Research Article

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Maximum Principles and ABP Estimates to Nonlocal Lane–Emden Systems and Some Consequences

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Abstract: This paper deals with maximum principles depending on the domain and ABP estimates associated to the following Lane–Emden system involving fractional Laplace operators:

$$\begin{cases} (-\Delta)^s u = \lambda\rho(x)|v|^{\alpha-1}v & \text{in } \Omega, \\ (-\Delta)^t v = \mu\tau(x)|u|^{\beta-1}u & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s, t \in (0,1)$, $\alpha, \beta > 0$ satisfy $\alpha\beta = 1$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, and ρ and τ are continuous functions on $\overline{\Omega}$ and positive in Ω. We establish some maximum principles depending on Ω. In particular, we explicitly characterize the measure of Ω for which the maximum principles corresponding to this problem hold in Ω. For this, we derived an explicit lower estimate of principal eigenvalues in terms of the measure of Ω. Aleksandrov–Bakelman–Pucci (ABP) type estimates for the above systems are also proved. We also show the existence of a viscosity solution for a nonlinear perturbation of the nonhomogeneous counterpart of the above problem with polynomial and exponential growths. As an application of the maximum principles, we measure explicitly how small $|\Omega|$ has to be to ensure the positivity of the obtained solutions.

Keywords: Nonlocal Lane–Emden system, Maximum Principles, ABP Estimates, Lower Estimate of Eigenvalue, Existence

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1 Introduction and Statements

We consider the nonlocal eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda\rho(x)u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\rho \in L^{n/s}(\Omega)$ is positive in Ω a.e., and the fractional Laplace operator $(-\Delta)^s$ defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) := C(n,s) \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$

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for all $x \in \mathbb{R}^n$ and $C(n,s)$ is a normalization constant, given by

$$C(n,s) := \frac{s2^{2s}\Gamma(\frac{2s+n}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)},$$

where Γ is the usual Gamma function. A natural space for the operator $(-\Delta)^s$ is the weighted L_1-space:

$$L_s := \Big\{u : \mathbb{R}^n \to \mathbb{R} : \|u\|_{L_s} = \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx < +\infty\Big\}.$$

Another space strongly connected to this operator is the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined by

$$\Big\{u \in L^2(\mathbb{R}^n) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\Big\},$$

that is, a Banach space endowed with the norm (see [9])

$$\|u\|_{H^s(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |u|^2\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{\frac{1}{2}}.$$

We denote by $\lambda_{1,s}(\rho,\Omega)$ the first eigenvalue of problem (1.1). In [25], Servadei and Valdinoci showed for $\rho = 1$ that

$$\lambda_{1,s}(\Omega) = \frac{C(n,s)}{2}\min_{\varphi\in X_0^s(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\int_\Omega |\varphi(x)|^2\,dx},$$

where $\lambda_{1,s}(\Omega) := \lambda_{1,s}(1,\Omega)$ and $X_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ is a Hilbert space with the norm $\|\cdot\|_{X_0^s(\Omega)}$ induced by the inner product (see [25])

$$\langle u, v\rangle_{X_0^s(\Omega)} = \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy.$$

For $\rho \in L^\infty(\Omega)$, the first eigenvalue $\lambda_{1,s}(\rho,\Omega)$ was studied in [8], and for $\rho \in L^{\frac{n}{s}}(\Omega)$ in [11].

It is well known that Aleksandrov–Bakelman–Pucci (ABP) type estimates to nonlocal operators are strictly related to maximum principles. The ABP estimate to the nonlocal operator $(-\Delta)^s$ (see [22, Theorem 2.3]) states the following: Let D be an open domain with $R(D) < +\infty$, where $R(D) > 0$ is the smallest constant such that

$$|B_{R(D)}(x) \setminus D| \geq \frac{1}{2}|B_{R(D)}(x)|$$

for all $x \in D$, where $|\cdot|$ stands for the Lebesgue measure of $\mathbb{R}^n$. Note that, if D is contained between two parallel hyperplanes at a distance d, then

$$R(D) \leq \frac{2^n d}{|B_1|},$$

where B_1 is the unit ball of $\mathbb{R}^n$. Assume that $f \in C(\overline{D})$ and $u \in C(\overline{D})$ satisfy $\sup_D u < \infty$ and

$$\begin{cases} (-\Delta)^s u \leq f(x) & \text{in } D,\\ u = 0 & \text{in } \mathbb{R}^n \setminus D,\end{cases}$$

in the viscosity sense (see [22, Definition 2.1]). Then

$$\sup_D u \leq C_s R(D)^{2s}\|f\|_{C(\overline{D})}, \tag{1.2}$$

where C_s is a positive constant. From this ABP estimate follows the maximum principle in domains (not necessarily bounded) for which $R(D)$ is small enough (see [22, Theorem 2.4]): Assume that $\rho \in L^\infty(D)$ and $u \in C(\overline{D})$ satisfy

$$\begin{cases} (-\Delta)^s u \geq \rho(x)u & \text{in } D,\\ u \geq 0 & \text{in } \mathbb{R}^n \setminus D,\end{cases}$$

in the viscosity sense with $\rho u \in C(\overline{D})$. Then there is a number $\overline{R}$ such that $R(D^-) \leq \overline{R}$ implies that each viscosity solution satisfies $u \geq 0$ in D, where $D^- := \{x \in D : u(x) < 0\}$.

Here we are interested in studying maximum principles depending on the domain and Aleksandrov–Bakelman–Pucci (ABP) type estimates to the nonlocal Lane–Emden system

$$\begin{cases} (-\Delta)^s u = \lambda\rho(x)|v|^{\alpha-1}v & \text{in } \Omega, \\ (-\Delta)^t v = \mu\tau(x)|u|^{\beta-1}u & \text{in } \Omega, \\ \quad u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where $s, t \in (0,1)$, $\alpha, \beta > 0$ satisfy $\alpha\beta = 1$, and ρ and τ are continuous functions on $\overline{\Omega}$ and positive in Ω.

The existence of, at least, one positive viscosity solution to problem (1.3) has been established when $\alpha\beta > 1$ in [15] for $s \neq t$, and in [16] for $s = t$. For $\alpha\beta < 1$, existence and uniqueness results of a positive viscosity solution was proved in [16] for $s = t$, and later in [18] for $s \neq t$. See the definitions of viscosity sub-supersolution and solution of problem (1.3) in the paper [18].

The weak formulation of system (1.3) is given by

$$\lambda \int_\Omega \rho(x)|v|^{\alpha-1}v\Phi(x)\,dx = \frac{C(n,s)}{2}\langle u, \Phi\rangle_{X_0^s(\Omega)} \tag{1.4}$$

and

$$\mu \int_\Omega \tau(x)|u|^{\beta-1}u\Psi(x)\,dx = \frac{C(n,t)}{2}\langle v, \Psi\rangle_{X_0^t(\Omega)} \tag{1.5}$$

for any $(\Phi, \Psi) \in H^s(\mathbb{R}^n) \times H^t(\mathbb{R}^n)$ with $\Phi = \Psi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. Note that, since ρ and τ are continuous, the notions of bounded and continuous weak solution and viscosity solution of (1.3) coincide (see [23]).

We say that $(\lambda, \mu) \in \mathbb{R}^2$ is a principal eigenvalue of system (1.3) if this admits a positive viscosity solution $(\varphi, \psi) \in (C(\mathbb{R}^n))^2$, which is called a principal eigenfunction corresponding to (λ, μ).

In [18], Leite and Montenegro proved that the set of principal eigenvalues of (1.3) makes up a smooth curve $\Lambda_1 : (0, \infty) \to \mathbb{R}^2$, $\Lambda_1(a) = (\lambda_1(a), \mu_1(a))$ with $\mu_1(a) = a\lambda_1(a)$, satisfying some important properties such as asymptotic behavior, local isolation, monotonicity and simplicity. They also furnished a min-max characterization for Λ_1. The proof of the min-max expression is more intricate since viscosity solutions are usually not classical.

We say that the weak maximum principle in Ω, denoted by (WMP), associated to (1.3) holds in Ω if for any viscosity supersolution $(u, v) \in (C(\mathbb{R}^n))^2$ such that $u, v \geq 0$ in $\mathbb{R}^n \setminus \Omega$, we obtain $u, v \geq 0$ in Ω. If furthermore either $u, v \equiv 0$ in Ω or at least $u > 0$ in Ω or $v > 0$ in Ω, then we say that the strong maximum principle in Ω, denoted by (SMP), holds in Ω. If $\lambda, \mu > 0$, then (SMP) in Ω means that either $u, v \equiv 0$ in Ω or $u, v > 0$ in Ω.

In [18, Theorem 1.3], Leite and Montenegro characterized completely the set of couples $(\lambda, \mu) \in \mathbb{R}^2$ such that (WMP) and (SMP) associated to (1.3) hold in Ω. More, precisely: Let $\mathcal{C}_1$ be the open region in the first quadrant below Λ_1, that is,

$$\mathcal{C}_1 := \{(\lambda_1(a)t, \mu_1(a)t) : a > 0 \text{ and } 0 < t < 1\}.$$

Then

$$(\lambda, \mu) \in \overline{\mathcal{C}_1} \setminus \Lambda_1 \quad \text{if and only if} \quad \text{(WMP) associated to (1.3) holds in } \Omega \quad \text{if and only if} \quad \text{(SMP) holds in } \Omega.$$

A topic of great interest theme within the elliptic PDEs theory concerns maximum principles for domains of small Lebesgue measure. For example, if $\lambda \in \mathbb{R}$ and $\mathcal{L}$ is a uniformly elliptic linear operator of second order, then there is a constant $\eta > 0$ such that $\mathcal{L} + \lambda$ satisfies weak and strong maximum principles in Ω whenever $|\Omega| < \eta$. This result was proved in two ways: by using the ABP estimate (see [3, Theorem 2.6]), and the Faber–Krahn inequality obtained by Faber [10] and Krahn [14] (see [19, Theorem 5.1], [5, Theorem 10.1], [1, Theorem 4.1], and [20, Proposition 8.6]). More recently, in [17], Leite and Montenegro established the maximum principles for domains of small Lebesgue measure for Lane–Emden systems.

Inspired by these ideas, we characterize some maximum principles corresponding to (1.3) depending on the domain.

Firstly, we classify when such (WMP) (or (SMP)) corresponding to (1.3) is satisfied in domains Ω (not necessarily of small measure) for which $R(\Omega)$ is sufficiently small. For this, we present the following lower bound, more specifically, a counterpart of [3, Lemma 4.1] to nonlocal Lane–Emden systems.

Lemma 1.1. *Let $\Lambda_1(a) = (\lambda_1(a), \mu_1(a))$ be the principal curve corresponding to* (1.3). *So for any $a > 0$, we get*

$$\lambda_1(a) \geq \frac{1}{a^{\frac{1}{\beta+1}} B \|\rho\|_{C(\overline{\Omega})}^{\frac{1}{\alpha+1}} \|\tau\|_{C(\overline{\Omega})}^{\frac{1}{\beta+1}}}, \tag{1.6}$$

$$\mu_1(a) \geq \frac{1}{a^{-\frac{1}{\alpha+1}} B \|\rho\|_{C(\overline{\Omega})}^{\frac{1}{\alpha+1}} \|\tau\|_{C(\overline{\Omega})}^{\frac{1}{\beta+1}}}, \tag{1.7}$$

where

$$B := \max\{C_s R(\Omega)^{2s}, C_t R(\Omega)^{2t}\}.$$

As a consequence of above lemma and [18, Theorem 1.3], we obtain the following theorem.

Theorem 1.2. *Let $\rho, \tau \in C(\overline{\Omega})$ and let $\alpha, \beta > 0$ be such that $\alpha\beta = 1$. Define*

$$\kappa_0 := \|\rho\|_{C(\overline{\Omega})}^{\frac{1}{\alpha+1}} \|\tau\|_{C(\overline{\Omega})}^{\frac{1}{\beta+1}}.$$

Then the following assertions are equivalent:
(i) $\lambda \geq 0$ *and* $\mu \geq 0$.
(ii) (WMP) *corresponding to* (1.3) *holds in* Ω *provided that*

$$R(\Omega) < \min\Big\{\Big(\frac{1}{C_s \mu^{\frac{1}{\beta+1}} \lambda^{\frac{1}{\alpha+1}} \kappa_0}\Big)^{\frac{1}{2s}}, \Big(\frac{1}{C_t \mu^{\frac{1}{\beta+1}} \lambda^{\frac{1}{\alpha+1}} \kappa_0}\Big)^{\frac{1}{2t}}\Big\}. \tag{1.8}$$

(iii) (SMP) *corresponding to* (1.3) *holds in* Ω *provided that* $R(\Omega)$ *satisfies* (1.8).

Secondly, we characterize when such (WMP) (or (SMP)) corresponding to (1.3) is satisfied in domains Ω of small Lebesgue measure. For this purpose, we shall obtain some explicit lower estimates.

When $\alpha = 1$, we get an explicit lower estimate of $\lambda_1(a)$ for any bounded domain Ω as follows.

Theorem 1.3. *Let $s, t \in (0, 1)$, $\alpha = 1$ (and so $\beta = 1$) and $\rho, \tau \in C(\overline{\Omega})$. Then, for every $\Omega \subset \mathbb{R}^n$ open and bounded, we have*

$$\lambda_1(a) \geq \frac{1}{C_0(1+a)} \lambda_{1,\tilde{r}}(B_1) |B_1|^{\frac{2\tilde{r}}{n}} |\Omega|^{-\frac{2\tilde{r}}{n}}, \tag{1.9}$$

where

$$C_0 := \max\{\|\rho\|_{C(\overline{\Omega})}, \|\tau\|_{C(\overline{\Omega})}\} \quad \textit{and} \quad \lambda_{1,\tilde{r}}(\Omega) := \min\{\lambda_{1,s}(\Omega), \lambda_{1,t}(\Omega)\}.$$

Then we obtain the following maximum principle.

Corollary 1.4. *Under the hypothesis of Theorem 1.3, the following assertions are equivalent:*
(i) $\lambda \geq 0$ *and* $\mu \geq 0$.
(ii) (WMP) *corresponding to* (1.3) *holds in* Ω *provided that*

$$|\Omega| < \Big(\frac{\lambda_{1,\tilde{r}}(B_1)}{C_0(\lambda+\mu)}\Big)^{\frac{n}{2\tilde{r}}} |B_1|. \tag{1.10}$$

(iii) (SMP) *corresponding to* (1.3) *holds in* Ω *provided that* $|\Omega|$ *satisfies* (1.10).

When $\alpha \neq 1$, the problem becomes much more delicate. In this case, we use the ideas of the proofs of [5, Theorem 10.1] and [17, Theorem 1.6], together with the following inequalities: For $n = 1$ and $\frac{1}{2} \leq s < 1$, define

$$H^{s,\frac{1}{s}}(\mathbb{R}^n) := \{u \in L^{\frac{1}{s}}(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^{\frac{1}{s}}(\mathbb{R}^n)\}, \quad \tilde{H}^{s,\frac{1}{s}}(\Omega) := \{u \in H^{s,\frac{1}{s}}(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

$$K_{1,s} := \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2}) 2^s \pi^{\frac{1}{2}}}, \qquad \alpha_{1,\frac{1}{s}} := \frac{1}{2} K_{1,s}^{-\frac{1}{1-s}}.$$

An inequality known as fractional Adams inequality (see [21, Theorem 1]) states that

$$\int_{\Omega} e^{\alpha_{1,\frac{1}{s}}|u|^{\frac{1}{1-s}}}\,dx \le c_{1,\frac{1}{s}}|\Omega| \tag{1.11}$$

for all $u \in \tilde{H}^{s,\frac{1}{s}}(\Omega)$ satisfying $\|(-\Delta)^{s/2}u\|_{L^{1/s}(\Omega)} \le 1$, where $c_{1,1/s}$ is a positive constant independent of Ω. Noticing that

$$|u|^p \le C(p)e^{\alpha_{1,\frac{1}{s}}|u|^{\frac{1}{1-s}}} \quad \text{for any } p \ge 1,$$

replacing u by $u/\|(-\Delta)^{s/2}u\|_{L^{1/s}(\Omega)}$ in inequality (1.11) and using the Hölder inequality, we have

$$\Big(\int_{\Omega}|u|^p\,dx\Big)^{\frac{2}{p}} \le c_{1,s}|\Omega|^{\frac{2}{p}+2s-1}\int_{\Omega}|(-\Delta)^{\frac{s}{2}}u|^2\,dx,$$

with $c_{1,s} = (C(p)c_{1,1/s})^{2/p}$ for any $p \ge 1$. Moreover, by [9, Proposition 3.6], we have

$$\Big(\int_{\Omega}|\varphi|^p\,dx\Big)^{\frac{2}{p}} \le \frac{1}{2}|\Omega|^{2/p+2s-1}c_{1,s}C(n,s)\|\varphi\|^2_{X_0^s(\Omega)} \tag{1.12}$$

for all $\varphi \in X_0^s(\Omega)$.

For $n > 2s$, we use the fractional Sobolev inequality for any $\varphi \in X_0^s(\Omega)$:

$$\|\varphi\|^2_{L^{2_s^*}(\Omega)} = \|\varphi\|^2_{L^{2_s^*}(\mathbb{R}^n)} \le c_{n,s}\|(-\Delta)^{\frac{s}{2}}\varphi\|^2_{L^2(\mathbb{R}^n)},$$

where $2_s^* = \frac{2n}{n-2s}$ and an explicit characterization of $c_{n,s}$ is well known; see [6, 7]. Furthermore, by [9, Proposition 3.6], we obtain

$$\|\varphi\|^2_{L^{2_s^*}(\Omega)} \le \frac{1}{2}c_{n,s}C(n,s)\|\varphi\|^2_{X_0^s(\Omega)} \tag{1.13}$$

for all $\varphi \in X_0^s(\Omega)$.

It will also be necessary to use an explicit lower estimate of $\lambda_{1,s}(\Omega)$ in terms of the Lebesgue measure of Ω for any $n \ge 1$. Here we will prove it in a more general context, for $\lambda_{1,s}(\rho,\Omega)$ and $\rho \in L^{n/s}(\Omega)$, as follows.

Theorem 1.5. *Let $s \in (0,1)$ and $\rho \in L^{n/s}(\Omega)$. Then, for every $\Omega \subset \mathbb{R}^n$ open and bounded, the following assertions hold:*
(i) *If $n > 2s$, then we have*

$$\lambda_{1,s}(\rho,\Omega) \ge \frac{(c_{n,s})^{-\frac{1}{2}}}{\|\rho\|_{L^{\frac{n}{s}}(\Omega)}}\lambda_{1,s}(B_1)^{\frac{1}{2}}|B_1|^{\frac{s}{n}}|\Omega|^{-\frac{s}{n}}. \tag{1.14}$$

(ii) *If $n = 1 \le 2s$, then we have*

$$\lambda_{1,s}(\rho,\Omega) \ge \frac{(c_{1,s})^{-\frac{2s}{1+s}}}{\|\rho\|_{L^{\frac{1}{s}}(\Omega)}}\lambda_{1,s}(B_1)^{\frac{1-s}{1+s}}|B_1|^{\frac{2s(1-s)}{1+s}}|\Omega|^{-s}.$$

Note that, if $\Omega = B_1$ and $\rho = 1$, then

$$\lambda_{1,s}(B_1) \ge \frac{1}{c_{n,s}|B_1|^{\frac{2s}{n}}} \quad \text{for any } n \ge 1.$$

Consequently, we deduce the following result.

Corollary 1.6. *Let $s \in (0,1)$ and $\rho \in L^{n/s}(\Omega)$, $n \ge 1$. Then, for every $\Omega \subset \mathbb{R}^n$ open and bounded, we have*

$$\lambda_{1,s}(\rho,\Omega) \ge \frac{|\Omega|^{-\frac{s}{n}}}{c_{n,s}\|\rho\|_{L^{\frac{n}{s}}(\Omega)}}.$$

Now, we define

$$\tilde{c} = \min\Big\{\frac{1}{c_{n,s}}, \frac{1}{c_{n,t}}\Big\}, \qquad C = \max\{1, \|\rho\|_{C(\overline{\Omega})}, \|\tau\|_{C(\overline{\Omega})}\},$$
$$\theta_s = \frac{2s(\alpha+1)+n(1-\alpha)}{2s(\alpha+1)}, \quad \overline{\theta_s} = \frac{2s(\beta+1)+n(1-\beta)}{2s(\beta+1)}.$$

Theorem 1.7. *Let $r = \min\{s, t\}$, let $\rho, \tau \in C(\overline{\Omega})$ and let $\Lambda_1(a) = (\lambda_1(a), \mu_1(a))$ be a principal eigenvalue of* (1.3). *Suppose*

$$|\Omega| \leq \min\Big\{1, \frac{1}{(c_{n,t})^{\frac{n}{2t\theta_t}}}\Big\}. \tag{1.15}$$

(i) *For $n > 2r$ and*

$$1 \leq \alpha < \frac{n+2r}{n-2r}, \tag{1.16}$$

we have

$$\lambda_1(a) + a^\alpha \lambda_1(a)^\alpha \geq \frac{\tilde{c}}{C^\alpha 2^{\alpha-1}} |\Omega|^{-\frac{2r\theta_r}{n}}. \tag{1.17}$$

(ii) *For $n = 1$, $\frac{1}{2} \leq s, t < 1$ and $\alpha \geq 1$, estimate* (1.17) *holds.*

In particular,

$$\lim_{|\Omega| \downarrow 0} \lambda_1(a) = \infty.$$

When $\alpha < 1$, we have $\beta > 1$, and by duality the next result follows.

Corollary 1.8. *Let $r = \min\{s, t\}$, let $\rho, \tau \in C(\overline{\Omega})$ and let $\Lambda_1(a) = (\lambda_1(a), \mu_1(a))$ be a principal eigenvalue of* (1.3). *Suppose*

$$|\Omega| \leq \min\Big\{1, \frac{1}{(c_{n,s})^{\frac{n}{2s\overline{\theta_s}}}}\Big\}.$$

(i) *For $n > 2r$ and $\frac{n-2r}{n+2r} < \alpha < 1$, we have*

$$\lambda_1(a)^\beta + a\lambda_1(a) \geq \frac{\tilde{c}}{C^\beta 2^{\beta-1}} |\Omega|^{-\frac{2r\overline{\theta_r}}{n}}. \tag{1.18}$$

(ii) *For $n = 1$, $\frac{1}{2} \leq s, t < 1$ and $0 < \alpha < 1$, estimate* (1.18) *holds.*

As an application of these explicit lower estimates and [18, Theorem 1.3], we obtain the following characterization of maximum principles.

Corollary 1.9. *Let $r = \min\{s, t\}$, let $\rho, \tau \in C(\overline{\Omega})$. Assume that $\alpha, \beta > 0$ for $n = 1$ and $\frac{1}{2} \leq s, t < 1$ and*

$$\frac{n-2r}{n+2r} < \alpha, \beta < \frac{n+2r}{n-2r} \tag{1.19}$$

for $n > 2r$. Then the following assertions are equivalent:

(i) $\lambda \geq 0$ *and* $\mu \geq 0$.

(ii) (WMP) *corresponding to* (1.3) *holds in Ω whenever*

$$|\Omega| < \begin{cases} \min\Big\{1, \dfrac{1}{(c_{n,t})^{\frac{n}{2t\theta_t}}}, \Big(\dfrac{\tilde{c}}{C^\alpha 2^{\alpha-1}(\lambda + \mu^\alpha)}\Big)^{\frac{n}{2r\theta_r}}\Big\} & \text{if } \alpha \geq 1, \\ \min\Big\{1, \dfrac{1}{(c_{n,s})^{\frac{n}{2s\overline{\theta_s}}}}, \Big(\dfrac{\tilde{c}}{C^\beta 2^{\beta-1}(\lambda^\beta + \mu)}\Big)^{\frac{n}{2r\overline{\theta_r}}}\Big\} & \text{if } 0 < \alpha < 1. \end{cases} \tag{1.20}$$

(iii) (SMP) *corresponding to* (1.3) *holds in Ω whenever $|\Omega|$ satisfies* (1.20).

Note that Theorem 1.2 and Corollary 1.9 complement each other. In fact, Theorem 1.2 applies to all $\alpha, \beta > 0$ and in domains Ω (not necessarily of small measure) for which $R(\Omega)$ is sufficiently small. On the other hand, Corollary 1.9 has a restriction in relation to α and β for $n > 2\min\{s, t\}$. However, it measures explicitly how small $|\Omega|$ has to be so that the maximum principles corresponding to (1.3) hold in Ω.

Now, we give an ABP estimate for supersolutions of the nonlocal Lane–Emden system (1.3).

Theorem 1.10. *Let $n > 2s$. Assume that $(u, v) \in (C(\mathbb{R}^n))^2$ is a viscosity supersolution of* (1.3) *and a viscosity solution of the problem*

$$\begin{cases} (-\Delta)^s u \leq \lambda\rho(x)|v|^{\alpha-1}v + f(x) & \text{in } \Omega, \\ (-\Delta)^t v \leq \mu\tau(x)|u|^{\beta-1}u + g(x) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.21}$$

where $(f,g) \in (C(\overline{\Omega}))^2$, $f,g \geq 0$ *in* Ω. *If* $(\lambda,\mu) \in \overline{\mathcal{C}_1} \setminus \Lambda_1$, *then the following ABP estimate holds in* Ω:

$$\sup_{\Omega} u + (\sup_{\Omega} v)^{\alpha} \leq A(\|f\|_{L^p(\Omega)} + \|g\|^{\alpha}_{L^p(\Omega)}), \tag{1.22}$$

where $p \geq \max\{\frac{n}{s}, \frac{n}{t}\}$ *and* $A > 0$ *is a constant independent of the* $(f,g) \in (C(\overline{\Omega}))^2$ *and of any viscosity solution* $(u,v) \in (C(\mathbb{R}^n))^2$.

Using Theorem 1.2 instead of [18, Theorem 1.3] in the proof of Theorem 1.10, we can extend this result to any $(\lambda,\mu) \in \mathbb{R}^2_+ = [0,\infty)^2$ whenever $R(\Omega)$ satisfies (1.8).

We will also present in this paper an important application of Corollary 1.9. For this, we show the existence of a viscosity solution to the following problem:

$$\begin{cases} (-\Delta)^s u = \lambda\rho(x)|v|^{\alpha-1}v + f(x) + F(x,u,v) & \text{in } \Omega, \\ (-\Delta)^t v = \mu\tau(x)|u|^{\beta-1}u + g(x) + G(x,u,v) & \text{in } \Omega, \\ \quad u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.23}$$

where f and g are functions defined on Ω and $F, G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying

$$0 \leq F(x,b,c) \leq D_1(|b|^{p_1} \exp(\sigma_1 |b|^{\frac{1}{1-s}}) + |c|^{p_2}), \tag{1.24}$$

$$0 \leq G(x,b,c) \leq D_2(|b|^{q_1} + |c|^{q_2} \exp(\sigma_2 |c|^{\frac{1}{1-t}})), \tag{1.25}$$

where $1 < p_1, p_2, q_1, q_2 < \infty$ and $\sigma_1, \sigma_2, D_1, D_2 \geq 0$ are constants.

We define

$$\begin{aligned}
a_{1,s} &:= \frac{2n - (n-2s)(\alpha+1)}{n(\alpha+1)}, \\
a_{2,s} &:= \frac{2n - (n-2s)(p_2+1)}{2n}, \\
a_{3,s} &:= \frac{n+2s}{2n}, \\
a_{4,s} &:= \frac{2n - (n-2s)(q_1+1)}{2n}, \\
a_5 &:= \frac{n - (n-2t)(q_2+1)}{2n}, \\
a_6 &:= \frac{n - (n-2s)(p_1+1)}{2n}, \\
a_7 &:= \frac{\alpha-\beta}{\alpha+1} + a_{1,t}, \\
a_8 &:= \frac{\alpha-\beta}{\alpha+1} + \beta a_{1,s}, \\
a_{9,s} &:= |\Omega|^{s-\frac{1}{2}} \Big(\frac{C(n,s)}{2}\Big)^{\frac{1}{2}}, \\
d_0 &:= \min\Big\{\frac{C(n,s)}{2}, \frac{C(n,t)}{2}\Big\}, \\
D_0 &:= \max\Big\{\frac{C(n,s)}{2}, \frac{C(n,t)}{2}\Big\}, \\
r_1 &:= \Big(\frac{d_0}{2^4 D_1 (c_{n,s} D_0)^{\frac{p_1+1}{2}} |\Omega|^{a_6+\frac{1}{2}} \mathcal{L}_s^{\frac{1}{2}}}\Big)^{\frac{1}{p_1-1}}, \\
r_2 &: = \Big(\frac{d_0}{2^4 D_2 (c_{n,t} D_0)^{\frac{q_2+1}{2}} |\Omega|^{a_5+\frac{1}{2}} \mathcal{L}_t^{\frac{1}{2}}}\Big)^{\frac{1}{q_2-1}}, \\
r_3 &:= \Big(\frac{(p_2+1)d_0}{2^4 D_1 (p_2 (c_{n,t} D_0)^{\frac{p_2+1}{2}} |\Omega|^{a_{2,t}} + (c_{n,s} D_0)^{\frac{p_2+1}{2}} |\Omega|^{a_{2,s}})}\Big)^{\frac{1}{p_2-1}}, \\
r_4 &:= \frac{1}{a_{9,s}} \Big(\frac{\alpha_{1,\frac{1}{s}}}{2\sigma_1}\Big)^{1-s},
\end{aligned}$$

$$r_5 := \Big(\frac{(q_1+1)d_0}{2^4 D_2(q_1(c_{n,s}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,s}} + (c_{n,t}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,t}})}\Big)^{\frac{1}{q_1-1}},$$
$$r_6 := \frac{1}{a_{9,t}}\Big(\frac{\alpha_{1,\frac{1}{t}}}{2\sigma_2}\Big)^{1-t},$$

where $\mathcal{L}_s = c_{1,\frac{1}{s}}$ if $n = 1$ and $\frac{1}{2} \le s < 1$, and $\mathcal{L}_s = 1$ otherwise. The last result gives explicit conditions to ensure the existence of a viscosity solution to (1.23).

Theorem 1.11. *Let $r = \min\{s, t\}$ and let $\alpha, \beta > 0$ be such that $\alpha\beta = 1$. Suppose that $\rho, \tau, f, g \in L^\infty(\Omega)$ and $F, G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying* (1.24) *and* (1.25). *Assume the following conditions:*
(i) $1 < p_1, p_2, q_1, q_2 < \infty$, $\sigma_1, \sigma_2 \ge 0$ *and* $\alpha \ge 1$ *for* $n = 1$ *and* $\frac{1}{2} \le s, t < 1$.
(ii) $1 < p_2, q_1, q_2 < \frac{n+2t}{n-2t}$, $1 < p_1 < \infty$, $\sigma_1 \ge 0$, $\sigma_2 = 0$, *and* α *satisfies* (1.16) *for* $n = 1$, $0 < t < \frac{1}{2}$ *and* $\frac{1}{2} \le s < 1$.
(iii) $1 < p_1, p_2, q_1 < \frac{n+2s}{n-2s}$, $1 < q_2 < \infty$, $\sigma_1 = 0$, $\sigma_2 \ge 0$, *and* α *satisfies* (1.16) *for* $n = 1$, $0 < s < \frac{1}{2}$ *and* $\frac{1}{2} \le t < 1$.
(iv) $1 < p_1, p_2, q_1, q_2 < \frac{n+2r}{n-2r}$, $\sigma_1 = \sigma_2 = 0$ *and* α *satisfies* (1.16) *for* $n > 2\max\{s, t\}$.

Assume also that

$$(\|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac{1}{2}} + \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac{1}{2}}) \le \frac{d_0 r_0}{20}, \tag{1.26}$$

where

$$r_0 = \begin{cases} \min\{r_1, r_2, r_3, r_4, r_5, r_6\} & \text{if } n = 1 \text{ and } \dfrac{1}{2} \le s, t < 1, \\ \min\{r_1, r_2, r_3, r_4, r_5\} & \text{if } n = 1,\ 0 < t < \dfrac{1}{2} \text{ and } \dfrac{1}{2} \le s < 1, \\ \min\{r_1, r_2, r_3, r_5, r_6\} & \text{if } n = 1,\ 0 < s < \dfrac{1}{2} \text{ and } \dfrac{1}{2} \le t < 1, \\ \min\{r_1, r_2, r_3, r_5\} & \text{if } n > 2\max\{s, t\}. \end{cases}$$

Suppose that

$$\lambda^* = \frac{d_0}{10\|\rho\|_{L^\infty(\Omega)}}\min\Big\{\frac{1}{c_{n,s}D_0|\Omega|^{a_{1,s}}}, \frac{r_0^{2-2\alpha}}{c_{n,t}^\alpha D_0^\alpha|\Omega|^{\alpha a_{1,t}}}\Big\}$$

and

$$\mu^* = \frac{d_0}{10\|\tau\|_{L^\infty(\Omega)}}\min\Big\{\frac{1}{c_{n,t}D_0|\Omega|^{a_7}}, \frac{r_0^{2-2\beta}}{(c_{n,s}D_0)^\beta|\Omega|^{a_8}}\Big\}.$$

Then, for every $\lambda \in [0, \lambda^*)$ and $\mu \in [0, \mu^*)$, system (1.23) *has a weak solution $(u, v) \in X_0^s(\Omega) \times X_0^t(\Omega)$. Furthermore, $(u, v) \in C^s(\mathbb{R}^n) \times C^t(\mathbb{R}^n)$ is a viscosity solution of problem* (1.23).

The values λ^* and μ^* can be obtained in a simpler way. For this, note that

$$\frac{1}{c_{n,s}D_0|\Omega|^{a_{1,s}}} \le \frac{r_0^{2-2\alpha}}{(c_{n,t}D_0)^\alpha|\Omega|^{\alpha a_{1,t}}} \quad \text{if and only if} \quad \frac{r_0^{2-2\beta}}{(c_{n,s}D_0)^\beta|\Omega|^{a_8}} \le \frac{1}{c_{n,t}D_0|\Omega|^{a_7}}.$$

Therefore, either

$$\lambda^* = \frac{d_0}{10\|\rho\|_{L^\infty(\Omega)}c_{n,s}D_0|\Omega|^{a_{1,s}}} \quad \text{and} \quad \mu^* = \frac{d_0 r_0^{2-2\beta}}{10\|\tau\|_{L^\infty(\Omega)}(c_{n,s}D_0)^\beta|\Omega|^{a_8}},$$

or

$$\lambda^* = \frac{d_0 r_0^{2-2\alpha}}{10\|\rho\|_{L^\infty(\Omega)}(c_{n,t}D_0)^\alpha|\Omega|^{\alpha a_{1,t}}} \quad \text{and} \quad \mu^* = \frac{d_0}{10\|\tau\|_{L^\infty(\Omega)}c_{n,t}D_0|\Omega|^{a_7}}.$$

By duality, Theorem 1.11 holds for all $\alpha, \beta > 0$ if $n = 1$ and $\frac{1}{2} \le s, t < 1$, and for all α, β satisfying (1.19) if $n > 2\min\{s, t\}$.

For the application of Corollary 1.9 mentioned above, we assume the hypotheses of Theorem 1.11. Suppose also that (f, g) is a pair of nonnegative functions and $\rho, \tau \in C(\overline{\Omega})$ are positive in Ω.

Let (u_0, v_0) be a viscosity solution of system (1.23). Then, by Corollary 1.9, we obtain $u_0, v_0 \ge 0$ in Ω whenever $|\Omega|$ satisfies (1.20). Moreover, if either $f, g \not\equiv 0$ in Ω or $f \not\equiv 0$ in Ω and $\mu > 0$ or $g \not\equiv 0$ in Ω and $\lambda > 0$, then $u_0, v_0 > 0$ in Ω provided that $|\Omega|$ satisfies (1.20). Thus, we measure explicitly how small $|\Omega|$ has to be to ensure the positivity of the obtained solutions in Theorem 1.11.

Our arguments are based on maximum and comparison principles related to nonlocal Lane–Emden systems obtained in [18]. Some results also play a fundamental role in our proofs such as ABP estimate for nonlocal operators (inequality (1.2)), fractional Faber–Krahn inequality (see [4, Theorem 3.5]), variational characterization of $\lambda_{1,s}(\Omega)$, fractional Adams inequality, fractional Sobolev inequality and [24, Proposition 1.4].

2 Proof of Lemma 1.1

Let (φ_a, ψ_a) be a principal eigenfunction of (1.3) associated to $(\lambda_1(a), \mu_1(a))$. Applying the ABP estimate for nonlocal operators (inequality (1.2)) to problem (1.3), we have

$$\begin{aligned}\|\varphi_a\|_{C(\overline{\Omega})} &= \sup_{\Omega} \varphi_a \\ &\leq B_1 \|\lambda_1(a)\rho(x)\psi_a^{\alpha}\|_{C(\overline{\Omega})} \\ &\leq \lambda_1(a) B_1 \|\rho\|_{C(\overline{\Omega})} \|\psi_a\|_{C(\overline{\Omega})}^{\alpha}\end{aligned}$$

and

$$\|\psi_a\|_{C(\overline{\Omega})} \leq \mu_1(a) B_2 \|\tau\|_{C(\overline{\Omega})} \|\varphi_a\|_{C(\overline{\Omega})}^{\beta},$$

where $B_1 := C_s R(\Omega)^{2s}$ and $B_2 := C_t R(\Omega)^{2t}$. Therefore, combining these two inequalities and using that $B = \max\{B_1, B_2\}$, $\mu_1(a) = a\lambda_1(a)$ and $\alpha\beta = 1$, we obtain inequality (1.6). Using again that $\mu_1(a) = a\lambda_1(a)$, we have that (1.7) is an immediate consequence of the lower bound (1.6), and so we end the proof.

3 Proof of Theorem 1.2

We first prove that (i) implies (ii). For this, let $\lambda \geq 0$ and $\mu \geq 0$. If either $\lambda = 0$ or $\mu = 0$, then assertion (ii) holds from Silvestre's maximum principle [27].

Now, assume that $\lambda > 0$ and $\mu > 0$. In this case, we take

$$\kappa_0 = \|\rho\|_{C(\overline{\Omega})}^{\frac{1}{\alpha+1}} \|\tau\|_{C(\overline{\Omega})}^{\frac{1}{\beta+1}}.$$

Note that $\kappa_0 > 0$. Thus, by taking $a = \frac{\mu}{\lambda}$ and using the lower bound (1.6) of Lemma 1.1 and the fact that $\alpha\beta = 1$, we have

$$\lambda_1(a) \geq \frac{1}{a^{\frac{1}{\beta+1}} B \|\rho\|_{C(\overline{\Omega})}^{\frac{1}{\alpha+1}} \|\tau\|_{C(\overline{\Omega})}^{\frac{1}{\beta+1}}} = \frac{1}{a^{\frac{1}{\beta+1}} B \kappa_0} > \lambda,$$

provided that $R(\Omega)$ satisfies (1.8). Then $(\lambda, \mu) \in \overline{\mathcal{C}_1} \setminus \Lambda_1$ for such domains. So, by [18, Theorem 1.3], we get that (ii) holds.

Finally, by [18, Theorem 1.3], we have that (ii) implies (i), and (ii) is equivalent to (iii). This finishes the derived proof.

4 Proof of Theorem 1.3

Let (φ_a, ψ_a) be a principal eigenfunction corresponding to $(\lambda_1(a), \mu_1(a))$, normalized so that

$$\|\varphi_a\|_{L^2(\Omega)}^2 + \|\psi_a\|_{L^2(\Omega)}^2 = 1.$$

Since (φ_a, ψ_a) is a positive viscosity solution of (1.3), applying equations (1.4) and (1.5) with $\Phi = \varphi_a$ and $\Psi = \psi_a$, we get

$$\lambda_1(a) \int_\Omega \rho(x)\psi_a\varphi_a \, dx = \frac{C(n,s)}{2} \|\varphi_a\|^2_{X_0^s(\Omega)}$$

and

$$\mu_1(a) \int_\Omega \tau(x)\varphi_a\psi_a \, dx = \frac{C(n,t)}{2} \|\psi_a\|^2_{X_0^t(\Omega)}.$$

Therefore, it follows from variational characterization of $\lambda_{1,s}(\Omega)$ and $\lambda_{1,t}(\Omega)$ that

$$\|\rho\|_{C(\overline{\Omega})}\lambda_1(a) \geq \frac{C(n,s)}{2}\|\varphi_a\|^2_{X_0^s(\Omega)} \geq \lambda_{1,s}(\Omega)\|\varphi_a\|^2_{L^2(\Omega)}$$

and

$$\|\tau\|_{C(\overline{\Omega})}a\lambda_1(a) \geq \frac{C(n,t)}{2}\|\psi_a\|^2_{X_0^t(\Omega)} \geq \lambda_{1,t}(\Omega)\|\psi_a\|^2_{L^2(\Omega)},$$

because $\mu_1(a) = a\lambda_1(a)$. Consequently, adding up these two inequalities shows that

$$\lambda_1(a) \geq \frac{1}{C_0(1+a)}\lambda_{1,\tilde{r}}(\Omega).$$

Moreover, by the fractional Faber–Krahn inequality (see [4, Theorem 3.5]), it is well known that

$$\lambda_{1,\tilde{r}}(\Omega) \geq \lambda_{1,\tilde{r}}(B_1)|B_1|^{\frac{2\tilde{r}}{n}}|\Omega|^{-\frac{2\tilde{r}}{n}}.$$

Finally, (1.9) holds. This ends the proof of the theorem.

5 Proof of Theorem 1.5

Let φ_1 denote a principal eigenfunction corresponding to $\lambda_{1,s}(\rho, \Omega)$. We first show assertion (i). Notice that

$$\frac{C(n,s)}{2}\|\varphi_1\|^2_{X_0^s(\Omega)} = \lambda_{1,s}(\rho,\Omega)\int_\Omega \rho(x)\varphi_1^2 \, dx.$$

Moreover, by using the Hölder inequality, we get

$$\int_\Omega \rho(x)\varphi_1^2 \, dx \leq \|\rho\|_{L^{\frac{n}{s}}(\Omega)}\|\varphi_1\|^2_{L^{\gamma+1}(\Omega)},$$

where $\gamma = \frac{n+s}{n-s}$. On the other hand, by the variational characterization of $\lambda_{1,s}(\Omega)$, inequality (1.13) and the interpolation inequality, we have

$$\frac{C(n,s)}{2}\frac{\|\varphi_1\|^2_{X_0^s(\Omega)}}{\|\varphi_1\|^2_{L^{\gamma+1}(\Omega)}} \geq (c_{n,s})^{\theta-1}\lambda_{1,s}(\Omega)^\theta,$$

where

$$\frac{1}{\gamma+1} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}.$$

Furthermore, by the fractional Faber–Krahn inequality, we obtain

$$\lambda_{1,s}(\Omega)^\theta \geq \lambda_{1,s}(B_1)^\theta|B_1|^{\frac{2s\theta}{n}}|\Omega|^{-\frac{2s\theta}{n}}.$$

So (1.14) occurs, and so we conclude the proof of assertion (i). Finally, the proof of assertion (ii) follows in a similar way by using the interpolation inequality with $\theta = \frac{n-s}{n+s}$. Then, instead of inequality (1.13), we invoke inequality (1.12). This finishes the proof.

6 Proof of Theorem 1.7

Firstly, we prove the case $n > 2\max\{s, t\}$. Let (φ_a, ψ_a) denote a principal eigenfunction corresponding to $(\lambda_1(a), \mu_1(a))$. Since

$$(-\Delta)^s \varphi_a = \lambda_1(a)\rho(x)\psi_a^\alpha$$

in the viscosity sense, applying equation (1.4) with $\Phi = \varphi_a$, we obtain

$$\lambda_1(a)\int_\Omega \rho(x)\psi_a^\alpha\varphi_a\,dx = \frac{C(n,s)}{2}\|\varphi_a\|^2_{X_0^s(\Omega)}.$$

Moreover, by using the Hölder and Young inequalities, we get

$$\int_\Omega \rho(x)\psi_a^\alpha\varphi_a\,dx \le C(\|\varphi_a\|^2_{L^{\alpha+1}(\Omega)} + \|\psi_a\|^{2\alpha}_{L^{\alpha+1}(\Omega)}).$$

Consequently,

$$\lambda_1(a)C(\|\varphi_a\|^2_{L^{\alpha+1}(\Omega)} + \|\psi_a\|^{2\alpha}_{L^{\alpha+1}(\Omega)}) \ge \frac{C(n,s)}{2}\|\varphi_a\|^2_{X_0^s(\Omega)}. \tag{6.1}$$

Similarly, it follows from

$$(-\Delta)^t \psi_a = \mu_1(a)\tau(x)\varphi_a^\beta$$

in the viscosity sense that

$$\mu_1(a)C|\Omega|^{\frac{\alpha-\beta}{\alpha+1}}(\|\varphi_a\|^{2\beta}_{L^{\alpha+1}(\Omega)} + \|\psi_a\|^2_{L^{\alpha+1}(\Omega)}) \ge \frac{C(n,t)}{2}\|\psi_a\|^2_{X_0^t(\Omega)}.$$

Thus, it follows from (1.15) that

$$\mu_1(a)^\alpha C^\alpha(\|\varphi_a\|^2_{L^{\alpha+1}(\Omega)} + \|\psi_a\|^{2\alpha}_{L^{\alpha+1}(\Omega)}) \ge \frac{1}{2^{\alpha-1}}\Big(\frac{C(n,t)}{2}\|\psi_a\|^2_{X_0^t(\Omega)}\Big)^\alpha. \tag{6.2}$$

Then, adding up inequalities (6.1) and (6.2) shows that

$$\lambda_1(a) + a^\alpha\lambda_1(a)^\alpha \ge \frac{1}{C^\alpha 2^{\alpha-1}}\Bigg(\frac{\frac{C(n,s)}{2}\|\varphi_a\|^2_{X_0^s(\Omega)} + (\frac{C(n,t)}{2}\|\psi_a\|^2_{X_0^t(\Omega)})^\alpha}{\|\varphi_a\|^2_{L^{\alpha+1}(\Omega)} + \|\psi_a\|^{2\alpha}_{L^{\alpha+1}(\Omega)}}\Bigg)$$

because $\mu_1(a) = a\lambda_1(a)$. On the other hand, by using the interpolation inequality, variational characterization of $\lambda_{1,s}(\Omega)$ and inequality (1.13), we derive

$$\frac{C(n,s)}{2}\frac{\|\varphi_a\|^2_{X_0^s(\Omega)}}{\|\varphi_a\|^2_{L^{\alpha+1}(\Omega)}} \ge (c_{n,s})^{\theta_s-1}\lambda_{1,s}(\Omega)^{\theta_s},$$

where $\frac{1}{\alpha+1} = \frac{\theta_s}{2} + \frac{1-\theta_s}{2_s^*}$. Furthermore, by Corollary 1.6, we obtain

$$(c_{n,s})^{\theta_s-1}\lambda_{1,s}(\Omega)^{\theta_s} \ge \frac{|\Omega|^{-\frac{2s\theta_s}{n}}}{c_{n,s}}.$$

It follows from (1.15) that (1.17) holds. The remaining cases follow in a similar way. This finishes the proof.

7 Proof of Theorem 1.10

Let $(\lambda,\mu) \in \overline{\mathcal{C}_1} \setminus \Lambda_1$ and let $(f,g) \in (C(\overline{\Omega}))^2$ be a nonnegative function. Let also $(u,v) \in (C(\mathbb{R}^n))^2$ be a viscosity supersolution of (1.3) satisfying (1.21) in the viscosity sense. Then, by [18, Theorem 1.3], we have that (u,v) is a nonnegative function. By [18, Corollary 1.1], there is a viscosity solution $(z,w) \in (C(\mathbb{R}^n))^2$ of the problem

$$\begin{cases} (-\Delta)^s z = \lambda\rho(x)|w|^{\alpha-1}w + f(x) & \text{in } \Omega, \\ (-\Delta)^t w = \mu\tau(x)|z|^{\beta-1}z + g(x) & \text{in } \Omega, \\ z = w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We affirm that there exists a constant $A > 0$ such that

$$\sup_{\Omega} z + (\sup_{\Omega} w)^{\alpha} \leq A(\|f\|_{L^p(\Omega)} + \|g\|^{\alpha}_{L^p(\Omega)}), \tag{7.1}$$

where $p \geq \max\{\frac{n}{s}, \frac{n}{t}\}$. In fact, proceeding by contradiction, assume that there is a sequence

$$(f_k, g_k)_{k\in\mathbb{N}} \in (C(\overline{\Omega}))^2$$

and corresponding viscosity solutions

$$(u_k, v_k)_{k\in\mathbb{N}} \in (C(\mathbb{R}^n))^2$$

such that

$$\|u_k\|_{C(\overline{\Omega})} + \|v_k\|^{\alpha}_{C(\overline{\Omega})} > k(\|f_k\|_{L^p(\Omega)} + \|g_k\|^{\alpha}_{L^p(\Omega)}).$$

We define

$$\begin{aligned}
\tilde{u}_k(x) &:= \frac{u_k(x)}{\|u_k\|_{C(\overline{\Omega})} + \|v_k\|^{\alpha}_{C(\overline{\Omega})}},\\
\tilde{v}_k(x) &:= \frac{v_k(x)}{(\|u_k\|_{C(\overline{\Omega})} + \|v_k\|^{\alpha}_{C(\overline{\Omega})})^{\beta}},\\
\tilde{f}_k(x) &:= \frac{f_k(x)}{\|u_k\|_{C(\overline{\Omega})} + \|v_k\|^{\alpha}_{C(\overline{\Omega})}},\\
\tilde{g}_k(x) &:= \frac{g_k(x)}{(\|u_k\|_{C(\overline{\Omega})} + \|v_k\|^{\alpha}_{C(\overline{\Omega})})^{\beta}}.
\end{aligned}$$

Therefore, $(\tilde{u}_k, \tilde{v}_k) \in (C(\mathbb{R}^n))^2$ is a viscosity solution of the system

$$\begin{cases}
(-\Delta)^s \tilde{u}_k = \lambda\rho(x)|\tilde{v}_k|^{\alpha-1}\tilde{v}_k + \tilde{f}_k(x) & \text{in } \Omega,\\
(-\Delta)^t \tilde{v}_k = \mu\tau(x)|\tilde{u}_k|^{\beta-1}\tilde{u}_k + \tilde{g}_k(x) & \text{in } \Omega,\\
\tilde{u}_k = \tilde{v}_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

and, furthermore,

$$\begin{cases}
\|\tilde{u}_k\|_{C(\overline{\Omega})} + \|\tilde{v}_k\|^{\alpha}_{C(\overline{\Omega})} = 1,\\
\|\tilde{f}_k\|_{L^p(\Omega)} + \|\tilde{g}_k\|^{\alpha}_{L^p(\Omega)} < \dfrac{1}{k}.
\end{cases} \tag{7.2}$$

Note that Ω is bounded. Then, by the Arzelá–Ascoli Theorem, up to a subsequence, we get the uniform convergence $\tilde{u}_k \to \tilde{u}$ and $\tilde{v}_k \to \tilde{v}$ locally in $\mathbb{R}^n$. Note also that

$$\begin{cases}
(-\Delta)^s(\tilde{u}_m - \tilde{u}_k) = \lambda\rho(x)(|\tilde{v}_m|^{\alpha-1}\tilde{v}_m - |\tilde{v}_k|^{\alpha-1}\tilde{v}_k) + \tilde{f}_m(x) - \tilde{f}_k(x) & \text{in } \Omega,\\
(-\Delta)^t(\tilde{v}_m - \tilde{v}_k) = \mu\tau(x)(|\tilde{u}_m|^{\beta-1}\tilde{u}_m - |\tilde{u}_k|^{\beta-1}\tilde{u}_k) + \tilde{g}_m(x) - \tilde{g}_k(x) & \text{in } \Omega,\\
\tilde{u}_m - \tilde{u}_k = \tilde{v}_m - \tilde{v}_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases} \tag{7.3}$$

in the viscosity sense. Thus, by using [24, Proposition 1.4] with $p \geq \max\{\frac{n}{s}, \frac{n}{t}\}$ in each equation of problem (7.3), we obtain

$$\begin{aligned}
\|\tilde{u}_m - \tilde{u}_k\|_{C^s(\mathbb{R}^n)} &\leq c_1(\|\tilde{v}_m^{\alpha} - \tilde{v}_k^{\alpha}\|_{L^p(\Omega)} + \|\tilde{f}_m - \tilde{f}_k\|_{L^p(\Omega)}),\\
\|\tilde{v}_m - \tilde{v}_k\|_{C^t(\mathbb{R}^n)} &\leq c_1(\|\tilde{u}_m^{\beta} - \tilde{u}_k^{\beta}\|_{L^p(\Omega)} + \|\tilde{g}_m - \tilde{g}_k\|_{L^p(\Omega)})
\end{aligned}$$

for some constant $c_1 > 0$ independent of k. Therefore, $\tilde{u}_k \to \tilde{u}$ in $C^s(\mathbb{R}^n)$ and $\tilde{v}_k \to \tilde{v}$ in $C^t(\mathbb{R}^n)$, and by (7.2) we have $\|\tilde{u}\|_{C(\overline{\Omega})} + \|\tilde{v}\|^{\alpha}_{C(\overline{\Omega})} = 1$. Moreover, $(\tilde{u}, \tilde{v}) \in (C(\mathbb{R}^n))^2$ is a positive viscosity solution of the problem

$$\begin{cases}
(-\Delta)^s \tilde{u} = \lambda\rho(x)|\tilde{v}|^{\alpha-1}\tilde{v} & \text{in } \Omega,\\
(-\Delta)^t \tilde{v} = \mu\tau(x)|\tilde{u}|^{\beta-1}\tilde{u} & \text{in } \Omega,\\
\tilde{u} = \tilde{v} = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

Then (λ, μ) is a principal eigenvalue of system (1.3). But this contradicts the assumption $(\lambda, \mu) \in \overline{\mathcal{C}_1} \setminus \Lambda_1$. Thus, we get a positive constant A such that (7.1) holds.

Note that the pairs $(u, v), (z, w) \in (C(\mathbb{R}^n))^2$ satisfy the following inequalities in the sense of viscosity:

$$\begin{cases} (-\Delta)^s u + \lambda\rho(x)|v|^{\alpha-1}v \le (-\Delta)^s z + \lambda\rho(x)|w|^{\alpha-1}w \\ (-\Delta)^t v + \mu\tau(x)|u|^{\beta-1}u \le (-\Delta)^t w + \mu\tau(x)|z|^{\beta-1}z \end{cases} \quad \text{in } \Omega,$$

and $u \le z$ and $v \le w$ in $\mathbb{R}^n \setminus \Omega$. Then, by [18, Theorem 1.4], we have $u \le z$ and $v \le w$ in Ω. Joining this fact with inequality (7.1), we obtain estimate (1.22). This concludes the proof.

8 Proof of Theorem 1.11

In this section, we focus on the proof of Theorem 1.11. For this, we will use the Galerkin method together with the following lemma.

Lemma 8.1. *Let $\phi : (\mathbb{R}^l, \|\cdot\|_l) \to (\mathbb{R}^l, \|\cdot\|_l)$ be a continuous function such that $\langle\phi(\xi), \xi\rangle \ge 0$ for every $\xi \in \mathbb{R}^l$ with $\|\xi\|_l = r_0$ for some $r_0 > 0$, where $\|x\|_l$ denotes a general norm in $\mathbb{R}^l$ and $\langle\cdot,\cdot\rangle^{1/2} = |\cdot|_2$. Then there exists z_0 in the closed ball $\overline{B}_{r_0}(0) := \{z \in \mathbb{R}^l : \|z\|_l \le r_0\}$ such that $\phi(z_0) = 0$.*

A proof of this lemma can be found in [13] for the particular case $\|x\|_l := |x|_2$, and in [2] for the general case.

Proof of Theorem 1.11. We define $E := X_0^s(\Omega) \times X_0^t(\Omega)$ induced with the norm

$$\|(u, v)\| := \|u\|_{X_0^s(\Omega)} + \|v\|_{X_0^t(\Omega)}.$$

Let $\mathcal{B}_s = \{w_{1,s}, w_{2,s}, \dots, w_{m,s}, \dots\}$ be an orthonormal Hilbertian basis of $X_0^s(\Omega)$. We also define

$$W_{m,s} := [w_{1,s}, w_{2,s}, \dots, w_{m,s}]$$

to be the space generated by $\{w_{1,s}, w_{2,s}, \dots, w_{m,s}\}$ with norm induced from $X_0^s(\Omega)$.

Clearly,

$$|\eta|_{m,s} + |\xi|_{m,t} = \Big\|\sum_{\imath=1}^m \eta_\imath w_{\imath,s}\Big\|_{X_0^s(\Omega)} + \Big\|\sum_{\imath=1}^m \xi_\imath w_{\imath,t}\Big\|_{X_0^t(\Omega)},$$

where

$$(\eta, \xi) = (\eta_1, \eta_2, \dots, \eta_m, \xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^{2m}$$

is a norm in $\mathbb{R}^{2m}$. Thus, the application

$$i : (\mathbb{R}^{2m}, |\cdot|_{m,s} + |\cdot|_{m,t}) \to (W_{m,s} \times W_{m,t}, \|(\cdot,\cdot)\|)$$

such that $i(\eta, \xi) = (u, v)$ is an isometric isomorphism, where

$$u = \sum_{\imath=1}^m \eta_\imath w_{\imath,s} \in W_{m,s} \quad \text{and} \quad v = \sum_{\imath=1}^m \xi_\imath w_{\imath,t} \in W_{m,t}.$$

Therefore, we consider the function $\phi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by

$$\phi(\eta, \xi) = (F_1(\eta, \xi), F_2(\eta, \xi), \dots, F_m(\eta, \xi), G_1(\eta, \xi), G_2(\eta, \xi), \dots, G_m(\eta, \xi)),$$

where

$$F_\kappa(\eta, \xi) = \frac{C(n, s)}{2}\langle u, w_{\kappa,s}\rangle_{X_0^s(\Omega)} - \lambda\int_\Omega \rho(x)|v|^{\alpha-1}v w_{\kappa,s}\,dx - \int_\Omega f(x)w_{\kappa,s}\,dx - \int_\Omega F(x, u, v)w_{\kappa,s}\,dx,$$

$$G_\kappa(\eta, \xi) = \frac{C(n, t)}{2}\langle v, w_{\kappa,t}\rangle_{X_0^t(\Omega)} - \mu\int_\Omega \tau(x)|u|^{\beta-1}u w_{\kappa,t}\,dx - \int_\Omega g(x)w_{\kappa,t}\,dx - \int_\Omega G(x, u, v)w_{\kappa,t}\,dx,$$

with $\kappa = 1, \dots, m$. Consequently,

$$\begin{aligned}\langle \phi(\eta,\xi),(\eta,\xi)\rangle = \frac{C(n,s)}{2}\|u\|^2_{X_0^s(\Omega)} - \lambda\int_\Omega \rho(x)|v|^{\alpha-1}vu\,dx - \int_\Omega f(x)u\,dx - \int_\Omega F(x,u,v)u\,dx \\ + \frac{C(n,t)}{2}\|v\|^2_{X_0^t(\Omega)} - \mu\int_\Omega \tau(x)|u|^{\beta-1}uv\,dx - \int_\Omega g(x)v\,dx - \int_\Omega G(x,u,v)v\,dx.\end{aligned}$$

Using the Hölder and Young inequalities, we derive

$$\int_\Omega \rho(x)|v|^{\alpha-1}vu\,dx \le \frac{1}{2}\|\rho\|_{L^\infty(\Omega)}\big(\|u\|^2_{L^{\alpha+1}(\Omega)} + \|v\|^{2\alpha}_{L^{\alpha+1}(\Omega)}\big)$$

and

$$\int_\Omega \tau(x)|u|^{\beta-1}uv\,dx \le \frac{1}{2}\|\tau\|_{L^\infty(\Omega)}|\Omega|^{\frac{\alpha-\beta}{\alpha+1}}\big(\|u\|^{2\beta}_{L^{\alpha+1}(\Omega)} + \|v\|^2_{L^{\alpha+1}(\Omega)}\big).$$

Now, from (1.24), (1.25) and the Young inequality, we get

$$\int_\Omega F(x,u,v)u \le D_1\bigg(\|u\|^{p_1+1}_{L^{2(p_1+1)}(\Omega)}\bigg(\int_\Omega \exp(2\sigma_1|u|^{\frac{1}{1-s}})\,dx\bigg)^{\frac12} + \frac{p_2}{p_2+1}\|v\|^{p_2+1}_{L^{p_2+1}(\Omega)} + \frac{1}{p_2+1}\|u\|^{p_2+1}_{L^{p_2+1}(\Omega)}\bigg)$$

and

$$\int_\Omega G(x,u,v)v \le D_2\bigg(\|v\|^{q_2+1}_{L^{2(q_2+1)}(\Omega)}\bigg(\int_\Omega \exp(2\sigma_2|v|^{\frac{1}{1-t}})\,dx\bigg)^{\frac12} + \frac{q_1}{q_1+1}\|u\|^{q_1+1}_{L^{q_1+1}(\Omega)} + \frac{1}{q_1+1}\|v\|^{q_1+1}_{L^{q_1+1}(\Omega)}\bigg).$$

Then

$$\begin{aligned}\langle \phi(\eta,\xi),(\eta,\xi)\rangle \ge \frac{C(n,s)}{2}\|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}\big(\|u\|^2_{L^{\alpha+1}(\Omega)} + \|v\|^{2\alpha}_{L^{\alpha+1}(\Omega)}\big) - \|f\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)} \\ - D_1\|u\|^{p_1+1}_{L^{2(p_1+1)}(\Omega)}\bigg(\int_\Omega \exp(2\sigma_1|u|^{\frac{1}{1-s}})\,dx\bigg)^{\frac12} - \frac{D_1p_2}{p_2+1}\|v\|^{p_2+1}_{L^{p_2+1}(\Omega)} - \frac{D_1}{p_2+1}\|u\|^{p_2+1}_{L^{p_2+1}(\Omega)} \\ + \frac{C(n,t)}{2}\|v\|^2_{X_0^t(\Omega)} - \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}|\Omega|^{\frac{\alpha-\beta}{\alpha+1}}\big(\|u\|^{2\beta}_{L^{\alpha+1}(\Omega)} + \|v\|^2_{L^{\alpha+1}(\Omega)}\big) - \|g\|_{L^\infty(\Omega)}\|v\|_{L^1(\Omega)} \\ - D_2\|v\|^{q_2+1}_{L^{2(q_2+1)}(\Omega)}\bigg(\int_\Omega \exp(2\sigma_2|v|^{\frac{1}{1-t}})\,dx\bigg)^{\frac12} - \frac{D_2q_1}{q_1+1}\|u\|^{q_1+1}_{L^{q_1+1}(\Omega)} - \frac{D_2}{q_1+1}\|v\|^{q_1+1}_{L^{q_1+1}(\Omega)}.\end{aligned}$$

From inequalities (1.12) and (1.13), we get

$$\begin{aligned}\langle \phi(\eta,\xi),(\eta,\xi)\rangle \ge d_0\|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}\big(c_{n,s}D_0|\Omega|^{a_{1,s}}\|u\|^2_{X_0^s(\Omega)} + (c_{n,t}D_0)^\alpha|\Omega|^{\alpha a_{1,t}}\|v\|^{2\alpha}_{X_0^t(\Omega)}\big) \\ - \|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac12}\|u\|_{X_0^s(\Omega)} - \frac{D_1p_2}{p_2+1}(c_{n,t}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,t}}\|v\|^{p_2+1}_{X_0^t(\Omega)} \\ - D_1(c_{n,s}D_0)^{\frac{p_1+1}{2}}|\Omega|^{a_6}\|u\|^{p_1+1}_{X_0^s(\Omega)}\bigg(\int_\Omega \exp(2\sigma_1|u|^{\frac{1}{1-s}})\,dx\bigg)^{\frac12} \\ - \frac{D_1}{p_2+1}(c_{n,s}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,s}}\|u\|^{p_2+1}_{X_0^s(\Omega)} + d_0\|v\|^2_{X_0^t(\Omega)} \\ - \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}\big(c_{n,t}D_0|\Omega|^{a_7}\|v\|^2_{X_0^t(\Omega)} + (c_{n,s}D_0)^\beta|\Omega|^{a_8}\|u\|^{2\beta}_{X_0^s(\Omega)}\big) \\ - \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac12}\|v\|_{X_0^t(\Omega)} \\ - D_2(c_{n,t}D_0)^{\frac{q_2+1}{2}}|\Omega|^{a_5}\|v\|^{q_2+1}_{X_0^t(\Omega)}\bigg(\int_\Omega \exp(2\sigma_2|v|^{\frac{1}{1-t}})\,dx\bigg)^{\frac12} \\ - \frac{D_2q_1}{q_1+1}(c_{n,s}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,s}}\|u\|^{q_1+1}_{X_0^s(\Omega)} - \frac{D_2}{q_1+1}(c_{n,t}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,t}}\|v\|^{q_1+1}_{X_0^t(\Omega)}.\end{aligned}$$

Now, suppose that $\|(u,v)\| = r_0$. Therefore,

$$\begin{aligned}\langle \phi(\eta,\xi),(\eta,\xi)\rangle \geq{}& \frac{d_0 r_0^2}{2} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}\big(c_{n,s}D_0|\Omega|^{a_{1,s}} r_0^2 + (c_{n,t}D_0)^\alpha |\Omega|^{\alpha a_{1,t}} r_0^{2\alpha}\big)\\ &- \|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac{1}{2}} r_0 - \frac{D_1 p_2}{p_2+1}(c_{n,t}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,t}} r_0^{p_2+1}\\ &- D_1(c_{n,s}D_0)^{\frac{p_1+1}{2}}|\Omega|^{a_6} r_0^{p_1+1}\Big(\int_\Omega \exp(2\sigma_1|u|^{\frac{1}{1-s}})\Big)^{\frac{1}{2}} - \frac{D_1}{p_2+1}(c_{n,s}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,s}} r_0^{p_2+1}\\ &- \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}\big(c_{n,t}D_0|\Omega|^{a_7} r_0^2 + (c_{n,s}D_0)^\beta|\Omega|^{a_8} r_0^{2\beta}\big) - \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac{1}{2}} r_0\\ &- D_2(c_{n,t}D_0)^{\frac{q_2+1}{2}}|\Omega|^{a_5} r_0^{q_2+1}\Big(\int_\Omega \exp(2\sigma_2|v|^{\frac{1}{1-t}})\Big)^{\frac{1}{2}}\\ &- \frac{D_2 q_1}{q_1+1}(c_{n,s}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,s}} r_0^{q_1+1} - \frac{D_2}{q_1+1}(c_{n,t}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,t}} r_0^{q_1+1}.\end{aligned}$$

Note that, if $n=1$ and $\frac{1}{2} \leq s < 1$, we have

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^{\frac{1}{s}}(\Omega)} \leq |\Omega|^{s-\frac{1}{2}}\Big(\frac{C(n,s)}{2}\Big)^{\frac{1}{2}}\|u\|_{X_0^s(\Omega)} \leq a_{9,s} r_0,$$

where

$$a_{9,s} = |\Omega|^{s-\frac{1}{2}}\Big(\frac{C(n,s)}{2}\Big)^{\frac{1}{2}}.$$

Then we derive

$$\int_\Omega \exp(2\sigma_1|u|^{\frac{1}{1-s}})\,dx = \int_\Omega \exp\Big(2\sigma_1(a_{9,s}r_0)^{\frac{1}{1-s}}\Big(\frac{u}{a_{9,s}\|(u,v)\|}\Big)^{\frac{1}{1-s}}\Big)\,dx. \tag{8.1}$$

Since $r_0 \leq r_4$, applying the fractional Adams inequality (1.11), we have

$$\begin{aligned}\int_\Omega \exp\Big(2\sigma_1(a_{9,s}r_0)^{\frac{1}{1-s}}\Big(\frac{u}{a_{9,s}\|(u,v)\|}\Big)^{\frac{1}{1-s}}\Big)\,dx &\leq \sup_{\|(-\Delta)^{\frac{s}{2}} z\|_{L^{\frac{1}{s}}(\Omega)}\leq 1}\int_\Omega \exp(2\sigma_1(a_{9,s}r_0)^{\frac{1}{1-s}}|z|^{\frac{1}{1-s}})\,dx\\ &\leq c_{1,\frac{1}{s}}|\Omega|.\end{aligned} \tag{8.2}$$

Analogously, if $n=1$ and $\frac{1}{2} \leq t < 1$, then we have

$$\int_\Omega \exp(2\sigma_2|v|^{\frac{1}{1-t}})\,dx \leq c_{1,\frac{1}{t}}|\Omega| \tag{8.3}$$

because $r_0 \leq r_6$.

It follows from (8.2), (8.3) and the definition of r_0 that

$$\begin{aligned}\langle \phi(\eta,\xi),(\eta,\xi)\rangle \geq{}& \frac{d_0 r_0^2}{2} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}\big(c_{n,s}D_0|\Omega|^{a_{1,s}} r_0^2 + (c_{n,t}D_0)^\alpha |\Omega|^{\alpha a_{1,t}} r_0^{2\alpha}\big)\\ &- \|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac{1}{2}} r_0 - \frac{D_1 p_2}{p_2+1}(c_{n,t}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,t}} r_0^{p_2+1}\\ &- D_1(c_{n,s}D_0)^{\frac{p_1+1}{2}}|\Omega|^{a_6+\frac{1}{2}} r_0^{p_1+1}\mathcal{L}_s^{\frac{1}{2}} - \frac{D_1}{p_2+1}(c_{n,s}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,s}} r_0^{p_2+1}\\ &- \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}\big(c_{n,t}D_0|\Omega|^{a_7} r_0^2 + (c_{n,s}D_0)^\beta|\Omega|^{a_8} r_0^{2\beta}\big)\\ &- \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac{1}{2}} r_0 - D_2(c_{n,t}D_0)^{\frac{q_2+1}{2}}|\Omega|^{a_5+\frac{1}{2}} r_0^{q_2+1}\mathcal{L}_t^{\frac{1}{2}}\\ &- \frac{D_2 q_1}{q_1+1}(c_{n,s}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,s}} r_0^{q_1+1} - \frac{D_2}{q_1+1}(c_{n,t}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,t}} r_0^{q_1+1}.\end{aligned}$$

Also, by the definition of r_0, we obtain

$$\begin{aligned}
\frac{d_0 r_0^2}{2^3} - D_1(c_{n,s}D_0)^{\frac{p_1+1}{2}}|\Omega|^{a_6+\frac{1}{2}} r_0^{p_1+1}\mathcal{L}_s^{\frac{1}{2}} &\geq \frac{d_0 r_0^2}{2^4},\\
\frac{d_0 r_0^2}{2^3} - \Big(\frac{D_1 p_2}{p_2+1}(c_{n,t}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,t}} + \frac{D_1}{p_2+1}(c_{n,s}D_0)^{\frac{p_2+1}{2}}|\Omega|^{a_{2,s}}\Big) r_0^{p_2+1} &\geq \frac{d_0 r_0^2}{2^4},\\
\frac{d_0 r_0^2}{2^3} - D_2(c_{n,t}D_0)^{\frac{q_2+1}{2}}|\Omega|^{a_5+\frac{1}{2}} r_0^{q_2+1}\mathcal{L}_t^{\frac{1}{2}} &\geq \frac{d_0 r_0^2}{2^4},\\
\frac{d_0 r_0^2}{2^3} - \Big(\frac{D_2 q_1}{q_1+1}(c_{n,s}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,s}} + \frac{D_2}{q_1+1}(c_{n,t}D_0)^{\frac{q_1+1}{2}}|\Omega|^{a_{4,t}}\Big) r_0^{q_1+1} &\geq \frac{d_0 r_0^2}{2^4}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle \phi(\eta,\xi),(\eta,\xi)\rangle \geq \frac{d_0 r_0^2}{4} &- \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}\big(c_{n,s}D_0|\Omega|^{a_{1,s}} r_0^2 + (c_{n,t}D_0)^{\alpha}|\Omega|^{\alpha a_{1,t}} r_0^{2\alpha}\big)\\
&- \|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac{1}{2}} r_0\\
&- \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}\big(c_{n,t}D_0|\Omega|^{a_7} r_0^2 + (c_{n,s}D_0)^{\beta}|\Omega|^{a_8} r_0^{2\beta}\big)\\
&- \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac{1}{2}} r_0.
\end{aligned}$$

Define

$$\begin{aligned}
\rho_1 &:= \frac{d_0 r_0^2}{20} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)} c_{n,s}D_0|\Omega|^{a_{1,s}} r_0^2,\\
\rho_2 &:= \frac{d_0 r_0^2}{20} - \frac{\lambda}{2}\|\rho\|_{L^\infty(\Omega)}(c_{n,t}D_0)^{\alpha}|\Omega|^{\alpha a_{1,t}} r_0^{2\alpha},\\
\rho_3 &:= \frac{d_0 r_0^2}{20} - \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)} c_{n,t}D_0|\Omega|^{a_7} r_0^2,\\
\rho_4 &:= \frac{d_0 r_0^2}{20} - \frac{\mu}{2}\|\tau\|_{L^\infty(\Omega)}(c_{n,s}D_0)^{\beta}|\Omega|^{a_8} r_0^{2\beta},\\
\rho_5 &:= \frac{d_0 r_0^2}{20} - \big(\|f\|_{L^\infty(\Omega)}|\Omega|^{a_{3,s}}(c_{n,s}D_0)^{\frac{1}{2}} + \|g\|_{L^\infty(\Omega)}|\Omega|^{a_{3,t}}(c_{n,t}D_0)^{\frac{1}{2}}\big) r_0.
\end{aligned}$$

Thus, $\rho_1, \rho_2 > 0$ for $\lambda < \lambda^*$, and $\rho_3, \rho_4 > 0$ for $\mu < \mu^*$. Note that $\rho_5 \geq 0$ if and only if inequality (1.26) occurs. Thus, for $\lambda < \lambda^*$ and $\mu < \mu^*$, we get

$$\langle \phi(\eta,\xi),(\eta,\xi)\rangle \geq \rho_1 + \dots + \rho_5 > 0,$$

where $(\xi,\eta) \in \mathbb{R}^{2m}$ satisfies $|\eta|_{m,s} + |\xi|_{m,t} = r_0$. Now, in order to apply Lemma 8.1, just prove the continuity of the function ϕ. For this, it is sufficient to show that the function $\tilde{F}_\kappa$ given by

$$\tilde{F}_\kappa(\eta,\xi) = \int_\Omega F(x,u,v) w_{\kappa,s}\,dx = \int_\Omega F\Big(x, \sum_{\iota=1}^m \eta_\iota w_{\iota,s}, \sum_{\iota=1}^m \xi_\iota w_{\iota,t}\Big) w_{\kappa,s}\,dx$$

is continuous because the continuity of the other parts of the function ϕ follows in a similar way.

Let us prove that $\tilde{F}_\kappa$ is a continuous function. Let $\eta^j \to \eta^0$ and $\xi^j \to \xi^0$, and in particular $\eta_i^j \to \eta_i^0$ and $\xi_i^j \to \xi_i^0$ for $i = 1, \dots, m$. Thus, there is $M \geq 1$ such that

$$|\eta^j| \leq M, \quad |\xi^j| \leq M \qquad \text{for all } j \in \mathbb{N}.$$

Letting $j \to +\infty$, we obtain

$$F\Big(x, \sum_{\iota=1}^m \eta_\iota^j w_{\iota,s}, \sum_{\iota=1}^m \xi_\iota^j w_{\iota,t}\Big) w_{\kappa,s} \to F\Big(x, \sum_{\iota=1}^m \eta_\iota^0 w_{\iota,s}, \sum_{\iota=1}^m \xi_\iota^0 w_{\iota,t}\Big) w_{\kappa,s}$$

in Ω a.e.

Let $\zeta = \max\{p_1, p_2\}$. So,

$$\begin{aligned}
&\Big|F\Big(x, \sum_{\imath=1}^{m} \eta_\imath^j w_{\imath,s}, \sum_{\imath=1}^{m} \xi_\imath^j w_{\imath,t}\Big) w_{\kappa,s}\Big| \\
&\quad \le D_1\Big(\Big|\sum_{\imath=1}^{m} \eta_\imath^j w_{\imath,s}\Big|^{p_1} \exp\Big(\sigma_1 \Big|\sum_{\imath=1}^{m} \eta_\imath^j w_{\imath,s}\Big|^{\frac{1}{1-s}}\Big)|w_{\kappa,s}| + \Big|\sum_{\imath=1}^{m} \xi_\imath^j w_{\imath,t}\Big|^{p_2} |w_{\kappa,s}|\Big) \\
&\quad \le D_1 m^{\zeta-1}\Big(\sum_{\imath=1}^{m} |\eta_\imath^j|^{p_1} |w_{\imath,s}|^{p_1} \exp\Big(\sigma_1 m^{\frac{s}{1-s}} M^{\frac{1}{1-s}} \sum_{\imath=1}^{m} |w_{\imath,s}|^{\frac{1}{1-s}}\Big)|w_{\kappa,s}| + \sum_{\imath=1}^{m} |\xi_\imath^j|^{p_2} |w_{\imath,t}|^{p_2} |w_{\kappa,s}|\Big) \\
&\quad \le D_1 m^{\zeta-1} M^{\zeta}\Big(\sum_{\imath=1}^{m} |w_{\imath,s}|^{p_1} \exp\Big(\sigma_1 m^{\frac{s}{1-s}} M^{\frac{1}{1-s}} \sum_{\imath=1}^{m} |w_{\imath,s}|^{\frac{1}{1-s}}\Big)|w_{\kappa,s}| + \sum_{\imath=1}^{m} |w_{\imath,t}|^{p_2} |w_{\kappa,s}|\Big),
\end{aligned}$$

where the right-hand side of this inequality is integrable and independent of j, by inequality (1.11) for $n = 1$ and $\frac{1}{2} \le s < 1$. By Lebesgue's dominated convergence theorem, we obtain $\tilde{F}_\kappa(\eta^j, \xi^j) \to \tilde{F}_\kappa(\eta^0, \xi^0)$ as $j \to \infty$. Therefore, $\tilde{F}_\kappa$ is a continuous function.

By Lemma 8.1, for every $m \in \mathbb{N}$ there exists $(\eta_0, \xi_0) \in \mathbb{R}^{2m}$ with

$$|\eta_0|_{m,s} + |\xi_0|_{m,t} \le r_0$$

such that $\phi(\eta_0, \xi_0) = 0$. Then there is $(u_m, v_m) \in W_{m,s} \times W_{m,t}$ satisfying

$$\|u_m\|_{X_0^s(\Omega)} + \|v_m\|_{X_0^t(\Omega)} \le r_0 \quad \text{for every } m \in \mathbb{N}, \tag{8.4}$$

and such that

$$\frac{C(n,s)}{2}\langle u_m, w_s\rangle_{X_0^s(\Omega)} = \lambda \int_\Omega \rho(x)|v_m|^{\alpha-1} v_m w_s \, dx + \int_\Omega f(x) w_s \, dx + \int_\Omega F(x, u_m, v_m) w_s \, dx$$

for all $w_s \in W_{m,s}$ and

$$\frac{C(n,t)}{2}\langle v_m, w_t\rangle_{X_0^t(\Omega)} = \mu \int_\Omega \tau(x)|u_m|^{\beta-1} u_m w_t \, dx + \int_\Omega g(x) w_t \, dx + \int_\Omega G(x, u_m, v_m) w_t \, dx$$

for all $w_t \in W_{m,t}$.

Notice that $W_{m,s} \subset X_0^s(\Omega)$ for all $m \in \mathbb{N}$, and r_0 does not depend on m. Therefore, the sequence (u_m, v_m) is bounded in E. Since E is a reflexive space, up to a subsequence still denoted by (u_m, v_m), there is $(u, v) \in E$ such that

$$u_m \rightharpoonup u \quad \text{weakly in } X_0^s(\Omega) \qquad \text{and} \qquad v_m \rightharpoonup v \quad \text{weakly in } X_0^t(\Omega),$$

that is,

$$\langle u_m, \Phi\rangle_{X_0^s(\Omega)} \to \langle u, \Phi\rangle_{X_0^s(\Omega)} \tag{8.5}$$

for any $\Phi \in X_0^s(\Omega)$ as $m \to \infty$, and

$$\langle v_m, \Psi\rangle_{X_0^t(\Omega)} \to \langle v, \Psi\rangle_{X_0^t(\Omega)} \tag{8.6}$$

for any $\Psi \in X_0^t(\Omega)$ as $m \to \infty$.

So,

$$u_m \to u \quad \text{in } L^{p_s}(\Omega) \qquad \text{and} \qquad v_m \to v \quad \text{in } L^{p_t}(\Omega), \tag{8.7}$$

where $p_s \in [1, 2_s^*)$ if $n > 2s$, and $p_s \in [1, \infty)$ if $n \le 2s$. and

$$u_m \to u \quad \text{and} \quad v_m \to v \qquad \text{a.e. in } \Omega. \tag{8.8}$$

Now, by the continuity of $F(x, s, t)$ and by (8.8), we get

$$F(x, u_m, v_m)^2 \to F(x, u, v)^2 \quad \text{a.e. in } \Omega.$$

By (1.12), (8.1), (8.2) and (8.4),

$$\int_\Omega |F(x, u_m, v_m)|^2 \, dx \le 2D_1^2\Big(\int_\Omega |u_m|^{2p_1} \exp(2\sigma_1 |u_m|^{\frac{1}{1-s}}) \, dx + \int_\Omega |v_m|^{2p_2} \, dx\Big) \le C(\mathcal{L}_s + 1),$$

where

$$C = C(n, s, t, p_1, p_2, q_1, q_2, D_1, D_2, \sigma_1, \sigma_2, |\Omega|) > 0.$$

Then

$$\|F(x, u_m, v_m)\|_{L^2(\Omega)} \leq C(\mathcal{L}_s + 1)^{\frac{1}{2}},$$

and [12, Theorem 13.44] leads to

$$F(x, u_m, v_m) \rightharpoonup F(x, u, v) \quad \text{weakly in } L^2(\Omega). \tag{8.9}$$

Similarly to $G(x, u_m, v_m)$, we have

$$G(x, u_m, v_m) \rightharpoonup G(x, u, v) \quad \text{weakly in } L^2(\Omega). \tag{8.10}$$

Let $k \in \mathbb{N}$. Thus, for every $m \geq k$, we obtain

$$\frac{C(n,s)}{2}\langle u_m, w_s\rangle_{X_0^s(\Omega)} = \lambda \int_\Omega \rho(x)|v_m|^{\alpha-1} v_m w_s\,dx + \int_\Omega f(x) w_s\,dx + \int_\Omega F(x, u_m, v_m) w_s\,dx \tag{8.11}$$

for all $w_s \in W_{k,s}$, and

$$\frac{C(n,t)}{2}\langle v_m, w_t\rangle_{X_0^t(\Omega)} = \mu \int_\Omega \tau(x)|u_m|^{\beta-1} u_m w_t\,dx + \int_\Omega g(x) w_t\,dx + \int_\Omega G(x, u_m, v_m) w_t\,dx \tag{8.12}$$

for all $w_t \in W_{k,t}$.

From (8.7), (8.9) and (8.10), we get

$$\begin{aligned}\lambda \int_\Omega \rho(x)|v_m|^{\alpha-1} v_m w_s\,dx &+ \int_\Omega f(x) w_s\,dx + \int_\Omega F(x, u_m, v_m) w_s\,dx \\ &\to \lambda \int_\Omega \rho(x)|v|^{\alpha-1} v w_s\,dx + \int_\Omega f(x) w_s\,dx + \int_\Omega F(x, u, v) w_s\,dx\end{aligned} \tag{8.13}$$

for all $w_s \in W_{k,s}$, and

$$\begin{aligned}\mu \int_\Omega \tau(x)|u_m|^{\beta-1} u_m w_t\,dx &+ \int_\Omega g(x) w_t\,dx + \int_\Omega G(x, u_m, v_m) w_t\,dx \\ &\to \mu \int_\Omega \tau(x)|u|^{\beta-1} u w_t\,dx + \int_\Omega g(x) w_t\,dx + \int_\Omega G(x, u, v) w_t\,dx\end{aligned} \tag{8.14}$$

for all $w_t \in W_{k,t}$ when $m \to \infty$. Now, we use (8.5), (8.6) and (8.11)–(8.14) to get

$$\frac{C(n,s)}{2}\langle u, w_s\rangle_{X_0^s(\Omega)} = \lambda \int_\Omega \rho(x)|v|^{\alpha-1} v w_s\,dx + \int_\Omega f(x) w_s\,dx + \int_\Omega F(x, u, v) w_s\,dx$$

for all $w_s \in W_{k,s}$, and

$$\frac{C(n,t)}{2}\langle v, w_t\rangle_{X_0^t(\Omega)} = \mu \int_\Omega \tau(x)|u|^{\beta-1} u w_t\,dx + \int_\Omega g(x) w_t\,dx + \int_\Omega G(x, u, v) w_t\,dx$$

for all $w_t \in W_{k,t}$. Then, since $[W_{k,s}]_{k\in\mathbb{N}}$ is dense in $X_0^s(\Omega)$, we obtain that $(u, v) \in E$ is a weak solution of problem (1.23). Applying [24, Proposition 1.4] inductively, we have $(u, v) \in C^s(\mathbb{R}^n) \times C^t(\mathbb{R}^n)$. Finally, by [26, Theorem 1], (u, v) is a viscosity solution of system (1.23). This ends the proof. □

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