Research Article

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Liouville Theorems for Fractional Parabolic Equations

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Abstract: In this paper, we establish several Liouville type theorems for entire solutions to fractional parabolic equations. We first obtain the key ingredients needed in the proof of Liouville theorems, such as narrow region principles and maximum principles for antisymmetric functions in unbounded domains, in which we remarkably weaken the usual decay condition $u \to 0$ at infinity to a polynomial growth on u by constructing proper auxiliary functions. Then we derive monotonicity for the solutions in a half space $\mathbb{R}^n_+ \times \mathbb{R}$ and obtain some new connections between the nonexistence of solutions in a half space $\mathbb{R}^n_+ \times \mathbb{R}$ and in the whole space $\mathbb{R}^{n-1} \times \mathbb{R}$ and therefore prove the corresponding Liouville type theorems. To overcome the difficulty caused by the nonlocality of the fractional Laplacian, we introduce several new ideas which will become useful tools in investigating qualitative properties of solutions for a variety of nonlocal parabolic problems.

Keywords: Liouville Type Theorems, Fractional Parabolic Equations, Entire Solutions, Monotonicity, Nonexistence of Solutions, Narrow Region Principles, Maximum Principle for Antisymmetric Functions

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1 Introduction

In this paper, we establish Liouville type theorems – nonexistence of positive solutions for the following fractional parabolic equations in both the whole space:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.1}$$

and the half space:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin \mathbb{R}^n_+ \times \mathbb{R}, \end{cases} \tag{1.2}$$

where $0 < s < 1$, f is a C^1 function, the range for time variable t is $(-\infty, \infty)$, and in this case, the solutions are referred to as entire solutions.

For each fixed $t \in \mathbb{R}$, the fractional Laplacian on x is defined by

$$(-\Delta)^s u(x,t) = C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dy = C_{n,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dy,$$

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where P.V. stands for the Cauchy principal value. It is easy to see that, for $u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_{2s}$, $(-\Delta)^s u$ is well defined, where

$$\mathcal{L}_{2s} = \Big\{ u(\cdot, t) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x,t)|}{1+|x|^{n+2s}}\,dx < +\infty \Big\}.$$

It is well known that the fractional Laplacian is a nonlocal operator, and as $s \to 1$, it approaches to the regular Laplacian $-\Delta$.

We exhibit some new monotonicity and Liouville type results for entire solutions of (1.1) and (1.2). It is well known that Liouville type theorems play crucial roles in the theory of PDEs, while monotonicity usually is a key tool to derive Liouville theorems. To this end, a number of systematic approaches have been established, such as the extension method [7], the method of moving planes [1, 2, 12, 15, 20], the method of moving spheres [13, 32], and the sliding methods [3, 43]. From the heuristic point of view, Liouville type theorems turn out to be equivalent to universal (initial or final) blow-up or decay estimates (see [37, 44] and the references therein).

For the elliptic equations involving either local or nonlocal operators, there have been numerous articles that dedicated to the study of Liouville type theorems, such as [5, 6, 8, 16, 17, 21, 22, 27, 31, 33, 36, 39, 40] and so on.

Entire solutions to semi-linear parabolic equations involving the regular Laplacian

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(u(x,t)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

play an important role in the dynamics of solutions to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(u(x,t)), & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

For example, the travelling wave solutions and the ω-limits sets of bounded solutions to (1.3) are comprised of entire solutions. In particular, the study of entire solutions is essential for a full understanding of the transient dynamics and the structures of the global attractors; see [25, 26]. For these reasons, entire solutions to reaction diffusion equations have been widely studied (see [24, 34, 41]).

For nonlinear parabolic equations involving local operators, such as $-\Delta$, there have also been a series of results in this respect.

Bidaut-Véron [4], Merle and Zaag [34], and Poláčik, Quittner and Souplet [37] proved that the only nonnegative bounded classical solution of the parabolic problem

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^p(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

is the trivial solution $u \equiv 0$ for $1 < p < \frac{n(n+2)}{(n-1)^2}$. Then Quittner generalized this result to $1 < p < \frac{n+2}{n-2}$ in [38]. This is also true if u^p is replaced by $f(u)$, where f is a decreasing continuous function and satisfies $f(c) = 0$ if and only if $c = 0$.

For the case of a half space, Kavian [29], and Levine and Meier [30] showed that the only nonnegative bounded classical solution of the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^p(x,t), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \in \{x_1 = 0\} \times \mathbb{R}^{n-1} \times \mathbb{R}, \end{cases}$$

is the trivial solution $u \equiv 0$ for $1 < p \le \frac{n+3}{n+1}$. Then Poláčik, Quittner and Souplet [37] and Xing [44] extended the range of the exponent p from $p \le \frac{n+3}{n+1}$ to $p < \frac{n^2-1}{(n-2)^2}$. Similar to the whole space case, the same conclusion holds if u^p is replaced by $f(u)$, where f is a decreasing continuous function and satisfies $f(c) = 0$ if and only if $c = 0$.

So far as we are aware, not much is known concerning the entire solutions to fractional parabolic equations. Due to the nonlocality of the fractional Laplacian, many traditional approaches for local elliptic operators do not work anymore in the nonlocal setting. However, these qualitative properties of solutions, in particular, the Liouville type theorems, are definitely important tools in the blow-up rate, a priori bounds, and optimal universal estimates of solutions to related initial and initial-boundary value nonlocal parabolic problems and so on. This is a motivation for the present paper.

In this paper, we prove Liouville type theorems for fractional parabolic problems (1.1) and (1.2) and establish some new connections between the solutions of (1.1) and (1.2). To this end, we first introduce some new ideas to develop a fractional parabolic version of the method of moving planes, in which the key ingredients are *narrow region principles* and *maximum principles for anti-symmetric functions* as we will state in the following two theorems.

For $x \in \mathbb{R}^n$, denote $x = (x_1, x')$, $x^\lambda = (2\lambda - x_1, x')$, where $x' \in \mathbb{R}^{n-1}$. Set

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad \text{and} \quad T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}.$$

Theorem 1.1 (Narrow Region Principle). *Let Ω be a bounded or unbounded narrow region in Σ_λ such that it is contained in $\{x \mid \lambda - 2l < x_1 < \lambda\}$ with small l. Suppose that $w(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ is uniformly bounded with respect to t and lower semi-continuous in x on $\bar{\Omega}$, and it satisfies*

$$w(x,t) \le o(1)|x|^\gamma \quad \textit{for any } 0 < \gamma < 2s \quad \textit{as } |x| \to +\infty,$$

and

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) + (-\Delta)^s w(x,t) = c(x,t)w(x,t), & (x,t) \in \Omega \times \mathbb{R}, \\ w(x,t) \ge 0, & (x,t) \in (\Sigma_\lambda \backslash \Omega) \times \mathbb{R}, \\ w(x^\lambda, t) = -w(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}. \end{cases} \tag{1.4}$$

If $c(x,t)$ is bounded from above, then for sufficiently small l, we have

$$w(x,t) \ge 0, \quad (x,t) \in \Sigma_\lambda \times \mathbb{R}. \tag{1.5}$$

Furthermore, the following strong maximum principle holds: either $w(x,t) > 0$ or $w(x,t) \equiv 0$, $(x,t) \in \Omega \times \mathbb{R}$.

In previous literature, for instance in [14], to establish a narrow region principle in an unbounded domain, one usually assumed that $\lim_{|x|\to+\infty} w(x,t) = 0$. Here, in Theorem 1.1, we remarkably weaken this condition and even allow $w(x,t)$ to go to infinity and only assume its growth rate does not exceed $|x|^\gamma$ $(0 < \gamma < 2s)$. To this end, we consider

$$\bar{w}(x,t) = \frac{e^{mt} w(x,t)}{h(x)}$$

with a suitable auxiliary function $h(x)$ satisfying

$$\lim_{|x|\to+\infty} h(x) = +\infty \quad \text{and} \quad \frac{(-\Delta)^s h(x)}{h(x)} \ge \frac{C}{l^{2s}}$$

for some positive constant C.

The main difficulty lies in how to seek such a function $h(x)$. For the Laplace operator, it is much easier to find such a function since we can choose $h(x) = \sin(ax_1)V(|x'|)$, where V is a radial function satisfying an ordinary differential equation. However, for the fractional Laplacian, due to the nonlocality, it seems impossible to use an ODE method to find such a function V. After many trials, we finally choose

$$h(x) = \left[\left(1 - \frac{(x_1 - (\lambda - l))^2}{l^2}\right)_+^s + 1\right](1 + |x'|^2)^{\frac{\gamma}{2}}.$$

This implies $\lim_{|x|\to+\infty} \bar{w}(x,t) = 0$. Consequently, when we use a contradiction argument to prove (1.5), $\bar{w}$ would be able to attain its negative minimum in the interior of Σ_λ.

For the regular Laplacian, the convenience is

$$\Delta w(x,t) = e^{-mt}(\Delta \bar{w}(x,t) \cdot h(x) + 2\nabla \bar{w}(x,t) \cdot \nabla h(x) + \bar{w}(x,t) \cdot \Delta h(x)). \tag{1.6}$$

At a minimum of $\bar{w}(x,t)$, the middle term on the right hand side vanishes since $\nabla \bar{w}(x,t) = 0$. This makes the analysis much easier. However, the fractional counter part of (1.6) is

$$(-\Delta)^s w(x,t) = e^{-mt}\left((-\Delta)^s \bar{w}(x,t) \cdot h(x) - 2C \int_{\mathbb{R}^n} \frac{(\bar{w}(x,t) - \bar{w}(y,t))(h(x) - h(y))}{|x-y|^{n+2s}}\, dy + \bar{w}(x,t) \cdot (-\Delta)^s h(x)\right).$$

At a minimum of $\bar{w}(x,t)$, the middle term on the right-hand side (the integral) neither vanishes nor has a definite sign. This is the main difficulty. To circumvent it, we combine the first two terms together to derive a good estimation.

Remark 1.2. In [10], the authors derive a narrow region principle for fractional elliptic equations in unbounded domains under the conditions that $w(x)$ is bounded. Here, by using our auxiliary function $h(x)$, we will be able to weaken this boundedness condition to a polynomial growth.

More precisely, as a byproduct, we can prove the following.

Proposition 1.3. *Let Ω be a bounded or unbounded narrow region in Σ_λ such that it is contained in*

$$\{x \mid \lambda - 2l < x_1 < \lambda\} \quad \text{with small } l.$$

Suppose that $w(x) \in C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_{2s}(\mathbb{R}^n)$ is lower semi-continuous on $\bar{\Omega}$, where

$$\mathcal{L}_{2s}(\mathbb{R}^n) = \Big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \;\Big|\; \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx < +\infty\Big\}.$$

Assume

$$w(x) \le o(1)|x|^\gamma \quad \text{for any } 0 < \gamma < 2s \quad \text{as } |x| \to +\infty,$$

and

$$\begin{cases} (-\Delta)^s w(x) + c(x)w(x) = 0, & x \in \Omega, \\ w(x) \ge 0, & x \in \Sigma_\lambda \backslash \Omega, \\ w(x^\lambda) = -w(x), & x \in \Sigma_\lambda. \end{cases}$$

If $c(x)$ is bounded from above, then we have $w(x) \ge 0$, $x \in \Sigma_\lambda$. Furthermore, the following strong maximum principle holds: either $w(x) > 0$ or $w(x) \equiv 0$, $x \in \Omega$.

Theorem 1.4 (Maximum Principle for Antisymmetric Functions). *Let Ω be a bounded or unbounded region in Σ_λ, and assume that the width of Ω in x_1 direction is bounded. Suppose that $w(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ is uniformly bounded with respect to t and lower semi-continuous in x on $\bar{\Omega}$, and it satisfies $w(x,t) \le o(1)|x|^\gamma$ for any $0 < \gamma < 2s$ as $|x| \to +\infty$, and*

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) + (-\Delta)^s w(x,t) = c(x,t)w(x,t), & (x,t) \in \Omega \times \mathbb{R}, \\ w(x,t) \ge 0, & (x,t) \in (\Sigma_\lambda \backslash \Omega) \times \mathbb{R}, \\ w(x^\lambda, t) = -w(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}. \end{cases} \tag{1.7}$$

If

$$c(x,t) \le 0 \quad \text{or} \quad c(x,t) > 0 \text{ is small}, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{1.8}$$

then $w(x,t) \ge 0$, $(x,t) \in \Sigma_\lambda \times \mathbb{R}$. Furthermore, the following strong maximum principle holds: either $w(x,t) > 0$ or $w(x,t) \equiv 0$, $(x,t) \in \Omega \times \mathbb{R}$.

Remark 1.5. In Theorem 1.4, if we replace the first equation in (1.7) by

$$\frac{\partial w}{\partial t}(x,t) + (-\Delta)^s w(x,t) \ge 0, \quad (x,t) \in \Omega \times \mathbb{R},$$

then the same conclusions as in Theorem 1.4 still hold.

Based on Theorem 1.1 and 1.4, we derive the strict monotonicity of solutions to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin \mathbb{R}^n_+ \times \mathbb{R}. \end{cases} \tag{1.9}$$

We also find some new connections between the existence of solutions of (1.9) and of the problem

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{1.10}$$

Theorem 1.6 (Monotonicity in a Half Space). *Assume $f\colon [0,\infty) \to \mathbb{R}$ is a C^1 function with $f(0) = 0$, $f'(0) \le 0$, and suppose $u \in (C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+}) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$.*

(i) *If u is a positive solution of (1.9) satisfying $u(x,t) \le o(1)|x|^\gamma$ for any $0 < \gamma < 2s$ as $|x| \to +\infty$, then it is increasing in x_1 and*

$$\frac{\partial u}{\partial x_1}(x,t) > 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

(ii) *If there is a positive bounded solution of (1.9), then there exists a positive bounded solution of (1.10).*

Combining Theorem 1.6 (ii) with the nonexistence result in the whole space obtained in [23], we derive the following nonexistence result in the half space.

Theorem 1.7 (Nonexistence). *Assume $n \ge 3$, $1 < p \le \frac{n-1+2s}{n-1}$. Then the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = u^p(x,t), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin \mathbb{R}^n_+ \times \mathbb{R}, \end{cases}$$

possesses no nontrivial nonnegative bounded solution that is in $(C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+}) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$.

Our approach can also be applied to derive Liouville type theorems for other fractional parabolic problems such as the following.

Theorem 1.8. *Assume $f\colon [0,\infty) \to \mathbb{R}$ is a C^1 function satisfying $f' \le 0$, and $u \in (C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ is a nonnegative bounded solution of*

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.11}$$

(i) *Then u is independent of spatial variables x_i $(1 \le i \le n)$, and hence $u_t = f(u(t))$ for all $t \in \mathbb{R}$.*
(ii) *Assume $f(c) = 0$ if and only if $c = 0$; then the only nonnegative bounded solution of (1.11) is the trivial solution $u \equiv 0$.*
(iii) *Assume $f(c) < 0$ for $c \ge 0$; then the nonnegative bounded solution of (1.11) does not exist.*

Such examples of $f(u)$ are $1 - e^u, -u^p, -e^u, \dots$. These kinds of results are also indispensable tools in the study of blow-up rate for indefinite problems (see [44]). For other results concerning nonlocal parabolic equations, please see e.g. [9, 18, 23, 28, 35].

In Section 2, we derive *narrow region principles* and *maximum principles for anti-symmetric functions*. In Section 3, we establish *the monotonicity and nonexistence in half spaces*. In Section 4, we prove Theorem 1.8.

2 Maximum Principles

Before presenting the proof of the narrow region principle, we first construct the auxiliary function $h(x)$ and evaluate

$$\frac{(-\Delta)^s h(x)}{h(x)}.$$

Lemma 2.1. *Let $x = (x_1, x')$, and $h(x) = f(x_1) g_\gamma(x')$, $0 < \gamma < 2s$, where*

$$f(x_1) = \Big(1 - \frac{x_1^2}{l^2}\Big)_+^s + 1, \quad \text{and} \quad g_\gamma(x') = (1 + |x'|^2)^{\frac{\gamma}{2}}.$$

Then there exists $C_1 > 0$ such that, for all sufficiently small l, we have

$$\frac{(-\Delta)^s h(x)}{h(x)} \ge \frac{C_1}{l^{2s}} \quad \text{for all } |x_1| < l. \tag{2.1}$$

Proof. First, if $|x_1| < l$, we have

$$(-\Delta)^s f(x_1) = (-\Delta)^s \Big[\Big(1 - \frac{x_1^2}{l^2}\Big)_+^s + 1\Big] = (-\Delta)^s \Big(1 - \frac{x_1^2}{l^2}\Big)_+^s = \frac{1}{l^2}(-\Delta)^s (1 - x_1^2)_+^s = \frac{C}{l^2}. \tag{2.2}$$

Also, $(-\Delta)^s g_\gamma(x')$ is bounded. Indeed, if $|x'| = 0$, then

$$\begin{aligned}(-\Delta)^s g_\gamma(x') &= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{1-(1+|y'|^2)^{\frac{\gamma}{2}}}{(|y'|^2+y_1^2)^{\frac{n+2s}{2}}}\,dy'\,dy_1 = C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{1-(1+|y'|^2)^{\frac{\gamma}{2}}}{(|y'|^2+|y'|^2t^2)^{\frac{n+2s}{2}}}|y'|\,dy'\,dt\\
&= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{1-(1+|y'|^2)^{\frac{\gamma}{2}}}{|y'|^{n-1+2s}(1+t^2)^{\frac{n+2s}{2}}}\,dy'\,dt = C\,\mathrm{P.V.}\int_{\mathbb{R}^{n-1}} \frac{1-(1+|y'|^2)^{\frac{\gamma}{2}}}{|y'|^{n-1+2s}}\,dy'\\
&= C\int_0^{+\infty} \frac{1-(1+r^2)^{\frac{\gamma}{2}}}{r^{1+2s}}\,dr < +\infty.\end{aligned}$$

If $|x'| \to +\infty$, then

$$\begin{aligned}(-\Delta)^s g_\gamma(x') &= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{g_\gamma(x')-g_\gamma(y')}{|x-y|^{n+2s}}\,dy = C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{g_\gamma(x')-g_\gamma(y')}{(|x'-y'|^2+(x_1-y_1)^2)^{\frac{n+2s}{2}}}\,dy\\
&= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{g_\gamma(x')-g_\gamma(y')}{(|x'-y'|^2+|x'-y'|^2t^2)^{\frac{n+2s}{2}}}|x'-y'|\,dt\,dy'\\
&= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{g_\gamma(x')-g_\gamma(y')}{|x'-y'|^{n-1+2s}}\cdot\frac{1}{(1+t^2)^{\frac{n+2s}{2}}}\,dt\,dy'\\
&= C\,\mathrm{P.V.}\int_{\mathbb{R}^{n-1}} \frac{g_\gamma(x')-g_\gamma(y')}{|x'-y'|^{n-1+2s}}\,dy' = C\,\mathrm{P.V.}\int_{\mathbb{R}^{n-1}} \frac{(1+|x'|^2)^{\frac{\gamma}{2}}-(1+|y'|^2)^{\frac{\gamma}{2}}}{|x'-y'|^{n-1+2s}}\,dy' \to 0.\end{aligned}$$

Therefore, $|(-\Delta)^s g_\gamma(x')| \le C$. Similarly, we have

$$\left|C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(y_1)(g_\gamma(x')-g_\gamma(y'))}{|x-y|^{n+2s}}\,dy\right| \le C. \tag{2.3}$$

Combining (2.2) and (2.3), we derive

$$\begin{aligned}(-\Delta)^s h(x) &= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(x_1)g_\gamma(x')-f(y_1)g_\gamma(y')}{|x-y|^{n+2s}}\,dy\\
&= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(x_1)g_\gamma(x')-f(y_1)g_\gamma(x')+f(y_1)g_\gamma(x')-f(y_1)g_\gamma(y')}{|x-y|^{n+2s}}\,dy\\
&= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{(f(x_1)-f(y_1))g_\gamma(x')+f(y_1)(g_\gamma(x')-g_\gamma(y'))}{|x-y|^{n+2s}}\,dy\\
&= g_\gamma(x')(-\Delta)^s f(x_1) + C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(y_1)(g_\gamma(x')-g_\gamma(y'))}{|x-y|^{n+2s}}\,dy \ge \frac{C}{l^{2s}}g_\gamma(x') - C.\end{aligned}$$

Therefore,

$$\frac{(-\Delta)^s h(x)}{h(x)} \ge \frac{1}{h(x)}\left(\frac{C}{l^{2s}}g_\gamma(x') - C\right) \ge \frac{C}{l^{2s}}.$$

This completes the proof of Lemma 2.1. □

Next we present the proof of the narrow region principle (Theorem 1.1).

Proof of Theorem 1.1. Let

$$h(x) = \left[\left(1 - \frac{(x_1-(\lambda-l))^2}{l^2}\right)_+^s + 1\right](1+|x'|^2)^{\frac{\gamma}{2}}, \quad 0<\gamma<2s, \quad \text{and} \quad \bar{w}(x,t) = \frac{e^{mt}w(x,t)}{h(x)},$$

where $m > 0$ is a constant.

By (1.4) and a direct calculation, $\bar{w}(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \bar{w}}{\partial t}(x,t) + \dfrac{1}{h(x)} C_{n,s} \,\mathrm{P.V.} \displaystyle\int_{\mathbb{R}^n} \dfrac{\bar{w}(x,t) - \bar{w}(y,t)}{|x-y|^{n+2s}} h(y)\,dy & \\ \quad = \Big(c(x,t) + m - \dfrac{(-\Delta)^s h(x)}{h(x)} \Big) \bar{w}(x,t), & (x,t) \in \Omega \times \mathbb{R}, \\ \bar{w}(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \backslash \Omega) \times \mathbb{R}, \\ \bar{w}(x,t) \to 0, & (x,t) \in \Sigma_\lambda \times \mathbb{R}, |x'| \to +\infty. \end{cases} \tag{2.4}$$

Next we prove that

$$\bar{w}(x,t) \geq \min\{0, \inf_{\Omega} \bar{w}(x,\bar{t})\}, \quad (x,t) \in \Omega \times [\bar{t}, T] \text{ for all } [\bar{t}, T] \subset \mathbb{R}. \tag{2.5}$$

If (2.5) is false, then there exists $(x_0, t_0) \in \Omega \times (\bar{t}, T]$ such that

$$\bar{w}(x_0, t_0) = \inf_{\Sigma_\lambda \times (\bar{t}, T]} \bar{w}(x,t) < \min\{0, \inf_{\Omega} \bar{w}(x, \bar{t})\},$$

and hence

$$\frac{\partial \bar{w}}{\partial t}(x_0, t_0) \leq 0,$$

while

$$\begin{aligned} \mathrm{P.V.} \int_{\mathbb{R}^n} & \frac{\bar{w}(x_0,t_0) - \bar{w}(y,t_0)}{|x_0-y|^{n+2s}} h(y)\,dy \\ &= \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{\bar{w}(x_0,t_0) - \bar{w}(y,t_0)}{|x_0-y|^{n+2s}} h(y)\,dy + \mathrm{P.V.} \int_{\Sigma_\lambda^c} \frac{\bar{w}(x_0,t_0) - \bar{w}(y,t_0)}{|x_0-y|^{n+2s}} h(y)\,dy \\ &= \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{\bar{w}(x_0,t_0) - \bar{w}(y,t_0)}{|x_0-y|^{n+2s}} h(y)\,dy + \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{\bar{w}(x_0,t_0) + \bar{w}(y,t_0)}{|x_0-y^\lambda|^{n+2s}} h(y^\lambda)\,dy \\ &\leq \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{\bar{w}(x_0,t_0) - \bar{w}(y,t_0)}{|x_0-y^\lambda|^{n+2s}} h(y^\lambda)\,dy + \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{\bar{w}(x_0,t_0) + \bar{w}(y,t_0)}{|x_0-y^\lambda|^{n+2s}} h(y^\lambda)\,dy \\ &= \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{2\bar{w}(x_0,t_0)}{|x_0-y^\lambda|^{n+2s}} h(y^\lambda)\,dy < 0, \end{aligned}$$

where we have used the fact that $|x_0 - y| \leq |x_0 - y^\lambda|$ and $h(y) \geq h(y^\lambda)$, $y \in \Sigma_\lambda$.

In addition, since $c(x,t)$ is bounded from above, we apply (2.1) and choose $m = \frac{C_1}{2l^{2s}}$,

$$\Big(c(x_0,t_0) + m - \frac{(-\Delta)^s h(x_0)}{h(x_0)} \Big) \bar{w}(x_0,t_0) > 0 \quad \text{if } l \text{ is sufficiently small.}$$

Therefore, we derive a contradiction from (2.4) and conclude (2.5). That is,

$$\bar{w}(x,t) \geq \min\{0, \inf_{\Omega} \bar{w}(x,\bar{t})\} = \min\Big\{0, e^{m\bar{t}} \inf_{\Omega} \frac{w(x,\bar{t})}{h(x)}\Big\} \geq -Ce^{m\bar{t}}.$$

Therefore, $w(x,t) \geq -Ce^{-m(t-\bar{t})} h(x)$. Since the above inequality holds for any $\bar{t}\ (< T) \in \mathbb{R}$, letting $\bar{t} \to -\infty$, we have $w(x,t) \geq 0$, $(x,t) \in \Omega \times \mathbb{R}$. It follows from $w(x,t) \geq 0$ in $(\Sigma_\lambda \backslash \Omega) \times \mathbb{R}$ that $w(x,t) \geq 0$, $(x,t) \in \Sigma_\lambda \times \mathbb{R}$.

Moreover, if there exists a point $(x^0, t^0) \in \Omega \times \mathbb{R}$ such that $w(x^0, t^0) = 0$, then

$$w(x^0,t^0) = \inf_{\Sigma_\lambda \times \mathbb{R}} w(x,t) = 0, \quad \text{and} \quad \frac{\partial w}{\partial t}(x^0,t^0) = 0, \quad (-\Delta)^s w(x^0,t^0) < 0.$$

Therefore, by (1.4), we have

$$0 > \frac{\partial w}{\partial t}(x^0,t^0) + (-\Delta)^s w(x^0,t^0) = c(x^0,t^0) w(x^0,t^0) = 0.$$

This is a contradiction, and thus the strong maximum principle holds. This completes the proof of Theorem 1.1. □

Next we present the proof of the maximum principle for anti-symmetric functions.

Proof of Theorem 1.4. Denote $\hat{a} = \sup\{|x_1|,\ x \in \Omega\}$. Let $x = (x_1, x')$, $a = \frac{1}{\hat{a}+1}$ and

$$h(x) = [(1 - a^2 x_1^2)_+^s + 1](1 + |bx'|^2)^{\frac{\gamma}{2}}, \quad 0 < \gamma < 2s.$$

It can be seen from the proof of Lemma 2.1 that we can choose a constant b associated with a in the above equality such that

$$\frac{(-\Delta)^s h(x)}{h(x)} \geq C_1 a^{2s}. \tag{2.6}$$

Denote

$$\bar{w}(x, t) = \frac{e^{mt} w(x, t)}{h(x)},$$

where $m > 0$ is a constant to be determined later. By (1.7) and a direct calculation, $\bar{w}(x, t)$ satisfies

$$\begin{cases} \frac{\partial \bar{w}}{\partial t}(x, t) + \frac{1}{h(x)} C_{n,s} \,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{\bar{w}(x, t) - \bar{w}(y, t)}{|x - y|^{n+2s}} h(y)\, dy & \\ \quad = \left(c(x, t) + m - \frac{(-\Delta)^s h(x)}{h(x)}\right) \bar{w}(x, t), & (x, t) \in \Omega \times \mathbb{R}, \\ \bar{w}(x, t) \geq 0, & (x, t) \in (\Sigma_\lambda \backslash \Omega) \times \mathbb{R}, \\ \bar{w}(x, t) \to 0, & (x, t) \in \Sigma_\lambda \times \mathbb{R}, |x'| \to +\infty. \end{cases} \tag{2.7}$$

Next we prove that

$$\bar{w}(x, t) \geq \min\{0, \inf_{\Omega} \bar{w}(x, \bar{t})\}, \quad (x, t) \in \Omega \times [\bar{t}, T] \text{ for all } [\bar{t}, T] \subset \mathbb{R}. \tag{2.8}$$

If (2.8) is not valid, there exists $(x_0, t_0) \in \Omega \times (\bar{t}, T]$ such that

$$\bar{w}(x_0, t_0) = \min_{\Sigma_\lambda \times (\bar{t}, T]} \bar{w}(x, t) < \min\{0, \inf_{\Omega} \bar{w}(x, \bar{t})\},$$

then

$$\frac{\partial \bar{w}}{\partial t}(x_0, t_0) \leq 0.$$

By (1.8), we may assume that

$$c(x_0, t_0) < \frac{C_1 a^{2s}}{2}.$$

Choosing $m = \frac{C_1 a^{2s}}{2}$, we derive from (2.6) and $\bar{w}(x_0, t_0) < 0$ that

$$\left(c(x_0, t_0) + m - \frac{(-\Delta)^s h(x_0)}{h(x_0)}\right) \bar{w}(x_0, t_0) > 0.$$

On the other hand, by a calculation similar to Theorem 1.1, we derive

$$\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{\bar{w}(x_0, t_0) - \bar{w}(y, t_0)}{|x_0 - y|^{n+2s}} h(y)\, dy < 0.$$

This is a contradiction with (2.7), and we thus arrive at (2.8). That is,

$$\bar{w}(x, t) \geq \min\{0, \inf_{\Omega} \bar{w}(x, \bar{t})\} = \min\left\{0, \inf_{\Omega} \frac{e^{m\bar{t}} w(x, \bar{t})}{h(x)}\right\} \geq -Ce^{m\bar{t}}.$$

It follows that $w(x, t) \geq -Ce^{-m(t-\bar{t})} h(x)$. Since the above inequality holds for any $\bar{t}\,(< T) \in \mathbb{R}$, letting $\bar{t} \to -\infty$, we have $w(x, t) \geq 0$, $(x, t) \in \Omega \times \mathbb{R}$, and thus conclude that $w(x, t) \geq 0$, $(x, t) \in \Sigma_\lambda \times \mathbb{R}$.

Moreover, if there exists a point $(x^0, t^0) \in \Omega \times \mathbb{R}$ such that $w(x^0, t^0) = 0$, then

$$w(x^0, t^0) = \inf_{\Sigma_\lambda \times \mathbb{R}} w(x, t) = 0, \quad \text{and} \quad \frac{\partial w}{\partial t}(x^0, t^0) = 0, \quad (-\Delta)^s w(x^0, t^0) < 0.$$

Therefore, by (1.7), we have

$$0 > \frac{\partial w}{\partial t}(x^0, t^0) + (-\Delta)^s w(x^0, t^0) = c(x^0, t^0) w(x^0, t^0) = 0.$$

This is a contradiction, and we thus obtain the strong maximum principle. This completes the proof of Theorem 1.4. □

3 Liouville Type Theorem in a Half Space

In this section, we first prove Theorem 1.6, the monotonicity of solutions to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin \mathbb{R}^n_+ \times \mathbb{R}, \end{cases}$$

by the method of moving planes, where f is a C^1 function satisfying $f(0) = 0$, $f'(0) \le 0$. We also establish the connection between the existence of solutions of the above problem and the following problem:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

For any $\lambda \in \mathbb{R}$ and $x = (x_1, x')$, $x' \in \mathbb{R}^{n-1}$, we denote $u_\lambda(x,t) = u(x^\lambda, t)$ and $w_\lambda(x,t) = u_\lambda(x,t) - u(x,t)$, where $x^\lambda = (2\lambda - x_1, x')$.

Proof of Theorem 1.6. To show the strict monotonicity of $u(x,t)$ in x_1-direction, we only need to prove that, for any $\lambda > 0$, we have $w_\lambda(x,t) > 0$, $(x,t) \in \Sigma_\lambda \times \mathbb{R}$.

(i) Let $\hat{\Sigma}_\lambda = \{x \in \mathbb{R}^n_+ \mid 0 < x_1 < \lambda\}$; then

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t), & (x,t) \in \hat{\Sigma}_\lambda \times \mathbb{R}, \\ w_\lambda(x,t) \ge 0, & (x,t) \in (\Sigma_\lambda \backslash \hat{\Sigma}_\lambda) \times \mathbb{R}, \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases}$$

where

$$c_\lambda(x,t) = \int_0^1 f'(su(x,t) + (1-s)u_\lambda(x,t))\, ds$$

is bounded. We divide the proof into three steps.

Step 1. We show that, for the sufficiently small λ, we have

$$w_\lambda(x,t) \ge 0, \quad (x,t) \in \Sigma_\lambda \times \mathbb{R}. \tag{3.1}$$

By Theorem 1.1 and taking $\Omega = \hat{\Sigma}_\lambda$, which is a narrow region, we derive $w_\lambda(x,t) \ge 0$, $(x,t) \in \hat{\Sigma}_\lambda \times \mathbb{R}$, then (3.1) follows from the above inequality since $w_\lambda(x,t) \ge 0$ in $(\mathbb{R}^n_+)^c \times \mathbb{R}$.

Step 2. Denote $\lambda_0 = \sup\{\lambda \mid w_\mu(x,t) \ge 0,\ (x,t) \in \Sigma_\mu \times \mathbb{R},\ \mu \le \lambda\}$. In this step, we prove that $\lambda_0 = +\infty$. If not, then $0 < \lambda_0 < +\infty$ and there exists a sequence $\lambda_k \ge \lambda_0$ such that

$$\lambda_k \to \lambda_0 \tag{3.2}$$

and $Z_k := \{(x,t) \in \Sigma_{\lambda_k} \times \mathbb{R} \mid w_{\lambda_k}(x,t) < 0\}$ is nonempty. Set

$$\begin{aligned} m_k := \sup\{u(y_1, x', t) \mid y_1 \in (0, \lambda_k),\ x' \in \mathbb{R}^{n-1},\ t \in \mathbb{R}, \\ \text{and there exists } x_1 \in (0, \lambda_k) \text{ such that } (x_1, x', t) \in Z_k\}. \end{aligned}$$

We consider the following two possibilities:

(a) $m_k \to 0$,

(b) passing to a subsequence, we have $m_k \ge \varepsilon_0$ for some $\varepsilon_0 > 0$.

First assume that case (a) holds. It can be seen from (1.9) that

$$\begin{cases} \dfrac{\partial w_{\lambda_k}}{\partial t}(x,t) + (-\Delta)^s w_{\lambda_k}(x,t) = c_{\lambda_k}(x,t) w_{\lambda_k}(x,t), & (x,t) \in \hat{\Sigma}_{\lambda_k} \times \mathbb{R}, \\ w_{\lambda_k}(x,t) \ge 0, & (x,t) \in (\Sigma_{\lambda_k} \backslash \hat{\Sigma}_{\lambda_k}) \times \mathbb{R}, \\ w_{\lambda_k}(x^{\lambda_k}, t) = -w_{\lambda_k}(x,t), & (x,t) \in \Sigma_{\lambda_k} \times \mathbb{R}, \end{cases}$$

where

$$c_{\lambda_k}(x,t)=\int_0^1 f'(su(x,t)+(1-s)u_{\lambda_k}(x,t))\,ds.$$

Denote $q_k := \sup_{(x,t)\in Z_k} c_{\lambda_k}(x,t)$. By the definition of c_{λ_k}, q_k and m_k, and $f'(0)\le 0$, we have $\overline{\lim}_{k\to+\infty} q_k \le 0$. By the maximum principle for antisymmetric functions (Theorem 1.4), we have $w_{\lambda_k}(x,t)\ge 0$ on Z_k. This of course contradicts the definition of Z_k, and therefore case (a) cannot occur.

Secondly, assume that case (b) holds, then there exist subsequences $x_1^k, y_1^k \in (0,\lambda_k)$, $z^k\in\mathbb{R}^{n-1}$, $t^k\in\mathbb{R}$ such that

$$w_{\lambda_k}(x_1^k,z^k,t^k)<0 \quad\text{and}\quad u(y_1^k,z^k,t^k)\ge\varepsilon_0. \tag{3.3}$$

We may assume that

$$x_1^k\to a,\quad y_1^k\to b \quad \text{for some } a,b\in[0,\lambda_0]. \tag{3.4}$$

Consider the functions

$$u^k(x,t):=u(x_1,x'+z^k,t+t^k),\qquad x=(x_1,x')\in\mathbb{R}^n,\ t\in\mathbb{R},$$

and define $w_{\lambda_k}^k(x,t):=u_{\lambda_k}^k(x,t)-u^k(x,t)$. By (3.3), we have

$$w_{\lambda_k}^k(x_1^k,0,0)<0 \quad\text{and}\quad u^k(y_1^k,0,0)\ge\varepsilon_0. \tag{3.5}$$

Since $u^k(x,t)$ is uniformly bounded, by the regularity estimates for the fractional parabolic equations [19], up to a subsequence (still denoted by u^k), as $k\to+\infty$, we have

$$\begin{aligned} &u^k(x,t)\to\tilde u(x,t),\quad (-\Delta)^s u^k(x,t)\to(-\Delta)^s\tilde u(x,t),\\ &w_{\lambda_k}^k(x,t)\to\tilde w_{\lambda_0}(x,t):=\tilde u_{\lambda_0}(x,t)-\tilde u(x,t)\ge 0,\quad (x,t)\in\Sigma_{\lambda_0}\times\mathbb{R}, \end{aligned} \tag{3.6}$$

and

$$\tilde w_{\lambda_0}(a,0,0)\le 0. \tag{3.7}$$

In addition, we know that $\tilde u(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial\tilde u}{\partial t}(x,t)+(-\Delta)^s\tilde u(x,t)=f(\tilde u(x,t)), & (x,t)\in\mathbb{R}_+^n\times\mathbb{R},\\ \tilde u(x,t)=0, & (x,t)\notin\mathbb{R}_+^n\times\mathbb{R},\\ \tilde u(x,t)\ge 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}. \end{cases} \tag{3.8}$$

By (3.4) and (3.5), we derive that

$$\tilde u(b,0,0)\ge\varepsilon_0>0. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\tilde u(x,t)>0,\quad (x,t)\in\mathbb{R}_+^n\times\mathbb{R}. \tag{3.10}$$

In fact, if (3.10) is false, then there exists a point $(x_0,t_0)\in\mathbb{R}_+^n\times\mathbb{R}$ such that

$$\tilde u(x_0,t_0)=0=\inf_{\mathbb{R}^n\times\mathbb{R}}\tilde u(x,t),$$

then

$$\frac{\partial\tilde u}{\partial t}(x_0,t_0)+(-\Delta)^s\tilde u(x_0,t_0)<0,$$

which contradicts the fact that $f(\tilde u(x_0,t_0))=f(0)=0$. Therefore, (3.10) holds.

Moreover, it can be deduced from (3.6) and (3.8) that $\tilde w_{\lambda_0}(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial\tilde w_{\lambda_0}}{\partial t}(x,t)+(-\Delta)^s\tilde w_{\lambda_0}(x,t)=f(\tilde u_{\lambda_0}(x,t))-f(\tilde u(x,t)), & (x,t)\in\Sigma_{\lambda_0}\times\mathbb{R},\\ \tilde w_{\lambda_0}(x,t)\ge 0, & (x,t)\in\Sigma_{\lambda_0}\times\mathbb{R},\\ \tilde w_{\lambda_0}(0,x',t)>0, & (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}, \end{cases}$$

and we conclude that

$$\tilde{w}_{\lambda_0}(x,t) > 0, \quad (x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}. \tag{3.11}$$

If (3.11) is not valid, then there exists a point $(x^*, t^*) \in \Sigma_{\lambda_0} \times \mathbb{R}$ such that $\tilde{w}_{\lambda_0}(x^*, t^*) = \inf_{\mathbb{R}^n \times \mathbb{R}} \tilde{w}_{\lambda_0}(x,t) = 0$. It follows that

$$\frac{\partial \tilde{w}_{\lambda_0}}{\partial t}(x^*, t^*) = 0 \quad \text{and} \quad (-\Delta)^s \tilde{w}_{\lambda_0}(x^*, t^*) < 0.$$

Therefore,

$$0 > \frac{\partial \tilde{w}_{\lambda_0}}{\partial t}(x^*, t^*) + (-\Delta)^s \tilde{w}_{\lambda_0}(x^*, t^*) = f(\tilde{u}_{\lambda_0}(x^*, t^*)) - f(\tilde{u}(x^*, t^*)) = 0.$$

This is a contradiction, and therefore, we derive (3.11).

Since $a \in [0, \lambda_0]$ in (3.4), combining (3.7) with (3.11), we must have $\tilde{w}_{\lambda_0}(a, 0, 0) = 0$ with $a = \lambda_0$, which indicates that

$$\lim_{k \to \infty} x_1^k = a = \lambda_0. \tag{3.12}$$

To proceed with the proof, we need the following Hopf lemma for antisymmetric functions, whose proof is similar to that for [42, Theorem 3.1]. However, for readers' convenience, we include it in the appendix.

Lemma 3.1 (Hopf's Lemma for Antisymmetric Functions). *Assume that $w_\lambda(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ is bounded and satisfies*

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \\ w_\lambda(x,t) \geq 0, & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases}$$

where $c_\lambda(x,t)$ is bounded from below. If there exists a point $x \in \Sigma_\lambda$ such that $w_\lambda(x,t) > 0$, $(x,t) \in \Sigma_\lambda \times \mathbb{R}$, then

$$\frac{\partial w_\lambda}{\partial x_1}(x^0, t_0) < 0 \quad \textit{for all } (x^0, t_0) \in T_\lambda \times \mathbb{R}.$$

Now we continue our proof in Step 2.

First, since $\tilde{u}(x,t) \equiv 0$ for $(x,t) \in \overline{\mathbb{R}^n_-} \times \mathbb{R}$, it follows from (3.10) that there exists a point $x \in \Sigma_{\lambda_0}$ such that $\tilde{w}_{\lambda_0}(x,t) > 0$ for all $t \in \mathbb{R}$. Second, $\tilde{w}_{\lambda_0}(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{w}_{\lambda_0}}{\partial t}(x,t) + (-\Delta)^s \tilde{w}_{\lambda_0}(x,t) = \tilde{c}_{\lambda_0}(x,t) \tilde{w}_{\lambda_0}(x,t), & (x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}, \\ \tilde{w}_{\lambda_0}(x,t) \geq 0, & (x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}, \\ \tilde{w}_{\lambda_0}(x^{\lambda_0}, t) = -\tilde{w}_{\lambda_0}(x,t), & (x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}, \end{cases}$$

where

$$\tilde{c}_{\lambda_0}(x,t) = \int_0^1 f'(s\tilde{u}(x,t) + (1-s)\tilde{u}_{\lambda_0}(x,t))\, ds.$$

Therefore, we derive from Hopf's lemma (Lemma 3.1) that

$$\frac{\partial \tilde{w}_{\lambda_0}}{\partial x_1}(x,t) < 0 \quad \text{for all } (x,t) \in T_{\lambda_0} \times \mathbb{R}.$$

It follows that

$$\frac{\partial \tilde{u}}{\partial x_1}(\lambda_0, 0, 0) = -\frac{1}{2} \frac{\partial \tilde{w}_{\lambda_0}}{\partial x_1}(\lambda_0, 0, 0) > 0.$$

Therefore, $\frac{\partial \tilde{u}}{\partial x_1}(x_1, 0, 0)$ is bounded from below by a positive constant in a neighborhood of λ_0, and this remains valid for $\frac{\partial u^k}{\partial x_1}(x_1, 0, 0)$, i.e., there exists a positive constant $\delta > 0$ such that, for all sufficiently large k,

$$\frac{\partial u}{\partial x_1}(x_1, z^k, t^k) = \frac{\partial u^k}{\partial x_1}(x_1, 0, 0) > 0, \quad x_1 \in [\lambda_0 - \delta, \lambda_0 + \delta].$$

This contradicts the fact that $w_{\lambda_k}(x_1^k, z^k, t^k) < 0$ in (3.3). Indeed, if k is sufficiently large, then $2\lambda_k - x_1^k > x_1^k$. Recall from (3.2) and (3.12) that $\lim_{k \to \infty} \lambda_k = \lim_{k \to \infty} x_1^k = \lambda_0$, and therefore, both of $2\lambda_k - x_1^k$ and x_1^k belong to $[\lambda_0 - \delta, \lambda_0 + \delta]$. Therefore, we conclude that $\lambda_0 = +\infty$.

Step 3. Combining Step 1 with Step 2, we derive that, for any $0 < \lambda < +\infty$,

$$w_\lambda(x,t) \geq 0, \quad (x,t) \in \Sigma_\lambda \times \mathbb{R}.$$

It follows that if there exists a point $(x^0,t^0) \in \Sigma_\lambda \times \mathbb{R}$ such that $w_\lambda(x^0,t^0) = 0 = \inf_{\mathbb{R}^n\times\mathbb{R}} w_\lambda(x,t)$, then

$$\frac{\partial w_\lambda}{\partial t}(x^0,t^0) + (-\Delta)^s w_\lambda(x^0,t^0) < 0,$$

which contradicts $f(u_\lambda(x^0,t^0)) - f(u(x^0,t^0)) = 0$. Therefore, for any $0 < \lambda < +\infty$,

$$w_\lambda(x,t) > 0, \quad (x,t) \in \Sigma_\lambda \times \mathbb{R}.$$

It follows that

$$\frac{\partial u}{\partial x_1}(x,t) > 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

Now we have verified that the solution u is strictly increasing along x_1 direction. Next we will use this property to construct an entire solution of the same equation in the whole space of dimension $n-1$: $\mathbb{R}^{n-1}\times\mathbb{R}$.

(ii) Let u be a positive bounded solution of (1.9). For $k = 1,2,\dots$, consider the functions

$$u_k(x',t) := u(k,x',t), \quad (x',t) \in \mathbb{R}^{n-1}\times\mathbb{R}.$$

Each of them solves the equation

$$\frac{\partial u_k}{\partial t}(x',t) + (-\Delta)^s_{\mathbb{R}^{n-1}} u_k(x',t) = f(u_k(x',t)), \quad (x',t) \in \mathbb{R}^{n-1}\times\mathbb{R}.$$

Since the sequence $u_k(x',t)$ is uniformly bounded, using parabolic estimates, one can show that there exists a subsequence of $u_k(x',t)$ converging uniformly on arbitrary compact set to a bounded nonnegative solution $\bar{u}(x',t)$ of

$$\frac{\partial \bar{u}}{\partial t}(x',t) + (-\Delta)^s_{\mathbb{R}^{n-1}} \bar{u}(x',t) = f(\bar{u}(x',t)), \quad (x',t) \in \mathbb{R}^{n-1}\times\mathbb{R}.$$

Here we have used the fact that $(-\Delta)^s_{\mathbb{R}^n}\bar{u}(x') = (-\Delta)^s_{\mathbb{R}^{n-1}}\bar{u}(x')$, which can be obtained by a direct calculation as shown in the appendix. This proves (ii) and completes the proof of Theorem 1.6. □

Now, based on Theorem 1.6, we are able to derive the Liouville type theorem of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = u^p(x,t), & (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin \mathbb{R}^n_+ \times \mathbb{R}, \end{cases}$$

by applying the following nonexistence result.

Lemma 3.2 ([23]). *Let $0 < \frac{\alpha n}{2s} \leq 1 + \sigma$, and assume $u \in (C^{1,1}_{\text{loc}} \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ satisfies*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = h(t)u^{1+\alpha}(x,t), & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{3.13}$$

where $u_0(x)$ is a nontrivial nonnegative and continuous function on $\mathbb{R}^n$, and $h(t)$ satisfies
(h_1) $h \in C[0,+\infty)$, $h \geq 0$,
(h_2) $c_0 t^\sigma \leq h(t) \leq c_1 t^\sigma$ *for sufficiently large t, where $c_0, c_1 > 0$ and $\sigma > -1$ are constants.*
Then the nonnegative solution $u(x,t)$ of (3.13) blows up for some $T_0 > 0$, and $u(x,t) = +\infty$ for every $t \geq T_0$ and $x \in \mathbb{R}^n$.

Proof of Theorem 1.7. As a consequence of Lemma 3.2, we derive that if $1 < p \leq \frac{n+2s}{n}$, then

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = u^p(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.14}$$

has no nontrivial nonnegative bounded solutions. Combining (3.14) with conclusion (ii) in Theorem 1.6, we arrive at Theorem 1.7. □

4 More Relevant Liouville Type Theorems

In this section, we employ the methods developed in the previous sections to prove Theorem 1.8.

Proof of Theorem 1.8. (i) We can deduce from (1.11) that $w_\lambda(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \\ w_\lambda(x^\lambda, t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases} \tag{4.1}$$

where

$$c_\lambda(x,t) = \int_0^1 f'(su(x,t) + (1-s)u_\lambda(x,t))\,ds$$

is bounded. Since $f' \le 0$, we have $c_\lambda(x,t) \le 0$, $(x,t) \in \Sigma_\lambda \times \mathbb{R}$. We choose the auxiliary function as

$$g(x) = |x - (\lambda+1)e_1|^\sigma, \quad \bar{w}_\lambda(x,t) = \frac{w_\lambda(x,t)}{g(x)},$$

where $e_1 = (1, 0, \dots, 0)$, and σ is a small positive number to be chosen as in the proof of [11, Theorem 1].

Obviously, $\bar{w}_\lambda(x,t)$ and $w_\lambda(x,t)$ have the same sign, and we have $\lim_{|x|\to+\infty} \bar{w}_\lambda(x,t) = 0$. For any fixed $t \in \mathbb{R}$, denote

$$\bar{w}_\lambda(x(t),t) = \inf_{x\in\Sigma_\lambda} \bar{w}_\lambda(x,t). \tag{4.2}$$

First we conclude that, for any fixed $t \in \mathbb{R}$, if $\bar{w}_\lambda(x(t),t) < 0$, then

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(t),t) \ge \frac{-C}{|x_1(t) - \lambda|^{2s}} \bar{w}_\lambda(x(t),t). \tag{4.3}$$

In fact, by a calculation similar to [11, formula (22)], we derive

$$(-\Delta)^s w_\lambda(x(t),t) \le \frac{C}{|x_1(t) - \lambda|^{2s}} w_\lambda(x(t),t).$$

Combining this with (4.1), we derive

$$\frac{\partial w_\lambda}{\partial t}(x(t),t) \ge \frac{-C}{|x_1(t) - \lambda|^{2s}} w_\lambda(x(t),t).$$

Therefore, (4.3) follows from the above inequality and the definition of $\bar{w}_\lambda(x,t)$ in (4.2).

For any fixed $t \in \mathbb{R}$, denote

$$m(t) := \bar{w}_\lambda(x(t),t) = \inf_{x\in\Sigma_\lambda} \bar{w}_\lambda(x,t).$$

To proceed with the proof, we need the following lemma.

Lemma 4.1. *For any fixed $t \in \mathbb{R}$, if $m(t) \le -m_0$, then*

$$\frac{C}{|x_1(t) - \lambda|^{2s}} > c_0 > 0, \tag{4.4}$$

where $x(t) = (x_1(t), \dots, x_n(t))$ is a minimum point of $\bar{w}_\lambda(x,t)$ in Σ_λ.

Proof. If (4.4) is not valid, then there exists a sequence of $\{t_k\}$, $k = 1, 2, \dots$, such that

$$m(t_k) \le -m_0 \tag{4.5}$$

and

$$\frac{C}{|x_1(t_k) - \lambda|^{2s}} \to 0, \quad k \to +\infty;$$

therefore, $|x_1(t_k)| \to +\infty$, $k \to +\infty$, and it follows that $m(t_k) = \bar{w}_\lambda(x(t_k), t_k) \to 0$, $k \to +\infty$, which contradicts (4.5) and hence completes the proof of Lemma 4.1. □

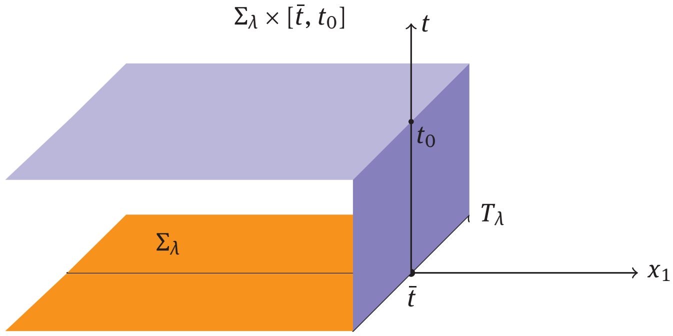

Figure 1: The domain of definition of $v(x, t)$.

Now we continue our proof. We want to show that

$$w_\lambda(x, t) \geq 0, \quad (x, t) \in \Sigma_\lambda \times \mathbb{R} \quad \text{for all } \lambda \in \mathbb{R}. \tag{4.6}$$

If (4.6) is false, then there exists $t_0 \in \mathbb{R}$ such that

$$-m_0 := m(t_0) = \bar{w}_\lambda(x(t_0), t_0) < 0. \tag{4.7}$$

To derive a contradiction with (4.7), for any $\bar{t} < t_0$, we construct a subsolution $z(t) = -\bar{M}e^{-c_0(t-\bar{t})}$, where c_0 is as defined in (4.4) and $-\bar{M} = \inf_{\Sigma_\lambda \times \mathbb{R}} \bar{w}_\lambda(x, t)$.

Next we prove that

$$\bar{w}_\lambda(x, t) \geq z(t), \quad (x, t) \in \overline{\Sigma_\lambda} \times [\bar{t}, t_0]. \tag{4.8}$$

Consider the function

$$v(x, t) = \bar{w}_\lambda(x, t) - z(t), \quad (x, t) \in \overline{\Sigma_\lambda} \times [\bar{t}, t_0],$$

whose domain of definition is shown in Figure 1.

First notice that, on the bottom of the cylinder $\Sigma_\lambda \times [\bar{t}, t_0]$, we have

$$v(x, t) = \bar{w}_\lambda(x, t) - z(t) = \bar{w}_\lambda(x, t) - (-\bar{M}) \geq 0, \quad (x, t) \in \Sigma_\lambda \times \{\bar{t}\},$$

and on the lateral surface of the cylinder $\Sigma_\lambda \times [\bar{t}, t_0]$, we have

$$v(x, t) = \bar{w}_\lambda(x, t) - z(t) = -z(t) \geq 0, \quad (x, t) \in T_\lambda \times [\bar{t}, t_0].$$

If (4.8) does not hold, then there exists a point $(x(\tilde{t}), \tilde{t}) \in \Sigma_\lambda \times (\bar{t}, t_0]$ such that

$$v(x(\tilde{t}), \tilde{t}) = \inf_{\overline{\Sigma_\lambda} \times (\bar{t}, t_0]} v(x, t) < 0 \tag{4.9}$$

and

$$\frac{\partial v}{\partial t}(x(\tilde{t}), \tilde{t}) \leq 0. \tag{4.10}$$

On one hand, from the definition of $v(x, t)$, we have

$$\bar{w}_\lambda(x(\tilde{t}), \tilde{t}) = \inf_{\Sigma_\lambda} \bar{w}_\lambda(x, \tilde{t}) < z(\tilde{t}) < 0.$$

Therefore, by (4.3), we have

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(\tilde{t}), \tilde{t}) \geq \frac{-C}{|x_1(\tilde{t}) - \lambda|^{2s}} \bar{w}_\lambda(x(\tilde{t}), \tilde{t}). \tag{4.11}$$

On the other hand, we obtain from (4.9) that $v(x(\tilde{t}), \tilde{t}) \leq v(x(t_0), t_0)$, i.e.,

$$\bar{w}_\lambda(x(\tilde{t}), \tilde{t}) - \bar{w}_\lambda(x(t_0), t_0) \leq z(\tilde{t}) - z(t_0) \leq 0$$

due to the monotonicity of $z(t)$. Therefore,

$$m(\tilde{t}) = \bar{w}_\lambda(x(\tilde{t}), \tilde{t}) \le \bar{w}_\lambda(x(t_0), t_0) = m(t_0) = -m_0. \tag{4.12}$$

Using Lemma 4.1, we derive from (4.12) that

$$\frac{C}{|x_1(\tilde{t}) - \lambda|^{2s}} > c_0 > 0.$$

Combining this with (4.11), we obtain

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(\tilde{t}), \tilde{t}) \ge -c_0 \bar{w}_\lambda(x(\tilde{t}), \tilde{t}).$$

Then, by (4.10), we derive

$$-c_0 z(\tilde{t}) = \frac{\partial z}{\partial t}(\tilde{t}) \ge \frac{\partial \bar{w}_\lambda}{\partial t}(x(\tilde{t}), \tilde{t}) \ge -c_0 \bar{w}_\lambda(x(\tilde{t}), \tilde{t}),$$

and consequently, $z(\tilde{t}) \le \bar{w}_\lambda(x(\tilde{t}), \tilde{t})$, which contradicts $v(x(\tilde{t}), \tilde{t}) < 0$. Therefore, we conclude that

$$\bar{w}_\lambda(x, t) \ge z(t) \ge z(\bar{t}), \quad (x, t) \in \overline{\Sigma_\lambda} \times [\bar{t}, t_0],$$

i.e., we derive (4.8).

Now let $\bar{t} \to -\infty$; since $z(\bar{t}) \to 0$, we have $\bar{w}_\lambda(x, t) \ge 0$, $(x, t) \in \overline{\Sigma_\lambda} \times (-\infty, t_0]$. This is a contradiction with the assumption (4.7) that $\bar{w}_\lambda(x(t_0), t_0) < 0$. As a consequence of the above results, we obtain (4.6). Therefore, we derive that $u(x, t)$ is increasing in x_1-direction due to the arbitrariness of λ, i.e.,

$$\frac{\partial u}{\partial x_1}(x, t) \ge 0.$$

Replacing x_1 by $-x_1$, we have

$$\frac{\partial u}{\partial x_1}(x, t) \le 0.$$

It follows that

$$\frac{\partial u}{\partial x_1}(x, t) = 0$$

and u is constant in x_1. Applying the same argument for other x_i, one shows that u is constant in all x_i $(1 \le i \le n)$. Hence u is a solution of $u_t = f(u(t))$ for all $t \in \mathbb{R}$. This proves (i).

Parts (ii) and (iii) follow from conclusion (i) and an elementary analysis; see [44, Propositions 3.10 and 3.11]. □

Corollary 4.2. *Assume $f\colon [0, \infty) \to \mathbb{R}$ is a decreasing C^1 function, and $u \in (C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+}) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ satisfies*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t) + (-\Delta)^s u(x, t) = f(u(x, t)), & (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x, t) = 0, & (x, t) \notin \mathbb{R}^n_+ \times \mathbb{R}. \end{cases} \tag{4.13}$$

(i) *Then each positive bounded solution u of* (4.13) *is increasing in x_1-direction,*

$$\frac{\partial u}{\partial x_1}(x, t) > 0, \quad (x, t) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

(ii) *If $f(c) < 0$ as $c \ge 0$, then the nonnegative bounded solution of* (4.13) *does not exist.*

Proof. For any given $0 < \lambda < +\infty$, let $\hat{\Sigma}_\lambda = \{x \in \mathbb{R}^n \mid 0 < x_1 < \lambda\}$. By (4.13), we have

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t}(x, t) + (-\Delta)^s w_\lambda(x, t) = c_\lambda(x, t) w_\lambda(x, t), & (x, t) \in \hat{\Sigma}_\lambda \times \mathbb{R}, \\ w_\lambda(x, t) \ge 0, & (x, t) \in (\Sigma_\lambda \backslash \hat{\Sigma}_\lambda) \times \mathbb{R}, \\ w_\lambda(x^\lambda, t) = -w_\lambda(x, t), & (x, t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases}$$

where

$$c_\lambda(x,t) = \int_0^1 f'(su(x,t) + (1-s)u_\lambda(x,t))\,ds$$

is bounded. Since $f' \leq 0$, we have $c_\lambda(x,t) \leq 0$, $(x,t) \in \hat{\Sigma}_\lambda \times \mathbb{R}$. Using Theorem 1.4, we derive $w_\lambda(x,t) > 0$, $(x,t) \in \hat{\Sigma}_\lambda \times \mathbb{R}$. Since $0 < \lambda < +\infty$ is arbitrary, we derive (i).

Since $u(x,t)$ is bounded in $\mathbb{R}^n \times \mathbb{R}$, by an argument similar to Theorem 1.6 (ii), the limit

$$\bar{u}(x',t) = \lim_{x_1 \to +\infty} u(x_1, x', t), \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

exists and satisfies

$$\frac{\partial \bar{u}}{\partial t}(x,t) + (-\Delta)^s \bar{u}(x,t) = f(\bar{u}(x,t)), \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

It follows from (iii) in Theorem 1.8 that the nonnegative bounded solution of (4.13) does not exist. This completes the proof of Corollary 4.2. □

A Appendix

Lemma A.1 (Hopf's Lemma for Antisymmetric Functions). *Assume that $w_\lambda(x,t) \in (C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{L}_{2s}) \times C^1(\mathbb{R})$ is bounded and satisfies*

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t) w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \\ w_\lambda(x,t) \geq 0, & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \\ w_\lambda(x^\lambda,t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases}$$

where $c_\lambda(x,t)$ is bounded from below if there exists a point $x \in \Sigma_\lambda$ such that

$$w_\lambda(x,t) > 0, \quad (x,t) \in \Sigma_\lambda \times \mathbb{R}. \tag{A.1}$$

Then

$$\frac{\partial w_\lambda}{\partial x_1}(\tilde{x},\tilde{t}) < 0 \quad \textit{for all } (\tilde{x},\tilde{t}) \in T_\lambda \times \mathbb{R}.$$

Proof. Without loss of generality, we may assume that $\lambda = 0$ and $\tilde{x} = 0$. If $\lambda = 0$, we denote $\Sigma_\lambda = \Sigma_0$, $T_\lambda = T_0$, $x^\lambda = x^0 = (-x_1, x')$, $y^\lambda = y^0 = (-y_1, y')$ and $w^\lambda(x,t) = w_0(x,t)$, $c_\lambda(x,t) = c_0(x,t)$. Let $\tilde{w}(x,t) = e^{mt} w_0(x,t)$, $m > 0$. Since $c_0(x,t)$ is bounded from below, we can choose m such that $m + c_0(x,t) \geq 0$. For fixed $\tilde{t}$, $\tilde{w}(x,t)$ satisfies

$$\frac{\partial \tilde{w}}{\partial t}(x,t) + (-\Delta)^s \tilde{w}(x,t) = (m + c_0(x,t))\tilde{w}(x,t) \geq 0, \quad (x,t) \in \Sigma_0 \times [\tilde{t}-1, \tilde{t}+1]. \tag{A.2}$$

By (A.1) and the continuity of w_0, there exists a set $D \subset\subset \Sigma_0$ and a positive constant c such that

$$w_0(x,t) > c, \quad (x,t) \in \Sigma_0 \times [\tilde{t}-1, \tilde{t}+1]. \tag{A.3}$$

Let D_0 be the reflection of D about the plane T_0 for any time t. Denote $g(x) = x_1 \zeta(x)$, where

$$\zeta(x) = \zeta(|x|) = \begin{cases} 1, & |x| < \varepsilon, \\ 0, & |x| \geq 2\varepsilon, \end{cases}$$

and $0 \leq \zeta(x) \leq 1$, $\zeta(x) \in C_0^\infty(B_{2\varepsilon}(0))$. Obviously, $g(x)$ is an antisymmetric function with respect to plane T_0, i.e., $g(-x_1, x_2, \ldots, x_n) = -g(x_1, x_2, \ldots, x_n)$.

Now we construct a subsolution $\underline{w}(x,t) = \chi_{D \cup D_0}(x)\tilde{w}(x,t) + \delta\eta(t)g(x)$, where

$$\chi_{D \cup D_0}(x) = \begin{cases} 1, & x \in D \cup D_0, \\ 0, & x \notin D \cup D_0, \end{cases}$$

and $\eta(t) \in C_0^\infty([\tilde{t}-1, \tilde{t}+1])$ satisfies

$$\eta(t) = \begin{cases} 1, & t \in [\tilde{t}-\frac{1}{2}, \tilde{t}+\frac{1}{2}], \\ 0, & t \notin [\tilde{t}-1, \tilde{t}+1]. \end{cases}$$

Since $g(x)$ is a $C_0^\infty(B_{2\varepsilon}(0))$ function, we have $(-\Delta)^s g(x) \in C^\infty(B_{2\varepsilon}(0))$. Then we apply the mean value theorem on $(-\Delta)^s g(x)$ at $\bar{x} = (0, x') \in T_0$ and $x = (x_1, x') \in B_{2\varepsilon}(0)$ to obtain

$$|(-\Delta)^s g(x)| = |(-\Delta)^s g(\bar{x}) + \nabla((-\Delta)^s g)(\xi) \cdot (x - \bar{x})| \le C_0 x_1, \tag{A.4}$$

where ξ lies between $\bar{x}$ and x, and $(-\Delta)^s g(\bar{x}) = 0$ due to the antisymmetry of $g(x)$.

By the definition of fractional Laplacian and (A.3), we derive that, for each fixed $t \in [\tilde{t}-1, \tilde{t}+1]$ and for any $x \in B_{2\varepsilon}(0) \cap \Sigma_0$, we have

$$\begin{aligned} (-\Delta)^s(\chi_{D\cup D_0}\tilde{w}(x,t)) &= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{\chi_{D\cup D_0}(x)\tilde{w}(x,t) - \chi_{D\cup D_0}(y)\tilde{w}(y,t)}{|x-y|^{n+2s}}\,dy \\ &= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{-\chi_{D\cup D_0}(y)\tilde{w}(y,t)}{|x-y|^{n+2s}}\,dy \\ &= C_{n,s}\,\mathrm{P.V.}\int_D \frac{-\tilde{w}(y,t)}{|x-y|^{n+2s}}\,dy + \int_D \frac{-\tilde{w}(y^0,t)}{|x-y^0|^{n+2s}}\,dy \\ &= C_{n,s}\,\mathrm{P.V.}\int_D \Big(\frac{1}{|x-y^0|^{n+2s}} - \frac{1}{|x-y|^{n+2s}}\Big)\tilde{w}(y,t)\,dy \\ &= -C_{n,s}x_1 \int_D \frac{2(n+2s)y_1}{\zeta(y)^{n+2s+2}}\tilde{w}(y,t)\,dy \le -C_1 x_1, \end{aligned} \tag{A.5}$$

where $\zeta(y)$ is some number between $|x-y|$ and $|x-y^0|$, C_1 is a positive constant, and the second inequality from the bottom holds due to the application of the mean value theorem to $h(z) = z^{-\frac{n+2s}{2}}$ over $[z_1, z_2]$ with $z_2 = |x-y|^2$ and $z_2 = |x-y^0|^2$.

For $(x,t) \in (B_{2\varepsilon}(0) \cap \Sigma_0) \times [\tilde{t}-1, \tilde{t}+1]$, by (A.4) and (A.5), we obtain

$$\begin{aligned} \frac{\partial \underline{w}}{\partial t} + (-\Delta)^s \underline{w}(x,t) &= \delta\eta'(t)g(x) + (-\Delta)^s(\chi_{D\cup D_0}\tilde{w}(x,t)) + \delta\eta(t)(-\Delta)^s g(x) \\ &\le \delta\eta'(t)g(x) - C_1x_1 + \delta\eta(t)C_0x_1. \end{aligned}$$

Hence, taking δ sufficiently small, we derive

$$\frac{\partial \underline{w}}{\partial t} + (-\Delta)^s \underline{w}(x,t) \le 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \Sigma_0) \times [\tilde{t}-1, \tilde{t}+1]. \tag{A.6}$$

Set $v(x,t) = \tilde{w}(x,t) - \underline{w}(x,t)$. Obviously, $v(x,t) = -v(x^0,t)$. From (A.2) and (A.6), we derive that $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t}(x,t) + (-\Delta)^s v(x,t) \ge 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \Sigma_0) \times [\tilde{t}-1, \tilde{t}+1].$$

Also, by the definition of $\underline{w}(x,t)$, we have

$$v(x,t) \ge 0, \quad (x,t) \in (\Sigma_0 \setminus (B_{2\varepsilon}(0) \cap \Sigma_0)) \times [\tilde{t}-1, \tilde{t}+1],$$

and $v(x, \tilde{t}-1) \ge 0$, $x \in \Sigma_0$. Now we apply the following lemma to $v(x,t)$.

Lemma A.2 (Maximum Principle for Antisymmetric Functions [14]). *Let Ω be a bounded domain in Σ_λ. Assume that $w_\lambda(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_{2s}) \times C^1([0,\infty))$ is lower semi-continuous in x on $\bar{\Omega}$ and satisfies*

$$\begin{cases} \frac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) \ge c_\lambda(x,t)w_\lambda(x,t), & (x,t) \in \Omega \times (0,\infty), \\ w_\lambda(x^\lambda,t) = -w_\lambda(x,t), & (x,t) \in \Sigma_\lambda \times [0,\infty), \\ w_\lambda(x,t) \ge 0, & (x,t) \in (\Sigma_\lambda \setminus \Omega) \times [0,\infty), \\ w_\lambda(x,0) \ge 0, & x \in \Omega. \end{cases}$$

If $c_\lambda(x,t)$ is bounded from above, then $w_\lambda(x,t) \ge 0$, $(x,t) \in \Omega \times [0,T]$ for all $T > 0$.

As an immediate consequence of this lemma with $c_0(x,t)=0$, we obtain

$$v(x,t)\geq 0,\quad (x,t)\in(B_{2\varepsilon}(0)\cap\Sigma_0)\times[t_0-1,t_0+1].$$

This implies that

$$e^{mt}w_0(x,t)-\delta g(x)\eta(t)\geq 0,\quad (x,t)\in(B_{2\varepsilon}(0)\cap\Sigma_0)\times[\tilde{t}-1,\tilde{t}+1].$$

It follows that

$$w_0(x,t)\geq e^{-mt}\delta x_1,\quad (x,t)\in(B_{\varepsilon}(0)\cap\Sigma_0)\times[\tilde{t}-\tfrac{1}{2},\tilde{t}+\tfrac{1}{2}].$$

Since $w_0(0,\tilde{t})=0$, we have

$$\frac{w_0(x,\tilde{t})-w_0(0,\tilde{t})}{x_1-0}\geq\delta e^{-m\tilde{t}}>0,\quad x\in B_{\varepsilon}(0)\cap\Sigma_0.$$

Therefore,

$$\frac{\partial w_0}{\partial x_1}(0,\tilde{t})<0.$$

This completes the proof of Lemma A.1. □

Lemma A.3. *Let $x=(x',x_n)$, where $x'\in\mathbb{R}^{n-1}$; then $(-\Delta)^s_{\mathbb{R}^n}u(x')=(-\Delta)^s_{\mathbb{R}^{n-1}}u(x')$.*

Proof. By the definition of the fractional Laplacian, we have

$$\begin{aligned}(-\Delta)^s_{\mathbb{R}^n}u(x')&=C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x')-u(y')}{|x-y|^{n+2s}}\,dy\\&=C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x')-u(y')}{(|x'-y'|^2+|x_1-y_1|^2)^{\frac{n+2s}{2}}}\,dy_1\,dy'\\&=C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x')-u(y')}{(|x'-y'|^2+(|x'-y'|t)^2)^{\frac{n+2s}{2}}}|x'-y'|\,dt\,dy'\\&=\int_0^{+\infty}\frac{1}{(1+t^2)^{\frac{n+2s}{2}}}\,dt\cdot C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^{n-1}}\frac{u(x')-u(y')}{|x'-y'|^{n+2s-1}}\,dy',\end{aligned}\tag{A.7}$$

where

$$C_{n,s}=\Big(\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{n+2s}}\,d\zeta\Big)^{-1},\tag{A.8}$$

and $\zeta=(\zeta_1,\zeta')$; please refer to [12].

Since

$$\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{n+2s}}\,d\zeta=\int_0^{+\infty}\frac{1}{(1+t^2)^{\frac{n+2s}{2}}}\,dt\int_{\mathbb{R}^{n-1}}\frac{1-\cos(2\pi\zeta_1)}{|\zeta'|^{n+2s}}\,d\zeta'.\tag{A.9}$$

Combining (A.7)–(A.9), we derive

$$\int_0^{+\infty}\frac{1}{(1+t^2)^{\frac{n+2s}{2}}}\,dt\cdot C_{n,s}=C_{n-1,s}.$$

Therefore,

$$(-\Delta)^s_{\mathbb{R}^n}u(x')=C_{n-1,s}\,\mathrm{P.V.}\int_{\mathbb{R}^{n-1}}\frac{u(x')-u(y')}{|x'-y'|^{n+2s-1}}\,dy'=(-\Delta)^s_{\mathbb{R}^{n-1}}u(x').$$

This completes the proof of Lemma A.3. □

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