|^{\alpha+1} = t^{\frac{\gamma+m}{2}}|u(\sigma t, \sqrt{t}x)|^{\alpha+1} \leq \frac{C}{(\sigma t)^{1-\varepsilon}} t^{\frac{\gamma-\gamma'}{2}} x_1 \cdots x_m(\sigma + |x|^2)^{-\frac{\gamma'+2m}{2}}.$$

By Corollary 2.3, we deduce that

$$e^{(1-\sigma)\Delta_m}(D^{\gamma+m}_{\sqrt{t}}|u(\sigma t)|^{\alpha+1}) \leq \frac{C}{(\sigma t)^{1-\varepsilon}} t^{\frac{\gamma-\gamma'}{2}} x_1 \cdots x_m(1 + |x|^2)^{-\frac{\gamma'+2m}{2}} \leq \frac{C}{(\sigma t)^{1-\varepsilon}} t^{\frac{\gamma-\gamma'}{2}}.$$

It follows that

$$|D^{\gamma+m}_{\sqrt{t}}(u(t) - e^{t\Delta_m}u_0)| \leq t\int_0^1 e^{(1-\sigma)\Delta_m}(D^{\gamma+m}_{\sqrt{t}}|u(\sigma t)|^{\alpha+1})\,d\sigma \leq Ct^{\frac{\gamma-\gamma'}{2}+\varepsilon}\int_0^1 \frac{d\sigma}{\sigma^{1-\varepsilon}} \leq Ct^{\varepsilon(1-\frac{\alpha(\gamma+m)}{2})}.$$

Since $\alpha(\gamma + m) > 2$, we see that $D^{\gamma+m}_{\sqrt{t}}(u(t) - e^{t\Delta_m}u_0) \to 0$ in $C_0(\Omega_m)$ as $t \to \infty$. This proves the result. □

Proof of Theorem 1.10. The three statements in this theorem follow from Proposition 5.1 and, respectively, [9, Corollary 4.2, p. 360, Corollary 1.3, p. 345 and Theorem 1.4, p. 345]. □

6 Nonlinear Asymptotic Behavior on Sectors

In this section, we consider equation (1.1) with nonnegative initial value $u_0 \in \mathcal{X}_{m,\gamma}$ in the case $\alpha < \frac{2}{\gamma+m}$, and our goal is to prove Theorem 1.11. First however, we need to show that the hypothesis on the initial condition u_0, which gives a lower bound for large $|x|$, implies a lower bound on the resulting solution at any fixed positive time. The key point is the behavior near the boundary. We prove the following result.

Proposition 6.1. *Let $u_0 \in \mathcal{X}_{m,\gamma}$ with $u_0 \geq 0$, and suppose that there exist $\rho > 0$ and $c > 0$ such that, for all $x \in \Omega_m$ with $|x| \geq \rho$, $u_0(x) \geq c\psi_0(x)$, where ψ_0 is given by* (1.6). *Let $u(t,\cdot) = u(t) = \mathcal{S}_m(t)u_0$ be the resulting solution of* (1.1) *as given by Theorem 1.2, and fix any $t_0 > 0$. It follows that $v_0 \equiv \mathcal{S}_m(t_0)u_0$ verifies the condition*

$$v_0(x) \geq \begin{cases} c' x_1 x_2 \cdots x_m, & 0 < |x| \leq 1, \\ c' x_1 x_2 \cdots x_m |x|^{-(\gamma+2m)}, & |x| \geq 1, \end{cases} \tag{6.1}$$

for some $c' > 0$, where the constant c' may depend on t_0.

We refer the reader to [8] for results of this type on a general domain. The present situation differs from that in [8] in that the sector Ω_m does not have the required regularity, and also that here we include the possibility that u_0 could be identically zero on a bounded subset of Ω_m. Unlike [8], our proof makes use of the explicit form of the kernel for the heat semigroup on Ω_m.

Proof. We first note that it suffices by comparison to prove this for the specific initial value

$$u_0(x) = \begin{cases} 0, & x \in \Omega_m,\ |x| < \rho, \\ c\psi_0(x), & x \in \Omega_m,\ |x| \geq \rho, \end{cases} \tag{6.2}$$

where ψ_0 is given by (1.6), and $\rho > 0$ is arbitrary. To accomplish this, we first prove that, for any fixed $t_0 > 0$, $v_0 = e^{t_0\Delta_m}u_0$ verifies (6.1), where u_0 is given by (6.2). For this purpose, since the estimate is linear, the value of $c > 0$ in (6.2) is of no importance.

Thus, we consider $e^{t\Delta_m}u_0$ on Ω_m given by (1.7) and (1.8), where u_0 is given by (6.2). Using the fact that $e^s - e^{-s} \geq 2s$ for all $s \geq 0$, we see that if $x, y \in \Omega_m$ and $1 \leq i \leq m$ (so that $x_i \geq 0$ and $y_i \geq 0$), then

$$e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}} = e^{-\frac{|x_i|^2}{4t}} e^{-\frac{|y_i|^2}{4t}} [e^{\frac{x_i y_i}{2t}} - e^{-\frac{x_i y_i}{2t}}] \geq \left[\frac{x_i y_i}{t}\right] e^{-\frac{|x_i|^2}{4t}} e^{-\frac{|y_i|^2}{4t}} \geq \left[\frac{x_i y_i}{t}\right] e^{-\frac{|x_i|^2}{2t}} e^{-\frac{|y_i|^2}{2t}}. \tag{6.3}$$

In addition, for $m+1 \leq j \leq N$, we have (since $(s-r)^2 \leq 2s^2 + 2r^2$)

$$e^{-\frac{|x_j-y_j|^2}{4t}} \geq e^{-\frac{|x_j|^2}{2t}} e^{-\frac{|y_j|^2}{2t}}.$$

It follows that

$$\begin{aligned} e^{t\Delta_m}u_0(x) &= \int_{\Omega_m} K_t(x,y)u_0(y)\,dy \\ &= (4\pi t)^{-\frac{N}{2}} \int_{\Omega_m} \prod_{j=m+1}^{N} e^{-\frac{|x_j-y_j|^2}{4t}} \prod_{i=1}^{m} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] u_0(y)\,dy \\ &\geq c x_1 x_2 \cdots x_m t^{-m} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \int_{\Omega_m} y_1 y_2 \cdots y_m e^{-\frac{|y|^2}{2t}} u_0(y)\,dy \\ &= c x_1 x_2 \cdots x_m t^{-m} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \int_{\substack{y \in \Omega_m \\ |y| \geq \rho}} y_1^2 y_2^2 \cdots y_m^2 e^{-\frac{|y|^2}{2t}} |y|^{-\gamma-2m}\,dy. \end{aligned} \tag{6.4}$$

This shows in particular that, for any $t > 0$, $e^{t\Delta_m}u_0$ satisfies (6.1), but only on any given bounded set in Ω_m.

We turn our attention to the case where $|x|$ is large. For $x \in \Omega_m$, let

$$\Omega_m(x) = \{y \in \Omega_m : 0 < x_i \leq y_i \leq \max[2x_i, 2],\ 1 \leq i \leq m,\ 0 < |x_i| \leq |y_i| \leq \max[2|x_i|, 2],\ m < i \leq N\}.$$

If $y \in \Omega_m(x)$, then

$$|y|^2 = \sum_{i=1}^{N} y_i^2 \le \sum_{i=1}^{N} \max[2|x_i|, 2]^2 \le \sum_{i=1}^{N} (4x_i^2 + 4) = 4|x|^2 + 4N.$$

Since this calculation is for large $|x|$, we may suppose that $|x|^2 \ge N$, and so we see that

$$y \in \Omega_m(x) \implies |y| \le 2\sqrt{2}|x| \le 4|x|.$$

Also, we want to use the specific formula in (6.2), so we impose $|x| \ge \rho$, where ρ is as in (6.2). Hence

$$y \in \Omega_m(x) \implies |y| \ge \rho.$$

We can now calculate

$$\begin{aligned}
e^{t\Delta_m} u_0(x) &= \int_{\Omega_m} K_t(x,y) u_0(y)\, dy \ge \int_{\Omega_m(x)} K_t(x,y) u_0(y)\, dy = \int_{\Omega_m(x)} K_t(x,y) y_1 y_2 \cdots y_m |y|^{-\gamma-2m}\, dy \\
&\ge 4^{-\gamma-2m} |x|^{-\gamma-2m} \int_{\Omega_m(x)} K_t(x,y) y_1 y_2 \cdots y_m\, dy \\
&= 4^{-\gamma-2m} |x|^{-\gamma-2m} (4\pi t)^{-\frac{N}{2}} \Big(\prod_{i=1}^{m} \int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i \Big) \\
&\quad \times \Big(\prod_{i=1+m}^{N} \int_{|x_i| \le |y_i| \le \max[2|x_i|,2]} e^{-\frac{|x_i-y_i|^2}{4t}}\, dy_i \Big) \\
&= 4^{-\gamma-2m} |x|^{-\gamma-2m} (4\pi t)^{-\frac{N}{2}} \Big(\prod_{i=1}^{m} \int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i \Big) \\
&\quad \times \Big(\prod_{i=1+m}^{N} \int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i \Big).
\end{aligned} \tag{6.5}$$

We first need to examine the integral

$$\int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i$$

for $1 \le i \le m$, under two different circumstances, $0 < x_i < 1$ and $x_i \ge 1$. Consider first the case $x_i \ge 1$. We have, since $y_i \ge x_i$,

$$\begin{aligned}
\int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i &\ge x_i \int_{x_i}^{2x_i} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i = x_i \int_0^{x_i} [e^{-\frac{|y_i|^2}{4t}} - e^{-\frac{|2x_i+y_i|^2}{4t}}]\, dy_i \\
&\ge x_i \int_0^{x_i} [e^{-\frac{|y_i|^2}{4t}} - e^{-\frac{|2x_i|^2}{4t}} e^{-\frac{|y_i|^2}{4t}}]\, dy_i = x_i \int_0^{x_i} e^{-\frac{|y_i|^2}{4t}} [1 - e^{-\frac{|2x_i|^2}{4t}}]\, dy_i \\
&\ge x_i [1 - e^{-\frac{1}{t}}] \int_0^1 e^{-\frac{|y_i|^2}{4t}}\, dy_i = C_t^1 x_i.
\end{aligned}$$

We next consider the case $x_i \le 1$. We have, by (6.3),

$$\begin{aligned}
\int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i &\ge \int_1^2 \Big[\frac{x_i y_i}{t}\Big] e^{-\frac{|x_i|^2}{2t}} e^{-\frac{|y_i|^2}{2t}} y_i\, dy_i = x_i e^{-\frac{|x_i|^2}{2t}} \int_1^2 \Big[\frac{y_i^2}{t}\Big] e^{-\frac{|y_i|^2}{2t}}\, dy_i \\
&\ge x_i e^{-\frac{1}{2t}} \int_1^2 \Big[\frac{y_i^2}{t}\Big] e^{-\frac{|y_i|^2}{2t}}\, dy_i = C_t^2 x_i.
\end{aligned}$$

We second need to examine the integral

$$\int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i$$

for $m+1 \le i \le N$, under two different circumstances, $0 < |x_i| < 1$ and $|x_i| \ge 1$. Consider first the case $|x_i| \ge 1$. We have, if in addition $x_i < 0$, that is $-x_i \ge 1$,

$$\begin{aligned}\int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i &= \int_{-x_i}^{-2x_i} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i \\ &\ge \int_{-x_i}^{-2x_i} [e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i = \int_0^{-x_i} e^{-\frac{|y_i|^2}{4t}}\, dy_i \\ &\ge \int_0^1 e^{-\frac{|y_i|^2}{4t}}\, dy_i = C_t^3.\end{aligned}$$

We have, if in addition $x_i > 0$, that is $x_i \ge 1$,

$$\begin{aligned}\int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i &= \int_{x_i}^{2x_i} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i \\ &\ge \int_{x_i}^{2x_i} [e^{-\frac{|x_i-y_i|^2}{4t}}]\, dy_i = \int_0^{x_i} e^{-\frac{|y_i|^2}{4t}}\, dy_i \\ &\ge \int_0^1 e^{-\frac{|y_i|^2}{4t}}\, dy_i = C_t^3.\end{aligned}$$

We next consider the case $|x_i| \le 1$. By the inequality $e^{-\frac{|x_j-y_j|^2}{4t}} \ge e^{-\frac{|x_j|^2}{2t}} e^{-\frac{|y_j|^2}{2t}}$, we have

$$\int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i \ge \int_1^2 [e^{-\frac{|x_i-y_i|^2}{4t}}]\, dy_i \ge \int_1^2 e^{-\frac{|x_i|^2}{2t}} e^{-\frac{|y_i|^2}{2t}}\, dy_i \ge e^{-\frac{1}{2t}} \int_1^2 e^{-\frac{|y_i|^2}{2t}}\, dy_i = C_t^4.$$

In all cases, we have

$$\int_{x_i}^{\max[2x_i,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} - e^{-\frac{|x_i+y_i|^2}{4t}}] y_i\, dy_i \ge C_t x_i \quad \text{for } 1 \le i \le m, \tag{6.6}$$

$$\int_{|x_i|}^{\max[2|x_i|,2]} [e^{-\frac{|x_i-y_i|^2}{4t}} + e^{-\frac{|x_i+y_i|^2}{4t}}]\, dy_i \ge C_t \qquad \text{for } m+1 \le i \le N, \tag{6.7}$$

whenever $x \in \Omega_m$. It therefore follows from (6.5), (6.6), (6.7) that, if $x \in \Omega_m$, then

$$e^{t\Delta_m} u_0(x) \ge C_t x_1 x_2 \cdots x_m |x|^{-\gamma-2m}, \quad |x| \ge \max[\sqrt{N}, \rho]. \tag{6.8}$$

Combining (6.4) and (6.8), we obtain that, for any fixed $t > 0$, $e^{t\Delta_m} u_0$ satisfies (6.1).

We next show the same result for $u(t,\cdot) = u(t) = \mathcal{S}_m(t)u_0$ being the resulting solution of (1.1), where u_0 is given by (6.2). To do so, set $w(t) = e^{\mu t}u(t)$, where $\mu = [c\rho^{\gamma+m}]^\alpha \ge \|u_0\|^\alpha_{L^\infty(\Omega_m)}$. Since $u(t) \le \|u_0\|_{L^\infty(\Omega_m)}$ for all $t > 0$, we have $u(t)^\alpha \le \mu$ for all $t > 0$. It follows that

$$w'(t) = e^{\mu t}u'(t) + e^{\mu t}\mu u(t) \ge e^{\mu t}u'(t) + e^{\mu t}u(t)^\alpha u(t) = e^{\mu t}\Delta u(t) = \Delta w(t).$$

Hence $w(t) \ge e^{t\Delta_m} w(0) = e^{t\Delta_m} u_0$. In other words, $u(t) \ge e^{-\mu t} e^{t\Delta_m} u_0$, which implies the desired result. $\square$

Remark 6.2. In addition to being well-posed in $C_0(\Omega_m)$, in $L^q(\Omega_m)$ for $1 \le q < \infty$, as noted in the introduction, and in $\mathcal{X}_{m,\gamma}$, as per Theorem 1.2, equation (1.1) is globally well-posed in $L^\infty(\Omega_m)$ in the following sense. For every $u_0 \in L^\infty(\Omega_m)$, there is a unique solution $u \in C((0,\infty); C_0^{b,u}(\Omega_m))$ of the integral equation (1.14), where $C_0^{b,u}(\Omega_m)$ denotes the closed subspace of $L^\infty(\Omega_m)$ of bounded, uniformly continuous functions on Ω_m which are zero on $\partial\Omega_m$, but not necessarily as $|x| \to \infty$. This solution has the following additional properties: the function u is a classical solution of (1.1) on $(0,\infty) \times \Omega_m$, $\|u(t) - e^{t\Delta_m}u_0\|_{L^\infty(\Omega_m)} \to 0$ as $t \to 0$, and $|u(t)| \le (\alpha t)^{1/\alpha}$ for all $t > 0$. One way to see this is first to establish the corresponding result on $L^\infty(\mathbb{R}^N)$, but of course with $C^{b,u}(\mathbb{R}^N)$ instead of $C_0^{b,u}(\Omega_m)$, and then to restrict to antisymmetric functions on $\mathbb{R}^N$. The result on $\mathbb{R}^N$ follows from standard arguments, i.e. contraction mapping, parabolic regularity, and comparison. We refer the reader to [2, Appendices B and C] for detailed information about $e^{t\Delta}$ on $C^{b,u}(\mathbb{R}^N)$. In particular, [2, Lemma B.1] establishes that $e^{t\Delta}h \in C^{b,u}(\mathbb{R}^N)$ for all $h \in L^\infty(\mathbb{R}^N)$, and [2, Theorem C.1], which is still valid for the nonlinear heat equation with absorption, establishes the necessary regularity.

Proposition 6.3. *Let $m \in \{1, 2, \dots, N\}$ and $\alpha > 0$. There exists a self-similar solution $V(t,x) = t^{-1/\alpha} g(\frac{x}{\sqrt{t}})$ of equation* (1.1) *such that $g \in C_0^{b,u}(\Omega_m)$, the space of bounded uniformly continuous functions on Ω_m which are zero on $\partial\Omega_m$, $g \ge 0$, and*

$$\alpha^{-1/\alpha} e^{\Delta_m} h \le g \le (\alpha\varepsilon)^{-1/\alpha} e^{(1-\varepsilon)\Delta_m} h \tag{6.9}$$

for all $0 < \varepsilon < 1$, where $h(x) = 1$ is the constant function on Ω_m.

The self-similar solution V is characterized by

$$V = \lim_{\lambda\to\infty} \Gamma_\lambda^{2/\alpha} v, \tag{6.10}$$

where v is the solution to (1.1) *with initial value $v_0 = h$, as described in Remark 6.2, the dilations $\Gamma_\lambda^{2/\alpha}$ are defined by* (1.17), *and where the limit* (6.10) *is uniform on compact subsets of* $(0,\infty) \times \overline{\Omega}_m$.

We observe that, in the case $m = 0$, the corresponding self-similar solution is $(\alpha t)^{-1/\alpha}$.

Proof. Throughout this proof, we let $h \in L^\infty(\Omega_m)$ denote the specific function $h(x) = 1$, $x \in \Omega_m$. It follows from (1.9) that

$$(e^{\lambda^2 t\Delta_m} h)(\lambda x) = (e^{t\Delta_m} h)(x) \tag{6.11}$$

Next we let $v = v(t,x)$ be the global solution of (1.1) or (1.14) with initial value $v_0 = h$, i.e. $v_0(x) = v(0,x) = 1$ for all $x \in \Omega_m$, as described in Remark 6.2. For all $\lambda > 0$, $v_\lambda(t,x) = \lambda^{2/\alpha} v(\lambda^2 t, \lambda x)$ is likewise a solution of (1.1) or (1.14), but with initial value $v_{0,\lambda}(x) = v_\lambda(0,x) = \lambda^{2/\alpha} v_0(\lambda x) = \lambda^{2/\alpha}$ for all $x \in \Omega_m$. Since $\lambda \to v_{0,\lambda}$ is an increasing function, by comparison so must be $\lambda \to v_\lambda$. Moreover, we know that

$$v_\lambda(t,x) \le (\alpha t)^{-1/\alpha} \tag{6.12}$$

so that the v_λ must converge to some function $V(t,x) \le (\alpha t)^{-1/\alpha}$, and in particular, $V(t) \in L^\infty(\Omega_m)$ for $t > 0$. Since each v_λ is a solution of the integral equation (1.14) on every interval $[\varepsilon, T] \subset (0,\infty)$, the same must be true for V by the monotone convergence theorem. Hence V is a solution of (1.1) and $V \in C((0,\infty); C_0^{b,u}(\Omega_m))$ since initial values in $L^\infty(\Omega_m)$ give rise to solutions of (1.14) in $C((0,\infty); C_0^{b,u}(\Omega_m))$ as per Remark 6.2. Note that, by parabolic regularity and standard compactness arguments, the convergence of the v_λ to V is uniform on compact subsets of $(0,\infty) \times \overline{\Omega}_m$. Moreover, V a self-similar solution, being the limit of the dilated solutions v_λ. Thus, we can write

$$V(t,x) = t^{-1/\alpha} g\Big(\frac{x}{\sqrt{t}}\Big),$$

where $g = V(1) \in C_0^{b,u}(\Omega_m)$ is the profile of V.

As for the behavior of g, we first observe that, for $t > 0$ and $\varepsilon > 0$, by (6.12),

$$v_\lambda(t+\varepsilon, \cdot) \le e^{t\Delta_m}(v_\lambda(\varepsilon)) \le (\alpha\varepsilon)^{-1/\alpha} e^{t\Delta_m} h.$$

Letting $\lambda \to \infty$, we see that, for $t > 0$ and $\varepsilon > 0$,

$$V(t+\varepsilon) \le e^{t\Delta_m} V(\varepsilon) \le (\alpha\varepsilon)^{-1/\alpha} e^{t\Delta_m} h$$

so that $g = V(1) \leq (\alpha\varepsilon)^{-1/\alpha} e^{(1-\varepsilon)\Delta_m} h$ for all small $0 < \varepsilon < 1$. Also, $V(2t) \leq (\alpha t)^{-1/\alpha} e^{t\Delta_m} h$. On the other hand, we claim that, for all $t > 0$,

$$V(t) \geq (\alpha t)^{-1/\alpha} e^{t\Delta_m} h. \tag{6.13}$$

To see this, we first show that

$$v(t) \geq (1 + \alpha t)^{-1/\alpha} e^{t\Delta_m} h. \tag{6.14}$$

Indeed, if we set $w(t) = (1 + \alpha t)^{1/\alpha} v(t)$ so that $w(0) = v(0) = h$, then, since $v(t) \leq (1 + \alpha t)^{-1/\alpha}$ (which follows in particular from Proposition 1.5 since $|e^{t\Delta_m} v_0| \leq 1$), we get that

$$w'(t) = (1 + \alpha t)^{1/\alpha - 1} v(t) + (1 + \alpha t)^{1/\alpha} v'(t) \geq (1 + \alpha t)^{1/\alpha} v(t)^{\alpha+1} + (1 + \alpha t)^{1/\alpha} v'(t) = \Delta w(t),$$

which implies that $w(t) \geq e^{t\Delta_m} w(0) = e^{t\Delta_m} h$. This proves (6.14). By (6.11), it follows that

$$v_\lambda(t, x) \geq \lambda^{2/\alpha} (1 + \alpha\lambda^2 t)^{-1/\alpha} (e^{t\lambda^2\Delta_m} h)(\lambda x) = (\lambda^{-2} + \alpha t)^{-1/\alpha} e^{t\Delta_m} h. \tag{6.15}$$

The lower bound (6.13) now follows by letting $\lambda \to \infty$ in (6.15). Hence $g = V(1) \geq \alpha^{-1/\alpha} e^{\Delta_m} h$.

Finally, we note the perhaps curious result that $V(2t) \leq (\alpha t)^{-1/\alpha} e^{t\Delta_m} h \leq V(t)$ for all $t > 0$. □

Proof of Theorem 1.11. By the hypotheses on u_0 and by Proposition 6.1, we have that, for $t_0 > 0$,

$$u(t_0, x) \geq c x_1 \cdots x_m \min[1, |x|^{-\gamma-2m}] \tag{6.16}$$

on Ω_m, and we know that $u(t_0) \in C_0(\Omega_m)$. Up to a translation in time and since we are concerned with the large time behavior, we may suppose that $u_0 \in \mathcal{X}_{m,\gamma} \cap C_0(\Omega_m)$, $u_0 \geq 0$, and it verifies (6.16).

In fact, it suffices to assume

$$u_0(x) = c x_1 \cdots x_m \min[1, |x|^{-\gamma-2m}]. \tag{6.17}$$

Indeed, suppose $u_0(x) = c x_1 \cdots x_m \min[1, |x|^{-\gamma-2m}] \leq v_0(x) \leq c'$ for some $c' > c$, and that $u(t, x)$, $v(t, x)$, and $w(t, x)$ are the solutions of (1.1) with initial values, respectively, u_0, v_0, and $w_0 \equiv c'$. We know by comparison that

$$t^{1/\alpha} u(t, x\sqrt{t}) \leq t^{1/\alpha} v(t, x\sqrt{t}) \leq t^{1/\alpha} w(t, x\sqrt{t}).$$

Hence if we prove that

$$\lim_{t\to\infty} t^{1/\alpha} u(t, x\sqrt{t}) = g(x)$$

uniformly on compact subsets of $\overline{\Omega}_m$, then clearly, since, by Proposition 6.3 along with a rescaling,

$$\lim_{t\to\infty} t^{1/\alpha} w(t, x\sqrt{t}) = g(x)$$

also uniformly on compact subsets of $\overline{\Omega}_m$, it follows that

$$\lim_{t\to\infty} t^{1/\alpha} v(t, x\sqrt{t}) = g(x)$$

uniformly on compact subsets of $\overline{\Omega}_m$. Thus, we now assume the initial value $u_0 \in \mathcal{X}_{m,\gamma}$ is given by (6.17), and we denote by $u(t) = \mathcal{S}_m(t) u_0$ the resulting solution of (1.1) given by Theorem 1.2.

We use a method introduced in [7]. Consider the space-time dilations defined by (1.17) with $\sigma = \frac{2}{\alpha}$,

$$u_\lambda(t, x) = \Gamma_\lambda^{2/\alpha} u(t, x) = \lambda^{2/\alpha} u(\lambda^2 t, \lambda x), \quad \lambda > 0,\ t > 0,\ x \in \Omega_m.$$

In particular, u_λ is the solution of (1.1) with initial data

$$u_{0,\lambda}(x) = D_\lambda^{2/\alpha} u_0(x) = \lambda^{2/\alpha} u_0(\lambda x) = c\lambda^{2/\alpha} x_1 \cdots x_m \min[\lambda^m, \lambda^{-\gamma-m} |x|^{-\gamma-2m}], \quad x \in \Omega_m. \tag{6.18}$$

Since $\frac{2}{\alpha} > \gamma + m$, it follows that $u_{0,\lambda}(x)$ is an increasing function in $\lambda > 0$ for all $x \in \Omega_m$. (It is the minimum of two functions which are obviously increasing in λ.) Consequently, the solutions $u_\lambda(t, x)$ are likewise increasing in $\lambda > 0$. We note also that the solutions $w_\lambda(t, x)$ are increasing in $\lambda > 0$ (as in the proof of Proposition 6.3), where w is the solution with initial value $w_0 \equiv c'$ as above.

Since

$$u_\lambda(t,x) \le w_\lambda(t,x) \le V(t,x) \quad \text{in } (0,\infty)\times\Omega_m,$$

where V is the self-similar solution of Proposition 6.3, it follows that the limit

$$\lim_{\lambda\to\infty} u_\lambda(t,x) = U(t,x) \le V(t,x) \tag{6.19}$$

exists and

$$u_\lambda(t,x) \le U(t,x) \tag{6.20}$$

for all $\lambda > 0$. Moreover, by parabolic regularity and standard compactness arguments, the limit (6.19) is uniform on compact subsets of $(0,\infty)\times\overline{\Omega}_m$.

We next wish to show that $U(t,x) = V(t,x)$ on $(0,\infty)\times\Omega_m$. For this, we need to obtain a lower bound for U. Let $A > 0$, and consider the family of truncated initial values

$$u^A_{0,\lambda}(x) = \min[u_{0,\lambda}(x), A], \quad x \in \Omega_m.$$

Let z^A_λ be the solution of (1.1) with initial data $u^A_{0,\lambda}$. By comparison principle and (6.20),

$$z^A_\lambda(t,x) \le u_\lambda(t,x) \le U(t,x) \quad \text{in } [0,\infty)\times\Omega_m$$

for every $\lambda > 0$ and $A > 0$. Moreover, it is clear from (6.18) that, for each fixed $A > 0$, the initial values $u^A_{0,\lambda}(x)$ are an increasing function of $\lambda > 0$, and so therefore must be the solutions $z^A_\lambda(t,x)$. Furthermore, the initial values satisfy the monotone limit

$$\lim_{\lambda\to\infty} u^A_{0,\lambda}(x) = A, \tag{6.21}$$

and the corresponding solutions converge in a monotone fashion to some function

$$\lim_{\lambda\to\infty} z^A_\lambda(t,x) = Z_A(t,x) \le U(t,x). \tag{6.22}$$

We next consider the integral equation satisfied by $z^A_\lambda(t)$, i.e. equation (1.14) with initial value $u^A_{0,\lambda}$. Using (6.21) and (6.22) along with the monotone convergence theorem, we see that $Z_A(t)$ satisfies

$$Z_A(t) = e^{t\Delta_m}A - \int_0^t e^{(t-s)\Delta_m}(|Z_A(s)|^\alpha Z_A(s))\,ds,$$

i.e. Z_A is the solution of (1.14) with initial value $Z_A(0) \equiv A$ on Ω_m.

We know by (the proof of) Proposition 6.3 that $\lim_{A\to\infty} Z_A(t) = V(t)$, which implies, along with (6.22), that $V(t,x) \le U(t,x)$. Thus, by (6.19), $V(t,x) = U(t,x)$.

Thus, we have shown that

$$\lim_{\lambda\to\infty} u_\lambda(t,x) = \lim_{\lambda\to\infty} \lambda^{2/\alpha}u(\lambda^2 t, \lambda x) = V(t,x) = t^{-1/\alpha} g\Big(\frac{x}{\sqrt{t}}\Big), \tag{6.23}$$

where the limit is uniform on compact subsets of $(0,\infty)\times\overline{\Omega}_m$. The result now follows first by setting $t = 1$ in (6.23), and then by replacing λ^2 by τ. □

7 Case of $\mathbb{R}^N$

In this section, we consider the extension of the results in the previous section on the sectors Ω_m to the case of antisymmetric functions on $\mathbb{R}^N$. Recall that if $\psi\colon \Omega_m \to \mathbb{R}$, then $\widetilde{\psi}$ denotes its pointwise extension to $\mathbb{R}^N$ which is antisymmetric with respect to $x_1, x_2, \dots, x_m$. Similarly, if $K \subset \overline{\Omega}_m$, then $\widetilde{K} \subset \mathbb{R}^N$ denotes its antisymmetric extension. Similar notation is used for spaces of functions, etc.

The following two results show the equivalence of various kinds of convergence on Ω_m to the corresponding convergence on $\mathbb{R}^N$.

Proposition 7.1. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let $(\psi_k)_{k\geq 1} \subset \mathcal{B}_{m,\gamma,M}$ and $\psi \in \mathcal{B}_{m,\gamma,M}$. The following are equivalent:*
(i) $\psi_k \to \psi$ *in* $\mathcal{B}^\star_{m,\gamma,M}$ *as* $k \to \infty$;
(ii) $\psi_k \to \psi$ *in* $\mathcal{D}'(\Omega_m)$ *as* $k \to \infty$;
(iii) $\widetilde{\psi_k} \to \widetilde{\psi}$ *in* $\mathcal{S}'(\mathbb{R}^N)$ *as* $k \to \infty$;
(iv) $\widetilde{\psi_k} \to \widetilde{\psi}$ *in* $\widetilde{\mathcal{B}^\star_{m,\gamma,M}}$ *as* $k \to \infty$;
(v) $\widetilde{\psi_k} \to \widetilde{\psi}$ *in* $\mathcal{D}'(\mathbb{R}^N \backslash \{0\})$ *as* $k \to \infty$;
(vi) $\widetilde{\psi_k} \to \widetilde{\psi}$ *in* $(\mathcal{B}^{\gamma+m}_M)^\star$ *as* $k \to \infty$.

Proof. From [9, Proposition 3.1 (i), p. 356] and [9, Proposition 5.1, p. 361], statements (i), (ii), (iii), and (iv) are equivalent. From [4, Proposition 2.1 (i), p. 1110], we have that (v) and (vi) are equivalent. It is clear that (v) implies (ii) and (iii) implies (v). This proves the result. □

Proposition 7.2. *Let $m \in \{1, \dots, N\}$, $0 < \gamma < N$ and $M > 0$. Let $(\psi_k)_{k\geq 1} \subset \mathcal{B}_{m,\gamma,M}$ and $\psi \in \mathcal{B}_{m,\gamma,M}$. The two following statements are equivalent:*
(i) $\psi_k \to \psi$ *in* $L^1_{\mathrm{loc}}(\Omega_m)$ *as* $k \to \infty$;
(ii) $\widetilde{\psi_k} \to \widetilde{\psi}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\})$ *as* $k \to \infty$.

Proof. (i) ⇒ (ii). It suffices to show $\int_{\rho\leq|x|\leq R} |\widetilde{\psi_k} - \widetilde{\psi}|\,dx \to 0$ for all $0 < \rho < R < \infty$. We know by assumption (i) that, for every $\delta > 0$,

$$\int_{\widetilde{K_\delta}} |\widetilde{\psi_k} - \widetilde{\psi}|\,dx = 2^m \int_{K_\delta} |\psi_k - \psi|\,dx \to 0 \quad \text{as } k \to \infty,$$

where $K_\delta = \{x \in \Omega_m : \rho \leq |x| \leq R, \operatorname{dist}(x, \partial\Omega_m) \geq \delta\}$. On the other hand,

$$\int_{K^c_\delta} |\psi_k - \psi|\,dx \leq 2M \int_{K^c_\delta} \psi_0\,dx \to 0 \quad \text{as } \delta \to 0,$$

where $K^c_\delta = \{x \in \Omega_m : \rho \leq |x| \leq R, \operatorname{dist}(x, \partial\Omega_m) \leq \delta\}$. Thus, given $\varepsilon > 0$, fix $\delta > 0$ so that

$$\int_{K^c_\delta} |\psi_k - \psi|\,dx \leq \frac{\varepsilon}{2^{m+1}}$$

for all $k \geq 1$, and then choose $k_0 > 0$ so that $\int_{K_\delta} |\psi_k - \psi|\,dx \leq \frac{\varepsilon}{2^{m+1}}$ for all $k \geq k_0$.

(ii) ⇒ (i). Let K be a compact of Ω_m. Then, by continuity of the reflection function, $\widetilde{K}$ is a compact of $\mathbb{R}^N \backslash \{0\}$ and $\int_K |\psi_k - \psi|\,dx = 2^{-m} \int_{\widetilde{K}} |\widetilde{\psi_k} - \widetilde{\psi}|\,dx \to 0$ as $k \to \infty$. Hence (i) holds. This establishes the result. □

In light of Propositions 7.1 and 7.2, Theorems 1.14, 1.16, 1.17, 1.18, and 1.19 are now immediate consequences of the analogous results on the sector Ω_m, either by re-interpretation as results about antisymmetric functions on $\mathbb{R}^N$ as described in [12, Section 3], or by simply re-doing the proofs essentially line for line but considering the antisymmetric extension to $\mathbb{R}^N$ of all the functions defined on Ω_m.

We wish, however, to specifically identify the self-similar solution on $\mathbb{R}^N$ which is the antisymmetric extension of the self-similar solution constructed in Proposition 6.3, as we think it is of sufficient independent interest.

Proposition 7.3. *Let $m \in \{1, 2, \dots, N\}$ and $\alpha > 0$. There exists a self-similar solution $V(t, x) = t^{-1/\alpha} g(\frac{x}{\sqrt{t}})$ of equation* (1.1) *such that $g \in C^{b,u}(\mathbb{R}^N)$, the space of bounded uniformly continuous functions on $\mathbb{R}^N$, g is antisymmetric in $x_1, x_2, \dots, x_m$, and*

$$\alpha^{-1/\alpha} e^{\Delta} h(x) \leq g(x) \leq (\alpha\varepsilon)^{-1/\alpha} e^{(1-\varepsilon)\Delta} h(x), \quad x \in \Omega_m,$$

for all $0 < \varepsilon < 1$, where $h \in L^\infty(\mathbb{R}^N)$ is the antisymmetric function such that $h(x) = 1$, $x \in \Omega_m$.

The self-similar solution V is characterized by

$$V = \lim_{\lambda\to\infty} \Gamma^{2/\alpha}_\lambda v,$$

where v is the solution to (1.1) *on $\mathbb{R}^N$ with initial value $v_0 = h$, as described in Remark 6.2, the dilations $\Gamma^{2/\alpha}_\lambda$ are defined by* (1.17)*, and where the limit* (6.10) *is uniform on compact subsets of* $(0, \infty) \times \mathbb{R}^N$.

A Appendix

We give here the proof of the parabolic version of the Kato's inequality, and we use it to establish a basic estimation used to prove Theorem 1.1. See also [10, Lemma A.1, p. 570].

Lemma A.1 (Kato's parabolic inequality). *Let $Q \subset \mathbb{R} \times \mathbb{R}^N$ be any open set. Let $u \in L^1_{\mathrm{loc}}(Q)$ be such that*

$$u_t - \Delta u = f \quad \text{in } \mathcal{D}'(Q),$$

with $f \in L^1_{\mathrm{loc}}(Q)$. Then $|u|_t - \Delta|u| \leq \operatorname{sign}(u)f$ in $\mathcal{D}'(Q)$, where

$$\operatorname{sign}(u) = \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0. \end{cases}$$

Proof. If $F\colon \mathbb{R} \to \mathbb{R}$ is a C^2 convex function and $z\colon Q \to \mathbb{R}$ a C^2 function, then

$$\begin{aligned}(\partial_t - \Delta)F(z) &= F'(z)\partial_t z - [F'(z)\Delta z + F''(z)|\nabla z|^2] \\ &= F'(z)(\partial_t - \Delta)z - F''(z)|\nabla z|^2 \\ &\leq F'(z)(\partial_t - \Delta)z.\end{aligned}$$

Mollify u to $u_k = \rho_k \star u$ such that $u_k \in C^\infty(Q)$, where ρ_k is a sequence of mollifiers. Note that $u_k \to u$ and $(\partial_t - \Delta)u_k \to (\partial_t - \Delta)u$ in $L^1_{\mathrm{loc}}(Q)$ as $k \to \infty$. It follows that $(\partial_t - \Delta)F(u_k) \leq F'(u_k)(\partial_t - \Delta)u_k$. We set $F(z) = \sqrt{\varepsilon^2 + z^2}$. We obtain then

$$(\partial_t - \Delta)F(u_k) \leq \frac{u_k}{F(u_k)}(\partial_t - \Delta)u_k. \tag{A.1}$$

By a simple calculation, we have $|F(u_k) - F(u)| \leq \big||u_k| - |u|\big| \leq |u_k - u|$; then $F(u_k) \to F(u)$ as $k \to \infty$ in $L^1_{\mathrm{loc}}(Q)$ as well as pointwise a.e., and $(\partial_t - \Delta)F(u_k) \to (\partial_t - \Delta)F(u)$ as $k \to \infty$ in $\mathcal{D}'(Q)$. Since $|\frac{u_k}{F(u_k)}| \leq 1$, then the dominated convergence theorem implies that $\frac{u_k}{F(u_k)} \to \frac{u}{F(u)}$ as $k \to \infty$ in $L^1_{\mathrm{loc}}(Q)$. Letting $k \to \infty$ in (A.1), we obtain

$$(\partial_t - \Delta)F(u) \leq \frac{u}{F(u)}(\partial_t - \Delta)u = \frac{u}{F(u)}f. \tag{A.2}$$

Since $F(u) \to |u|$ uniformly as $\varepsilon \to 0$ such that $(\partial_t - \Delta)F(u) \to (\partial_t - \Delta)|u|$ in $\mathcal{D}'(Q)$. Also, $\frac{u}{F(u)} \to \frac{u}{|u|}$ in $L^1_{\mathrm{loc}}(Q)$ (again by the dominated convergence theorem). By letting $\varepsilon \to 0$ in (A.2), we obtain $(\partial_t - \Delta)|u| \leq \operatorname{sign}(u)f$. This completes the proof. □

We have the following result, which is an application of Kato's inequality.

Corollary A.2. *Let $X = C_0(\Omega_m)$ or $L^p(\Omega_m)$ for some $1 \leq p < \infty$. Let $u, v \in C((0,\infty), X)$ be two solutions of equation (1.1) with initial values, respectively, $u_0, v_0 \in X$. Then $|u(t) - v(t)| \leq e^{t\Delta_m}|u_0 - v_0|$ for all $t > 0$.*

Proof. Denote by w the unique solution with initial value $w_0 = \frac{|u_0 - v_0|}{2} \in X$.

Let $z = |u - v|$. Applying Lemma A.1 with $Q = (0,\infty) \times \Omega_m$ and $f = |v|^\alpha v - |u|^\alpha u \in C(Q)$, we have

$$z_t - \Delta z + \big||u|^\alpha u - |v|^\alpha v\big| \leq 0.$$

Since $\big||u|^\alpha u - |v|^\alpha v\big| \geq 2^{-\alpha}|u - v|^{\alpha+1} = 2^{-\alpha}z^{\alpha+1}$, we deduce that $z_t - \Delta z + 2^{-\alpha}z^{\alpha+1} \leq 0$. Let $\overline{z} = \frac{z}{2}$. Then

$$\overline{z}_t - \Delta\overline{z} + \overline{z}^{\alpha+1} \leq 0 = w_t - \Delta w + w^{\alpha+1}.$$

Since $\overline{z}(0) = w(0)$, it follows from the comparison principle that $\overline{z} \leq w$. Since

$$w(t) = e^{t\Delta_m}w_0 - \int_0^t e^{(t-s)\Delta_m}(w(s)^{\alpha+1})\,ds \leq e^{t\Delta_m}w_0,$$

the result follows. □

Finally, we give two results which we found during our research for this article, and which we believe have an independent interest, but which ultimately were not needed for the proofs of the main results.

Consider the eigenvalue problem, on some domain $B \subset \mathbb{R}^N$, $-\Delta H = \Lambda H$, where $\Lambda \in \mathbb{R}$. We look for a solution of the form

$$H(x) = x_1 x_2 \cdots x_m Q(r), \tag{A.3}$$

where $r = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. We note that, for $1 \le i \le m$,

$$\begin{aligned}\partial_i H(x) &= x_1 x_2 \cdots \hat{x}_i \cdots x_m Q(r) + x_1 x_2 \cdots x_m Q'(r) \frac{\partial r}{\partial x_i} \\ &= x_1 x_2 \cdots \hat{x}_i \cdots x_m Q(r) + x_1 x_2 \cdots x_m Q'(r) \frac{x_i}{r},\end{aligned}$$

where $\hat{x}_i$ means that x_i is missing from the product, and

$$\begin{aligned}\partial_i^2 H(x) &= 2 x_1 x_2 \cdots \hat{x}_i \cdots x_m Q'(r) \frac{x_i}{r} + x_1 x_2 \cdots x_m \Big[Q''(r) \Big(\frac{x_i}{r}\Big)^2 + Q'(r) \frac{r^2 - x_i^2}{r^3} \Big] \\ &= 2 x_1 x_2 \cdots x_m \frac{Q'(r)}{r} + x_1 x_2 \cdots x_m \Big[Q''(r) \Big(\frac{x_i}{r}\Big)^2 + Q'(r) \frac{r^2 - x_i^2}{r^3} \Big],\end{aligned}$$

and if $m < i \le N$, then

$$\partial_i^2 H(x) = x_1 x_2 \cdots x_m \Big[Q''(r) \Big(\frac{x_i}{r}\Big)^2 + Q'(r) \frac{r^2 - x_i^2}{r^3} \Big].$$

It follows that

$$\begin{aligned}\Delta H(x) = \sum_{i=1}^N \partial_i^2 H(x) &= x_1 x_2 \cdots x_m \Big[\frac{2m}{r} Q'(r) + Q''(r) + \frac{N-1}{r} Q'(r) \Big] \\ &= x_1 x_2 \cdots x_m \Big[Q''(r) + \frac{N+2m-1}{r} Q'(r) \Big].\end{aligned} \tag{A.4}$$

Proposition A.3. *Let $B_1 = \{x \in \Omega_m : |x| < 1\} \subset \mathbb{R}^N$, and let $\Lambda > 0$ be the lowest eigenvalue of $-\Delta$ on B_1 with Dirichlet boundary conditions. It follows that there exists an eigenfunction $H_1 : \overline{B_1} \to [0, \infty)$ of the form* (A.3), *where $Q : [0, 1] \to [0, \infty)$ is decreasing with $Q(0) = 1$ and $Q(1) = 0$ and $r = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. Moreover, the value of $\Lambda > 0$ is precisely the lowest eigenvalue of $-\Delta$ on the unit ball in $\mathbb{R}^{N+2m}$ with Dirichlet boundary conditions, and its corresponding eigenfunction is precisely $Q(r')$, where $r' = (x_1^2 + x_2^2 + \cdots + x_{N+2m}^2)^{1/2}$.*

Proof. Let $Q(r')$, where $r' = (x_1^2 + x_2^2 + \cdots + x_{N+2m}^2)^{1/2}$, denote the radially symmetric, radially decreasing, nonnegative eigenfunction of $-\Delta$ on the unit ball in $\mathbb{R}^{N+2m}$ [normalized so that $Q(0) = 1$], with eigenvalue $\Lambda > 0$. In particular, the function $Q : [0, 1] \to [0, \infty)$ satisfies the differential equation

$$-\Big[Q''(s) + \frac{N+2m-1}{s} Q'(s) \Big] = \Lambda Q(s), \quad 0 < s \le 1. \tag{A.5}$$

Let $H : \overline{B_1} \to \mathbb{R}^+$ be given by (A.3), where $r = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. It follows from (A.4) and (A.5) that $-\Delta H = \Lambda H$ on B_1 and that $H(x) = 0$ for all $x \in \partial B_1$. Since $H(x) > 0$ for all $x \in B_1$, it follows that Λ is the lowest eigenvalue of $-\Delta$ on B_1. □

Let us now give a remark about the elliptic equation verified by ψ_0.

Remark A.4. Let $N \ge 1$, $m \in \{0, 1, \ldots, N\}$, $0 < \gamma < N$, and ψ_0 be given by (1.6). Then

$$-\Delta \psi_0 = (\gamma + 2m)(N - 2 - \gamma) \frac{\psi_0}{|x|^2}$$

for all $x \in \Omega_m$.

Proof. By (1.6), the function ψ_0 can be written in the form (A.3), that is,

$$\psi_0(x) = x_1 \cdots x_m Q(r),$$

with $Q(r) = c_{m,\gamma} r^{-\gamma-2m}$, $r = (x_1^2 + \cdots + x_m^2 + \cdots + x_N^2)^{1/2}$, where $c_{m,\gamma} = \gamma(\gamma+2)\cdots(\gamma+2m-2)$. For such a Q, we have

$$\begin{aligned} Q''(r) + \frac{N+2m-1}{r} Q'(r) &= c_{m,\gamma}(\gamma+2m)[\gamma+2m+1-(N+2m-1)]r^{-\gamma-2m-2} \\ &= c_{m,\gamma}(\gamma+2m)(\gamma+2-N)r^{-\gamma-2m-2}. \end{aligned}$$

The result follows then by (A.4). □

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