Research Article

Alessio Porretta*

A Note on the Sobolev and Gagliardo–Nirenberg Inequality when $p > N$

https://doi.org/10.1515/ans-2020-2086
Received December 29, 2019; accepted February 28, 2020

Abstract: It is known that the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is embedded into $L^{Np/(N-p)}(\mathbb{R}^N)$ if $p < N$ and into $L^\infty(\mathbb{R}^N)$ if $p > N$. There is usually a discontinuity in the proof of those two different embeddings since, for $p > N$, the estimate $\|u\|_\infty \leq C\|Du\|_p^{N/p}\|u\|_p^{1-N/p}$ is commonly obtained together with an estimate of the Hölder norm. In this note, we give a proof of the L^∞-embedding which only follows by an iteration of the Sobolev–Gagliardo–Nirenberg estimate $\|u\|_{N/(N-1)} \leq C\|Du\|_1$. This kind of proof has the advantage to be easily extended to anisotropic cases and immediately exported to the case of discrete Lebesgue and Sobolev spaces; we give sample results in case of finite differences and finite volumes schemes.

Keywords: Sobolev spaces, Gagliardo–Nirenberg Inequality, Discrete Sobolev Inequalities

MSC 2010: 35A23

Communicated by: Julián López-Gómez and Patrizia Pucci
Dedicated to Laurent Véron, a mathematical gentleman, with esteem and friendship

1 Introduction

Let $\mathbb{R}^N$ denote the euclidean N-dimensional space, and assume that $N \geq 2$. In its basic form, the celebrated Sobolev inequality [13] asserts that, for every $1 \leq p < N$,

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C_S\|Du\|_{L^p(\mathbb{R}^N)}, \quad p^* = \frac{Np}{N-p}, \tag{1.1}$$

for every C^1 function u with compact support in $\mathbb{R}^N$. It is common knowledge that the inequality for $p > 1$ can be easily deduced from the case $p = 1$, sometimes called the Gagliardo inequality. This latter one, in its general form, reads as

$$\|u\|^N_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq \prod_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^1(\mathbb{R}^N)} \tag{1.2}$$

and was proved independently by Gagliardo [9] and Nirenberg [11].

It is well known that (1.1) fails to be true if $p = N$ and $p^* = \infty$. However, for $p > N$, it holds that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq K_p[\|u\|_{L^p(\mathbb{R}^N)} + \|Du\|_{L^p(\mathbb{R}^N)}] \tag{1.3}$$

for every C^1 function u with compact support in $\mathbb{R}^N$.

The usual proof of (1.3) goes together with the Morrey estimate (see e.g. [1, 7]), which states the embedding of $W^{1,p}(\mathbb{R}^N)$, for $p > N$, into the space of Hölder continuous functions [10]. Even if the Morrey embedding gives, of course, a fundamental piece of information, it seems a natural question whether (1.3) could not be

***Corresponding author: Alessio Porretta,** Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy, e-mail: porretta@mat.uniroma2.it

obtained itself from the scale of Sobolev inequalities, rather than as a byproduct of an estimate on the oscillation of u. This question is especially motivated by the application to discrete numerical schemes for PDEs since discrete-type Sobolev inequalities are more efficiently proved by using only scaling arguments.

The purpose of this note is to give a proof of (1.3) which relies *only* on the (recursive application of) Gagliardo inequality (1.2). In particular, we wish to give evidence to the following two remarks:

(a) Inequality (1.3) can be directly obtained from (1.2) through an iteration scheme.

(b) This approach preserves the natural form of (1.2) and easily extends, for instance, to anisotropic cases and discrete versions.

To be precise, the natural generalized form of inequality (1.2) that we prove in this paper for the case $p > N$ is the following one.

Theorem 1.1. *Let p_i, $i = 1 \dots N$, be such that $\sum_{i=1}^N \frac{1}{p_i} < 1$. For every $r \geq 1$, there exists a constant C, only depending on p_i, N and r, such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\left(\prod_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\mathbb{R}^N)}\right)^{\frac{\theta}{N}} \|u\|_{L^r(\mathbb{R}^N)}^{1-\theta}, \qquad \theta := \frac{N}{N + r(1 - \sum_{i=1}^N \frac{1}{p_i})}, \tag{1.4}$$

for every C^1 function u with compact support in $\mathbb{R}^N$.

As we said before, the interest here lies in the proof of (1.4), which is obtained with an elementary iteration from the case $p = 1$, using only algebraic steps and Hölder inequalities. In this approach, inequality (1.3) for the case $p > N$ follows in the same spirit as inequality (1.1) for the case $p < N$, up to replacing a finite with an infinite iteration. In fact, in order to get at the sup-norm, one needs a limit as $q \to \infty$ of the embedding in $L^q(\mathbb{R}^N)$, obtained by applying (1.2) to increasing powers of u, like in Moser-type iterations. The convergence of the iteration and scaling arguments, which fix the precise form of the embedding estimate, are the only ingredients required. In particular, compared to the usual proof (see [1, 7]), we do not make any use of the geometry of the underlying euclidean space since we only rely on the starting inequality (1.2).

Apart from the pedagogical interest of this proof, we think that it may have an interest in cases where the structure of the state space is more complex than the euclidean flat case. As a motivation, and an application of our approach, we consider the case of discrete-type inequalities which are needed in numerical schemes for partial differential equations.

An extensive literature now exists about discrete-type Sobolev inequalities; we mention in particular [3, 4, 6, 8] (and many other references cited therein) for the case of finite volumes. However, the case $p > N$, $p^* = \infty$ is often outside the range of those results, despite the fact that, in numerical analysis, the case of low dimension ($N = 2$, $N = 3$) is very relevant. At the end of this note, we provide a discrete-type version of (1.3) for finite volume schemes, see Theorem 4.1, which is obtained as in the continuous case with an iteration method. Notice that, in the finite volumes setting, it is very convenient to start with inequality (1.2), applied to BV functions, a very natural frame for piecewise constant functions. We hope that the kind of discrete Gagliardo–Nirenberg inequality proved in Theorem 4.1 may have an interest for people working in numerical analysis: indeed, a special case for $N = 2$ was needed in our recent paper [12], for a finite difference scheme used to show numerical hypocoercivity of the Kolmogorov equation.

2 The Iteration Scheme

We start by showing that the iteration of the Gagliardo–Nirenberg–Sobolev inequality (1.2) is convergent and leads to the estimate of the sup-norm. This is the main technical step in our approach.

Lemma 2.1. *Let $p > N$. For every $r \geq 1$, there exist constants α, β, C, only depending on p, N and r, such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\left(\prod_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)}\right)^{\alpha} \|u\|_{L^r(\mathbb{R}^N)}^{\beta} \tag{2.1}$$

for every C^1 function u with compact support in $\mathbb{R}^N$.

Proof. Given $\gamma > p$, we apply (1.2) to u^γ obtaining

$$\|u^\gamma\|^N_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \le \gamma^N \prod_{i=1}^N \|u^{\gamma-1}\frac{\partial u}{\partial x_i}\|_{L^1(\mathbb{R}^N)} \le \gamma^N \bigg(\prod_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)\bigg(\int u^{(\gamma-1)p'}\,dx\bigg)^{\frac{N}{p'}}. \tag{2.2}$$

Since $\gamma > p$ and $p > N$, we have $\gamma < (\gamma-1)p' < \gamma\frac{N}{N-1}$, so we interpolate

$$\bigg(\int u^{(\gamma-1)p'}\,dx\bigg)^{\frac{N}{p'}} \le \bigg(\int u^{\gamma\frac{N}{N-1}}\,dx\bigg)^{\frac{\theta N}{p'}}\bigg(\int u^\gamma\bigg)^{\frac{N(1-\theta)}{p'}},$$

where $(\gamma-1)p' = \theta\gamma\frac{N}{N-1} + (1-\theta)\gamma$, which means

$$\theta = \frac{(\gamma-1)p' - \gamma}{\gamma\frac{1}{N-1}}.$$

We deduce from (2.2)

$$\bigg(\int u^{\gamma\frac{N}{N-1}}\,dx\bigg)^{N-1-\frac{\theta N}{p'}} \le \gamma^N\bigg(\prod_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)\bigg(\int u^\gamma\bigg)^{\frac{N(1-\theta)}{p'}},$$

which yields

$$\bigg(\int u^{\gamma\frac{N}{N-1}}\,dx\bigg)^{\frac{N-1}{\gamma N}} \le \bigg\{\gamma^N\bigg(\prod_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)\bigg\}^{\frac{(N-1)p'}{\gamma N(p'(N-1)-\theta N)}}\bigg(\int u^\gamma\bigg)^{\frac{(N-1)(1-\theta)}{\gamma(p'(N-1)-\theta N)}}. \tag{2.3}$$

We use the value of θ in terms of γ, N, p and the two exponents in the right term become respectively

$$\frac{(N-1)p'}{\gamma N(p'(N-1)-\theta N)} = \frac{p'}{N}\bigg[\frac{1}{\gamma(N-(N-1)p') + Np'}\bigg] = \frac{1}{N}\bigg[\frac{1}{\gamma(1-\frac{N}{p}) + N}\bigg],$$

$$\frac{(N-1)(1-\theta)}{\gamma(p'(N-1)-\theta N)} = \frac{1}{\gamma}\bigg[\frac{\gamma(N-(N-1)p') + (N-1)p'}{\gamma(N-(N-1)p') + Np'}\bigg] = \frac{1}{\gamma}\bigg[1 - \frac{1}{\gamma(1-\frac{N}{p}) + N}\bigg].$$

Henceforth, we set the iteration scheme. For $r > p$, we define the recursive sequence

$$\begin{cases} \gamma_n = \Big(\dfrac{N}{N-1}\Big)\gamma_{n-1}, \\ \gamma_0 = r \end{cases} \quad \rightsquigarrow \quad \gamma_n = r\Big(\frac{N}{N-1}\Big)^n,$$

and we define

$$\sigma_n := \frac{1}{\gamma_n(1-\frac{N}{p}) + N}.$$

With the above notations, using (2.3) with $\gamma = \gamma_{n-1}$, we have

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le \bigg\{\gamma_{n-1}^N\bigg(\prod_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)\bigg\}^{\frac{\sigma_{n-1}}{N}} \|u\|^{1-\sigma_{n-1}}_{L^{\gamma_{n-1}}(\mathbb{R}^N)}, \tag{2.4}$$

which holds for every $n \ge 1$. To shorten notations, we define

$$C_{n-1} := \bigg\{\gamma_{n-1}^N\bigg(\prod_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)\bigg\}^{\frac{\sigma_{n-1}}{N}}$$

so that (2.4) takes the form

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le C_{n-1}\|u\|^{1-\sigma_{n-1}}_{L^{\gamma_{n-1}}(\mathbb{R}^N)} \quad \text{for all } n \ge 1.$$

We can now iterate this estimate, and we get

$$\begin{aligned} \|u\|_{L^{\gamma_n}(\mathbb{R}^N)} &\le C_{n-1}\|u\|^{1-\sigma_{n-1}}_{L^{\gamma_{n-1}}(\mathbb{R}^N)} \\ &\le C_{n-1}(C_{n-2}\|u\|^{1-\sigma_{n-2}}_{L^{\gamma_{n-2}}(\mathbb{R}^N)})^{1-\sigma_{n-1}} \\ &\le C_{n-1}C_{n-2}^{1-\sigma_{n-1}}(C_{n-3}\|u\|^{1-\sigma_{n-3}}_{L^{\gamma_{n-3}}(\mathbb{R}^N)})^{(1-\sigma_{n-1})(1-\sigma_{n-2})} \\ &\le C_{n-1}C_{n-2}^{1-\sigma_{n-1}}C_{n-3}^{(1-\sigma_{n-1})(1-\sigma_{n-2})}\dots\|u\|_r^{\prod_{j=0}^{n-1}(1-\sigma_j)}. \end{aligned}$$

Finally, we deduce the estimate

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le C_{n-1} \prod_{k=0}^{n-2} C_k^{\prod_{j=k+1}^{n-1}(1-\sigma_j)} \|u\|_r^{\prod_{j=0}^{n-1}(1-\sigma_j)} \quad \text{for all } n \ge 1. \tag{2.5}$$

We observe that

$$\prod_{j=k}^{n-1}(1-\sigma_j) = \exp\biggl(\sum_{j=k}^{n-1} \log(1-\sigma_j)\biggr),$$

and since, by definition of σ_n and γ_n,

$$\log(1-\sigma_j) \sim -\sigma_j \sim -\frac{1}{r(1-\frac{N}{p})}\Bigl(\frac{N}{N-1}\Bigr)^{-j},$$

the above sum is convergent and there exists some $c_0 > 0$ such that

$$0 < c_0 \le \prod_{j=k}^{n-1}(1-\sigma_j) \le 1 \quad \text{for all } k \le n-1 \text{ and all } n \ge 1. \tag{2.6}$$

We recall that, by definition of C_k and γ_k, we have

$$C_k = \biggl\{\gamma_k^N\biggl(\prod_{i=1}^{N}\Bigl\|\frac{\partial u}{\partial x_i}\Bigr\|_{L^p(\mathbb{R}^N)}\biggr)\biggr\}^{\frac{\sigma_k}{N}} = A^{\sigma_k}\Bigl(\frac{N}{N-1}\Bigr)^{k\sigma_k}, \quad \text{where} \quad A := r\biggl(\prod_{i=1}^{N}\Bigl\|\frac{\partial u}{\partial x_i}\Bigr\|_{L^p(\mathbb{R}^N)}\biggr)^{\frac{1}{N}}.$$

Hence

$$\prod_{k=0}^{n-2} C_k^{\prod_{j=k+1}^{n-1}(1-\sigma_j)} = (A)^{\sum_{k=0}^{n-2}\sigma_k\prod_{j=k+1}^{n-1}(1-\sigma_j)}\Bigl(\frac{N}{N-1}\Bigr)^{\sum_{k=0}^{n-2}k\sigma_k\prod_{j=k+1}^{n-1}(1-\sigma_j)}.$$

Using (2.6) and $\sigma_k = O((\frac{N}{N-1})^{-k})$, we notice that the above quantity is bounded uniformly with respect to n. Therefore, from (2.5), we deduce that

$$\limsup_{n\to\infty}\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le C\biggl(\prod_{i=1}^{N}\Bigl\|\frac{\partial u}{\partial x_i}\Bigr\|_{L^p(\mathbb{R}^N)}\biggr)^{\alpha}\|u\|_{L^r(\mathbb{R}^N)}^{\beta}$$

for some positive constants C, α, β only depending on r, p, N. Since u has compact support, the left-hand side converges to $\|u\|_{L^\infty(\mathbb{R}^N)}$ and (2.1) is proved, at least for $r > p$. In addition, the above iteration clearly shows that the exponent β is smaller than one (see (2.6)). Since, for any $s < r$, we have

$$\|u\|_r \le \|u\|_s^{\frac{s}{r}}\|u\|_\infty^{1-\frac{s}{r}},$$

the estimate is immediately extended to all Lebesgue spaces $L^s(\mathbb{R}^N)$, with values $s \le p$, with an estimate

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\biggl(\prod_{i=1}^{N}\Bigl\|\frac{\partial u}{\partial x_i}\Bigr\|_{L^p(\mathbb{R}^N)}\biggr)^{\alpha}\|u\|_{L^s(\mathbb{R}^N)}^{\beta}$$

for possibly different C, α, β. □

As we now show, there is no need of a difficult inspection of the above iteration argument in order to detect the values of α and β. Once inequality (2.1) is obtained, the precise value of α and β can be easily found through scaling arguments.

Lemma 2.2. *Assume that* (2.1) *holds true. Then we have*

$$\alpha = \frac{p}{r(p-N)+Np} \quad \textit{and} \quad \beta = \frac{r(p-N)}{r(p-N)+Np}. \tag{2.7}$$

Proof. First of all, replacing u with λu, $\lambda > 0$, we deduce from (2.1) that

$$N\alpha + \beta = 1. \tag{2.8}$$

Secondly, we take $u = u_R := \zeta(\frac{x}{R})$, where $\zeta \in C_c^1(\mathbb{R}^N)$ is a cut-off function such that $0 \le \zeta \le 1$, $\zeta(x) \equiv 0$ if $|x| > 2$ and $\zeta(x) \equiv 1$ if $|x| \le 1$. Applying (2.1) to u_R, we get

$$1 \le C\bigg(\prod_{i=1}^{N}\bigg\|\frac{\partial\zeta}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)^{\alpha}\|\zeta\|^{\beta}_{L^r(\mathbb{R}^N)}R^{\frac{N\beta}{r}}R^{N\alpha(\frac{N}{p}-1)},$$

and since R is arbitrary, this implies

$$\frac{\beta}{r} + \alpha\Big(\frac{N}{p} - 1\Big) = 0. \tag{2.9}$$

Putting together (2.8) and (2.9) gives (2.7). □

Collecting the above lemmas, we deduce the embedding of $W^{1,p}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ $(r \ge 1)$ into $L^\infty(\mathbb{R}^N)$.

Corollary 2.1. *Let $p > N$. For every $r \ge 1$, there exists a constant C, only depending on p, N and r, such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\bigg(\prod_{i=1}^{N}\bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)^{\frac{p}{r(p-N)+Np}}\|u\|^{\frac{r(p-N)}{r(p-N)+Np}}_{L^r(\mathbb{R}^N)} \tag{2.10}$$

for every C^1 function u with compact support in $\mathbb{R}^N$.

Remark 2.1. If we simply use $\|\frac{\partial u}{\partial x_i}\|_{L^p(\mathbb{R}^N)} \le \|Du\|_{L^p(\mathbb{R}^N)}$, then (2.10) implies

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|Du\|^{\frac{Np}{r(p-N)+Np}}_{L^p(\mathbb{R}^N)}\|u\|^{\frac{r(p-N)}{r(p-N)+Np}}_{L^r(\mathbb{R}^N)}, \tag{2.11}$$

which is one standard form of the so-called Gagliardo–Nirenberg inequality.

Remark 2.2. If $r = p$, then (2.10) reads as

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\bigg(\prod_{i=1}^{N}\bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^p(\mathbb{R}^N)}\bigg)^{\frac{1}{p}}\|u\|^{\frac{p-N}{p}}_{L^p(\mathbb{R}^N)},$$

which implies (1.3) by Young's inequality.

3 The Anisotropic Case

The same strategy used before can be applied to the more general anisotropic case.

Lemma 3.1. *Let p_i, $i = 1\dots N$, be such that $\sum_{i=1}^{N}\frac{1}{p_i} < 1$. For every $r \ge 1$, there exist constants α, β, C, only depending on p_i, N and r, such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\bigg(\prod_{i=1}^{N}\bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^{p_i}(\mathbb{R}^N)}\bigg)^{\alpha}\|u\|^{\beta}_{L^r(\mathbb{R}^N)} \tag{3.1}$$

for every C^1 function u with compact support in $\mathbb{R}^N$.

Proof. Let $\{s_i\}_{1\le i\le N}$ be real numbers and $s := \sum_{i=1}^{N} s_i$. We start from the inequality

$$\|u\|^{s}_{L^{\frac{s}{N-1}}(\mathbb{R}^N)} \le \prod_{i=1}^{N} s_i\bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^{p_i}(\mathbb{R}^N)}\|u\|^{s_i-1}_{L^{(s_i-1)p_i'}(\mathbb{R}^N)} \tag{3.2}$$

Inequality (3.2) is the usual starting point for the anisotropic Sobolev inequality; see e.g. [2, 14]. For the sake of completeness, let us recall how this is obtained, using once more the Gagliardo argument. Since

$$\begin{aligned}|u(x)|^{s_i} &\le s_i\int_{-\infty}^{x_i}|u(x_1,\dots,x_{i-1},t,x_{i+1},\dots)|^{(s_i-1)}\frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots)\,dt\\ &\le s_i\bigg(\int_{\mathbb{R}}\bigg|\frac{\partial u}{\partial x_i}(x)\bigg|^{p_i}dx_i\bigg)^{\frac{1}{p_i}}\bigg(\int_{\mathbb{R}}|u(x)|^{(s_i-1)p_i'}\,dx_i\bigg)^{\frac{1}{p_i'}},\end{aligned}$$

we have

$$|u(x)|^{\frac{s}{N-1}} \le \prod_{i=1}^{N} f_i(x_{-i}),$$

where $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots)$ and

$$f_i := s_i^{\frac{1}{N-1}} \Big(\int_{\mathbb{R}} \Big| \frac{\partial u}{\partial x_i}(x) \Big|^{p_i} \, dx_i \Big)^{\frac{1}{p_i(N-1)}} \Big(\int_{\mathbb{R}} |u(x)|^{(s_i-1)p_i'} \, dx_i \Big)^{\frac{1}{p_i'(N-1)}}.$$

Applying the Gagliardo inequality (see e.g. [5])

$$\Big\| \prod_{i=1}^{N} f_i \Big\|_{L^1(\mathbb{R}^N)} \le \prod_{i=1}^{N} \|f_i\|_{L^1(\mathbb{R}^{N-1})},$$

we get

$$\int |u(x)|^{\frac{s}{N-1}} \, dx \le \prod_{i=1}^{N} s_i^{\frac{1}{N-1}} \int_{\mathbb{R}^{N-1}} \Big(\int_{\mathbb{R}} \Big| \frac{\partial u}{\partial x_i}(x) \Big|^{p_i} \, dx_i \Big)^{\frac{1}{p_i(N-1)}} \Big(\int_{\mathbb{R}} |u(x)|^{(s_i-1)p_i'} \, dx_i \Big)^{\frac{1}{p_i'(N-1)}} \, dx_{-i}$$

which implies, by the Hölder inequality,

$$\int |u(x)|^{\frac{s}{N-1}} \, dx \le \prod_{i=1}^{N} s_i^{\frac{1}{N-1}} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^{p_i}(\mathbb{R}^N)}^{\frac{1}{N-1}} \| |u|^{s_i-1} \|_{L^{p_i'}(\mathbb{R}^N)}^{\frac{1}{N-1}}.$$

This proves (3.2). Now we choose $\{s_i\}$ so that the $N-1$ conditions

$$(s_i - 1)p_i' = (s_j - 1)p_j' \quad \text{for all } i \ne j \tag{3.3}$$

are satisfied. Thanks to (3.3), and since $\sum_i \frac{1}{p_i} < 1$, we claim that

$$\frac{s}{N} < (s_i - 1)p_i' < \frac{s}{N-1} \quad \text{for all } i = 1, \dots, N \tag{3.4}$$

provided

$$s_i > 1 + \frac{N}{p_i' \sum_{j=1}^{N} \frac{1}{p_j}}. \tag{3.5}$$

Indeed, using (3.3), and the standard conjugate relation for p_j', we have

$$\begin{aligned} s &= s_i + \sum_{j \ne i} (s_j - 1) + N - 1 = s_i + (s_i - 1)p_i' \sum_{j \ne i} \frac{1}{p_j'} + N - 1 \\ &= s_i + (s_i - 1)p_i' \sum_{j \ne i} \Big(1 - \frac{1}{p_j} \Big) + N - 1 \\ &= s_i + (N-1)(1 + (s_i - 1)p_i') - (s_i - 1)p_i' \sum_{j \ne i} \frac{1}{p_j} \\ &= (N-1)(s_i - 1)p_i' + s_i \Big(1 - p_i' \sum_{j \ne i} \frac{1}{p_j} \Big) + N - 1 + p_i' \sum_{j \ne i} \frac{1}{p_j}. \end{aligned} \tag{3.6}$$

Since the condition $\sum_j \frac{1}{p_j} < 1$ implies $p_i' \sum_{j \ne i} \frac{1}{p_j} < 1$, we immediately deduce that

$$s > (N-1)(s_i - 1)p_i',$$

which gives the right-hand inequality in (3.4). In addition, we compute

$$\begin{aligned} s - N(s_i - 1)p_i' &= -s_i \Big(p_i' - 1 + p_i' \sum_{j \ne i} \frac{1}{p_j} \Big) + p_i' + N - 1 + p_i' \sum_{j \ne i} \frac{1}{p_j} \\ &= -s_i p_i' \sum_j \frac{1}{p_j} + N + p_i' \sum_j \frac{1}{p_j}, \end{aligned}$$

and therefore $s < N(s_i - 1)p_i'$ provided (3.5) holds true.

Let us suppose, by now, that (3.5) is satisfied. Then we have shown that (3.4) holds true, and we can use the interpolation inequality to deduce from (3.2)

$$\|u\|^s_{L^{\frac{s}{N-1}}(\mathbb{R}^N)} \le \prod_{i=1}^{N} s_i \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}(\mathbb{R}^N)} \Big(\int |u|^{\frac{s}{N}}\Big)^{\frac{1-\theta_i}{p_i'}} \Big(\int |u|^{\frac{s}{N-1}}\Big)^{\frac{\theta_i}{p_i'}}, \tag{3.7}$$

where $(s_i - 1)p_i' = \theta_i \frac{s}{N-1} + (1-\theta_i)\frac{s}{N}$, which means

$$\theta_i = \frac{N-1}{s}[N(s_i-1)p_i' - s].$$

Simplifying (3.7), we obtain

$$\Big(\int |u|^{\frac{s}{N-1}}\Big)^{N-1-\sum_i \frac{\theta_i}{p_i'}} \le \Big\{\prod_{i=1}^{N} s_i \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}(\mathbb{R}^N)}\Big\} \Big(\int |u|^{\frac{s}{N}}\Big)^{\sum_i \frac{1-\theta_i}{p_i'}}.$$

Since

$$\sum \frac{\theta_i}{p_i'} = \frac{N-1}{s}\Big(s\sum_j \frac{1}{p_j} - N^2\Big), \quad \sum \frac{1-\theta_i}{p_i'} = \frac{N}{s}\Big[s\Big(1-\sum_j \frac{1}{p_j}\Big) + N(N-1)\Big],$$

we get

$$\|u\|_{L^{\frac{s}{N-1}}(\mathbb{R}^N)} \le \Big\{\prod_{i=1}^{N} s_i \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}(\mathbb{R}^N)}\Big\}^{\frac{1}{s(1-\sum_j \frac{1}{p_j})+N^2}} \|u\|_{L^{\frac{s}{N}}(\mathbb{R}^N)}^{\frac{s(1-\sum_j \frac{1}{p_j})+N(N-1)}{s(1-\sum_j \frac{1}{p_j})+N^2}}. \tag{3.8}$$

We set henceforth

$$\gamma_n := r\Big(\frac{N}{N-1}\Big)^n,$$

and we apply (3.8) with $s = N\gamma_{n-1}$. Notice that the $N-1$ conditions (3.3) and the choice $s = N\gamma_{n-1}$ yield a unique choice of the $\{s_i\}_{i=1,\dots,N}$ used above[1]. In addition, we have that s_i depends on n and goes to infinity as $n \to \infty$; this makes sure that condition (3.5) is satisfied for n large. Indeed, by taking r sufficiently large, we can suppose that this condition holds for all $n \ge 1$.

With the above choice, (3.8) reads as the recursive estimate

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le \Big\{\prod_{i=1}^{N} s_i \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}(\mathbb{R}^N)}\Big\}^{\frac{\sigma_{n-1}}{N}} \|u\|^{1-\sigma_{n-1}}_{L^{\gamma_{n-1}}(\mathbb{R}^N)} \quad \text{for all } n \ge 1,$$

where we have set

$$\sigma_n := \frac{1}{\gamma_n(1-\sum_j \frac{1}{p_j}) + N}.$$

Defining as well

$$C_{n-1} := \Big\{\prod_{i=1}^{N} s_i \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}(\mathbb{R}^N)}\Big\}^{\frac{\sigma_{n-1}}{N}},$$

the above estimate takes the form

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le C_{n-1}\|u\|^{1-\sigma_{n-1}}_{L^{\gamma_{n-1}}(\mathbb{R}^N)} \quad \text{for all } n \ge 1,$$

exactly as in Lemma 2.1. Therefore, we obtain again the estimate

$$\|u\|_{L^{\gamma_n}(\mathbb{R}^N)} \le C_{n-1}\prod_{k=0}^{n-2} C_k^{\prod_{j=k+1}^{n-1}(1-\sigma_j)} \|u\|_r^{\prod_{j=0}^{n-1}(1-\sigma_j)} \quad \text{for all } n \ge 1, \tag{3.9}$$

and we conclude with similar arguments: as in the isotropic case, there exists $c_0 > 0$ such that

$$0 < c_0 \le \prod_{j=k}^{n-1}(1-\sigma_j) \le 1 \quad \text{for all } k \le n-1 \text{ and all } n \ge 1,$$

1 In fact, the computations in (3.6) give the explicit relation between s and s_i as $s = s_i p_i'(N - \sum_j \frac{1}{p_j}) + p_i' \sum_j \frac{1}{p_j} - N(p_i' - 1)$.

and since we have $s_i \le \theta\gamma_{n-1}$ for some constant θ only depending on p_i, N, we estimate

$$C_{n-1} \le \left\{(\theta\gamma_{n-1})^N \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^N)} \right\}^{\frac{\sigma_{n-1}}{N}}.$$

Therefore, we conclude, exactly as in Lemma 2.1, that there exist constants $\alpha, C > 0$ such that

$$C_{n-1} \prod_{k=0}^{n-2} C_k^{\prod_{j=k+1}^{n-1}(1-\sigma_j)} \le C \left\{ \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^N)} \right\}^{\alpha},$$

and passing to the limit in (3.9), we finally obtain (3.1), at least for r sufficiently large.

The conclusion is then extended to any value $r \ge 1$ as in Lemma 2.1, with a straightforward interpolation argument. □

Finally, the same proof as in Lemma 2.2 gives the unique values of α, β. Thus the proof of Theorem 1.1 is concluded.

4 Discrete Inequalities

Discrete inequalities similar to those proved in the previous sections can be obtained with the same approach. In order to keep things simpler, we only consider the isotropic case as in Section 2. We start with the easier case of finite difference schemes.

4.1 Finite Difference Schemes

A discrete inequality as (2.10) was obtained in [12] for a finite difference scheme approximating the Kolmogorov equation in $\mathbb{R}^2$. In that situation, we proved that, for any $p > 2$ and $r \ge 1$, there exists a constant C, only depending on p, r, such that

$$\|f\|_\infty \le C(\|D_x f\|_p \|D_y f\|_p)^{\frac{p}{r(p-2)+2p}} \|f\|_r^{\frac{r(p-2)}{r(p-2)+2p}}$$

for every $f \in \ell_h^s(\mathbb{R}^2)$ with $s = \min(2, r)$, where x, y are coordinates in $\mathbb{R}^2$, $f = (f_{ij})$ is a function defined on the scheme and D_x, D_y are finite difference versions of the derivatives.

Here we give an extension of this kind of estimate to N-dimensional finite difference schemes. To this purpose, let us consider a uniform grid on $\mathbb{R}^N$ with mesh step h, and let $P = (i_1 h, \dots, i_N h)$ denote a generic point in $\mathbb{R}_h^N$, with $i_1, \dots, i_N \in \mathbb{Z}$. The values of a function f at P are denoted by $f_{i_1,\dots,i_N}$, and the natural Lebesgue space ℓ_h^p is defined as

$$f \in \ell_h^p \iff \|f\|_{\ell_h^p} := \Big(h^N \sum_{i_1,\dots,i_N} |f_{i_1,\dots,i_N}|^p\Big)^{\frac{1}{p}} < \infty.$$

For a function f defined on the scheme, the discrete derivatives can be defined, for example, as follows:

$$D_{x_k} f(P) := \frac{f_{i_1,\dots,i_{k-1},i_k,i_{k+1},\dots,i_N} - f_{i_1,\dots,i_{k-1},i_k-1,i_{k+1},\dots,i_N}}{h}.$$

Of course, right-sided derivatives, or centered derivatives, could be used alternatively.

A discrete Gagliardo inequality can be the starting point here. Assume that f has compact support. Since, in any direction k, one has

$$|f_{i_1,\dots,i_N}| \le \sum_{j=-\infty}^{i_k} |f_{i_1,\dots,i_{k-1},j,i_{k+1},\dots,i_N} - f_{i_1,\dots,i_{k-1},j-1,i_{k+1},\dots,i_N}|,$$

by taking power $\frac{1}{N-1}$ and N copies of this inequality in the N directions, one obtains

$$|f_{i_1,\dots,i_N}|^{\frac{N}{N-1}} \le \prod_{k=1}^{N}\Bigg(\sum_{j=-\infty}^{\infty} |f_{i_1,\dots,i_{k-1},j,i_{k+1},\dots,i_N} - f_{i_1,\dots,i_{k-1},j-1,i_{k+1},\dots,i_N}|\Bigg)^{\frac{1}{N-1}}.$$

Then, exactly as in the continuous case, integrating and using the Hölder inequality (and scaling the powers of h), one gets

$$\|f\|^{N}_{\ell_h^{\frac{N}{N-1}}} \le \prod_{k=1}^{N}\|D_{x_k}f\|_{\ell_h^1} \tag{4.1}$$

for any f which has compact support. Inequality (4.1) is the discrete equivalent of (1.2). Now, applying (4.1) to $|f|^\gamma$ and using that

$$|D_{x_k}|f|^\gamma| \le \gamma(|f|^{\gamma-1} + |f_{i_1,\dots,i_{k-1},i_k-1,i_{k+1},\dots,i_N}|^{\gamma-1})|D_{x_k}f|,$$

one obtains with the Hölder inequality

$$\|f\|^{\gamma}_{\ell_h^{\frac{\gamma N}{N-1}}} \le c\gamma\Bigg(\prod_{k=1}^{N}\|D_{x_k}f\|_{\ell_h^p}\Bigg)^{\frac{1}{N}}\|f\|^{\gamma-1}_{\ell_h^{(\gamma-1)p'}}.$$

The iteration scheme then follows exactly as in Lemma 2.1 and provides with the inequality

$$\|f\|_\infty \le c\Bigg(\prod_{k=1}^{N}\|D_{x_k}f\|_{\ell_h^p}\Bigg)^{\alpha}\|f\|^{\beta}_{\ell_h^r}$$

for some $\alpha, \beta > 0$ and for every f with compact support. The consistency of the scheme implies that the same inequality be true for C^1 functions with compact support, so the values α, β are fixed once again by Lemma 2.2. Finally, we end up with the inequality

$$\|f\|_\infty \le c\Bigg(\prod_{k=1}^{N}\|D_{x_k}f\|_{\ell_h^p}\Bigg)^{\frac{\theta}{N}}\|f\|^{1-\theta}_{\ell_h^r}, \quad \theta := \frac{Np}{Np + r(p-N)}, \tag{4.2}$$

which is proved to hold (with a standard density argument) for every $f \in \ell_h^s$, $s = \min(p, r)$.

Remark 4.1. We point out that different choices could as well be done for the discrete derivatives D_{x_k} in different directions. The proof adapts easily, for example, to left, right or centered choices. For instance, in [12], we obtained (4.2) using the choice $D_x f := \frac{f_{i,j}-f_{i-1,j}}{h}$, $D_y f := \frac{f_{i,j+1}-f_{i,j-1}}{2h}$ in order to match the anisotropic behavior of the Kolmogorov equation $\partial_t f - \partial_{xx} f - x\partial_y f = 0$ in $\mathbb{R}^2$.

4.2 Finite Volume Schemes

Let us now show similar discrete inequalities in the more general context of finite volume schemes. We follow here [8] for the reference functional setting. An admissible mesh of $\mathbb{R}^N$ is given by:

- A family $\mathcal{M}$ of control volumes, denoted by K, which are bounded convex polyhedral subsets of $\mathbb{R}^N$ with positive measure, and such that $\mathcal{M}$ realizes a locally finite partition of $\mathbb{R}^N$.
- A family $\mathcal{E}$ of relatively open subsets of hyperplanes of $\mathbb{R}^N$, denoted by σ, with positive $N-1$ measure, which represent the faces of each volume K. Indeed, for each K, there exists $\mathcal{E}_K \subset \mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$, and $\mathcal{E} = \bigcup_{K\in\mathcal{M}}\mathcal{E}_K$. We also assume that the cardinality of $\mathcal{E}_K$ is uniformly bounded for all $K \in \mathcal{M}$ (i.e. there is a uniform bound on the number of faces of the control volumes).
- A family of points $\{x_K\}_{K\in\mathcal{M}}$ such that x_K belongs to the interior of K, and for any two neighboring cells K, L, the line through x_K, x_L, denoted by $[x_K, x_L]$, intersects and is orthogonal to the face $\sigma_{KL} := \partial K \cap \partial L$.

The size of the mesh is defined as $h := \sup_{K\in\mathcal{M}} \operatorname{diam}(K)$ and supposed to be finite. For any x_K, x_L, their distance is denoted as

$$d_\sigma := |x_K - x_L| \quad \text{for } \sigma = \sigma_{KL}.$$

Notice that the above condition on the line $[x_K, x_L]$ implies $d_\sigma \geq d(x_K, \sigma_{KL})$. The mesh is called regular if those distances are equivalent (uniformly in the mesh), namely,

$$\text{there exists } c_0 > 0 \text{ such that } d_\sigma \leq c_0 d(x_K, \sigma) \text{ for all } \sigma \in \mathcal{E}_K \text{ and all } K \in \mathcal{M}. \tag{4.3}$$

Functions u defined on the scheme are nothing but a collection of real numbers $(u_K)_{K\in\mathcal{M}}$, and clearly identified with the space of measurable functions in $\mathbb{R}^N$ which are piecewise constant on $\mathcal{M}$. In particular, we set

$$\ell^p(\mathcal{M}) := \Big\{u = \sum_{K\in\mathcal{M}} u_K \mathbf{1}_K,\ u_K \in \mathbb{R} : \sum_{K\in\mathcal{M}} |u_K|^p |K| < \infty\Big\}$$

with its natural norm

$$\|u\|_{\ell^p} := \Big(\sum_{K\in\mathcal{M}} |u_K|^p |K|\Big)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty, \quad \|u\|_{\ell^\infty} := \sup_{K\in\mathcal{M}} |u_K|.$$

The discrete version of the seminorm of $W^{1,p}$ is given by the following:

$$|u|_{1,p} := \Big(\sum_{\sigma\in\mathcal{E},\sigma=K|L} |\sigma| d_\sigma \Big(\frac{|u_K - u_L|}{d_\sigma}\Big)^p\Big)^{\frac{1}{p}}, \tag{4.4}$$

where, here and later, we denote by $|\cdot|$ the Lebesgue measure, used for both $N-1$ and N-dimensional sets, which will be clear in the context. In (4.4), we assume, without loss of generality, that $d_\sigma > 0$ for all $\sigma = K|L$ (see also [8]).

We now establish a discrete version of (2.11). The following result may be seen as a complement of similar embedding estimates proved in [3, 4] for $p < N$.

Theorem 4.1. *Let $\mathcal{M}$ be a discrete regular mesh as defined above. Let $p > N$. For any $r \geq 1$, there exists a constant C, only depending on p, r, N, and c_0 given by* (4.3), *such that*

$$\|u\|_{\ell^\infty} \leq C|u|_{1,p}^\theta \|u\|_{\ell^r}^{1-\theta}, \quad \theta := \frac{Np}{r(p-N) + Np},$$

for every $u \in \ell^q$, $q = \min(p, r)$.

Proof. Without loss of generality, we assume that u has compact support, i.e. $u_K \neq 0$ for only a finite number of $K \in \mathcal{M}$ (the general case is recovered by density). We recall the Sobolev inequality for functions of bounded variation

$$\|v\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq C_N \|v\|_{BV(\mathbb{R}^N)},$$

where C_N only depends on N. If applied to $u \in \ell^1(\mathcal{M})$ (piecewise constant functions, compactly supported, belong to $BV(\mathbb{R}^N)$), the previous inequality reads as

$$\|u\|_{\ell^{\frac{N}{N-1}}} \leq C_N \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K,\sigma=K|L} |\sigma_{KL}||u_K - u_L|.$$

We apply this inequality to $|u|^\gamma$ and use that $||u_K|^\gamma - |u_L|^\gamma| \leq \gamma(|u_K|^{\gamma-1} + |u_L|^{\gamma-1})|u_K - u_L|$. We get

$$\|u\|_{\ell^{\frac{\gamma N}{N-1}}}^\gamma \leq C_N \frac{\gamma}{2} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K,\sigma=K|L} |\sigma_{KL}|(|u_K|^{\gamma-1} + |u_L|^{\gamma-1})|u_K - u_L|.$$

Assuming $\gamma > p$ and using the Hölder inequality, we obtain (we shorten the notations in the summation index, where $\sigma = K|L$ is omitted)

$$\|u\|_{\ell^{\frac{\gamma N}{N-1}}}^\gamma \leq C_N \frac{\gamma}{2} \Big(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| d_\sigma (|u_K|^{\gamma-1} + |u_L|^{\gamma-1})^{p'}\Big)^{\frac{1}{p'}} \Big(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| d_\sigma^{1-p} |u_K - u_L|^p\Big)^{\frac{1}{p}}.$$

Using condition (4.3) on the mesh, and the definition of discrete Sobolev seminorm, we deduce that

$$\|u\|_{\ell^{\frac{\gamma N}{N-1}}}^\gamma \leq C c_0^{\frac{1}{p'}} \gamma \Big(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| d(x_K, \sigma) |u_K|^{(\gamma-1)p'}\Big)^{\frac{1}{p'}} |u|_{1,p} \leq C c_0^{\frac{1}{p'}} \gamma \Big(\sum_{K\in\mathcal{M}} |K||u_K|^{(\gamma-1)p'}\Big)^{\frac{1}{p'}} |u|_{1,p},$$

where C denotes possibly different constants only depending on N, p. Thus we obtain

$$\|u\|^{\gamma}_{\ell^{\frac{\gamma N}{N-1}}} \leq C c_0^{\frac{1}{p'}} \gamma \|u\|^{\gamma-1}_{\ell^{(\gamma-1)p'}} |u|_{1,p},$$

which is the discrete equivalent of (2.2) in Lemma 2.1. Starting from this inequality, the iteration scheme can be applied without changes; eventually, this leads to the estimate

$$\|u\|_{l^\infty} \leq \hat{C} |u|^{\alpha}_{1,p} \|u\|^{\beta}_{\ell^r}$$

for some $\alpha, \beta > 0$, and some $\hat{C}$ depending on N, p, r, c_0. The consistency of the scheme implies that the same inequality should hold for smooth functions with compact support; hence the values of $\alpha, \beta > 0$ are fixed by Lemma 2.2. □

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