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Research Article

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Sharp Hardy Identities and Inequalities on Carnot Groups

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Abstract: In this paper we establish general weighted Hardy identities for several subelliptic settings including Hardy identities on the Heisenberg group, Carnot groups with respect to a homogeneous gauge and Carnot–Carathéodory metric, general nilpotent groups, and certain families of Hörmander vector fields. We also introduce new weighted uncertainty principles in these settings. This is done by continuing the program initiated by [N. Lam, G. Lu and L. Zhang, Factorizations and Hardy's-type identities and inequalities on upper half spaces, *Calc. Var. Partial Differential Equations* **58** (2019), no. 6, Paper No. 183; N. Lam, G. Lu and L. Zhang, Geometric Hardy's inequalities with general distance functions, *J. Funct. Anal.* **279** (2020), no. 8, Article ID 108673] of using the Bessel pairs introduced by [N. Ghoussoub and A. Moradifam, *Functional Inequalities: New Perspectives and New Applications*, Math. Surveys Monogr. 187, American Mathematical Society, Providence, 2013] to obtain Hardy identities. Using these identities, we are able to improve significantly existing Hardy inequalities in the literature in the aforementioned subelliptic settings. In particular, we establish the Hardy identities and inequalities in the spirit of [H. Brezis and J. L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Complut. Madrid* **10** (1997), 443–469] and [H. Brezis and M. Marcus, Hardy's inequalities revisited. Dedicated to Ennio De Giorgi, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **25** (1997), no. 1–2, 217–237] in these settings.

Keywords: Bessel Pair, Hardy Inequality, Hardy–Sobolev Inequality, Weights, Carnot Groups, Carnot–Carathéodory Metric

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1 Introduction

Let G denote a Carnot group with sub-Laplacian $\mathcal{L} = \sum_{j=1}^{m} X_j^2$, horizontal gradient $\nabla_H = (X_1, \dots, X_m)$, homogeneous dimension Q, and $\mathcal{L}$-gauge d (see preliminaries for precise definitions). In this paper, we employ factorization of differential operators and Bessel pairs to obtain weighted Hardy identities on Carnot groups G of the form

$$\int_{B_R(0)} V(d)\left|\nabla_H\left(\frac{u}{\varphi(d)}\right)\right|^2 \varphi^2(d)\,dx = \int_{B_R(0)} V(d)|\nabla_H u|^2\,dx - \int_{B_R(0)} W(d)u^2|\nabla_H d|^2\,dx \tag{1.1}$$

for $u \in C_0^\infty(B_R(0) \setminus \{0\})$, where (V, W) is a pair of positive C^1-functions on $(0, R)$ which satisfy the condition that there exists a positive C^1-solution $\varphi : (0, R) \to \mathbb{R}$ to the differential equation

$$(V\varphi')' + W\varphi = 0 \quad \text{on } (0, R).$$

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Such a pair of functions is called a Bessel pair. Here, dx is a Haar measure on G, and $B_R(0)$ is the gauge ball determined by d with radius R and center being the origin 0. This result is contained in Theorem 3.2 below. As a particular important application, we obtain weighted Hardy identities on the Heisenberg group $H_N = \mathbb{R}^{2N+1}$, the simplest and most important nonabelian Carnot group. It is important to note that, for an appropriately chosen Bessel pair, our results recover the Hardy identity corresponding to known Hardy inequalities on the Heisenberg group and more general Carnot groups, thereby strongly improving these known inequalities by finding exact remainder terms.

In addition to these identities, we also obtain Hardy identities of the form

$$\begin{aligned}\int_{0<d_K<R} V(d_K)\Big|\nabla_H\Big(\frac{u}{\varphi(d_K)}\Big)\Big|^2 \varphi(d_K)^2\,dx &= \int_{0<d_K<R} V(d_K)|\nabla_H u|^2\,dx - \int_{0<d_K<R} W(d_K)u^2\,dx\\ &\quad - \int_{0<d_K<R} V(d_K)|u|^2\varphi^{-1}(d_K)\varphi'(d_K)\Big[\mathcal{L}d_K - \frac{\alpha-1}{d_K}\Big]\,dx,\end{aligned} \tag{1.2}$$

where ∇_H is the horizontal gradient with respect to certain smooth vector fields $X_1, \dots, X_m$, and d_K is the Carnot–Carathéodory distance (also known CC distance) to a closed subset K with respect to this system of vector fields. A key point here is that, by [24], the CC distance function d_K satisfies the horizontal eikonal equation $|\nabla_H d_K| = 1$, unlike a general homogeneous gauge d for which $|\nabla_H d| = 1$ is typically false. This is the content of Theorem 3.16.

We recall that, on the Heisenberg group $H_N = \mathbb{R}^{2N+1}$, the sharp Hardy inequality

$$\int_{H_N} |\nabla_H u|^2\,dz\,dt \geq \frac{(Q-2)^2}{4}\int_{H_N} \frac{|u|^2}{d^2}|\nabla_H d|^2\,dz\,dt, \quad u \in C_0^\infty(H_N \setminus \{0\}), \tag{1.3}$$

was established by Garofalo and Lanconelli in [14]. The L^p-Hardy inequality on the Heisenberg group for $1 < p < Q = 2N + 2$ was established by Niu, Zhang and Wang [26] by using the Picone's identity for the p-subLaplacian (see also Zhang and Niu [28])

$$\int_{H_N} |\nabla_H u|^p\,dz\,dt \geq \Big(\frac{Q-p}{p}\Big)^p \int_{H_N} \frac{|u|^p}{d^p}|\nabla_H d|^p\,dz\,dt, \quad u \in C_0^\infty(H_N \setminus \{0\}).$$

When d is replaced by the Carnot–Carathéodory metric d_{cc}, then there holds

$$\int_{H_N} |\nabla_H u|^2\,dz\,dt \geq c\int_{H_N} \frac{|u|^2}{d_{\mathrm{cc}}^2}\,dz\,dt, \quad u \in C_0^\infty(H_N \setminus \{0\}), \tag{1.4}$$

for some constant $c > 0$. See for example [13, 22, 27]. Using the fact that d_{cc} and d are equivalent metrics, one may also obtain a Hardy inequality of the form (1.4) with d_{cc} replaced by d, though the best constant appears to be unknown in either case. Nevertheless, while (1.3) is sharp, there are no extremizers, and so it is natural to try to improve this inequality by adding nontrivial nonnegative terms to the right-hand side of (1.3). As mentioned above, we in fact determine exact remainder terms so that (1.3) becomes

$$\int_{H_N} |\nabla_H u|^2\,dz\,dt = \frac{(Q-2)^2}{4}\int_{H_N} \frac{|u|^2}{d^2}|\nabla_H d|^2\,dz\,dt + \text{remainder rerms.}$$

Therefore, we obtain a strict improvement of the known Hardy inequality (1.3), and the counterparts for Carnot groups. This is the content of Theorem 3.6.

The approach used here is inspired by the recent work of Lam, Lu and Zhang in [20] where they used factorization of differential operators and Bessel pairs to prove, among other things, various Hardy identities on the half-space

$$\mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}.$$

Their main identity theorem is stated as follows.

Theorem A. *Let $0 < R \leq \infty$, and let V and W be positive C^1-functions on $(0, R)$. If $(r^{N+1}V, r^{N+1}W)$ is a Bessel pair on $(0, R)$ with positive solution φ, then there holds*

$$\int\limits_{B_R(0)\cap\mathbb{R}^N_+} V(|x|)|\nabla u|^2\,dx - \int\limits_{B_R(0)\cap\mathbb{R}^N_+} \Big[W(|x|) - \frac{V'(|x|)}{|x|}\Big]|u|^2\,dx = \int\limits_{B_R(0)\cap\mathbb{R}^N_+} V(|x|)\Big|\nabla\Big(\frac{u}{\varphi}\frac{1}{x_N}\Big)\Big|\varphi^2 x_N^2\,dx$$

for $u \in C_0^\infty(B_R(0) \cap \mathbb{R}^N_+)$,

In addition to this theorem, Lam, Lu and Zhang also proved the following Hardy identities on strips of the form $A_{2R} = \mathbb{R}^{N-1} \times (0, 2R)$ which imply in particular Hardy identities on the half-space $\mathbb{R}^N_+$ with the Euclidean norm $|x|$ replaced by the distance function $d_\infty(x) = x_N$, the height from the boundary. Their result is stated as follows.

Theorem B. *Let $0 < R \leq \infty$, let V and W be positive C^1-functions on $(0, R)$, and let $d_R = \min\{x_N, 2R - x_N\}$ be a distance function to the boundary of A_{2R}. If (V, W) is a Bessel pair on $(0, R)$ with positive solution φ, then for $u \in C_0^\infty(A_{2R})$ there holds*

$$\int\limits_{A_{2R}} V(d_R(x))|\nabla u|^2\,dx - \int\limits_{A_{2R}} W(d_R(x))|u|^2\,dx = \int\limits_{A_{2R}} V(d_R(x))\Big|\nabla\Big(\frac{u}{\varphi(d_R(x))}\Big)\Big|^2 \varphi^2(d_R(x))\,dx$$

and

$$\int\limits_{A_{2R}} V(d_R(x))\Big|\frac{\partial u}{\partial x_N}\Big|^2\,dx - \int\limits_{A_{2R}} W(d_R(x))|u|^2\,dx = \int\limits_{A_{2R}} V(d_R(x))\Big|\frac{\partial}{\partial x_N}\Big(\frac{u}{\varphi(d_R(x))}\Big)\Big|^2 \varphi^2(d_R(x))\,dx.$$

Actually, similar half-space Hardy inequalities have been established for the Heisenberg group. We recall the following result of Luan and Yang given in [23].

Theorem C. *Let $H_N^+ = \{(z, t) \in H_N : t > 0\}$ denote the upper half-space of H_N. Then there holds for $u \in C_0^\infty(H_N^+)$,*

$$\int\limits_{H_N^+} |\nabla_H u|^2\,dz\,dt \geq \int\limits_{H_N^+} \frac{|z|^2}{t^2}|u|^2\,dz\,dt \tag{1.5}$$

and this inequality is sharp and without extremizers.

See [18] for similar Hardy inequalities on different Heisenberg half spaces.

In the spirit of Theorems A, B and C, we establish the Hardy identities on strips

$$\int\limits_{A_{2R}} V(d_R)\Big|\nabla_H\Big(\frac{u}{\varphi(d_R)}\Big)\Big|^2 \varphi^2(d_R)\,dz\,dt = \int\limits_{A_{2R}} V(d_R)|\nabla u|^2\,dz\,dt - \frac{1}{4}\int\limits_{A_{2R}} W(d_R)|z|^2|u|^2\,dz\,dt,$$

$$\int\limits_{A_{2R}} V\varphi^2(d_R)\Big|\frac{\partial}{\partial t}\Big(\frac{u}{\varphi(d_R)}\Big)\Big|^2\,dz\,dt = \int\limits_{A_{2R}} V(d_R)\Big|\frac{\partial u}{\partial t}\Big|^2\,dz\,dt - \int\limits_{A_{2R}} W(d_R)|u|^2\,dz\,dt,$$

and half-gauge balls

$$\int\limits_{B_R(0)\cap H_N^+} V\Big|\nabla\Big(\frac{u}{\varphi}\frac{1}{t}\Big)\Big|^2\varphi^2 t^2\,dz\,dt = \int\limits_{B_R(0)\cap H_N^+} V|\nabla u|^2\,dz\,dt - \int\limits_{B_R(0)\cap H_N^+} \Big[W - \frac{V'}{d}\Big]|u|^2|\nabla d|^2\,dz\,dt.$$

Here $d_R(z, t) = \min\{t, 2R - t\}$ and $d_\infty(z, t) = t$ is the vertical height function. The identity on strips implies in particular improvements of (1.5). These identities are the contents of Theorems 3.8 and 3.12.

We note that, in [23], (1.5) has an interesting improved version by adding to the right side a nonnegative nontrivial term with weight given in terms of the homogeneous gauge d; however, they do not obtain exact remainder terms. Moreover, this inequality is not coordinate independent, and so the best constant appearing in front of $\int |x|^2 t^{-2}|u|^2\,dx$ depends on the coordinates chosen for the left invariant vector fields on H_N. In particular, our choice of coordinates are different and the best constant turns out to be $\frac{1}{16}$ for our choice of coordinates.

We also mention that, our Hardy identities of the form (1.2) for the general distance function d_K are in the spirit of the following Hardy identities for general distance functions (for example, the distance to a subset) on Euclidean space obtained by Lam, Lu and Zhang in [21].

Theorem D. *Let $0 < R \leq \infty$, let V and W be positive C^1-functions on $(0, R)$, and let d be a distance function. Assume that for some $\alpha \in \mathbb{R}$, $\Delta d(x) - \frac{\alpha-1}{d(x)}$ exists on $\{0 < d(x) < R\}$ in the sense of distributions and $(r^{\alpha-1}V, r^{\alpha-1}W)$ is a Bessel pair on $(0, R)$ with positive solution φ. Then, for $u \in C_0^\infty(\{0 < d(x) < R\})$,*

$$\int\limits_{0<d(x)<R} V(d(x))|\nabla u(x)|^2\,dx - \int\limits_{0<d(x)<R} W(d(x))|u(x)|^2\,dx$$

$$= \int\limits_{0<d(x)<R} V(d(x))\varphi^2(d(x))\Big|\nabla\Big(\frac{u(x)}{\varphi(d(x))}\Big)\Big|^2\,dx - \int\limits_{0<d(x)<R} V(d(x))|u(x)|^2\Big[\Delta d(x) - \frac{\alpha-1}{d(x)}\Big]\frac{\varphi'(d(x))}{\varphi(d(x))}\,dx.$$

Among the particularly interesting applications of the Hardy identity (1.1) are Hardy identities and inequalities which improve Hardy inequalities of the form

$$\int\limits_{B_R(0)} |\nabla u|^2\,dx \geq \int\limits_{B_R(0)} \frac{|u|^2}{d^2}|\nabla d|^2\,dx$$

by adding positive nontrivial L^2-terms to the right-hand side. In fact, Brezis and Vázquez established the remarkable inequality

$$\int\limits_{\Omega} |\nabla u|^2\,dx - \Big(\frac{N-2}{2}\Big)^2 \int\limits_{\Omega} \frac{|u|^2}{|x|^2}\,dx \geq \frac{z_0^2\omega_N^{\frac{2}{N}}}{|\Omega|^{\frac{2}{N}}} \int\limits_{\Omega} |u|^2\,dx,$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is any bounded domain, $u \in H_0^1(\Omega)$, and z_0 is the first zero of the Bessel function J_0. (See also [4] for the distance function the boundary.) We introduce similar inequalities on Carnot groups (see Theorem 3.6) by establishing identities of the form

$$\int\limits_{B_R(0)} |\nabla_H u|^2\,dx - \Big(\frac{N-2}{2}\Big)^2 \int\limits_{B_R(0)} \frac{|u|^2}{d^2}\,dx = \frac{z_0^2}{R^2} \int\limits_{B_R(0)} |u|^2\,dx + \int\limits_{B_R(0)} \Big|\nabla_H\Big(\frac{ud^{\frac{Q}{2}}}{J_{0;R}(d)}\Big)\Big|^2 \frac{J_{0;R}(d)^2}{d^Q}\,dx,$$

where $J_{0;R}(r) = J_0(\frac{z_0 r}{R})$. This is done by choosing an appropriate Bessel pair (V, W) and applying (1.1). We note that such Brezis–Vázquez-type inequalities [5] on Carnot groups were observed first by Kombe in [19].

We take the opportunity to mention some forthcoming works which continue the program of establishing Hardy identities by use of Bessel pairs, including the work in this paper and the work of Lam, Lu and Zhang already mentioned. First, we mention that Duy, Lam and Lu establish recently in [7] the following L^p-Hardy identities by use of p-Bessel pairs.

Theorem E. *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, and let A and W be positive C^1-functions on $(0, R)$. If $(r^{N-1}V, r^{N-1}W)$ is a p-Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R(0) \setminus \{0\})$,*

$$\int\limits_{B_R(0)} V(|x|)|\nabla u|^p\,dx - \int\limits_{B_R(0)} W(|x|)|u|^p\,dx = \int\limits_{B_R(0)} V(|x|)C_p\Big(\nabla u, \varphi\nabla\Big(\frac{u}{\varphi}\Big)\Big)\,dx$$

and

$$\int\limits_{B_R(0)} V(|x|)|\mathcal{R}u|^p\,dx - \int\limits_{B_R(0)} W(|x|)|u|^p\,dx = \int\limits_{B_R(0)} V(|x|)C_p\Big(\mathcal{R}u, \varphi\mathcal{R}\Big(\frac{u}{\varphi}\Big)\Big)\,dx.$$

Here φ is the positive solution of

$$(r^{N-1}V(r)|y'|^{p-2}y')' + r^{N-1}W(r)|y|^{p-2}y = 0$$

on $(0, R)$. Moreover, $\mathcal{R}$ is the radial derivative, and C_p is defined by

$$C_p(x, y) = |x|^p - |x-y|^p - p|x-y|^{p-2}(x-y)\cdot y.$$

We also point out here that we have established in [10] the L^p-Hardy identities for general distance functions in the Euclidean setting in which we generalize Theorem D to the L^p-setting. We have also established in [12] the L^p-Hardy identities for domains in Euclidean space. Our L^p-Hardy identities from [10] for general distance functions is stated as follows.

Theorem F. *Let $p > 1$, $0 < R \leq \infty$, let d be a distance function on $\mathbb{R}^N$, and let $V, W \in C^1(0, R)$ be positive. Assume that for some $\alpha \in \mathbb{R}$, $\Delta d(x) - \frac{\alpha-1}{d(x)}$ exists on $\{0 < d(x) < R\}$ in the sense of distributions, and that $(r^{\alpha-1}V, r^{\alpha-1}W)$ is a p-Bessel pair on $(0, R)$ with positive solution φ. Then, for $u \in C_0^\infty(\{0 < d(x) < \infty\})$, there holds*

$$\begin{aligned}\int\limits_{0<d(x)<R} V(d(x))|\nabla u|^p\,dx - \int\limits_{0<d(x)<R} W(d(x))|u|^p\,dx \\ = \int\limits_{0<d(x)<R} V(d(x))C_p\Big(\nabla u, \varphi(d(x))\nabla\Big(\frac{u}{\varphi(d(x))}\Big)\Big)\,dx \\ - \int\limits_{0<d(x)<R} V(d(x))|u|^p\varphi(d(x))^{1-p}|\varphi'(d(x))|^{p-2}\varphi'(d(x))\Big[\Delta d(x) - \frac{\alpha-1}{d(x)}\Big]\,dx\end{aligned}$$

and

$$\begin{aligned}\int\limits_{0<d(x)<R} V(d(x))|\nabla d\cdot\nabla u|^p\,dx - \int\limits_{0<d(x)<R} W(d(x))|u|^p\,dx \\ = \int\limits_{0<d(x)<R} V(d(x))C_p\Big(\nabla d\cdot\nabla u, \varphi(d(x))\nabla d\cdot\nabla\Big(\frac{u}{\varphi(d(x))}\Big)\Big)\,dx \\ - \int\limits_{0<d(x)<R} V(d(x))|u|^p\varphi(d(x))^{1-p}|\varphi'(d(x))|^{p-2}\varphi'(d(x))\Big[\Delta d(x) - \frac{\alpha-1}{d(x)}\Big]\,dx.\end{aligned}$$

Another important geometric inequality is the Hardy uncertainty principle given by

$$\frac{|Q+\lambda-2|}{2}\int\limits_G u^2|\nabla_H d|^2\,dx \leq \Big(\int\limits_G u^2 d^2|\nabla_H d|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_G |\nabla_H u|^2\,dx\Big)^{\frac{1}{2}}. \tag{1.6}$$

The uncertainty principle for the Heisenberg group may be found in [14]. Using factorization, we also obtain a new family of inequalities which generalize (1.6) to a family of uncertainty principles whose weights are given in terms of Bessel pairs. These inequalities are of the form

$$\frac{1}{2}\int\limits_{B_R(0)} u^2|\nabla_H d|^2\Big[W + V\Big(\frac{\varphi'}{\varphi}\Big)^2\Big]\,dx \leq \Big(\int\limits_{B_R(0)} u^2|\nabla_H d|^2 V(\frac{\varphi'}{\varphi})^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B_R(0)} V|\nabla_H u|^2\,dx\Big)^{\frac{1}{2}}$$

and are the content of Theorem 3.13. These inequalities appear to be entirely new even for Euclidean space, and they give a new interpretation of the classical Hardy uncertainty principle.

We recall that Ghoussoub and Moradifam introduced in [15] (see also their book [16]) the notion of Bessel pairs to enhance and unify several improved Hardy-type inequalities in the literature for the Euclidean setting. They established the following important characterization theorem.

Theorem G. *Assume $N \geq 1$, let $0 < R \leq \infty$, and let V and W be C^1-functions on $(0, R)$ such that*

$$\int_0^R \frac{1}{r^{N-1}V(r)}dr = \infty \quad \text{and} \quad \int_0^R r^{N-1}V(r)dr < \infty.$$

If $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R(0))$ there holds

$$\int\limits_{B_R(0)} V(|x|)|\nabla u|^2 \geq \int\limits_{B_R(0)} W(|x|)|u|^2\,dx. \tag{1.7}$$

Also, if (1.7) holds for all $u \in C_0^\infty(B_R(0))$, then $(r^{N-1}V, r^{N-1}cW)$ is a Bessel pair on $(0, R)$ for some $c > 0$.

It is worth mentioning that these results extend straightforwardly to more general settings. We describe two such generalizations now. The first is the case of simply connected nilpotent Lie groups G where we specify a system of left invariant vector fields $X = \{X_1, \dots, X_m\}$ which bracket generate the Lie algebra of G. Then, as was shown by Nagel, Ricci and Stein in [25], the sub-Laplacian $\mathcal{L} = \sum_{j=1}^m X_j^2$ has a global fundamental solution $\Gamma_X(x, y)$. Then (1.1) holds if we take $\nabla_H = (X_1, \dots, X_m)$, $d(x) = d_X(x) = \Gamma_X(x, 0)^{2-q}$ for some $q > 2$, and $B_R(0) = \{d_X(x) < R\}$.

The second is the case that $X = \{X_1, \dots, X_m\}$ is a system of Hörmander vector fields homogeneous of degree -1 (see the preliminaries for precise details). In fact, recently it was shown in [1] that, for these kinds of vector fields, the Hörmander operator $\mathcal{L} = \sum_{j=1}^m X_j^2$ admits a global fundamental solution $\Gamma_X(x, y)$, and one can, similar to the nilpotent case, take $d(x) = \Gamma_X(x, 0)^{2-q}$, where q is the homogeneous dimension arising from the family of dilations used to define X. Analogous Hardy identities may be formulated just as in the nilpotent case just mentioned. These identities are the contents of Theorem 3.20.

Finally, we mention that the L^p-weighted Hardy's identities and inequalities (for $p \neq 2$) with the p-Bessel pairs on Carnot groups and for Hörmander vector fields have been established in [11]. This is inspired by the earlier works of the authors on L^p-weighted Hardy's identities and inequalities (for $p \neq 2$) with the p-Bessel pairs in Euclidean spaces [7, 10, 12].

Organization. The paper is outlined as follows. We first recall in the Preliminaries section facts about Hörmander vector fields, the Heisenberg group, and general Carnot groups. In the following section, we first state and prove the main factorization lemma. From here, we establish the main Hardy identities on Carnot groups, and then apply this result by choosing specific Bessel pairs. Next, we establish another factorization lemma and use it to establish Hardy identities on the Heisenberg half-space and half-gauge ball. Next, we use the factorization lemma to state and prove general weighted uncertainty principles as well as some applications. Then we establish Hardy identities for general distance functions d_K, where d_K is the CC distance to a closed set K. Lastly, we state and prove Hardy identities and specific inequalities for more general settings including certain vector fields and simply connected nilpotent Lie groups.

For convenience of the reader, we point out explicitly the main results of the paper. Firstly, in Theorem 3.2 we prove the main Hardy identity for Carnot groups. Secondly, in Theorem 3.6, we apply Theorem 3.2 to obtain a Hardy identity in the spirit of Brezis–Vázquez. Thirdly, in Theorem 3.8 and Theorem 3.12, we prove Hardy identities related to the Heisenberg half space. Fourthly, in Theorem 3.13, we introduce a new class of uncertainty principles for groups. Lastly, in Theorems 3.16 and 3.20, we prove Hardy identities for the more general settings with respect to the CC metric and global fundamental solution, respectively.

Remark 1.1. We employ the following notational conventions. Firstly, if a theorem is the result of a referenced paper, then that theorem is labeled by a Latin letter (e.g., such as Theorem A). Secondly, for a general Carnot group, its measure will generally be denoted by dx; however, in the case of the Heisenberg group, we opt to use $dz\,dt$ when it is more natural. Secondly, general points shall be denoted by x, and the Lebesgue measure shall be denoted by dx. In particular, a point in the Heisenberg group may be denoted by $(x_1, \dots, x_N, y_1, \dots, y_N, t)$ or (z, t) or x, and its measure will always be denoted by dx or $dz\,dt$. Lastly, in general, d shall denote a homogeneous gauge, d_X shall denote a power of the fundamental solution corresponding to a system of vector fields X, d_{cc} shall denote the Carnot–Carathéodory (CC) metric with respect to a given system of vector fields, and d_K shall denote the CC distance to a closed subset K.

2 Preliminaries

We begin by recalling facts about general vector fields, and then proceed to recalling facts and properties about the Heisenberg group and more general Carnot groups. Thus, let $X = \{X_1, \dots, X_m\}$ denote a collection of smooth vector fields on $\mathbb{R}^N$, i.e.,

$$X_j = \sum_{i=1}^N a_{ij}(x) \frac{\partial}{\partial x_i}, \quad j = 1, \dots, m,$$

with $a_{ij} \in C^\infty(\mathbb{R}^N)$. A subunit curve for X is a Lipschitz continuous curve $\gamma : [0, T] \to \mathbb{R}^N$, $T > 0$, such that there exist measurable functions $h_1, \dots, h_m : [0, T] \to \mathbb{R}$ satisfying

$$\gamma'(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)), \quad \sum_{j=1}^{m} h_j^2(t) \leq 1$$

for almost every $t \in [0, T]$. The Carnot–Carathéodory distance between two points $x, y \in \mathbb{R}^N$ is then defined to be

$$d_{\mathrm{cc}}(x, y) = \inf\{T \geq 0 : \text{there exists a subunit curve } \gamma : [0, T] \to \mathbb{R}^N \text{ with } \gamma(0) = x,\ \gamma(T) = y\}.$$

In case the vector fields $X_1, \dots, X_m$ are such that the Lie algebra generated by X is N-dimensional at every point in $\mathbb{R}^N$, then X is said to satisfy Hörmander's condition. The Chow–Rashevsky theorem asserts that, if X satisfies Hörmander's condition, then $d_{\mathrm{cc}}(x, y) < \infty$ for all $x, y \in \mathbb{R}^N$. In particular, d_{cc} becomes a metric on $\mathbb{R}^N$.

Associated with X is the Hörmander sum of squares operator $\mathcal{L}_X = \sum_{j=1}^{m} X_j^2$, and, in case X satisfies the Hörmander condition, it is a well-known result of Hörmander that $\mathcal{L}_X$ is hypoelliptic. We also define the horizontal gradient ∇_H and divergence as follows:

$$\nabla_H u = (X_1 u, \dots, X_m u), \quad \operatorname{div}_H(a_1, \dots, a_m) = \sum_{j=1}^{m} X_j^* a_j,$$

where $a = (a_1, \dots, a_m) : \mathbb{R}^N \to \mathbb{R}^m$ is any sufficiently regular function, and X_j^* is the formal adjoint of X_j.

Now, suppose for convenience that $\mathcal{L}_X$ has a global fundamental solution $\Gamma_X(x, y)$. Then motivated by the fact that, for $N \geq 3$, the function $\Gamma_{\mathrm{eu}}(x, y) = \frac{1}{N(N-2)\omega_N} |x - y|^{2-N}$, where ω_N is the volume of the Euclidean unit ball, is the fundamental solution for the Euclidean Laplacian, and $|x| = \Gamma_{\mathrm{eu}}(x, 0)^{\frac{1}{2-Q}}$ defines the Euclidean norm, we may define the function $d_X(x) = \Gamma_X(x, 0)^{\frac{1}{2-N}}$ and use it to define the corresponding ball

$$B_{X,R}(0) = \{x \in \mathbb{R}^N : d_X(x) < R\}.$$

An important family of Hörmander vector fields are the left-invariant vector fields of the Heisenberg group $H_N = \mathbb{R}^{2N+1}$. Recall that H may be defined as $\mathbb{R}^{2N+1}$ equipped with the group law

$$(a, b, t)(a', b', t') = \Big(a + a', b + b', t + t' + \frac{1}{2}(\langle a, b' \rangle - \langle a', b \rangle)\Big),$$

where $a, b, a', b' \in \mathbb{R}^N$ are horizontal coordinates, and $t, t' \in \mathbb{R}$ are central coordinates. At a given point $(z, t) = (x_1, \dots, x_N, y_1, \dots, y_N, t) \in H_N$, the corresponding left invariant vector fields are given by

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t} \tag{2.1}$$

for $i = 1, \dots, N$. These vector fields satisfy the Heisenberg relation

$$[X_i, Y_j] = \delta_{ij} T, \quad i, j = 1, \dots, N,$$

and so the Lie algebra $\mathfrak{h}$ of H is a graded nilpotent Lie algebra of step 2. The corresponding sub-Laplacian is given by

$$\mathcal{L} = \sum_{j=1}^{m} (X_j^2 + Y_j^2) = \Delta_{\mathbb{R}^{2N}} + \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{m} \Big(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\Big) \frac{\partial}{\partial t}, \tag{2.2}$$

and the horizontal gradient is given as above.

The Heisenberg group is a homogeneous group in the sense that it has a family of anisotropic dilations $\delta_\lambda : H \to H$ given by $\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$ for $\lambda > 0$. The corresponding homogeneous dimension of H is given by $Q = 2N + 2$. Moreover, H has a homogeneous gauge given by

$$d(z, t) = (|z|^4 + 16t^2)^{\frac{1}{4}}, \tag{2.3}$$

and this norm is such that d^{2-Q} is a scalar multiple of the fundamental solution of $\mathcal{L}$. In particular, for an appropriate $c > 0$, the balls $B_{X,cR}(0)$ are exactly the balls gauge balls $B_R(0) = \{x \in H_N : d(x) < R\}$. The Haar measure, which will be denoted by dx or $dz\,dt$, on H is given by the Lebesgue measure $dx = dz\,dt$ on $\mathbb{R}^{2N+1}$. The Haar measure and homogeneous gauge enjoy the following homogeneity properties:

$$d(\delta_\lambda(z, t)) = \lambda^Q \, dz\,dt, \quad d(\delta_\lambda(z, t)) = \lambda d(z, t).$$

The Heisenberg group belongs to a large family of nilpotent groups, called the Carnot groups, which are important in many fields of analysis and have a rich geometric structure. We recall Carnot groups now. First, recall that a stratified Lie algebra $\mathfrak{g}$ of step r is a Lie algebra with subspaces $V_1, \dots, V_r$ satisfying

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r \quad \text{and} \quad [V_1, V_i] = V_{i+1}, \quad i = 1, \dots, r-1, \quad [V_1, V_r] = 0.$$

If G is a simply connected Lie group with a stratified Lie algebra $\mathfrak{g}$, then G is called a Carnot group. In fact, if $N_j = \dim V_j$, and using the exponential map, we may identify G with $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ so that each point $g \in G$ is identified with a point $x = (x^{(1)}, \dots, x^{(r)})$ in $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ so that $x^{(j)} \in \mathbb{R}^{N_j}$; such an identification will always be made. Just as for the Heisenberg group, the Carnot groups have a natural family of anisotropic dilations $\delta_\lambda : G \to G$ given by

$$\delta_\lambda(x) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}).$$

The homogeneous dimension Q of the group is defined to be $Q = \sum_{j=1}^{m} j \dim V_j$. Let dx denote a Haar measure on G, which can be taken to be the Lebesgue measure on $\mathbb{R}^{N_1} \times \cdots \mathbb{R}^{N_r}$. The left-invariant vector fields of the first layer on G may be written in the following way:

$$X_j = \frac{\partial}{\partial x_j} + \sum_{h=2}^{r} \sum_{k=1}^{N_h} a_{j,k}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \frac{\partial}{\partial x_k^{(h)}}, \quad j = 1, \dots, N_1,$$

where $a_{j,k}$ are degree $h-1$ polynomials which are homogeneous with respect to δ_λ. These vector fields generate the Lie algebra of G and so every left invariant vector on G may be obtained from taking linear combinations of brackets of the X_j. The horizontal gradient, horizontal divergence, and sub-Laplacian of G are defined as before.

We now recall the notions of symmetric homogeneous norms and gauges in the following definitions.

Definition 2.1. A nonnegative continuous function d on G is called a symmetric homogeneous norm provided
(1) $d(\delta_\lambda(x)) = \lambda d(x)$ for all $\lambda > 0$, $x \in G$,
(2) $d(x) > 0$ iff $x \neq 0$,
(3) $d(x^{-1}) = d(x)$ for every $x \in G$.

Definition 2.2. Suppose d is a symmetric homogeneous norm on G which is smooth away from the origin. If d satisfies $\mathcal{L} d^{2-Q} = 0$ away from the origin, then d is called an $\mathcal{L}$-gauge. An $\mathcal{L}$-radial function is any function $u : G \setminus \{0\} \to \mathbb{R}$ satisfying $u = f \circ d$ for a suitable $f : (0, \infty) \to \mathbb{R}$ and a given $\mathcal{L}$-gauge d.

Note that the Heisenberg homogeneous gauge (2.3) satisfies this definition. Moreover, it is well known that (e.g., see [3]), if $\Gamma(x, y)$ denotes the fundamental solution of $\mathcal{L}$, then any $\mathcal{L}$-gauge d is a scalar multiple of $\Gamma^{\frac{1}{2-Q}}(x, 0)$ away from the origin (see for example [3]). This fact allows for a nice formula for $\mathcal{L}$ acting on radial functions, and we recall this useful formula in the following lemma.

Lemma A. *If $f(d)$ is a smooth radial function on $G \setminus \{0\}$, then*

$$\mathcal{L}(f(d)) = |\nabla_H d|^2 \Big(f''(d) + \frac{Q-1}{d} f'(d) \Big).$$

The proof may be found in [3, Proposition 5.4.3].

In fact, these properties of G and its fundamental solution hold in a more general setting. Indeed, if $\{Y_1, \dots, Y_m\}$ is a system of $C^\infty(\mathbb{R}^n)$ homogeneous linearly independent vector fields with homogeneous degree 1 with respect to an anisotropic dilation

$$\delta_\lambda : (x_1, \dots, x_n) \mapsto (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n),$$

where $1 = \sigma_1 \leq \cdots \leq \sigma_n$, and if this system satisfies Hörmander's finite rank condition at $x = 0$, then, by [1], their sum of squares operator $\mathcal{L}_Y = Y_1^2 + \cdots + Y_m^2$ admits a global fundamental solution Γ_Y satisfying
(1) $\mathcal{L}_{Y,y} \Gamma_Y(x, y) = \delta_x$,
(2) Γ_Y is strictly positive,
(3) $\Gamma(\delta_\lambda(x); \delta_\lambda(y)) = \lambda^{2-q} \Gamma(x; y)$ for all $x \neq y$ and $\lambda > 0$,
(4) $\Gamma(x; y) = \Gamma(y; x)$ for $x \neq y$.

Here, $q = \sum_{j=1}^{n} \sigma_j$ is the homogeneous dimension of the system, and by being homogeneous of degree 1 it is understood that

$$Y_j(f \circ \delta_\lambda) = \lambda (Y_j f) \circ \delta_\lambda$$

for all $\lambda > 0$, $f \in C^\infty(\mathbb{R}^N)$, and $j = 1, \ldots, m$. Following the proof of [3, Proposition 5.4.3], it is easy to see that $d_Y = \Gamma^{\frac{1}{2-q}}$ satisfies

$$\mathcal{L}_Y d_Y = (q-1)\frac{|\nabla_Y d_Y|^2}{d_Y},$$

where $\nabla_Y = (Y_1, \ldots, Y_m)$. Moreover, if $f(d_Y)$ is a radial function with respect to d_Y, then there holds

$$\mathcal{L}_Y(f(d_Y)) = |\nabla_Y d|^2 \Big(f''(d_Y) + \frac{q-1}{d_Y} f'(d_Y) \Big).$$

3 Main Results and Proofs

3.1 Hardy Identities on Carnot Groups

We begin by introducing our main factorization lemma which our main results rely on. While this lemma is stated for $\mathcal{L}$, ∇_H and div_H on G, it is worth mentioning that this lemma holds for more general operators M, M^*, the formal adjoint M^*, and M^*M replacing ∇_H, div_H, and $\mathcal{L}$, respectively.

Lemma 3.1. *Let $\Omega \subset G$ be a domain, let $V, \varphi : \Omega \to \mathbb{R}_{>0}$ be $C^1(\Omega)$, and let $s \in \mathbb{R}$. Define*

$$T_s = \sqrt{V}\nabla_H - s\sqrt{V}\varphi^{-1}\nabla_H\varphi, \quad T_s^+ = -\operatorname{div}_H(\sqrt{V}\cdot) - s\sqrt{V}\varphi^{-1}\nabla_H\varphi\cdot,$$

where T_s^+ is the formal adjoint of T_s, and T_s acts on compactly supported smooth functions supported in Ω. Then

$$T_s^+T_s u = -\operatorname{div}_H(V\nabla_H u) + s(u\varphi^{-1}\nabla_H V \cdot \nabla_H\varphi - Vu\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi + Vu\varphi^{-1}\mathcal{L}\varphi) + s^2 Vu\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi.$$

Moreover, if $T = T_1$, then

$$\begin{aligned} T^+Tu &= -\operatorname{div}_H(V\nabla_H u) + u\varphi^{-1}\nabla_H V\cdot\nabla_H\varphi + Vu\varphi^{-1}\mathcal{L}\varphi, \\ |Tu|^2 &= V\varphi^2\big|\nabla_H(u\varphi^{-1})\big|^2. \end{aligned} \tag{3.1}$$

Proof. First compute

$$\begin{aligned} -\operatorname{div}_H(\sqrt{V}(\sqrt{V}\nabla_H u)) &= -\operatorname{div}_H(V\nabla_H u), \\ -\operatorname{div}_H(sV\varphi^{-1}u\nabla_H\varphi) &= -s(\varphi^{-1}u\nabla_H\varphi\cdot\nabla_H V - Vu\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi + V\varphi^{-1}\nabla_H u\cdot\nabla_H\varphi + V\varphi^{-1}u\mathcal{L}\varphi), \\ -s\sqrt{V}\varphi^{-1}\nabla_H\varphi\cdot(\sqrt{V}\nabla_H u) &= -sV\varphi^{-1}\nabla_H\varphi\cdot\nabla_H u, \\ -s\sqrt{V}\varphi^{-1}\nabla_H\varphi\cdot(-s\sqrt{V}\varphi^{-1}\nabla_H\varphi) &= s^2V\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi. \end{aligned}$$

Collecting these identities together, we get

$$T_s^+T_s u = -\operatorname{div}_H(V\nabla_H u) + s(u\varphi^{-1}\nabla_H V\cdot\nabla_H\varphi - Vu\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi + Vu\varphi^{-1}\mathcal{L}\varphi) + s^2Vu\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi,$$

as desired. Evaluating at $s = 1$ provided the desired identity (3.1).

Next, by expanding

$$\begin{aligned} |Tu|^2 &= (\sqrt{V}\nabla_H u - u\sqrt{V}\varphi^{-1}\nabla_H\varphi)\cdot(\sqrt{V}\nabla_H u - u\sqrt{V}\varphi^{-1}\nabla_H\varphi) \\ &= V|\nabla_H u|^2 - 2V\varphi^{-1}u\nabla_H\varphi\cdot\nabla_H u + u^2V\varphi^{-2}\nabla_H\varphi\cdot\nabla_H\varphi \end{aligned}$$

we see that

$$\begin{aligned} V\varphi^2|\nabla_H(u\varphi^{-1})|^2 &= V\varphi^2(\varphi^{-1}\nabla_H u - \varphi^{-2}u\nabla_H\varphi)\cdot(\varphi^{-1}\nabla_H u - \varphi^{-2}u\nabla_H\varphi) \\ &= V|\nabla_H u|^2 - 2\varphi^{-1}u\nabla_H\varphi\cdot\nabla_H u + \varphi^{-2}u^2\nabla_H\varphi\cdot\varphi_H\varphi \\ &= |Tu|^2, \end{aligned}$$

which is the desired identity. □

Now, if there is a third positive C^1-function W such that (V, W) is a Bessel pair with positive solution φ on an interval $(0, R)$, then Lemma 3.1 may be applied to the functions $V(d(x))$ and $\varphi(d(x))$ to establish the following Hardy identities. We remark that similar results were obtained in [17, 26] where they assume the weights satisfy a general differential inequality; however, our approach with Bessel pairs provides a fairly simple way to produce many interesting weights.

Theorem 3.2. *Let G be a Carnot group with symmetric homogeneous gauge d, and let $B_R(0)$ denote the d-gauge ball of radius R for a given $0 < R \le \infty$. Let V and W be positive C^1-functions on $(0, R)$. If $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair on $(0, R)$ with positive solution φ, then for $u \in C_0^\infty(B_R(0) \setminus \{0\})$ there holds*

$$\int_{B_R(0)} V(d)\left|\nabla_H\left(\frac{u}{\varphi(d)}\right)\right|^2 \varphi(d)^2\,dx = \int_{B_R(0)} V(d)|\nabla_H u|^2\,dx - \int_{B_R(0)} W(d)u^2|\nabla_H d|^2\,dx.$$

In particular, the following Hardy's inequality holds:

$$\int_{B_R(0)} V(d)|\nabla_H u|^2\,dx \ge \int_{B_R(0)} W(d)|u|^2|\nabla_H d|^2\,dx.$$

Proof. First observe that

$$\nabla_H V(d) = V'(d)\nabla_H d, \quad \nabla_H \varphi(d) = \varphi'(d)\nabla_H d.$$

Next, by using the factorization lemma (Lemma 3.1), we have

$$\begin{aligned}\int_{B_R(0)} V\varphi^2|\nabla_H(u\varphi^{-1})|^2\,dx &= \int_{B_R(0)} |Tu|^2\,dx\\ &= \int_{B_R(0)} uT^+Tu\,dx\\ &= -\int_{B_R(0)} u\,\mathrm{div}_H(V\nabla_H u)\,dx + \int_{B_R(0)} u^2\varphi^{-1}\nabla_H V\cdot\nabla_H\varphi\,dx + \int_{B_R(0)} Vu^2\varphi^{-1}\mathcal{L}\varphi\,dx.\end{aligned}$$

Next, we use the divergence theorem and Lemma A to obtain

$$\begin{aligned}\int_{B_R(0)} V\varphi^2|\nabla_H(u\varphi^{-1})|^2\,dx &= \int_{B_R(0)} V|\nabla_H u|^2\,dx + \int_{B_R(0)} u^2V'\varphi^{-1}\varphi'|\nabla_H d|^2\,dx\\ &\quad + \int_{B_R(0)} Vu^2\varphi^{-1}\left(\varphi'' + \frac{Q-1}{d}\varphi'\right)|\nabla_H d|^2\,dx\\ &= \int_{B_R(0)} V|\nabla_H u|^2\,dx + \int_{B_R(0)} u^2\varphi^{-1}\left(V\varphi'' + \frac{Q-1}{d}V\varphi' + V'\varphi'\right)|\nabla_H d|^2\,dx.\end{aligned}$$

Then, using that $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair, we have that

$$\frac{Q-1}{r}V(r)\varphi'(r) + V'(r)\varphi'(r) + V(r)\varphi''(r) = -W(r)\varphi(r),$$

and so

$$\int_{B_R(0)} V\varphi^2|\nabla_H(u\varphi^{-1})|^2\,dx = \int_{B_R(0)} V|\nabla_H u|^2\,dx - \int_{B_R(0)} u^2\varphi^{-1}W|\nabla_H d|^2\,dx,$$

which is the desired identity. From

$$\int_{B_R(0)} V(d)\left|\nabla\left(\frac{u}{\varphi(d)}\right)\right|^2 \varphi(d)^2\,dx \ge 0$$

the last inequality of the theorem follows. □

Remark 3.3. One of the main strengths of the identities given in Theorem 3.2 is that it allows one to check for the existence of extremizers of

$$\int_{B_R(0)} V(d)|\nabla_H u|^2\,dx \geq \int_{B_R(0)} W(d)u^2|\nabla_H d|^2\,dx$$

provided that this inequality is in fact sharp. Indeed, if

$$\int_{B_R(0)} V(d)|\nabla_H u|^2\,dx = \int_{B_R(0)} W(d)u^2|\nabla_H d|^2\,dx,$$

then

$$\int_{B_R(0)} V(d)\Big|\nabla_H\Big(\frac{u}{\varphi(d)}\Big)\Big|^2 \varphi(d)^2\,dx = 0.$$

This implies that

$$u = \varphi(d),$$

and so the existence of extremizers is dictated by the integrability of φ in the appropriate weighted Sobolev space.

Remark 3.4. Throughout, we assume that $u \in C_0^\infty(G \setminus \{0\})$. This assumption may be weakened to assuming $u \in C_0^\infty(G)$ provided that the origin is a removable singularity with respect to the weighted Sobolev norm

$$\|u\|^2 = \int_{B_R(0)} V(d)|\nabla_H u|^2\,dx + \int_{B_R(0)} W(d)|u|^2|\nabla_H d|^2\,dx.$$

On the other hand, if either V or W is too singular, then $u \in C_0^\infty(G \setminus \{0\})$ may be necessary for integrability.

As mentioned above, by choosing specific Bessel pairs and using Theorem 3.2, we may improve geometric inequalities in the sub-elliptic setting. We now state several such applications and include their respective proofs which depict the appropriate choice of Bessel pair.

The first identity we present is the Hardy identity with power weights which imply the classical weighted Hardy inequalities on Carnot groups.

Corollary 3.5. *For all $u \in C_0^\infty(G \setminus \{0\})$, and all $\lambda \in \mathbb{R}$, there holds*

$$\int_G d^{-\lambda}|\nabla_H u|^2\,dx - \frac{(Q-2-\lambda)^2}{4}\int_G d^{-\lambda-2}|u|^2|\nabla_H d|^2\,dx = \int_G d^{2-Q}|\nabla_H(d^{\frac{Q-2-\lambda}{2}}u)|^2\,dx.$$

Proof. The identity follows from applying Theorem 3.2 to the Bessel pair

$$(r^{Q-1}V, r^{Q-1}W) = \Big(r^{Q-1-\lambda}, \frac{(Q-2-\lambda)^2}{4}r^{Q-3-\lambda}\Big),$$

which has as a positive solution $\varphi(r) = r^{-\frac{Q-2-\lambda}{2}}$. ☐

The following corollary establishes the Hardy identity which implies the Brezis–Vázquez-type inequality in our setting.

Theorem 3.6. *Let G be a Carnot group with symmetric homogeneous gauge d, and let $B_R(0)$ denote the d-gauge ball of radius R for a given $0 < R < \infty$. Then, for $u \in C_0^\infty(B_R(0) \setminus \{0\})$, there holds*

$$\int_{B_R(0)} |\nabla_H u|^2\,dx - \Big(\frac{Q-2}{2}\Big)^2 \int_{B_R(0)} \frac{|u|^2}{d^2}\,dx = \frac{z_0^2}{R^2}\int_{B_R(0)} |u|^2\,dx + \int_{B_R(0)} \Big|\nabla_H \frac{ud^{\frac{Q}{2}}}{J_{0;R}(d)}\Big|^2 \frac{J_{0;R}(d)^2}{d^Q}\,dx.$$

Proof. The identity follows from applying Theorem 3.2 to the Bessel pair

$$(r^{Q-1}V, r^{Q-1}W) = \Big(r^{Q-1}, r^{Q-1}\Big(\Big(\frac{N-2}{2}\Big)^2 r^{-2} + \frac{z_0^2}{R^2}\Big)\Big)$$

which has as a positive solution

$$\varphi(r) = r^{\frac{Q}{2}}J_0\Big(\frac{rz_0}{R}\Big) = r^{-\frac{Q}{2}}J_{0;R}(r).$$ ☐

3.2 Hardy Identities on the Heisenberg Half Space

Define the Heisenberg half-space $H_N^+ = \{(z,t) \in H_N : t > 0\}$ and the half-gauge ball $B_R^+(0) = B_R(0) \cap H_N^+$. In this subsection, we introduce general Hardy identities in terms of Bessel pairs on the half-space H_N^+ and half-gauge ball B_N^+, and, for specific choices of the Bessel pair, we recover and find exact remainder terms for the half-space inequalities given by Luan and Yang in [23]. To begin, first define the strip

$$A_{2R} = \{(z,t) \in H_N : t \in (0, 2R)\},$$

and define the vertical height function $d_R(x,t) = \min\{t, 2R - t\}$, and set $d_\infty(z,t) = t$. Using Lemma 3.1, we will obtain Hardy inequalities on A_{2R} in this section.

To obtain Hardy identities in terms of the vertical derivative $\frac{\partial}{\partial t}$, the following factorization lemma specialized to the vertical derivative $T = \frac{\partial}{\partial t}$ on H_N is needed. In fact, the following factorization is effectively used in [20], but for completion and to state the factorization in the same generality as Lemma 3.1, we state and prove the following lemma.

Lemma 3.7. *Let*

$$S = \sqrt{V}\frac{\partial}{\partial t} - \sqrt{V}\varphi^{-1}\frac{\partial\varphi}{\partial t}, \quad S^+ = -\frac{\partial}{\partial t}(\sqrt{V}\cdot) - \sqrt{V}\varphi^{-1}\frac{\partial\varphi}{\partial t},$$

where S^+ is the formal adjoint of S, and S acts on compactly supported functions with appropriate domains. Then

$$S^+Su = -\frac{\partial}{\partial t}\Big(V\frac{\partial u}{\partial t}\Big) + \varphi^{-1}\frac{\partial V}{\partial t}\frac{\partial\varphi}{\partial t}u + V\varphi^{-1}\frac{\partial^2\varphi}{\partial t^2}u, \quad |Su|^2 = V\varphi^2\Big|\frac{\partial}{\partial t}\Big(\frac{u}{\varphi}\Big)\Big|^2.$$

Proof. First compute

$$\frac{\partial}{\partial t}\Big(V\varphi^{-1}\frac{\partial\varphi}{\partial t}u\Big) = \varphi^{-1}\frac{\partial V}{\partial t}\frac{\partial\varphi}{\partial t}u - \varphi^{-2}V\Big(\frac{\partial\varphi}{\partial t}\Big)^2u + V\varphi^{-1}\frac{\partial^2\varphi}{\partial t^2}u + V\varphi^{-1}\frac{\partial\varphi}{\partial t}\frac{\partial u}{\partial t},$$

from which we obtain

$$\begin{aligned} S^+Su &= -\frac{\partial}{\partial t}\Big(V\frac{\partial u}{\partial t}\Big) + \varphi^{-1}\frac{\partial V}{\partial t}\frac{\partial\varphi}{\partial t}u - \varphi^{-2}V\Big(\frac{\partial\varphi}{\partial t}\Big)^2u + V\varphi^{-1}\frac{\partial^2\varphi}{\partial t^2}u + V\varphi^{-1}\frac{\partial\varphi}{\partial t}\frac{\partial u}{\partial t} \\ &\quad - V\varphi^{-1}\frac{\partial\varphi}{\partial t}\frac{\partial u}{\partial t} + V\varphi^{-2}\Big(\frac{\partial\varphi}{\partial t}\Big)^2u \\ &= -\frac{\partial}{\partial t}\Big(V\frac{\partial u}{\partial t}\Big) + \varphi^{-1}\frac{\partial V}{\partial t}\frac{\partial\varphi}{\partial t}u + V\varphi^{-1}\frac{\partial^2\varphi}{\partial t^2}u, \end{aligned}$$

as desired. Lastly, computing

$$|Su|^2 = V\Big(\frac{\partial u}{\partial t}\Big)^2 + V\varphi^{-2}\Big(\frac{\partial\varphi}{\partial t}\Big)^2\Big(\frac{\partial u}{\partial t}\Big)^2 - 2V\varphi^{-1}\frac{\partial\varphi}{\partial t}u$$

and

$$\Big|\frac{\partial}{\partial t}\Big(\frac{u}{\varphi}\Big)\Big|^2 = \varphi^{-2}\Big(\frac{\partial u}{\partial t}\Big)^2 + \varphi^{-4}u^2\Big(\frac{\partial\varphi}{\partial t}\Big)^2 - 2\varphi^{-3}u\frac{\partial\varphi}{\partial t}\frac{\partial u}{\partial t},$$

we find

$$|Su|^2 = V\varphi^2\Big|\frac{\partial}{\partial t}\Big(\frac{u}{\varphi}\Big)\Big|^2.$$

□

We may now state the main Hardy identity for strips.

Theorem 3.8. *Let $0 < R \le \infty$, and let V and W be positive C^1-functions on $(0,R)$. If $(r^{Q+1}V, r^{Q+1}W)$ is a Bessel pair on $(0,R)$ with positive solution φ, then for $u \in C_0^\infty(A_{2R})$, there holds*

$$\int\limits_{A_{2R}} V(d_R)\Big|\nabla_H\Big(\frac{u}{\varphi(d_R)}\Big)\Big|^2\varphi^2(d_R)\,dz\,dt = \int\limits_{A_{2R}} V(d_R)|\nabla u|^2\,dz\,dt - \frac{1}{4}\int\limits_{A_{2R}} W(d_R)|z|^2|u|^2\,dz\,dt$$

and

$$\int\limits_{A_{2R}} V\varphi^2\Big|\frac{\partial}{\partial t}\Big(\frac{u}{\varphi}\Big)\Big|^2\,dz\,dt = \int\limits_{A_{2R}} V(d_R)\Big|\frac{\partial u}{\partial t}\Big|^2\,dz\,dt - \int\limits_{A_{2R}} W(d_K)|u|^2\,dz\,dt,$$

Proof. First observe that

$$\mathcal{L}\varphi(d_R) = \varphi''(d_R)|z|^2, \quad |\nabla_H d_R|^2 = \frac{1}{4}|z|^2,$$

and so, by Lemma 3.1, we have that

$$\begin{aligned} T^+Tu &= -\operatorname{div}_H(V\nabla_H u) + \frac{1}{4}u\frac{\varphi'(d_R)}{\varphi(d_R)}|z|^2 V'(d_R) + \frac{1}{4}uV(d_R)\frac{\varphi''(d_R)}{\varphi(d_R)}|z|^2 \\ &= -\operatorname{div}_H(V\nabla_H u) + \frac{1}{4}u|z|^2\frac{1}{\varphi(d_R)}[\varphi'(d_R)V'(d_R) + \varphi''(d_R)V(d_R)]. \end{aligned}$$

Therefore, using that (V, W) is a Bessel pair and the divergence theorem, we find

$$\begin{aligned} \int_{A_{2R}} V(d_R)\left|\nabla_H\left(\frac{u}{\varphi(d_R)}\right)\right|^2 \varphi^2(d_R)\,dz\,dt &= \int_{A_{2R}} |Tu|^2\,dz\,dt \\ &= \int_{A_{2R}} V(d_R)|\nabla_H u|^2\,dz\,dt - \frac{1}{4}\int_{A_{2R}} W(d_R)|z|^2|u|^2\,dz\,dt, \end{aligned}$$

as desired.

To obtain the second identity, one may simply appeal to [20, Theorem 1.3]; we include the following short proof for completeness. Observe

$$\frac{\partial V}{\partial t}\frac{\partial \varphi}{\partial t} = V'(d_K)\varphi'(d_K), \quad \frac{\partial^2\varphi}{\partial t^2} = \varphi''(d_K),$$

and so, using Lemma 3.7, we find

$$S^+Su = -\frac{\partial}{\partial t}\left(V\frac{\partial u}{\partial t}\right) + u\varphi^{-1}[V'\varphi' + V\varphi''].$$

Therefore, using that (V, W) is a Bessel pair and integration by parts, we find

$$\int_{A_{2R}} V\varphi^2\left|\frac{\partial}{\partial t}\left(\frac{u}{\varphi}\right)\right|^2 dz\,dt = \int_{A_{2R}} |Su|^2\,dz\,dt = \int_{A_{2R}} V(d_R)\left|\frac{\partial u}{\partial t}\right|^2 dz\,dt - \int_{A_{2R}} W(d_K)|u|^2\,dz\,dt,$$

as desired. □

Using Theorem 3.8, we may obtain the following Hardy identity which improves and recovers (1.5).

Corollary 3.9. *For $u \in C_0^\infty(H_N^+)$, there holds*

$$\int_{H_N^+} |\nabla_H t^{\frac{\alpha}{2}} u|^2 t\,dz\,dt = \int_{H_N^+} t^{\alpha+1}|\nabla_H u|^2\,dz\,dt - \frac{\alpha^2}{16}\int_{H_N^+} |z|^2 t^{\alpha-1}|u|^2\,dz\,dt$$

and

$$\int_{H_N^+} \left|\frac{\partial}{\partial t}(t^{\frac{\alpha}{2}}u)\right|^2 t\,dz\,dt = \int_{H_N^+} t^{\alpha+1}\left|\frac{\partial u}{\partial t}\right|^2 dz\,dt - \frac{\alpha^2}{4}\int_{H_N^+} t^{\alpha-1}|u|^2\,dz\,dt.$$

In particular, we have the algebraically interesting Hardy inequalities

$$\int_{H_N^+} t^{\alpha+1}|Tu|^2\,dz\,dt = \int_{H_N^+} t^{\alpha+1}|[X_j, Y_j]u|^2\,dz\,dt \geq \frac{\alpha^2}{4}\int_{H_N^+} t^{\alpha-1}|u|^2\,dz\,dt.$$

Proof. The identity follows from applying Theorem 3.8 to the Bessel pair

$$(V, W) = \left(r^{\alpha+1}, \frac{\alpha^2}{4}r^{\alpha-1}\right)$$

with positive solution $\varphi = r^{-\frac{\alpha}{2}}$ on $(0, \infty)$. □

Remark 3.10. We point out that to obtain Hardy identity from [23], one needs to take $\alpha = -1$, and the change of variable $t \mapsto 4t$ is needed since they use the vector fields

$$X_j = \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t},$$

compare this with (2.1).

We now state and prove the Hardy identities for the half ball $B_R^+(0)$. We will again need a new factorization lemma.

Lemma 3.11. *Let V, φ, Y be sufficiently smooth and define*

$$\begin{aligned} U &= \sqrt{V}\nabla_H - \sqrt{V}\varphi^{-1}\nabla_H\varphi + \sqrt{V}Y^{-1}\nabla_H Y, \\ U^+ &= -\operatorname{div}_H(\sqrt{V}\cdot) - \sqrt{V}\varphi^{-1}\nabla_H\varphi \cdot + \sqrt{V}Y^{-1}\nabla_H Y \cdot . \end{aligned}$$

where U^+ is the formal adjoint of U, and U acts on compactly supported smooth functions with appropriate domains. Then

$$\begin{aligned} U^+Uu &= -\operatorname{div}_H(V\nabla_H u) + u\varphi^{-1}\nabla_H V \cdot \nabla_H\varphi + Vu\varphi^{-1}\Delta\varphi + 2VuY^{-2}|\nabla_H Y|^2 - VuY^{-1}\Delta Y \\ &\quad - uY^{-1}\nabla_H V \cdot \nabla_H Y - 2Vu\varphi^{-1}Y^{-1}\nabla_H\varphi \cdot \nabla_H Y, \\ |Uu|^2 &= V\varphi^2 Y^{-2}|\nabla_H(u\varphi^{-1}Y)|^2. \end{aligned}$$

Proof. If T denotes the operator in Lemma 3.1, then

$$U = T + \sqrt{V}Y^{-1}\nabla_H Y, \quad U^+ = T^+ + \sqrt{V}Y^{-1}\nabla_H Y.$$

Consequently, there holds

$$\begin{aligned} U^+Uu &= (T^+ + \sqrt{V}Y^{-1}\nabla_H Y)(T + \sqrt{V}Y^{-1}\nabla_H Y)u \\ &= T^+Tu + T^+(u\sqrt{V}Y^{-1}\nabla_H Y) + \sqrt{V}Y^{-1}\nabla_H Y \cdot (Tu) + uVY^{-2}|\nabla_H Y|^2. \end{aligned} \tag{3.2}$$

Since we already have T^+T from Lemma 3.1, we need only compute the middle two terms

$$\begin{aligned} T^+(u\sqrt{V}Y^{-1}\nabla_H Y) &= -\operatorname{div}_H(VuY^{-1}\nabla_H Y) - uVY^{-1}\varphi^{-1}\nabla_H\varphi \cdot \nabla_H Y \\ &= uY^{-1}\nabla_H V \cdot \nabla_H Y - VY^{-1}\nabla_H u \cdot \nabla_H Y + VuY^{-2}|\nabla_H Y|^2 - VuY^{-1}\mathcal{L}Y \\ &\qquad - uVY^{-1}\varphi^{-1}\nabla_H\varphi \cdot \nabla_H Y, \\ \sqrt{V}Y^{-1}\nabla_H Y \cdot (Tu) &= VY^{-1}\nabla_H Y \cdot \nabla_H u - uV\varphi^{-1}Y^{-1}\nabla_H Y \cdot \nabla_H\varphi. \end{aligned} \tag{3.3}$$

Therefore, using that

$$T^+Tu = -\operatorname{div}_H(V\nabla_H u) + u\varphi^{-1}\nabla_H V \cdot \nabla_H\varphi + Vu\varphi^{-1}\mathcal{L}\varphi,$$

we may combine (3.2) and (3.3) to obtain the desired identity for U^+U.

The computation for $|Uu|^2$ is similar to the computation in Lemma 3.1. □

We remark that Lemma 3.1 may be obtained from Lemma 3.7 by taking $Y = 0$.

With this factorization lemma established, we are now prepared to state and prove the following Hardy identity on the half-gauge ball $B_R^+(0)$.

Theorem 3.12. *Let $0 < R \le \infty$, and let V and W be positive C^1-functions on $(0, R)$. If $(r^{Q+1}V, r^{Q+1}W)$ is a Bessel pair on $(0, R)$, then for $u \in C_0^\infty(B_R \cap H_N^+)$ there holds*

$$\int\limits_{B_R(0)\cap H_N^+} V\left|\nabla\left(\frac{u}{\varphi}\frac{1}{t}\right)\right|^2 \varphi^2 t^2 \,dz\,dt = \int\limits_{B_R(0)\cap H_N^+} V|\nabla u|^2 \,dz\,dt - \int\limits_{B_R(0)\cap H_N^+} \left[W - \frac{V'}{d}\right]|u|^2|\nabla d|^2 \,dz\,dt.$$

Proof. This theorem will follow by taking $Y = \frac{1}{t}$ in Lemma 3.11. To see this, first compute

$$\mathcal{L}\frac{1}{t} = \frac{|z|^2}{2t^3}, \quad \nabla_H d \cdot \nabla_H \frac{1}{t} = -\frac{1}{t^2}\nabla_H d \cdot \nabla_H t = -\frac{1}{t}|\nabla_H d|^2.$$

We then compute

$$
\begin{aligned}
u\varphi^{-1}\nabla_H V\cdot\nabla_H\varphi &= u\varphi^{-1}\varphi' V'|\nabla_H d|^2,\\
Vu\varphi^{-1}\mathcal{L}\varphi &= Vu\varphi^{-1}|\nabla_H d|^2\Big(\varphi'' + \frac{Q-1}{d}\varphi'\Big),\\
2Vut^2\Big|\nabla_H\frac{1}{t}\Big|^2 &= \frac{1}{2}Vu\frac{|z|^2}{t^2},\\
Vut\mathcal{L}t &= \frac{1}{2}Vu\frac{|z|^2}{t^2},\\
ut\nabla_H V\cdot\nabla_H\frac{1}{t} &= -uV'|\nabla_H d|^2,\\
2Vu\varphi^{-1}t\nabla_H\varphi\cdot\nabla_H\frac{1}{t} &= -2Vu\varphi^{-1}\varphi'|\nabla_H d|^2.
\end{aligned}
$$

Collecting everything together, we have

$$
\begin{aligned}
U^+Uu &= -\operatorname{div}_H(V\nabla_H u) + uV'|\nabla_H d|^2 + u\varphi^{-1}|\nabla_H d|^2\Big[V\varphi'' + V'\varphi' + \frac{Q+1}{d}V\varphi'\Big]\\
&= -\operatorname{div}_H(V\nabla_H u) + uV'|\nabla_H d|^2 + uW|\nabla_H d|^2,
\end{aligned}
$$

where, to get the last equality, we used the fact that $(r^{Q+1}V, r^{Q+1}W)$ is a Bessel pair. At last,

$$
\begin{aligned}
\int\limits_{B_R(0)\cap H_N^+} V\Big|\nabla\Big(\frac{u}{\varphi}\frac{1}{t}\Big)\Big|^2\varphi^2t^2\,dz\,dt &= \int\limits_{B_R(0)\cap H_N^+} |Uu|^2\,dz\,dt\\
&= \int\limits_{B_R(0)\cap H_N^+} V|\nabla u|^2\,dz\,dt - \int\limits_{B_R(0)\cap H_N^+} \Big[W - \frac{V'}{d}\Big]|u|^2|\nabla d|^2\,dz\,dt,
\end{aligned}
$$

as desired. □

3.3 Weighted Uncertainty Principles

Lemma 3.1 may also be used to obtain Caffarelli–Kohn–Nirenberg-type inequalities with more general weights than simple power weights. The following results are new even for the Euclidean setting.

Theorem 3.13. *If $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair on $(0,R)$ with solution φ, then for $u\in C_0^\infty(B_R(0)\setminus\{0\})$ there holds*

$$
\begin{aligned}
\frac{1}{2}\int\limits_{B_R(0)} u^2|\nabla_H d|^2\Big[W(d) + V(d)\Big(\frac{\varphi'(d)}{\varphi(d)}\Big)^2\Big]\,dx &\le \Big(\int\limits_{B_R(0)} u^2|\nabla_H d|^2 V(d)\Big(\frac{\varphi'(d)}{\varphi(d)}\Big)^2\,dx\Big)^{\frac{1}{2}}\\
&\quad\times\Big(\int\limits_{B_R(0)} V(d)|\nabla_H u|^2\,dx\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Proof. The following proof is motivated by Costa's work [6]. (See also [8, 9].) Using Lemma 3.1, we find

$$
\begin{aligned}
\int\limits_{B_R(0)} |T_s u|^2\,dx &= -\int\limits_{B_R(0)} \bar{u}\operatorname{div}_H(V\nabla_H u)\,dx + s^2\int\limits_{B_R(0)} |u|^2V\Big(\frac{\varphi'}{\varphi}\Big)^2|\nabla_H d|^2\,dx\\
&\quad + s\int\limits_{B_R(0)} |u|^2\Big[V'\frac{\varphi'}{\varphi} - V\Big(\frac{\varphi'}{\varphi}\Big)^2 + V\frac{1}{\varphi}\Big(\varphi'' + \frac{Q-1}{d}\varphi'\Big)\Big]|\nabla_H d|^2\,dx.
\end{aligned}
$$

Then, using that $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair, we find

$$
\int\limits_{B_R(0)} |T_s u|^2\,dx = \int\limits_{B_R(0)} |\nabla_H u|^2V\,dx + s^2\int\limits_{B_R(0)} |u|^2V\Big(\frac{\varphi'}{\varphi}\Big)^2|\nabla_H d|^2\,dx - s\int\limits_{B_R(0)} |u|^2\Big[W + V\Big(\frac{\varphi'}{\varphi}\Big)^2\Big]|\nabla_H d|^2\,dx.
$$

At last, by using that $As^2 + Bs + C \geq 0$ for all $s \in \mathbb{R}$ implies $\frac{1}{2}|B| \leq A^{\frac{1}{2}} C^{\frac{1}{2}}$, with

$$A = \int_{B_R(0)} |u|^2 V\Big(\frac{\varphi'}{\varphi}\Big)^2 |\nabla_H d|^2 \, dx,$$
$$B = -\int_{B_R(0)} u^2 |\nabla_H d|^2 \Big[W + V\Big(\frac{\varphi'}{\varphi}\Big)^2\Big] \, dx,$$
$$C = \int_{B_R(0)} V|\nabla_H u|^2 \, dx,$$

we obtain

$$\frac{1}{2}\int_{B_R(0)} u^2 |\nabla_H d|^2 \Big[W + V\Big(\frac{\varphi'}{\varphi}\Big)^2\Big] \, dx \leq \Big(\int_{B_R(0)} u^2 |\nabla_H d|^2 V\Big(\frac{\varphi'}{\varphi}\Big)^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{B_R(0)} V|\nabla_H u|^2 \, dx\Big)^{\frac{1}{2}},$$

as desired. ☐

By taking

$$(V, W) = (r^\lambda, |Q + \lambda - 1| r^{\lambda-2}),$$

then $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair with positive solution $\varphi = r^{1-\lambda-Q}$ on $(0, \infty)$. Using this Bessel pair in Theorem 3.13, we recover the weighted Hardy uncertainty principles as a corollary

$$\frac{|Q + \lambda - 2|}{2} \int_G u^2 d^\lambda |\nabla_H d|^2 \, dx \leq \Big(\int_G u^2 d^{\lambda+2} |\nabla_H d|^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_G |\nabla_H u|^2 d^\lambda \, dx\Big)^{\frac{1}{2}}$$

for all $\lambda \in \mathbb{R}$ and $u \in C_0^\infty(G \setminus \{0\})$. In fact, one may show that if

$$V = r^{-2b}, \quad V\Big(\frac{\varphi'}{\varphi}\Big)^2 = \alpha r^{-2a}$$

were to hold for some $a, b, \alpha \in \mathbb{R}$, then W changes sign on $(0, \infty)$ unless $a = b + 1$. Therefore, it is reasonable to interpret Corollary 3.14 as a general family of uncertainty principles.

Now, using that $W > 0$ and $V(\frac{\varphi'}{\varphi})^2 \geq 0$, Theorem 3.13 may be used to obtain general weighted Hardy-type inequalities of the following form.

Corollary 3.14. *Suppose $V > 0$ and $\varphi \geq 0$ are C^1-functions on $(0, R)$ such that*

$$(r^{Q-1} V \varphi')' < 0 \quad \text{on } (0, R).$$

Then, for all $u \in C_0^\infty(B_R(0) \setminus \{0\})$, there holds

$$\frac{1}{4}\int_{B_R(0)} |u|^2 V(d)\Big(\frac{\varphi'(d)}{\varphi(d)}\Big)^2 |\nabla_H d|^2 \, dx \leq \int_{B_R(0)} |\nabla_H u|^2 V(d) \, dx.$$

Proof. If $(r^{Q-1} V\varphi')' < 0$ for a given pair (V, φ), then taking

$$W = -\frac{1}{r^{Q-1}\varphi}(r^{Q-1} V \varphi')',$$

we obtain a Bessel pair $(r^{Q-1}V, r^{Q-1}W)$. Since $W > 0$ and $V(\frac{\varphi'}{\varphi})^2 \geq 0$, there holds

$$\int_{B_R(0)} u^2 |\nabla_H d|^2 \Big[W + V\Big(\frac{\varphi'}{\varphi}\Big)^2\Big] \, dx \geq \int_{B_R(0)} u^2 |\nabla_H d|^2 V\Big(\frac{\varphi'}{\varphi}\Big)^2 \, dx,$$

and so Theorem 3.13 implies

$$\frac{1}{2}\int_{B_R(0)} u^2 |\nabla_H d|^2 V\Big(\frac{\varphi'}{\varphi}\Big)^2 \, dx \leq \Big(\int_{B_R(0)} u^2 |\nabla_H d|^2 V\Big(\frac{\varphi'}{\varphi}\Big)^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{B_R(0)} V|\nabla_H u|^2 \, dx\Big)^{\frac{1}{2}}.$$

Rearranging terms gives the desired inequality. ☐

In the case that $V = 1$, Corollary 3.14 may be used to obtain the following sufficient condition for a certain Hardy-type inequality to hold. For simplicity, we state it for Euclidean space only.

Corollary 3.15. *Let $w(r)$ be a nonnegative C^1-function on $(0, R)$. Then, for*

$$\frac{1}{4}\int_{B_R(0)} |u|^2 w(|x|)\,dx \leq \int_{B_R(0)} |\nabla u|^2\,dx$$

to hold for $u \in C_0^\infty(B_R(0) \setminus \{0\})$, it is sufficient for w to satisfy the differential inequality

$$\Big(r^{N-1} w^{\frac{1}{2}} e^{\int w^{\frac{1}{2}}\,dr}\Big)' < 0. \tag{3.4}$$

This inequality is equivalent to

$$\frac{N-1}{r} w + \frac{1}{2} w' + w^{\frac{3}{2}} < 0. \tag{3.5}$$

Proof. Setting $w = (\frac{\varphi'}{\varphi})^2$, we see that

$$\varphi = Ce^{\int w^{\frac{1}{2}}\,dr}, \quad \varphi' = Cw^{\frac{1}{2}} e^{\int w^{\frac{1}{2}}\,dr}$$

for some $C > 0$. The result follows from Corollary 3.14.

To see why (3.5) is equivalent, first observe that

$$\varphi' = w^{\frac{1}{2}}\varphi, \quad \varphi'' = \frac{1}{2} w^{-\frac{1}{2}}\varphi + w^{\frac{1}{2}}\varphi' = \frac{1}{2} w^{-\frac{1}{2}}\varphi + w\varphi.$$

Consequently, (3.4) holds if and only if

$$0 > (r^{N-1}\varphi')' = (N-1)r^{N-2}\varphi' + r^{N-1}\varphi'' = (N-1)r^{N-2} w^{\frac{1}{2}}\varphi + \frac{1}{2} w^{-\frac{1}{2}}\varphi + w\varphi,$$

and whence (3.5). □

It is interesting to note that the Hardy potential $w(r) = (N-2)^2 r^{-2}$ does not satisfy (3.5). However, the class of weights satisfying (3.5) is not small. Indeed, for every Bessel pair (V, W) with positive solution φ, one may simply take $w = (\varphi'/\varphi)^2$. It is also worth mentioning that the more singular w is at the origin, the more likely (3.5) is to hold.

3.4 General Distance Functions in Carnot–Carathéodory Spaces

We first recall a result for a general class of vector fields given in [24] by Monti and Serra-Cassano. In preparation, we state three cases in which their result (and hence our results) apply. Note that Case A is satisfied by any Carnot group, and hence the Heisenberg group.

Case A. Let $X_1, \dots, X_m \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $m < N$, satisfy Hörmander's condition and be of the form

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{N} a_{ij}(x)\frac{\partial}{\partial x_i}, \quad j = 1, \dots, m,$$

where $a_{ij} \in C^\infty(\mathbb{R}^N)$.

Case B. Let $X_1, \dots, X_N \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ be of the form

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = p_2(x_1)\frac{\partial}{\partial x_2}, \quad \dots, \quad X_N = p_N(x_1, \dots, x_{N-1})\frac{\partial}{\partial x_N},$$

where $p_j \in C^\infty(\mathbb{R}^{j-1})$, $j = 1, \dots, N$, are functions vanishing on a set of null $(j-1)$-dimensional Lebesgue measure.

Case C. Let $X_1, \dots, X_m \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and these vector fields span $\mathbb{R}^N$ at every point.

Their theorem may now be stated.

Theorem H. *Let* $(\mathbb{R}^N, d)$ *be a CC complete space induced by* $X_1, \dots, X_m$ *belonging to any one of the cases A, B or C. Let* $K \subset \mathbb{R}^N$ *be a closed set, and let* d_K *denote the CC distance from K. Then* d_K *satisfies the horizontal eikonal equation*

$$|\nabla_H d_K(x)| = 1$$

for almost every $x \in \mathbb{R}^N \setminus K$.

In particular, if $K = \{0\}$ is the origin, and if $d_{\mathrm{cc}} = d_{\{0\}}$ is the CC-distance to the origin, then d_{cc} satisfies the eikonal equation $|\nabla_H d_{\mathrm{cc}}| = 1$, in contrast with the homogeneous gauges considered above. Using this fact and Lemma 3.1, we may establish the following Hardy identity theorem for general distance functions arising from cases A, B and C above.

Theorem 3.16. *Let* $(\mathbb{R}^N, d)$ *be a CC complete space induced by* $X_1, \dots, X_m$ *satisfying any one of the cases A, B or C. Let* $K \subset \mathbb{R}^N$ *be a closed set, and let* d_K *denote the CC distance from K. Let* $\Omega_R = \{x \in H : 0 < d_K(x) < R\}$ *for a given* $0 < R \le \infty$, *and let V and W be positive* C^1*-functions on* $(0, R)$. *If* $(r^{\alpha-1}V, r^{\alpha-1}W)$ *is a Bessel pair on* $(0, R)$ *with positive solution* φ *for some* $\alpha \in \mathbb{R}$, *then for* $u \in C_0^\infty(\Omega_R \setminus \{0\})$ *there holds*

$$\begin{aligned}\int_{\Omega_R} V(d_K)\Big|\nabla_H\Big(\frac{u}{\varphi(d_K)}\Big)\Big|^2 \varphi(d_K)^2\,dx &= \int_{\Omega_R} V(d_K)|\nabla_H u|^2\,dx - \int_{\Omega_R} W(d_K)u^2\,dx \\ &\quad - \int_{\Omega_R} V(d_K)|u|^2\varphi^{-1}(d_K)\varphi'(d_K)\Big[\mathcal{L}d_K - \frac{\alpha-1}{d_K}\Big]\,dx.\end{aligned}$$

In particular, if

$$\varphi'(d_K)\Big[\mathcal{L}d_K - \frac{\alpha-1}{d_K}\Big] \le 0 \tag{3.6}$$

in the sense of distributions, then

$$\int_{\Omega_R} V(d_K)|\nabla_H u|^2\,dx \ge \int_{\Omega_R} W(d_K)|u|^2\,dx.$$

Proof. First observe that

$$\begin{aligned}\nabla_H V(d_K) &= V'(d_K)\nabla_H d_K, \\ \nabla_H \varphi(d_K) &= \varphi'(d_K)\nabla_H d_K, \\ \nabla_H V(d_K)\cdot\nabla_H\varphi(K) &= V'(d_K)\varphi'(d_K), \\ \mathcal{L}\varphi(d_K) &= \sum_{j=1}^m X_j^2\varphi(d_K) = \varphi''(d_K) + \varphi'(d_K)\mathcal{L}d_K,\end{aligned} \tag{3.7}$$

where we have invoked Theorem H to use $|\nabla_H d_K| = 1$. Now, by using the factorization lemma (Lemma 3.1), we have

$$\begin{aligned}\int_{\Omega_R} V\varphi^2|\nabla_H(u\varphi^{-1})|^2\,dx &= \int_{\Omega_R} |Tu|^2\,dx \\ &= \int_{\Omega_R} uT^+Tu\,dx \\ &= -\int_{\Omega_R} u\,\mathrm{div}_H(V\nabla_H u)\,dx + \int_{\Omega_R} u^2\varphi^{-1}\nabla_H V\cdot\nabla_H\varphi\,dx + \int_{\Omega_R} Vu^2\varphi^{-1}\mathcal{L}\varphi\,dx.\end{aligned}$$

Next, we use the divergence theorem and (3.7) to obtain

$$\begin{aligned}\int_{\Omega_R} V\varphi^2|\nabla_H(u\varphi^{-1})|^2\,dx &= \int_{\Omega_R} V|\nabla_H u|^2\,dx + \int_{\Omega_R} u^2V'\varphi^{-1}\varphi'\,dx + \int_{\Omega_R} Vu^2\varphi^{-1}(\varphi'' + \varphi'\mathcal{L}d_K)\,dx \\ &= \int_{\Omega_R} V|\nabla_H u|^2\,dx + \int_{\Omega_R} u^2\varphi^{-1}\Big(V\varphi'' + \frac{\alpha-1}{d}V\varphi' - \frac{\alpha-1}{d}V\varphi' + \varphi'\mathcal{L}d_K + V'\varphi'\Big)\,dx.\end{aligned}$$

Then, using that $(r^{\alpha-1}V, r^{\alpha-1}W)$ is a Bessel pair, we have that

$$\frac{\alpha-1}{r}V(r)\varphi'(r)+V'(r)\varphi'(r)+V(r)\varphi''(r)=-W(r)\varphi(r),$$

and so

$$\begin{aligned}\int_{\Omega_R} V(d_K)\varphi^2\big|\nabla_H(u\varphi^{-1}(d_K))\big|^2\,dx &= \int_{\Omega_R} V(d_K)|\nabla_H u|^2\,dx-\int_{\Omega_R} u^2W(d_K)\,dx\\ &\quad-\int_{\Omega_R} V(d_K)|u|^2\varphi^{-1}(d_K)\varphi'(d_K)\Big[\mathcal{L}d_K-\frac{\alpha-1}{d_K}\Big]\,dx.\end{aligned}$$

which is the desired identity. □

It is also worth mentioning that verifying if (3.6) holds is difficult in general. So, for sake of concreteness, we state applications for Theorem 3.16 for the particular case when the vector fields are the left-invariant vector fields on the Heisenberg group H_N, and the distance is the CC distance to the center Z of H_N (this is shown below). We note that, in this case, Ω_R is the infinite cylinder of radius R with axis being Z. First we establish the general Hardy identity for this setting, and then its applications.

Corollary 3.17. *Let $Z=\{(z,t)\in H: z=0\}$ denote the center of the Heisenberg group $H=\mathbb{R}^{2N+1}$, and let $d_Z(x)$ denote the CC distance from a point $x\in H$ to Z. Let $\Omega_R=\{x\in H: d_Z(x)<R\}$ for a given $0<R\le\infty$, and let V and W be positive C^1-functions on $(0,R)$. If $(r^{2N-1}V, r^{2N-1}W)$ is a Bessel pair on $(0,R)$ with positive solution φ for some $\alpha\in\mathbb{R}$, then for $u\in C_0^\infty(\Omega_R\setminus Z)$ there holds*

$$\int_{\Omega_R} V(|z|)\Big|\nabla_H\Big(\frac{u}{\varphi(|z|)}\Big)\Big|^2\varphi(|z|)^2\,dx=\int_{\Omega_R} V(|z|)|\nabla_H u|^2\,dx-\int_{\Omega_R} W(|z|)u^2\,dx. \tag{3.8}$$

Proof. First, the CC distance $d_Z(x)$ from a point $x\in H_N$ to the Heisenberg group center Z is computed. To begin, let $d_{\mathrm{eu}}(x,y)$ denote the Euclidean distance between two points $x,y\in\mathbb{R}^{2N+1}$ when considered as an Euclidean space, and let

$$d_{\mathrm{eu},Z}(x)=\inf\{d_{\mathrm{eu}}(x,y): y\in Z\}$$

denote the Euclidean distance from a point $x\in\mathbb{R}^{2N+1}$ to the Heisenberg group center Z. It is well known that the CC distance from a point $(z,0)\in H_N$ to the origin $(0,0)$ is given by

$$d_{\mathrm{cc}}((z,0),(0,0))=|z|.$$

Due to left-invariance, the distance from any point (z,t) to $(0,t)$ is thus $|z|$.

Now, it is also well known that, for two points $x,y\in\mathbb{R}^{2N+1}$, there holds $d_{\mathrm{eu}}(x,y)\le d_{\mathrm{cc}}(x,y)$, and so $d_{\mathrm{eu},Z}(x)\le d_Z(x)$. Given that

$$d_{\mathrm{cc}}((z,t),(0,t))=|z|=d_{\mathrm{eu},Z}((z,t),(0,t))\le d_Z(z,t),$$

it follows that $d_Z(z,t)=|z|$ for all $(z,t)\in H_N$. In particular, Ω_R is the infinite cylinder of radius R with axis being Z.

Next, using (2.2), we compute

$$\mathcal{L}d_K(z,t)=\Delta_{\mathbb{R}^{2N}}|z|=\frac{2N-1}{|z|}=\frac{2N-1}{d_{\mathrm{cc}}(z,t)}.$$

It is also easy to verify directly that $|\nabla_H d_Z|=1$. With these observations, we find that

$$-\int_{\Omega_R} V(d_{\mathrm{cc}})|u|^2\varphi^{-1}(d_{\mathrm{cc}})\varphi'(d_{\mathrm{cc}})\Big[\mathcal{L}d_{\mathrm{cc}}-\frac{2N-1}{d_{\mathrm{cc}}}\Big]\,dx=0.$$

The identity in (3.8) then follows by applying Theorem 3.16 with $\alpha=2N$. □

We use this theorem to establish the following weighted Hardy identities for d_Z.

Corollary 3.18. *For all $u \in C_0^\infty(H_N \setminus Z)$, and all $\lambda \in \mathbb{R}$, there holds*

$$\int_{H_N} d_Z^{-\lambda}|\nabla_H u|^2\,dx - \frac{(2N-2-\lambda)^2}{4}\int_{H_N} d_Z^{-\lambda-2}|u|^2\,dx = \int_{H_N} d_Z^{2-2N}\Big|\nabla_H\Big(d_Z^{\frac{2N-2-\lambda}{2}}u\Big)\Big|^2\,dx.$$

Proof. The identity follows from applying Corollary 3.18 to the Bessel pair

$$(r^{2N-1}V, r^{2N-1}W) = \Big(r^{2N-1-\lambda}, \frac{(2N-2-\lambda)^2}{4}r^{Q-3-\lambda}\Big),$$

which has as a positive solution $\varphi(r) = r^{-\frac{2N-2-\lambda}{2}}$. ☐

Recalling that on $\mathbb{R}^N$, the sharp Hardy inequality takes the form

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx,$$

and so it is apparent that the Hardy inequality on $\mathbb{R}^2$ reduces to the trivial inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq 0.$$

Regardless of this, one may obtain critical Hardy inequalities of the form

$$\int_{\Omega_R} \frac{1}{|x|^{\alpha-2}}|\nabla u|^2\,dx - \frac{1}{4}\int_{\Omega_R} \frac{|u(x)|^2}{|x|^2|\log\frac{R}{|x|}|^2}\,dx = \int_{\Omega_R} \frac{1}{|x|^{\alpha-2}}\log\frac{R}{|x|}\Big|\nabla\Big(\frac{u(x)}{\sqrt{\log\frac{R}{|x|}}}\Big)\Big|^2\,dx;$$

see [21] and the references therein. Similar to the Euclidean case, if we take $\lambda = 2N-2$ in Corollary 3.18, then we obtain trivial identities. Therefore, we establish analogous critical Hardy identities for the distance function d_K to the center of H_K.

Corollary 3.19. *For all $u \in C_0^\infty(\Omega_R \setminus Z)$, and all $\lambda \in \mathbb{R}$, there holds*

$$\int_{\Omega_R} d_K^{2-2N}|\nabla_H u(x)|^2\,dx - \frac{1}{4}\int_{\Omega_R} \frac{|u(x)|^2}{d_Z(x)^{2N}|\log\frac{R}{d_K(x)}|^2}\,dx = \int_{\Omega_R} \frac{1}{d_K(x)^{2N-2}}\Big|\nabla\Big(\frac{u(x)}{\sqrt{\log\frac{R}{|x|}}}\Big)\Big|^2\log\frac{R}{d_K(x)}\,dx.$$

Proof. The identity follows from applying Corollary 3.18 to the Bessel pair

$$(r^{2N-1}V, r^{2N-1}W) = \Big(r, \frac{1}{4r|\log\frac{r}{R}|^2}\Big),$$

i.e., with

$$(V, W) = \Big(r^{2-2N}, \frac{1}{4r^{2N}|\log\frac{r}{R}|^2}\Big)$$

with solution $\varphi = \sqrt{|\log\frac{r}{R}|}$. ☐

3.5 Hardy Identities for Nilpotent Groups and Hörmander Vector fields

In this subsection we state the analogous results for Theorem 3.2 in the more general setting of simply connected nilpotent groups and homogeneous Hörmander vector fields. The main point here is that the appropriate sub-Laplacians in these settings have a global fundamental solutions whose powers generalize the homogeneous gauges of Carnot groups. Thus, first consider a simply connected nilpotent Lie group G, where we specify a system of left invariant vector fields $X_1, \ldots, X_m$ which bracket generate the Lie algebra of G. Indeed, as was shown by Nagel, Ricci and Stein in [25], the sub-Laplacian $\mathcal{L} = \sum_{j=1}^m X_j^2$ has a global fundamental solution $\Gamma(x, y)$. Then, by taking $d_X(x) = \Gamma(x, 0)^{2-q}$ for some q, we find

$$\mathcal{L}(d_X) = |\nabla_H d_X|^2 \frac{q-1}{d_X},$$

where $\nabla_H = (X_1, \dots, X_m)$ is the horizontal gradient corresponding to the system $X_1, \dots, X_m$.

Similarly, if $X = \{X_1, \dots, X_m\}$ is a system of Hörmander vector fields homogeneous of degree -1, then, by [2], the Hörmander operator $\mathcal{L} = \sum_{j=1}^m X_j^2$ admits a fundamental solution $\Gamma_X(x, y)$. Moreover, we also define $d_X(x) = \Gamma_X(x, 0)^{\frac{1}{2-q}}$ and observe

$$\mathcal{L}(d_X) = |\nabla_H d_X|^2 \frac{q-1}{d_X}$$

also holds.

As was defined for general Hörmander vector fields in the preliminaries, we shall take for a suitable domain the sets

$$B_{X,R}(0) = \{x \in \mathbb{R}^N : d_X(x) < R\}.$$

We now state the two main Hardy identities in this setting. Of course the Bessel pairs used in the previous section may be used in the following theorem, and so the analogous results also hold in this setting.

Theorem 3.20. *Let $X_1, \dots, X_m$ denote either a system of homogeneous Hörmander vector fields on $\mathbb{R}^N$, or a bracket generating collection of left invariant vector fields on a simply connected nilpotent group. Next, let $d_X(x) = \Gamma_X(x, 0)^{\frac{1}{2-q}}$, where $\Gamma_X(x, y)$ is the fundamental solution of $\mathcal{L} = \sum_{j=1}^m X_j^2$, and let $B_{X,R}(0) = \{d_X < R\}$. Lastly, let V and W be positive C^1-functions on $(0, R)$. If $(r^{Q-1}V, r^{Q-1}W)$ is a Bessel pair on $(0, R)$ with positive solution φ, then for $u \in C_0^\infty(B_{X,R} \setminus \{0\})$ there holds*

$$\int\limits_{B_{X,R}(0)} V(d_X) \left|\nabla_H \left(\frac{u}{\varphi(d_X)}\right)\right|^2 \varphi(d_X)^2 \, dx = \int\limits_{B_{X,R}(0)} V(d_X)|\nabla_H u|^2 \, dx - \int\limits_{B_{X,R}(0)} W(d_X) u^2 |\nabla_H d_X|^2 \, dx. \tag{3.9}$$

In particular,

$$\int\limits_{B_{X,R}(0)} V(d_X)|\nabla_H u|^2 \, dx \geq \int\limits_{B_{X,R}(0)} W(d_X)|u|^2 |\nabla_H d|^2 \, dx.$$

Proof. The proof is identical to the proof of Theorem 3.2. □

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