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Research Article

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Non-Degeneracy of Peak Solutions to the Schrödinger–Newton System

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Abstract: We are concerned with the following Schrödinger–Newton problem:

$$-\varepsilon^2\Delta u + V(x)u = \frac{1}{8\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3}\frac{u^2(\xi)}{|x-\xi|}\,d\xi\bigg)u, \quad x\in\mathbb{R}^3.$$

For ε small enough, we prove the non-degeneracy of the positive solution to the above problem, that is, the corresponding linear operator

$$\mathcal{L}_\varepsilon(\eta) = -\varepsilon^2\Delta\eta(x) + V(x)\eta(x) - \frac{1}{8\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3}\frac{u_\varepsilon^2(\xi)}{|x-\xi|}\,d\xi\bigg)\eta(x) - \frac{1}{4\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3}\frac{u_\varepsilon(\xi)\eta(\xi)}{|x-\xi|}\,d\xi\bigg)u_\varepsilon(x)$$

is non-degenerate, i.e., $\mathcal{L}_\varepsilon(\eta_\varepsilon) = 0 \Rightarrow \eta_\varepsilon = 0$ for small $\varepsilon > 0$. The main tools are the local Pohozaev identities and the blow-up analysis. This may be the first non-degeneracy result on the peak solutions to the Schrödinger–Newton system.

Keywords: Schrödinger–Newton System, Non-Degeneracy, Pohozaev Identity

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1 Introduction and Main Results

The Schrödinger–Newton system describing the quantum mechanics of a polaron at rest,

$$\begin{cases} \dfrac{\varepsilon^2}{2m}\Delta u - V(x)u + \psi u = 0, & x\in\mathbb{R}^3,\\ \Delta\psi + 4\pi\tau|u|^2 = 0, & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

was derived by R. Penrose [13] as a model of self-gravitating matter, in which quantum state reduction is understood as a gravitational phenomenon. Here, the interacts with a matter density is given by the square of the wave function u , which is the solution of the Schrödinger equation. In addition, an electric field is generated by a potential $V(x)$. In (1.1), ψ is the gravitational potential, ε is the Planck constant, $\tau = Gm^2$ and G is the Newton's constant of gravitation.

Let

$$u(x) \mapsto \frac{u}{4\varepsilon\sqrt{\pi\tau m}}, \quad V(x) \mapsto \frac{1}{2m}V(x), \quad \psi(x) \mapsto \frac{1}{2m}\psi(x).$$

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Then system (1.1) can be written, maintaining the original notations, as

$$\begin{cases} \varepsilon^2\Delta u - V(x)u + \psi u = 0, & x \in \mathbb{R}^3, \\ \varepsilon^2\Delta\psi + \dfrac{|u|^2}{2} = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

The second equation in (1.2) can be solved explicitly with respect to ψ, so that the system can be turned into the following single nonlocal equation:

$$-\varepsilon^2\Delta u + V(x)u = \frac{1}{8\pi\varepsilon^2}\Big(\int_{\mathbb{R}^3} \frac{u^2(\xi)}{|x-\xi|}\,d\xi\Big)u, \quad x \in \mathbb{R}^3, \tag{1.3}$$

which has been investigated extensively. Especially, when $\varepsilon = 1$ and $V(x) = 1$, the existence and uniqueness of the ground states for (1.3) was obtained by variational methods [7, 9, 12], while the non-degeneracy was proved in [15, 17].

Theorem A (cf. [8, 17]). *For any fixed $a \in \mathbb{R}^3$ satisfying $V(a) > 0$, there exists a unique radial solution U_a of the problem*

$$\begin{cases} -\Delta u + V(a)u = \dfrac{1}{8\pi}\Big(\displaystyle\int_{\mathbb{R}^3} \frac{u^2(\xi)}{|x-\xi|}\,d\xi\Big)u & \text{in } \mathbb{R}^3, \\ u(x) > 0 \quad \text{in } \mathbb{R}^3, \qquad u(0) = \max\limits_{x\in\mathbb{R}^3} u(x). \end{cases}$$

The solution U_a is strictly decreasing and

$$\lim_{|x|\to\infty} U_a(x)e^{|x|}|x| = \lambda_0 > 0, \qquad \lim_{|x|\to\infty} \frac{U_a'(x)}{U_a(x)} = -1$$

for some constant $\lambda_0 > 0$. Moreover, if $\phi(x) \in H^1(\mathbb{R}^3)$ solves the linearized equation

$$-\Delta\phi(x) + V(a)\phi(x) = \frac{1}{8\pi}\Big(\int_{\mathbb{R}^3} \frac{U_a^2(\xi)}{|x-\xi|}\,d\xi\Big)\phi(x) + \frac{1}{4\pi}\Big(\int_{\mathbb{R}^3} \frac{U_a(\xi)\phi(\xi)}{|x-\xi|}\,d\xi\Big)U_a(x),$$

then $\phi(x)$ is a linear combination of $\partial U_a/\partial x_j$, $j = 1, 2, 3$.

If ε is small and $V(x)$ is not a constant, the existence of solutions with ground states for (1.3) was proved by [10]. Then Wei and Winter [17] considered the existence of multiple solutions concentrating at k points of the local minimum points of $V(x)$. We also refer to [3, 14, 16] and the references therein for the existence of solutions with concentration in other cases. The uniqueness result of concentrating solutions can be found in [11], by using local Pohozaev type of identity and blow-up analysis, which was recently developed in [1, 4, 6].

Theorem B (cf. [17]). *Suppose that $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$ are non-degenerate critical points of $V(x)$. There exists a positive solution $\{u_\varepsilon\}_{\varepsilon>0}$ concentrating at $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$, i.e., there exist $\{x_{i,\varepsilon}\}_{\varepsilon>0} \subset \mathbb{R}^3$ such that*

$$u_\varepsilon = \sum_{i=1}^{k} U_i\Big(\frac{x - x_{i,\varepsilon}}{\varepsilon}\Big) + w_\varepsilon, \tag{1.4}$$

where $|x_{i,\varepsilon} - a_i| = o(1)$ for $i = 1, \dots, k$, and $\|w_\varepsilon\|_\varepsilon = o(\varepsilon^{\frac{3}{2}})$.

It is well known that the non-degeneracy of the solutions is of fundamental importance when dealing with the orbital stability or instability result of the corresponding time-dependent equations. Especially, when $V(x)$ is not a constant, apart from the existence and uniqueness results, whether the positive solution is non-degenerate is still unknown.

We assume that $V(x)$ is a bounded C^1 function satisfying $\inf_{x\in\mathbb{R}^3} V(x) > 0$. Define the following Sobolev space H_ε:

$$H_\varepsilon := \Big\{u(x) \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x))\,dx < \infty\Big\},$$

and the corresponding norm

$$\|u\|_\varepsilon = (u(x), u(x))_\varepsilon^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x))\, dx \Big)^{\frac{1}{2}}.$$

For any $\eta \in H^1(\mathbb{R}^3)$, we define

$$\mathcal{L}_\varepsilon(\eta) = -\varepsilon^2 \Delta \eta(x) + V(x)\eta(x) - \frac{1}{8\pi\varepsilon^2} \Big(\int_{\mathbb{R}^3} \frac{u_\varepsilon^2(\xi)}{|x-\xi|}\, d\xi \Big) \eta(x) - \frac{1}{4\pi\varepsilon^2} \Big(\int_{\mathbb{R}^3} \frac{u_\varepsilon(\xi)\eta(\xi)}{|x-\xi|}\, d\xi \Big) u_\varepsilon(x),$$

where u_ε is obtained by Theorem B.

Our main results are as follows.

Theorem 1.1. *Suppose that* $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$ *($k \geq 1$) are non-degenerate critical points of* $V(x)$*. Let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be a positive solution to* (1.3) *concentrating at* $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$*. If* η_ε *is a solution to* $\mathcal{L}_\varepsilon(\eta_\varepsilon) = 0$*, then* $\eta_\varepsilon = 0$ *for small* $\varepsilon > 0$.

Inspired by Guo, Musso, Peng and Yan [5], we apply local Pohozaev identity and the blow-up analysis to obtain $\eta_\varepsilon(x) = o(1)$ near the non-degenerate critical points. Especially, we point out that, distinct from the classical Schrödinger equations, the corresponding local Pohozaev identity of the Schrödinger–Newton system would have two terms involving volume integral due to the nonlocal term. Moreover, the asymptotic behavior of the concentrating points to Schrödinger–Newton problem is quite different from that of the classical Schrödinger equation. Hence, we should deal with the non-degeneracy of the single-peak and multi-peak solutions separately.

Organization of the Paper. In Section 2, we obtain some estimates needed in the proof of Theorem 1.1, especially including the local Pohozaev identities. The main result on the non-degeneracy of the one-peak solutions will be proved in Section 3, while the result of multi-peak solutions will be obtained in Section 4.

2 The Basic Estimates

We first recall the following known results.

Proposition 2.1 ([11]). *Suppose that* $u_\varepsilon(x)$ *is a positive solution of equation* (1.3) *concentrating at different points* $a_1, \dots, a_k$ *with* $k \geq 1$*. Then, for any fixed* $R \gg 1$*, there exist* $\theta > 0$ *and* $C > 0$ *such that*

$$u_\varepsilon(x) \leq C e^{-\frac{\theta|x - x_{l,\varepsilon}|}{\varepsilon}} \quad \text{for } l = 1, \dots, k \text{ and } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}),$$

$$|\nabla u_\varepsilon(x)| \leq C e^{-\frac{\theta R}{\varepsilon}} \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}).$$

Corollary 2.2. *Suppose that* $u_\varepsilon(x)$ *is a solution of equation* (1.3) *as in Proposition 2.1. Then the followings statements hold:*

(1) *For any fixed* $R \gg 1$*, there exists* $\theta_1 > 0$ *such that*

$$u_\varepsilon(x), |\nabla u_\varepsilon(x)| = O(e^{-\theta_1 R}) \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}). \tag{2.1}$$

(2) *For any fixed* $d > 0$*, there exists* $\theta_2 > 0$ *such that*

$$u_\varepsilon(x), |\nabla u_\varepsilon(x)| = O(e^{-\frac{\theta_2}{\varepsilon}}) \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{d}(x_{j,\varepsilon}). \tag{2.2}$$

Proposition 2.3. *Let $u(x)$ be the solution of* (1.3), $\mathcal{L}_\varepsilon(\eta) = 0$. *Then we have the following* local Pohozaev identities:

$$\int_\Omega \frac{\partial V(x)}{\partial x^i} u^2(x)\,dx = -2\varepsilon^2 \int_{\partial\Omega} \frac{\partial u(x)}{\partial \nu}\frac{\partial u(x)}{\partial x_i}\,d\sigma + \int_{\partial\Omega} (\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x))\nu_i(x)\,d\sigma \\ - \frac{1}{8\pi\varepsilon^2}\int_{\partial\Omega}\int_{\mathbb{R}^3} \frac{u^2(\xi)u^2(x)}{|x-\xi|}\nu_i(x)\,d\xi\,d\sigma + \frac{1}{8\pi\varepsilon^2}\int_\Omega\int_{\mathbb{R}^3} u^2(\xi)u^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\,d\xi\,dx \tag{2.3}$$

and

$$\int_\Omega \frac{\partial V(x)}{\partial x^i} u(x)\eta(x)\,dx = -\varepsilon^2 \int_{\partial\Omega} \frac{\partial u(x)}{\partial \nu}\frac{\partial \eta(x)}{\partial x_i} + \frac{\partial \eta(x)}{\partial \nu}\frac{\partial u(x)}{\partial x_i}\,d\sigma \\ + \int_{\partial\Omega} (\varepsilon^2\langle\nabla u(x),\nabla\eta(x)\rangle + V(x)u(x)\eta(x))\nu_i(x)\,d\sigma \\ - \frac{1}{8\pi\varepsilon^2}\int_{\partial\Omega}\int_{\mathbb{R}^3}\Big(\frac{u^2(\xi)u(x)\eta(x)}{|x-\xi|} + \frac{u^2(x)u(\xi)\eta(\xi)}{|x-\xi|}\Big)\nu_i(x)\,d\xi\,d\sigma \\ - \frac{1}{8\pi\varepsilon^2}\int_\Omega\int_{\mathbb{R}^3}\Big(u^2(\xi)u(x)\eta(x)\frac{x^i-\xi^i}{|x-\xi|^3} + u(\xi)\eta(\xi)u^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\Big)\,d\xi\,dx, \tag{2.4}$$

where Ω is a bounded open domain of $\mathbb{R}^3$, $i = 1,2,3$, $\nu(x) = (\nu_1(x),\nu_2(x),\nu_3(x))$ is the outward unit normal of $\partial\Omega$ and x^i, ξ^i are the i-th components of x, ξ.

Proof. Identity (2.3) is obtained by multiplying $\frac{\partial u(x)}{\partial x^i}$ on both sides of (1.3) and integrating on Ω. While the identity in (2.4) is obtained by multiplying $\frac{\partial \eta(x)}{\partial x^i}$ and $\frac{\partial u(x)}{\partial x^i}$ on both sides of (1.3) and $\mathcal{L}_\varepsilon(\eta) = 0$, respectively, and integrating on Ω. We omit the details. □

Let

$$F_1(x) = \frac{1}{8\pi\varepsilon^2}\int_{\mathbb{R}^3}\frac{u_\varepsilon^2(\xi)}{|x-\xi|}\,d\xi, \qquad F_2(x) = \frac{1}{4\pi\varepsilon^2}\int_{\mathbb{R}^3}\frac{u_\varepsilon(x)u_\varepsilon(\xi)}{|x-\xi|}\eta_\varepsilon(\xi)\,d\xi. \tag{2.5}$$

Proposition 2.4. *For $\eta_\varepsilon(x)$ satisfying $\mathcal{L}_\varepsilon(\eta) = 0$, we have*

$$\|\eta_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{3}{2}}). \tag{2.6}$$

Proof. From

$$-\varepsilon^2\Delta\eta(x) + V(x)\eta(x) - F_1(x)\eta(x) - F_2(x) = 0,$$

we have

$$\|\eta_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^3} F_1(x)\eta_\varepsilon^2(x)\,dx + \int_{\mathbb{R}^3} F_2(x)\eta_\varepsilon(x)\,dx. \tag{2.7}$$

Next, by the Hardy–Littlewood–Sobolev inequality, Hölder's inequality and the fact $|\eta_\varepsilon(x)| \le 1$, we know

$$\Big|\int_{\mathbb{R}^3} F_1(x)\eta_\varepsilon^2(x)\,dx\Big| \le C\varepsilon^{-2}\Big(\int_{\mathbb{R}^3}|u_\varepsilon(\xi)|^{\frac{12}{5}}\,d\xi\Big)^{\frac{5}{6}}\cdot\Big(\int_{\mathbb{R}^3}|\eta_\varepsilon(x)|^{\frac{12}{5}}\,dx\Big)^{\frac{5}{6}} \\ \le C\varepsilon^{-2}\Big(\int_{\mathbb{R}^3}|u_\varepsilon(\xi)|^{\frac{12}{5}}\,d\xi\Big)^{\frac{5}{6}}\cdot\Big(\int_{\mathbb{R}^3}|\eta_\varepsilon(x)|^2\,dx\Big)^{\frac{5}{6}} \\ \le C\varepsilon^{\frac{1}{2}}\|u_\varepsilon\|_\varepsilon^2\|\eta_\varepsilon\|_\varepsilon^{\frac{5}{3}} \le C\varepsilon^{-\frac{5}{2}}\|\eta_\varepsilon\|_\varepsilon^{\frac{5}{3}} \le C\varepsilon^3 + \frac{1}{4}\|\eta_\varepsilon\|_\varepsilon^2 \tag{2.8}$$

and

$$\Big|\int_{\mathbb{R}^3} F_2(x)\eta_\varepsilon(x)\,dx\Big| \le C\varepsilon^3 + \frac{1}{4}\|\eta_\varepsilon\|_\varepsilon^2. \tag{2.9}$$

Then (2.7), (2.8) and (2.9) imply (2.6). □

Proposition 2.5. *Suppose that* $u_\varepsilon(x) = \sum_{l=1}^k U_{a_l}(\frac{x-x_{l,\varepsilon}}{\varepsilon}) + w_\varepsilon(x)$ *is a positive solution of equation* (1.3) *and that* $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$ *are the different non-degenerate critical points of* $V(x)$ *with* $k \geq 1$. *Then it holds*

$$\|w_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{7}{2}}) + O\Big(\varepsilon^{\frac{3}{2}} \max_{j=1,\dots,k} |x_{j,\varepsilon} - a_j|^2\Big). \tag{2.10}$$

Proof. We postpone the proof to the Appendix. □

3 Proof of Theorem 1.1 with $k = 1$

When $k = 1$, we have the following modified estimate.

Proposition 3.1. *Let* $u_\varepsilon(x)$ *be the solution of* (1.3) *concentrating at a non-degenerate critical point* $a_1 \in \mathbb{R}^3$ *of* $V(x)$. *Then it holds*

$$|x_{1,\varepsilon} - a_1| = o(\varepsilon). \tag{3.1}$$

Proof. This result can be found in [11], but we sketch the proof for being self-enclosed. First, for the small fixed constant $\bar{d} > 0$, taking $u(x) = u_\varepsilon(x)$ and $\Omega = B_d(x_{1,\varepsilon})$ in the Pohozaev identity (2.3) with any $d \in (\bar{d}, 2\bar{d})$, we have, for $i = 1, 2, 3$,

$$\int\limits_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} u_\varepsilon^2(x)\,dx = \int\limits_{\partial B_d(x_{1,\varepsilon})} B(x)\,d\sigma + \frac{1}{8\pi\varepsilon^2} \int\limits_{B_d(x_{1,\varepsilon})} \int\limits_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\,d\xi\,dx, \tag{3.2}$$

where

$$B(x) = -2\varepsilon^2 \frac{\partial u_\varepsilon(x)}{\partial \nu} \frac{\partial u_\varepsilon(x)}{\partial x_i} + \bigg(\varepsilon^2 |\nabla u_\varepsilon(x)|^2 + u_\varepsilon^2(x)\bigg(V(x) - \frac{1}{8\pi\varepsilon^2} \int\limits_{\mathbb{R}^3} \frac{u_\varepsilon^2(\xi)}{|x-\xi|}\,d\xi\bigg)\bigg)\nu_i(x).$$

Next, since

$$\frac{\partial V(x)}{\partial x^i} = \frac{\partial V(x)}{\partial x^i} - \frac{\partial V(a_j)}{\partial x^i} = \sum_{l=1}^3 (x^l - a_j^l) \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} + o(|x - a_j|) \quad \text{for } i = 1, 2, 3,$$

for any $d \in (\bar{d}, 2\bar{d})$, (1.4) gives

$$\begin{aligned}\int\limits_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} u_\varepsilon^2(x)\,dx &= \int\limits_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} U_{a_1}^2\Big(\frac{x - x_{1,\varepsilon}}{\varepsilon}\Big)\,dx + o(\varepsilon^4 + \varepsilon^3 |x_{1,\varepsilon} - a_1|) \\ &= \varepsilon^3 \Big(\int\limits_{\mathbb{R}^3} U_{a_1}^2(x)\,dx\Big) \sum_{l=1}^3 \frac{\partial^2 V(a_1)}{\partial x^i \partial x^l}(x_{1,\varepsilon}^l - a_1^l) + o(\varepsilon^4 + \varepsilon^3 |x_{1,\varepsilon} - a_1|),\end{aligned} \tag{3.3}$$

where $x_{1,\varepsilon}^l$, a_1^l are the l-th components of $x_{1,\varepsilon}$, a_1. On the other hand, for any fixed d, we have

$$\bigg|\int\limits_{\partial B_d(x_{1,\varepsilon})} \varepsilon^2 \frac{\partial u_\varepsilon(x)}{\partial \nu} \frac{\partial u_\varepsilon(x)}{\partial x_i}\,d\sigma\bigg| \leq C\varepsilon^2 \int\limits_{\partial B_d(x_{1,\varepsilon})} |\nabla u_\varepsilon(x)|^2\,d\sigma.$$

Then from (A.4), for any fixed d, we find

$$\bigg|\int\limits_{\partial B_d(x_{1,\varepsilon})} B(x)\,d\sigma\bigg| \leq C \int\limits_{\partial B_d(x_{1,\varepsilon})} [\varepsilon^2 |\nabla u_\varepsilon(x)|^2 + (u_\varepsilon(x))^2]\,d\sigma.$$

So using (2.2), (A.1) and (2.10), there exists $d_\varepsilon \in (\bar{d}, 2\bar{d})$ such that

$$\int\limits_{\partial B_{d_\varepsilon}(x_{1,\varepsilon})} B(x)\,d\sigma = O(e^{-\frac{\eta}{\varepsilon}} + \|w_\varepsilon\|_\varepsilon^2) = O(\varepsilon^7 + \varepsilon^3 |x_{1,\varepsilon} - a_1|^4). \tag{3.4}$$

Also for any $d \in (\bar{d}, 2\bar{d})$, by symmetry and (2.2), we deduce

$$\frac{1}{8\pi\varepsilon^2} \int\limits_{B_d(x_{1,\varepsilon})} \int\limits_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\,d\xi\,dx = \frac{1}{8\pi\varepsilon^2} \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\,d\xi\,dx + O(e^{-\frac{\eta}{\varepsilon}}) = O(e^{-\frac{\eta}{\varepsilon}}). \tag{3.5}$$

Let $d = d_\varepsilon$ in (3.2). Then (3.3), (3.4) and (3.5) imply

$$\sum_{l=1}^{3} \frac{\partial^2 V(a_1)}{\partial x^i \partial x^l}(x_{1,\varepsilon}^l - a_1^l) = o(|x_{1,\varepsilon} - a_1|) + o(\varepsilon),$$

which gives (3.1). □

Next, we prove Theorem 1.1 by contradiction. Suppose that there exists $\varepsilon_m \to 0$ satisfying

$$\|\eta_{\varepsilon_m}\|_{L^\infty} = 1, \quad \mathcal{L}_{\varepsilon_m}\eta_{\varepsilon_m} = 0.$$

For simplicity, we will omit the subscript m, replacing ε_m by ε.

Lemma 3.2. *Let $\eta_{1,\varepsilon}(x) = \eta_\varepsilon(\varepsilon x + x_{1,\varepsilon})$. By taking a subsequence, if necessary, it holds*

$$\eta_{1,\varepsilon}(x) \to \sum_{i=1}^{3} a_{1,i} \frac{\partial U_{a_1}(x)}{\partial x^i} \tag{3.6}$$

uniformly in $C^1(B_R(0))$ for any $R > 0$, where $\eta_\varepsilon(x)$ is the solution to $\mathcal{L}_\varepsilon(\eta) = 0$, and $a_{1,i}$ ($i = 1, 2, 3$) are some constants.

Proof. Since $\|\eta_{1,\varepsilon}\|_{L^\infty(\mathbb{R}^3)} \le 1$, by the regularity theory, we know

$$\eta_{1,\varepsilon}(x) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad \|\eta_{1,\varepsilon}\|_{C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)} \le C \quad \text{for some } \alpha \in (0,1).$$

So we may assume that

$$\eta_{1,\varepsilon}(x) \to \eta_1(x) \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^3).$$

By direct calculations, we have

$$-\Delta\eta_{1,\varepsilon}(x) = -V(\varepsilon x + x_{1,\varepsilon})\eta_{1,\varepsilon}(x) + A_1(\varepsilon x + x_{1,\varepsilon})\eta_{1,\varepsilon}(x) + A_2(\varepsilon x + x_{1,\varepsilon}).$$

Since from (A.4) and (A.5), we have

$$A_1(\varepsilon x + x_{1,\varepsilon}) = \frac{1}{8\pi}\bigg(\int_{\mathbb{R}^3} \frac{U_{a_1}^2(\xi)}{|x-\xi|}\,d\xi\bigg) + o(1), \quad x \in B_{\frac{d}{\varepsilon}}(0),$$

and

$$A_2(\varepsilon x + x_{1,\varepsilon}) = \frac{U_{a_1}(x)}{4\pi}\bigg(\int_{\mathbb{R}^3} \frac{U_{a_1}(\xi)\eta_{1,\varepsilon}(\xi)}{|x-\xi|}\,d\xi\bigg) + o(1), \quad x \in B_{\frac{d}{\varepsilon}}(0).$$

Next, for any given $\Phi(x) \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (-\Delta\eta_{1,\varepsilon}(x) + V(\varepsilon x + x_{1,\varepsilon})\eta_{1,\varepsilon}(x))\Phi(x)\,dx - \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U_{a_1}^2(\xi)}{|x-\xi|}\eta_{1,\varepsilon}(x)\Phi(x)\,d\xi\,dx \\ &\quad - \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U_{a_1}(\xi)\eta_{1,\varepsilon}(\xi)}{|x-\xi|} U_{a_1}(x)\Phi(x)\,d\xi\,dx = o(1)\|\Phi\|_{H^1(\mathbb{R}^3)}. \end{aligned} \tag{3.7}$$

Letting $\varepsilon \to 0$ in (3.7) and using the elliptic regularity theory, we find that $\eta_1(x)$ satisfies

$$-\Delta\eta_1(x) + V(a_1)\eta_1(x) = \frac{1}{8\pi}\bigg(\int_{\mathbb{R}^3} \frac{U_{a_1}^2(\xi)}{|x-\xi|}\,d\xi\bigg)\eta_1(x) + \frac{1}{4\pi}\bigg(\int_{\mathbb{R}^3} \frac{U_{a_1}(\xi)\eta_1(\xi)}{|x-\xi|}\,d\xi\bigg)U_{a_1}(x) \quad \text{in } \mathbb{R}^3.$$

By Theorem A, it holds that $\eta_1(x) = \sum_{i=1}^3 a_{1,i}\frac{\partial U_{a_1}(x)}{\partial x^i}$, which means (3.6). □

Lemma 3.3. *Let $a_{1,i}$ be as in Lemma 3.2. Then we have*

$$a_{1,i} = 0 \quad \textit{for all } i = 1, 2, 3.$$

Proof. Recall the Pohozaev identity (2.4). First, by use of Corollary 2.2, for some $\gamma > 0$ it holds that

$$
\begin{aligned}
&-\varepsilon^2 \int_{\partial B_d(x_{1,\varepsilon})} \frac{\partial u_\varepsilon(x)}{\partial \nu}\frac{\partial \eta_\varepsilon(x)}{\partial x_i} + \frac{\partial \eta_\varepsilon(x)}{\partial \nu}\frac{\partial u_\varepsilon(x)}{\partial x_i}\, d\sigma \\
&\quad + \int_{\partial B_d(x_{1,\varepsilon})} (\varepsilon^2 \langle \nabla u_\varepsilon(x), \nabla \eta_\varepsilon(x)\rangle + V(x) u_\varepsilon(x)\eta_\varepsilon(x))\nu_i(x)\, d\sigma = O(e^{-\frac{\gamma}{\varepsilon}}).
\end{aligned}
$$

Moreover, combined with Hardy–Littlewood–Sobolev inequality, we know that

$$
-\frac{1}{8\pi\varepsilon^2}\int_{\partial B_d(x_{1,\varepsilon})}\int_{\mathbb{R}^3}\Big(\frac{u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)}{|x-\xi|} + \frac{u_\varepsilon^2(x)u_\varepsilon(\xi)\eta_\varepsilon(\xi)}{|x-\xi|}\Big)\nu_i(x)\, d\xi\, d\sigma = O(e^{-\frac{\gamma}{\varepsilon}}).
$$

By symmetry,

$$
\begin{aligned}
&-\frac{1}{8\pi\varepsilon^2}\int_{B_d(x_{1,\varepsilon})}\int_{\mathbb{R}^3}\Big(u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)\frac{x^i-\xi^i}{|x-\xi|^3} + u_\varepsilon(\xi)\eta_\varepsilon(\xi)u_\varepsilon^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\Big)\, d\xi\, dx \\
&= \frac{1}{8\pi\varepsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)\frac{x^i-\xi^i}{|x-\xi|^3} + u_\varepsilon(\xi)\eta_\varepsilon(\xi)u_\varepsilon^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\Big)\, d\xi\, dx + O(e^{-\frac{\gamma}{\varepsilon}}) \\
&= O(e^{-\frac{\gamma}{\varepsilon}}).
\end{aligned}
$$

To sum up,

$$
\text{RHS of (2.4)} = O(e^{-\frac{\gamma}{\varepsilon}}).
$$

On the other hand,

$$
\begin{aligned}
\int_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} u_\varepsilon(x)\eta_\varepsilon(x)\, dx &= \sum_{j=1}^{3}\int_{B_d(x_{1,\varepsilon})} \frac{\partial^2 V(a_1)}{\partial x^i \partial x^j}(x^j - a_1^j)u_\varepsilon(x)\eta_\varepsilon(x)\, dx \\
&\quad + o\Big(\int_{B_d(x_{1,\varepsilon})} |x-a_1| u_\varepsilon(x)\eta_\varepsilon(x)\, dx\Big).
\end{aligned}
$$

We estimate

$$
\begin{aligned}
\int_{B_d(x_{1,\varepsilon})} (x^j - a_1^j)u_\varepsilon(x)\eta_\varepsilon(x)\, dx &= \varepsilon^4\Big(\int_{B_{\frac{d}{\varepsilon}}(0)} \Big(x^j + \frac{x_{1,\varepsilon}^j - a_1^j}{\varepsilon}\Big)U_{a_1}(x)\Big(\sum_{i=1}^{3} a_{1,i}\frac{\partial U_{a_1}(x)}{\partial x^i} + o(1)\Big)\, dx\Big) \\
&= a_{1,j}\varepsilon^4 \int_{\mathbb{R}^3} x^j U_{a_1}(x)\frac{\partial U_{a_1}(x)}{\partial x^j}\, dx + o(\varepsilon^4) \\
&= -\frac{a_{1,j}}{2}\varepsilon^4 \int_{\mathbb{R}^3} U_{a_1}^2(x)\, dx + o(\varepsilon^4).
\end{aligned}
$$

Hence

$$
\sum_{j=1}^{3}\int_{B_d(x_{1,\varepsilon})} \frac{\partial^2 V(a_1)}{\partial x^i \partial x^j} a_{1,j}\, dx = o(1).
$$

Since, by assumption, a_1 is a non-degenerate critical point of the potential $V(x)$, we find that $a_{1,j} = 0$ for $j = 1, 2, 3$. □

Lemma 3.4. *For any fixed $R > 0$, it holds*

$$
\eta_\varepsilon(x) = o(1), \quad x \in B_{R\varepsilon}(x_{1,\varepsilon}).
$$

Proof. Lemma 3.2 and Lemma 3.3 show that for any fixed $R > 0$, one has $\eta_{1,\varepsilon}(x) = o(1)$ in $B_R(0)$. Also, we know $\eta_{1,\varepsilon}(x) = \eta_\varepsilon(\varepsilon x + x_{1,\varepsilon})$. Then $\eta_\varepsilon(x) = o(1)$, $x \in B_{R\varepsilon}(x_{1,\varepsilon})$. □

Similar to Proposition 2.1 with $k = 1$, we have the following estimate.

Lemma 3.5. *For large $R > 0$ there exist $\theta > 0$ and $C > 0$ such that*

$$\eta_\varepsilon(x) \le Ce^{-\frac{\theta|x-x_{1,\varepsilon}|}{\varepsilon}} \quad \text{for } l = 1, \dots, k \text{ and } x \in \mathbb{R}^3 \setminus B_{R\varepsilon}(x_{1,\varepsilon}),$$
$$|\nabla\eta_\varepsilon(x)| \le Ce^{-\frac{\theta R}{\varepsilon}} \quad \text{for } x \in \mathbb{R}^3 \setminus B_{R\varepsilon}(x_{1,\varepsilon}).$$

Proof of Theorem 1.1 with $k = 1$. By contradiction, we suppose that $\|\eta_\varepsilon\|_{L^\infty}$ and $\mathcal{L}_\varepsilon\eta_\varepsilon = 0$. From Lemma 3.4 and Lemma 3.5 one has $\eta_\varepsilon(x) = o(1)$ for all $x \in \mathbb{R}^3$, which contradicts with $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. As a result, we obtain that $\eta_\varepsilon = 0$ for ε small enough. □

4 Proof of Theorem 1.1 with $k > 1$

The case of $k \ge 2$ is distinct from that of $k = 1$, which can be seen from the following known result.

Lemma 4.1 ([11]). *Let $u_\varepsilon(x)$ be the solution of (1.3) concentrating at k, $k \ge 2$, different non-degenerate critical points $\{a_1, \dots, a_k\} \subset \mathbb{R}^3$ of $V(x)$. Then it holds*

$$|x_{j,\varepsilon} - a_j| = O(\varepsilon) \quad \text{for } j = 1, 2, \dots, k.$$

Furthermore, there exist $j_0 \in \{1, \dots, k\}$, $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\varepsilon \le |x_{j_0,\varepsilon} - a_{j_0}| \le C_2\varepsilon.$$

Lemma 4.2. *Let $\eta_{j,\varepsilon}(x) = \eta_\varepsilon(\varepsilon x + x_{j,\varepsilon})$ for $j = 1, 2, \dots, k$ and $k \ge 2$. Then, by taking a subsequence if necessary, it holds*

$$\eta_{j,\varepsilon}(x) \to \sum_{i=1}^{3} a_{j,i}\frac{\partial U_{a_j}(x)}{\partial x^i}$$

uniformly in $C^1(B_R(0))$ for any $R > 0$, where $a_{j,i}$, $i = 1, 2, 3$, are some constants.

Proof. Following the proof of Lemma 3.2, we can obtain Lemma 4.2 similarly. □

Lemma 4.3. *Let $a_{j,i}$ be as in Proposition 4.2. Then we have $a_{j,i} = 0$ for all $j = 1, \dots, k$, $i = 1, 2, 3$.*

Proof. Similar as in the proof of Lemma 3.3, we also recall Pohozaev identity (2.4). By use of Corollary 2.2 and the Hardy–Littlewood–Sobolev inequality, for some $\gamma > 0$ it holds that

$$-\varepsilon^2 \int_{\partial B_d(x_{j,\varepsilon})} \frac{\partial u_\varepsilon(x)}{\partial \nu}\frac{\partial \eta_\varepsilon(x)}{\partial x_i} + \frac{\partial \eta_\varepsilon(x)}{\partial \nu}\frac{\partial u_\varepsilon(x)}{\partial x_i}\,d\sigma + \int_{\partial B_d(x_{j,\varepsilon})} (\varepsilon^2\langle\nabla u_\varepsilon(x), \nabla\eta_\varepsilon(x)\rangle + V(x)u_\varepsilon(x)\eta_\varepsilon(x))\nu_i(x)\,d\sigma$$
$$- \frac{1}{8\pi\varepsilon^2}\int_{\partial B_d(x_{j,\varepsilon})}\int_{\mathbb{R}^3}\Big(\frac{u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)}{|x-\xi|} + \frac{u_\varepsilon^2(x)u_\varepsilon(\xi)\eta_\varepsilon(\xi)}{|x-\xi|}\Big)\nu_i(x)\,d\xi\,d\sigma = O(e^{-\frac{\gamma}{\varepsilon}}).$$

Next e claim that

$$-\frac{1}{8\pi\varepsilon^2}\int_{B_d(x_{j,\varepsilon})}\int_{\mathbb{R}^3}\Big(u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)\frac{x^i-\xi^i}{|x-\xi|^3} + u_\varepsilon(\xi)\eta_\varepsilon(\xi)u_\varepsilon^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\Big)\,d\xi\,dx = o(\varepsilon^4), \tag{4.1}$$

which then gives that

$$\text{RHS of (2.4)} = o(\varepsilon^4).$$

Indeed, set

$$A_1 = \frac{1}{8\pi\varepsilon^2}\int_{B_d(x_{j,\varepsilon})}\int_{\mathbb{R}^3} u_\varepsilon^2(\xi)u_\varepsilon(x)\eta_\varepsilon(x)\frac{x^i-\xi^i}{|x-\xi|^3}\,d\xi\,dx,$$
$$A_2 = \frac{1}{8\pi\varepsilon^2}\int_{B_d(x_{j,\varepsilon})}\int_{\mathbb{R}^3} u_\varepsilon(\xi)\eta_\varepsilon(\xi)u_\varepsilon^2(x)\frac{x^i-\xi^i}{|x-\xi|^3}\,d\xi\,dx,$$
$$W_{j,\varepsilon} = (x) = \sum_{l=1,\,l\ne j}^{k} U_{a_l}\Big(\frac{x-x_{l,\varepsilon}}{\varepsilon}\Big).$$

Now, A_1 can be written as

$$A_1 = A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4},$$

where

$$A_{1,1} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) u_\varepsilon(\xi)\eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

$$A_{1,2} = \frac{1}{4\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) w_\varepsilon(x) u_\varepsilon(\xi)\eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

$$A_{1,3} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} (w_\varepsilon(x))^2 u_\varepsilon(\xi)\eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

$$A_{1,4} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} W_{j,\varepsilon}(x)(2u_\varepsilon(x) - W_{j,\varepsilon}(x)) u_\varepsilon(\xi)\eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

while A_2 can be written as follows:

$$A_2 = A_{2,1} + A_{2,2} + A_{2,3},$$

where

$$A_{2,1} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) \eta_\varepsilon(x)(u_\varepsilon(\xi))^2 \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

$$A_{2,2} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} W_{j,\varepsilon}(x)\eta_\varepsilon(x)(u_\varepsilon(\xi))^2 \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

$$A_{2,3} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_{j,\varepsilon})} \int_{\mathbb{R}^3} w_\varepsilon(x)\eta_\varepsilon(x)(u_\varepsilon(\xi))^2 \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx.$$

Then in view of the fact that (by the Hardy–Littlewood–Sobolev inequalities) for any $u_1, u_2, u_3, u_4 \in H_\varepsilon$ and $0 < \lambda \le 2$,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u_1(\xi)u_2(\xi)u_3(x)u_4(x) \cdot |x-\xi|^{-\lambda}\, d\xi\, dx \le C\varepsilon^{-\lambda}\|u_1\|_\varepsilon\|u_2\|_\varepsilon\|u_3\|_\varepsilon\|u_4\|_\varepsilon, \tag{4.2}$$

and there exist two positive constants d_1 and η such that, for $j = 1, 2, \dots, k$,

$$U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) = O(e^{-\frac{\eta}{\varepsilon}}) \quad \text{for } x \in \mathbb{R}^3 \setminus B_d(x_{j,\varepsilon}) \text{ and } 0 < d < d_1. \tag{4.3}$$

Then we know that (2.6) and (B.5) imply

$$A_{1,3} = O(\varepsilon^{-4}\|w_\varepsilon\|_\varepsilon^2\|u_\varepsilon + u_\varepsilon\|_\varepsilon\|\eta_\varepsilon\|_\varepsilon) = O(\varepsilon^6), \quad A_{1,4} = O(e^{-\frac{\eta}{\varepsilon}}), \quad A_{2,2} = O(e^{-\frac{\eta}{\varepsilon}}).$$

Moreover, we could apply similar argument as in [11] to estimate that

$$A_{1,1} = G_1 + \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) w_\varepsilon(\xi)\eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx + o(\varepsilon^4),$$

where

$$G_1 = \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx,$$

and

$$A_{1,2} = \frac{1}{2\pi\varepsilon^2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) w_\varepsilon(x) U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx + o(\varepsilon^4),$$

$$A_{2,1} = G_2 - \frac{1}{2\pi\varepsilon^2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big) w_\varepsilon(x) U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x-\xi|^3}\, d\xi\, dx + o(\varepsilon^4),$$

where

$$G_2 = -\frac{1}{8\pi\varepsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\Big)U_{a_j}\Big(\frac{\xi-x_{j,\varepsilon}}{\varepsilon}\Big)\eta_\varepsilon(x)\frac{x^i-\xi^i}{|x-\xi|^3}\,d\xi\,dx,$$

and

$$A_{2,3} = -\frac{1}{8\pi\varepsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\Big)w_\varepsilon(\xi)\eta_\varepsilon(\xi)\frac{x^i-\xi^i}{|x-\xi|^3}\,d\xi\,dx + o(\varepsilon^4).$$

To sum up, $A_1 + A_2 = o(\varepsilon^4)$, which implies the claim (4.1).

On the other hand, similar estimate to that in Lemma 3.3, we have

$$\begin{aligned}
\int_{B_d(x_{j,\varepsilon})}\frac{\partial V(x)}{\partial x^i}u_\varepsilon(x)\eta_\varepsilon(x)\,dx &= \sum_{l=1}^{3}\int_{B_d(x_{j,\varepsilon})}\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l}(x^l-a_j^l)u_\varepsilon(x)\eta_\varepsilon(x)\,dx + o\Big(\int_{B_d(x_{j,\varepsilon})}|x-a_j|u_\varepsilon(x)\eta_\varepsilon(x)\,dx\Big)\\
&= \sum_{l=1}^{3}\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l}\varepsilon^4\Big(\int_{B_{\frac{d}{\varepsilon}}(0)}\Big(x^l+\frac{x_{j,\varepsilon,}^l-a_j^l}{\varepsilon}\Big)U_{a_j}(x)\Big(\sum_{i=1}^{3}a_{j,i}\frac{\partial U_{a_j}(x)}{\partial x^i}+o(1)\Big)\,dx\Big)+o(\varepsilon^4)\\
&= \sum_{l=1}^{3}\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l}a_{j,l}\varepsilon^4\int_{\mathbb{R}^3}x^lU_{a_j}(x)\frac{\partial U_{a_j}(x)}{\partial x^l}\,dx + o(\varepsilon^4)\\
&= -\sum_{l=1}^{3}\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l}\frac{a_{j,l}}{2}\varepsilon^4\int_{\mathbb{R}^3}U_{a_j}^2(x)\,dx + o(\varepsilon^4).
\end{aligned}$$

Hence

$$\sum_{l=1}^{3}\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l}a_{j,l} = o(1).$$

Since, again by assumption, a_j is a non-degenerate critical point of the potential $V(x)$, we find that $a_{j,l} = 0$ for $l = 1, 2, 3$. □

Lemma 4.4. *For any fixed $R > 0$, it holds*

$$\eta_\varepsilon(x) = o(1), \quad x \in \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}).$$

Proof. Similar to the proof of Lemma 3.4, Lemma 4.2 and Lemma 4.3 give the result. □

Similar to Proposition 2.1, we also have the following estimate.

Lemma 4.5. *For large $R > 0$ there exist $\theta > 0$ and $C > 0$ such that*

$$\eta_\varepsilon(x) \le Ce^{-\frac{\theta|x-x_{l,\varepsilon}|}{\varepsilon}} \quad \text{for } l = 1,\dots,k \text{ and } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}),$$

$$|\nabla\eta_\varepsilon(x)| \le Ce^{-\frac{\theta R}{\varepsilon}} \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}).$$

Proof of Theorem 1.1. By contradiction, we suppose that $\|\eta_\varepsilon\|_{L^\infty}$ and $\mathcal{L}_\varepsilon\eta_\varepsilon = 0$. From Lemmas 4.4 and 4.5, $\eta_\varepsilon(x) = o(1)$, for all $x \in \mathbb{R}^3$, which contradicts with $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. As a result, we obtain that $\eta_\varepsilon = 0$ for ε small enough. □

A Estimates on A_1, A_2

Lemma A.1. *Suppose $f_\varepsilon \in L^1(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, for any fixed small $\overline{d} > 0$ independent of ε and x_ε. Then there exists a small constant $d_\varepsilon \in (\overline{d}, 2\overline{d})$ such that*

$$\int_{\partial B_{d_\varepsilon}(x_\varepsilon)}|f_\varepsilon(x)|\,d\sigma \le \frac{1}{\overline{d}}\int_{\mathbb{R}^3}|f_\varepsilon(x)|\,dx. \tag{A.1}$$

Proof. First, for any fixed small $\overline{d} > 0$ and x_ε,

$$\int_{\overline{d}}^{2\overline{d}} \int_{\partial B_r(x_\varepsilon)} |f_\varepsilon(x)|\, d\sigma\, dr = \int_{B_{2\overline{d}}(x_\varepsilon)\setminus B_{\overline{d}}(x_\varepsilon)} |f_\varepsilon(x)|\, dx \le \int_{\mathbb{R}^3} |f_\varepsilon(x)|\, dx. \tag{A.2}$$

Also $\int_{\partial B_r(x_\varepsilon)} |f_\varepsilon(x)|\, d\sigma$ is continuous with respect to r. By the mean value theorem of integrals, there exists a constant $d_\varepsilon \in (\overline{d}, 2\overline{d})$ such that

$$\int_{\overline{d}}^{2\overline{d}} \int_{\partial B_r(x_\varepsilon)} |f_\varepsilon(x)|\, d\sigma\, dr = d_\varepsilon \int_{\partial B_r(x_\varepsilon)} |f_\varepsilon(x)|\, d\sigma. \tag{A.3}$$

Then (A.2) and (A.3) imply (A.1). □

Using the notations from (2.5), we have the following estimates.

Lemma A.2. *For any fixed $R > 0$, it holds*

$$A_1(x) = o(1)\cdot R + O\Big(\frac{1}{R}\Big) \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}) \tag{A.4}$$

and

$$A_2(x) = O(e^{-\theta' R}) \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}) \text{ and some } \theta' > 0. \tag{A.5}$$

Proof. First, we know

$$\Big\{\xi : |x-\xi| \le R\frac{\varepsilon}{2}\Big\} \subset \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{\frac{R\varepsilon}{2}}(x_{j,\varepsilon}) \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon}),$$

and $\|u_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{3}{2}})$. Then by (2.1), for $x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon})$, it holds

$$\begin{aligned} A_1(x) &= \frac{1}{8\pi\varepsilon^2} \int_{|x-\xi|\le \frac{R\varepsilon}{2}} (u_\varepsilon(\xi))^2 |x-\xi|^{-1}\, d\xi + \frac{1}{8\pi\varepsilon^2} \int_{|x-\xi| > \frac{R\varepsilon}{2}} (u_\varepsilon(\xi))^2 |x-\xi|^{-1}\, d\xi \\ &= O\Big(\varepsilon^{-2} \int_{|x-\xi|\le \frac{R\varepsilon}{2}} (w_\varepsilon(\xi))^2 |x-\xi|^{-1}\, d\xi\Big) + O(e^{-2\theta R} R^2) + O\Big(\frac{1}{R}\Big). \end{aligned} \tag{A.6}$$

Also, by Hölder's inequality, we have

$$\begin{aligned} \int_{|x-\xi|\le \frac{R\varepsilon}{2}} (w_\varepsilon(\xi))^2 |x-\xi|^{-1}\, d\xi &= O\bigg(\Big(\int_{\mathbb{R}^3} (w_\varepsilon(\xi))^6\, d\xi\Big)^{\frac{1}{3}} \cdot \Big(\int_{|x-\xi|\le \frac{R\varepsilon}{2}} |x-\xi|^{-\frac{3}{2}}\, d\xi\Big)^{\frac{2}{3}}\bigg) \\ &= R \cdot O(\varepsilon^{-1} \|w_\varepsilon(\xi)\|_\varepsilon^2) = o(\varepsilon^2)\cdot R. \end{aligned} \tag{A.7}$$

Then (A.6) and (A.7) imply (A.4).

Next for $x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_{j,\varepsilon})$, we have

$$A_2(x) = O(e^{-\theta R}) \cdot \varepsilon^{-2} \int_{\mathbb{R}^3} u_\varepsilon(\xi) |x-\xi|^{-1} \cdot |\eta_\varepsilon(\xi)|\, d\xi \tag{A.8}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} u_\varepsilon(\xi)|x-\xi|^{-1} \cdot |\eta_\varepsilon(\xi)|\, d\xi &= \int_{|x-\xi|\le \frac{R\varepsilon}{2}} u_\varepsilon(\xi)|x-\xi|^{-1} \cdot |\eta_\varepsilon(\xi)|\, d\xi + O((R\varepsilon)^{-1} \|u_\varepsilon + u_\varepsilon\|_\varepsilon \|\eta_\varepsilon\|_\varepsilon) \\ &= O(\|(u_\varepsilon(\cdot) + u_\varepsilon(\cdot)\|_\varepsilon) \cdot \Big(\int_{|x-\xi|\le \frac{R\varepsilon}{2}} |x-\xi|^{-2}\, d\xi\Big)^{\frac{1}{2}} + O(R^{-1}\varepsilon^2) \\ &= O((R^{\frac{1}{2}} + R^{-1})\varepsilon^2). \end{aligned} \tag{A.9}$$

Then (A.8) and (A.9) imply (A.5). □

Lemma A.3. *For any fixed small $d > 0$, it holds*

$$A_1(x) = \frac{1}{8\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)|x-\xi|^{-1}\,d\xi\bigg) + o(1) \quad \text{in } B_d(x_{j,\varepsilon}) \tag{A.10}$$

and

$$A_2(x) = \frac{1}{4\pi}\cdot U_{a_j}\Big(\frac{x - x_{j,\varepsilon}}{\varepsilon}\Big)\bigg(\int_{\mathbb{R}^3} U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)\eta_\varepsilon(\xi)|x-\xi|^{-1}\,d\xi\bigg) + o(1) \quad \text{in } B_d(x_{j,\varepsilon}). \tag{A.11}$$

Proof. For $x \in B_d(x_{j,\varepsilon})$, we have

$$\begin{aligned}
&\bigg|A_1(x) - \frac{1}{8\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3} U_{a_j}^2\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)|x-\xi|^{-1}\,d\xi\bigg)\bigg| \\
&= O\bigg(\varepsilon^{-2}\int_{\mathbb{R}^3} |w_\varepsilon(\xi)|\cdot\Big(u_\varepsilon(\xi) + U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)\Big)|x-\xi|^{-1}\,d\xi\bigg) + O(e^{-\frac{\eta}{\varepsilon}}) \\
&= O\bigg(\varepsilon^{-2}\int_{|x-\xi|\le C} |w_\varepsilon(\xi)|\cdot\Big(u_\varepsilon(\xi) + U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)\Big)|x-\xi|^{-1}\,d\xi\bigg) \\
&\quad + O\bigg(\varepsilon^{-2}\|w_\varepsilon(\cdot)\|_\varepsilon\cdot\Big\|u_\varepsilon(\cdot) + U_{a_j}\Big(\frac{\cdot - x_{j,\varepsilon}}{\varepsilon}\Big)\Big\|_\varepsilon\bigg) + O(e^{-\frac{\eta}{\varepsilon}}),
\end{aligned} \tag{A.12}$$

where C is a fixed constant.

On the other hand, by Hölder's inequality, we know

$$\begin{aligned}
&\int_{|x-\xi|\le C} |w_\varepsilon(\xi)|\cdot\Big(u_\varepsilon(\xi) + U_{a_j}\Big(\frac{\xi - x_{j,\varepsilon}}{\varepsilon}\Big)\Big)|x-\xi|^{-1}\,d\xi \\
&= O\bigg(\|w_\varepsilon(\cdot)\|_{L^6(\mathbb{R}^3)}\cdot\Big\|u_\varepsilon(\cdot) + U_{a_j}\Big(\frac{\cdot - x_{j,\varepsilon}}{\varepsilon}\Big)\Big\|_{L^2(\mathbb{R}^3)}\bigg(\int_{|x-\xi|\le C} |x-\xi|^{-3}\,d\xi\bigg)^{\frac{1}{3}}\bigg) \\
&= O\bigg(\varepsilon^{-1}\|w_\varepsilon(\cdot)\|_\varepsilon\cdot\Big\|u_\varepsilon(\cdot) + U_{a_j}\Big(\frac{\cdot - x_{j,\varepsilon}}{\varepsilon}\Big)\Big\|_\varepsilon\bigg) = o(\varepsilon^2).
\end{aligned} \tag{A.13}$$

Then (A.12) and (A.13) imply (A.10). Similar to the estimates of (A.10), combining Proposition 3.1, we deduce (A.11). □

B Estimates of the Term w_ε

We denote

$$R_\varepsilon(x) = \sum_{l=1}^{k} U_{a_l}\Big(\frac{x - x_{l,\varepsilon}}{\varepsilon}\Big), \quad W_{j,\varepsilon}(x) = \sum_{l=1, l\ne j}^{k} U_{a_l}\Big(\frac{x - x_{l,\varepsilon}}{\varepsilon}\Big).$$

Let $M_\varepsilon(x, w_\varepsilon(x))$ be as follows:

$$M_\varepsilon(x, w_\varepsilon(x)) := -\varepsilon^2\Delta w_\varepsilon(x) + G(x, w_\varepsilon(x)),$$

where

$$G(x, w_\varepsilon(x)) = V(x)w_\varepsilon(x) - \frac{1}{8\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3}\frac{(R_\varepsilon(\xi))^2}{|x-\xi|}\,d\xi\bigg)w_\varepsilon(x) + \frac{1}{4\pi\varepsilon^2}\bigg(\int_{\mathbb{R}^3}\frac{R_\varepsilon(\xi)w_\varepsilon(\xi)}{|x-\xi|}\,d\xi\bigg)R_\varepsilon(x).$$

Let $u_\varepsilon(x) = R_\varepsilon(x) + w_\varepsilon(x)$ be the solution of (1.3). Then

$$M_\varepsilon(x, w_\varepsilon(x)) = N(x, w_\varepsilon(x)) + l_\varepsilon(x),$$

where

$$N(x, w_\varepsilon(x)) = \frac{1}{8\pi\varepsilon^2}\bigg(\int\limits_{\mathbb{R}^3} \frac{w_\varepsilon^2(\xi)}{|x-\xi|}\,d\xi\bigg)(R_\varepsilon(x) + w_\varepsilon(x)) + \frac{w_\varepsilon(x)}{4\pi\varepsilon^2}\int\limits_{\mathbb{R}^3} \frac{R_\varepsilon(\xi)w_\xi(\xi)}{|x-\xi|}\,d\xi$$

and

$$l_\varepsilon(x) = \frac{W_{j,\varepsilon}(x)}{8\pi\varepsilon^2}\int\limits_{\mathbb{R}^3} \frac{W_{j,\varepsilon}(\xi)U_{a_j}(\frac{\xi-x_{j,\varepsilon}}{\varepsilon})}{|x-\xi|}\,d\xi + \sum_{j=1}^{k}(V(a_j) - V(x))U_{a_j}\Big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\Big).$$

Proposition B.1. *Let $u_\varepsilon(x) = R_\varepsilon(x) + w_\varepsilon(x)$ be the solution of* (1.3). *Then there exists a constant $\bar\rho > 0$ independent of ε such that*

$$\int\limits_{\mathbb{R}^3} M_\varepsilon(x, w_\varepsilon(x))w_\varepsilon(x)\,dx \geq \bar\rho\|w_\varepsilon\|_\varepsilon^2. \tag{B.1}$$

Proof. Similar to the proof of [2, Proposition 3.1], we can prove (B.1) by the contradiction argument and blow-up analysis. For the more details, one can refer to [1, 2]. □

Proof of Proposition 2.5. First, from Proposition B.1, we know

$$\|w_\varepsilon\|_\varepsilon^2 \leq C\int\limits_{\mathbb{R}^3} N(x, w_\varepsilon(x))w_\varepsilon(x)\,dx + C\int\limits_{\mathbb{R}^3} l_\varepsilon(x)w_\varepsilon(x)\,dx. \tag{B.2}$$

Next, using Theorem B and (4.2), we deduce

$$\begin{aligned}\int\limits_{\mathbb{R}^3} N(x, w_\varepsilon(x))w_\varepsilon(x)\,dx &= \frac{1}{8\pi\varepsilon^2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{w_\varepsilon^2(\xi)}{|x-\xi|}(R_\varepsilon(x) + w_\varepsilon(x))w_\varepsilon(x)\,dx\,d\xi\\ &\quad + \frac{1}{4\pi\varepsilon^2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{R_\varepsilon(\xi)w_\xi(\xi)}{|x-\xi|}w_\varepsilon^2(x)\,dx\,d\xi\\ &= O(\varepsilon^{-3}\|w_\varepsilon\|_\varepsilon^3 \cdot \|w_\varepsilon + R_\varepsilon\|_\varepsilon) = o(1)\|w_\varepsilon\|_\varepsilon^2.\end{aligned} \tag{B.3}$$

Also from (4.3) and

$$V(a_j) - V(x) = -\sum_{i=1}^{3}\sum_{l=1}^{3}(x^i - a_j^i)(x^l - a_j^l)\frac{\partial^2 V(a_j)}{\partial x^i\partial x^l} + o(|x-a_j|^2),$$

we have

$$\begin{aligned}\int\limits_{\mathbb{R}^3} l_\varepsilon(x)w_\varepsilon(x)\,dx &= \sum_{j=1}^{k}\int\limits_{\mathbb{R}^3}(V(a_j) - V(x))U_{a_j}\Big(\frac{x-x_{j,\varepsilon}}{\varepsilon}\Big)w_\varepsilon(x)\,d\xi\,dx\\ &\quad - \frac{1}{8\pi\varepsilon^2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} W_{j,\varepsilon}(\xi)U_{a_j}\Big(\frac{\xi-x_{j,\varepsilon}}{\varepsilon}\Big)W_{j,\varepsilon}(x)w_\varepsilon(x)|x-\xi|^{-1}\,d\xi\\ &= O\Big(\varepsilon^{\frac{3}{2}}\|w_\varepsilon\|_\varepsilon\Big(\varepsilon^2 + \max_{j=1,\dots,k}|x_{j,\varepsilon} - a_j|^2\Big) + e^{-\frac{\eta}{\varepsilon}}\Big).\end{aligned} \tag{B.4}$$

Then (B.2), (B.3) and (B.4) imply (2.10). □

Proposition B.2. *Let $u_\varepsilon(x)$ be a positive solution of* (1.3) *as in Proposition 2.5. Then it holds*

$$\|w_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{7}{2}}). \tag{B.5}$$

Proof. It follows from the results of Proposition 3.1 and Proposition 2.5 directly. □

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References

[1] D. Cao, S. Li and P. Luo, Uniqueness of positive bound states with multi-bump for nonlinear Schrödinger equations, *Calc. Var. Partial Differential Equations* **54** (2015), no. 4, 4037–4063.

[2] D. Cao and S. Peng, Semi-classical bound states for Schrödinger equations with potentials vanishing or unbounded at infinity, *Comm. Partial Differential Equations* **34** (2009), no. 10–12, 1566–1591.

[3] S. Cingolani, S. Secchi and M. Squassina, Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities, *Proc. Roy. Soc. Edinburgh Sect. A* **140** (2010), no. 5, 973–1009.

[4] Y. Deng, C.-S. Lin and S. Yan, On the prescribed scalar curvature problem in $\mathbb{R}^N$, local uniqueness and periodicity, *J. Math. Pures Appl. (9)* **104** (2015), no. 6, 1013–1044.

[5] Y. Guo, M. Musso, S. Peng and S. Yan, Non-degeneracy of multi-bubbling solutions for the prescribed scalar curvature equations and applications, *J. Funct. Anal.* **279** (2020), no. 6, Article ID 108553.

[6] Y. Guo, S. Peng and S. Yan, Local uniqueness and periodicity induced by concentration, *Proc. Lond. Math. Soc. (3)* **114** (2017), no. 6, 1005–1043.

[7] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Studies in Appl. Math.* **57** (1976/77), no. 2, 93–105.

[8] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Grad. Stud. Math. 14, American Mathematical Society, Providence, 2001.

[9] P.-L. Lions, The Choquard equation and related questions, *Nonlinear Anal.* **4** (1980), no. 6, 1063–1072.

[10] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), no. 4, 223–283.

[11] P. Luo, S. Peng and C. Wang, Uniqueness of positive solutions with concentration for the Schrödinger–Newton problem, *Calc. Var. Partial Differential Equations* **59** (2020), no. 2, Paper No. 60.

[12] G. P. Menzala, On regular solutions of a nonlinear equation of Choquard's type, *Proc. Roy. Soc. Edinburgh Sect. A* **86** (1980), no. 3–4, 291–301.

[13] R. Penrose, Quantum computation, entanglement and state reduction, *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **356** (1998), no. 1743, 1927–1939.

[14] S. Secchi, A note on Schrödinger–Newton systems with decaying electric potential, *Nonlinear Anal.* **72** (2010), no. 9–10, 3842–3856.

[15] P. Tod and I. M. Moroz, An analytical approach to the Schrödinger–Newton equations, *Nonlinearity* **12** (1999), no. 2, 201–216.

[16] G. Vaira, Existence of bound states for Schrödinger–Newton type systems, *Adv. Nonlinear Stud.* **13** (2013), no. 2, 495–516.

[17] J. Wei and M. Winter, Strongly interacting bumps for the Schrödinger–Newton equations, *J. Math. Phys.* **50** (2009), no. 1, Article ID 012905.