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Consensus of multi-agent systems with randomly occurring nonlinearities via uncertain pinning control under switching topologies

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Abstract
This paper is focused on the consensus problem of multi-agent systems via uncertain pinning control under switching topologies. The stochastic disturbances and randomly occurring nonlinearities are proposed to describe more realistic systems. The communication topology is modeled by a directed graph and it is divided into two cases, the consensus problem is discussed in these two cases. In addition, there exist some uncertain pinning connections between the followers and leader due to switching topologies, the distributed control protocol is designed to satisfy the follower asymptotically converge to the leader. By constructing suitable multiple Lyapunov functions and utilizing tools of M-matrix theory, some sufficient consensus criteria are deduced to reach this goal. Finally, two examples are given to verify the correctness of the proposed method.

Keywords Consensus · Switching topologies · Stochastic disturbances · Randomly occurring nonlinearities

1 Introduction

Over the last decades, the consensus problem of multi-agent systems has received extensive attention and applied in many fields such as distributed computation, cooperative surveillance, satellite attitude control, UAV formation and so on[1–4]. It is worth noting that scholars have found some interesting phenomena like the phase transitions in physical, swarming of insects,

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clustering of fish and the lead flight of birds, all of which are formed through the information transmission and exchange between individuals to form regular group behavior. In real life, we refer to entities that have independent thoughts and can interact with the surrounding environment as agents. The problem of consensus is that all agents converge to a certain state with the evolution of time, which is of great practical significance and theoretical value as the basis for cooperative coordination and control between agents[5, 6]. In [7], the authors discuss the problem of consensus under fixed topologies through event-triggered control. By using matrix inequality theory, the problem of sample-data consensus is studied in [8]. Some other relevant research progresses were surveyed in [9–11].

It is worth pointing out that the dynamic behavior of multi-agent systems is often influenced by environmental disturbances like the uncertainty of the external environment, the emergence of external interference and network-induced random failures [12–14], the internal dynamic model of each agent is generally dynamic, so random occurring nonlinearity is inevitable[15]. At this time, the nonlinear disturbance may undergo random mutations, and the nonlinear dynamic behavior may occur in the form of probability [15]. Recently, the problem of robust filtering has been studied for sensor networks with missing measurements and randomly occurring nonlinearities [16]. In [17], the author studied the stability problem for networked systems with randomly occurring nonlinearities under adaptive event-triggered control. In addition, the stability analysis has also been studied for stochastic neural networks with randomly occurring nonlinearities, see [18]. According to information, the randomly occurring nonlinearities are rarely considered in multi-agent systems.

In many practical applications, stochastic perturbation is ubiquitous [19, 20]. At the same time, dynamic behavior is often disturbed by various disturbances due to modeling errors, measurement errors or external disturbances. Since the influence of various disturbances on the stability of the system is very large, how to study and solve the effects of stochastic perturbations has become an interesting and important topic. Up to now, some basic work about the problem of consensus with stochastic disturbances has been discussed in [21, 22]. As far as we know, multi-agent system with stochastic disturbances and randomly occurring nonlinearities are rarely considered.

On the other hand, in the context of multi-agent systems, the communication topology makes a significant role in achieving consensus. The primary purpose in multi-agent systems is proposing appropriate conditions on the communication topologies such that consensus can be achieved. A particularly interesting topic is that previous literature has often been discussed under the premise of fixed topologies and the connection topology always satisfies some connection conditions. For example, in [9], the authors discuss the problem of Leader-following consensus under fixed topologies, which requires the communication topology to include a directed spanning tree. Furthermore, in practice, the system is often subject to some external environment interference, resulting in real-time changes in the connection structure. Therefore, inspired by this observation, research for the consensus problem under switching topologies is more meaningful.

It is well known that the choice of control protocol is particularly important for the study of consensus problem. Common types of control protocols include feedback control, sampling control, event-triggered control, intermittent control, pinning control and so on. Among them, pinning control is often used in multi-agent systems, it only needs to control a part of the agent [23, 24]. When the topology structure is unfixed, this leads to the selection of the pinning node unfixed. Motivated by this observation, the problem of consensus is explored via uncertain pinning control under switching topologies. Currently, tracking consensus problem has been discussed under switching topologies by event triggering control see [25]. In [26], for heterogeneous linear systems under switching topologies, the tracking control problem is

studied. However, as yet, under switching topology, the problem of consensus with stochastic disturbances and randomly occurring nonlinearities via uncertain pinning control has not been studied, this is very theoretical and practical significance.

Motivated by the previous discussion, we focus on the consensus problem via uncertain pinning control under switching topologies. Furthermore, the consensus will be discussed by designing uncertain pinning control protocol and constructing appropriate multiple Lyapunov function. The consensus criterion is deduced by using LMI toolbox and M-matrix theory. Finally, two simulation examples are given to verify the effectiveness of the proposed results.

The structure of the paper is arranged as follows. Some basic notations and algebraic graph theory are reviewed, the preliminaries are shown in Sect. 2. In Sect. 3, two main theorem results are obtained. Two simulation examples are given to show the correctness of the proposed method in Sect. 4. Finally, conclusions and research outlook are given in Sect. 5.

2 Preliminaries and problem formulation

2.1 Preliminaries

2.1.1 Notations

Notations: $\mathbb{N}$ and $\mathbb{R}^n$ represent positive natural numbers set and n-dimensional Euclidean Spaces, respectively. $\|\cdot\|$ indicates the Euclidian norm, $\otimes$ represents the Kronecker product, I_n denotes the n-dimensional identity matrix, $A_{m\times n}$ refers to $m \times n$ matrix. "T" denotes vector transposition, symbol "$*$" represents the symmetric part, $\lambda_{max}(P)$ refers to the largest eigenvalue of matrix P. $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation. $Prob\{\pi\}$ refers to the probability of event π occurring. If not specified, the matrix has a compatible dimension.

2.1.2 Graph theory

Let $\mathcal{G} = (\mathcal{V}, \varepsilon, \mathcal{A})$ be a directed graph, which contains the set of nodes $\mathcal{V} = \nu_1, \nu_2, ..., \nu_N$ and the set of directed edges $\varepsilon \subseteq \mathcal{V} \times \mathcal{V}$. $\mathcal{A} = [a_{ij}]_{N\times N}$ is the adjacent matrix and satisfies $a_{ij} > 0$ if $(\nu_i, \nu_j) \in \varepsilon$; otherwise, $a_{ij} = 0$. The Laplacian matrix $\mathcal{L} = [l_{ij}]_{N\times N}$ satisfies $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$. A directed graph is called to contain a spanning tree, and there is a directed path from one vertex to every other vertices.

Furthermore, the communication topologies are considered as time-varying digraph. $\mathcal{G} = \{\mathcal{G}^1, \mathcal{G}^2, ..., \mathcal{G}^s\}$, $s \geq 1$ represents the set of directed topologies. Define $\sigma(t) : [0, +\infty) \to S = \{1, 2, ..., s\}$. $0 = t_1 < t_2 < ...$ represent the switching instants of interval $[0, t)$, assume that there exist a intervals $[t_k, t_{k+1})$, $k \in N$, and $\tau_0 \leq t_{k+1} - t_k \leq \tau_1$, with $\tau_0 > 0$ indicates the dwell time, in which the topology structure is fixed. Obviously, the topology graph $\mathcal{G}^{\sigma(t)}$ satisfies that $\mathcal{G}^{\sigma(t)} \in \mathcal{G}$ when $t \geq 0$.

2.2 Problem formulation

In this subsection, the multi-agent system with N followers are considered. The dynamics behavior of each agent is shown as

$$dx_i(t) = \Big(A(t)x_i(t) + \alpha(t)f(x_i(t), t) + Bu_i(t)\Big)dt + \varrho_i(x_i(t), t)d\omega(t),\ i = 1, 2, ..., N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^{m\times n}$ represent the state variables and the control input of the i-th agent. $B \in \mathbb{R}^{n\times m}$ is a constant matrices. $f(x_i(t), t) \in \mathbb{R}^n$ is a nonlinear function. $\varrho_i(x_i(t), t) \in \mathbb{R}^n$ is the noise intensity function, $\omega(t)$ is a one-dimensional Brownian motion and satisfying $\mathbb{E}\{\omega(t)\} = 0$. Define $A(t) = A + \delta(t)\Delta A(t)$, and $\Delta A(t) = FH(t)G$, where $\Delta A(t)$ denotes the parameter uncertainties with F and G are constant matrices. $H(t)$ is time-varying unknown matrix and satisfies $H^T(t)H(t) \leq I,\ \forall t \geq 0$. $\alpha(t), \delta(t)$ are stochastic variables used to describe the white snoise sequences that satisfy the following Bernoulli-distribution laws:

$$\begin{aligned} &Prob\{\alpha(t) = 1\} = \mathbb{E}\{\alpha(t)\} = \alpha,\ Prob\{\alpha(t) = 0\} = 1 - \mathbb{E}\{\alpha(t)\} = 1 - \alpha, \\ &Prob\{\delta(t) = 1\} = \mathbb{E}\{\delta(t)\} = \delta,\ Prob\{\delta(t) = 0\} = 1 - \mathbb{E}\{\delta(t)\} = 1 - \delta, \end{aligned} \tag{2}$$

where $\alpha, \delta \in [0, 1]$ are two known real constants. Clearly, it is easy to conclude from (2) that $\mathbb{E}\{(\alpha(t) - \alpha)^2\} = \alpha(1 - \alpha)$ and $\mathbb{E}\{(\delta(t) - \delta)^2\} = \delta(1 - \delta)$ hold. Meanwhile, assuming that the $\alpha(t)$, $\delta(t)$ and $\omega(t)$ are independent of each other.

Remark 1 The variables $\alpha(t), \delta(t)$ describe the probability of distribution. In [1], the state estimation problem of sensor networks is investigated by using the method of Event-triggered control. In addition, the authors study the finite-time synchronization problem for neural networks in [2]. But so far, under switching topologies, it has not been considered simultaneously for multi-agent system. $\alpha(t) f(x_i(t), t)$ and $\delta(t)\Delta A(t)$ are used to describe randomly occurring nonlinearities and uncertainties in the system.

In this paper, Eq. (1) is set as the dynamic of the follower, assuming the leader is labeled $i = 0$, and the dynamics behavior of the leader is shown as

$$dx_0(t) = \Big(A(t)x_0(t) + \alpha(t) f(x_0(t), t)\Big)dt + \varrho_0(x_0(t), t)d\omega(t), \tag{3}$$

where $x_0(t) \in \mathbb{R}^n$ is the state of the leader, $f(x_0(t), t) \in \mathbb{R}^n$ is a nonlinear function. $\varrho_0(x_0(t), t) \in \mathbb{R}^n$ describes the noise intensity of the leader.

Remark 2 The stochastic Brownian motions affect the dynamics of both the leader and all the follower agents. As discussed in introduction, $\varrho_i(x_i(t), t) \in \mathbb{R}^n$, $i = 0, 1, 2, ..., N$, in our model is used to describe the external stochastic disturbances, which is essentially different from the deterministic cases.

Definition 1 [27] The consensus problem for systems (1) and (3) can be solved, if an appropriate control protocol $u_i(t)$ can be designed to satisfy the following equation for any initial conditions:

$$\lim_{t\to\infty} \mathbb{E}\{\| x_i(t) - x_0(t) \|\} = 0,\ \forall\, i = 1, 2, ..., N. \tag{4}$$

In order to achieve consensus between systems (1) and (3), the following control protocol is introduced, which is related to the state information of both the leader and neighboring agents.

$$u_i(t) = \bar{m}K \sum_{j=1, j\neq i}^{N} a_{ij}^{\sigma(t)}(x_j(t) - x_i(t)) - \bar{m}d_i(t)K(x_i(t) - x_0(t)),\ i = 1, 2, ..., N, \tag{5}$$

where $\bar{m}$ is the coupling strength, $d_i(t) \geq 0$ denotes the pinning strategy with positive weight $d_i(t) > 0$ indicates that the leader information can be received by the i-th agent at time t. $K \in \mathbb{R}^{m\times n}$ is the feedback matrix and $a_{ij}^{\sigma(t)}$ is the element of the matrix $\mathcal{A}^{\sigma(t)}$.

Let $e_i(t) = x_i(t) - x_0(t)$ for each agent i, the error dynamics of $e_i(t)$ resulting from (1), (3) and (5) can be obtained as

$$\begin{aligned} de_i(t) =& \Big(A(t)e_i(t) + \alpha \bar{f}(e_i(t), t) + (\alpha(t) - \alpha)\bar{f}(e_i(t), t) - \bar{m}BK \sum_{j=1}^{N} l_{ij}^{\sigma(t)} e_j(t) \\ & - \bar{m}d_i(t)BKe_i(t) \Big) dt + \bar{\varrho}_i(e_i(t), t)d\omega(t), \ i = 1, 2, ..., N, \end{aligned} \tag{6}$$

where $\bar{f}(e_i(t), t) = f(x_i(t), t) - f(x_0(t), t)$, $\bar{\varrho}_i(e_i(t), t) = \varrho_i(x_i(t), t) - \varrho_0(x_0(t), t)$. $\mathcal{L}^{\sigma(t)} = [l_{ij}^{\sigma(t)}]_{N\times N}$ is the Laplacian matrix, and $\bar{\mathcal{L}}^{\sigma(t)} = [\bar{l}_{ij}^{\sigma(t)}]_{N\times N} = \mathcal{L}^{\sigma(t)} + diag\{d_1(t), ..., d_N(t)\}$ is defined as an augmented Laplacian matrix. Then, Eq. (6) can be rewritten as the following compact matrix form

$$\begin{aligned} de(t) =& \Big((I_N \otimes A(t))e(t) + \alpha F(e(t), t) + (\alpha(t) - \alpha)F(e(t), t) \\ & - \bar{m}(\bar{\mathcal{L}}^{\sigma(t)} \otimes BK)e(t) \Big) dt + \bar{\rho}(e(t), t)d\omega(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} & e(t) = (e_1^T(t), ..., e_N^T(t))^T, F(e(t), t) = (\bar{f}^T(e_1(t), t), ..., \bar{f}^T(e_N(t), t))^T, \\ & \bar{\rho}(e(t), t) = (\bar{\rho}_1^T(e_1(t), t), ..., \bar{\rho}_N^T(e_N(t), t))^T. \end{aligned}$$

Assumption 1 There exists a constant ω that satisfies the following Lipschitz condition:

$$\| f(x(t), t) - f(y(t), t) \| \leq \omega \| x(t) - y(t) \|, \tag{8}$$

for all $x(t), y(t) \in \mathbb{R}^n, t \geq 0$.

Assumption 2 There exists a known constant matrix Γ such that the function $\varrho_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, i = 1, 2, ..., N$, satisfies the following Lipschitz condition:

$$trace\{(\varrho_i(\mu, t) - \varrho_0(\nu, t))^T(\varrho_i(\mu, t) - \varrho_0(\nu, t))\} \leq \| \Gamma(\mu - \nu) \|^2, \ \ i = 1, 2, ..., N, \tag{9}$$

where $\mu, \nu \in \mathbb{R}^n$.

Assumption 3 A directed graph $\mathcal{G}$ is assumed to contain a directed spanning tree, if there exists a directed path from one vertex to other vertices.

Definition 2 [28] A real matrix M is called a non-singular M-matrix if its non-diagonal elements are all non-positive, and each eigenvalue satisfying has a positive real part.

Definition 3 [29] There exist real numbers $\tau_\alpha > 0$ and $N_0 > 0$ that satisfy the following inequality:

$$N_\sigma(T, t) \leq N_0 + \frac{t - T}{\tau_\alpha}, \quad \forall t \geq T \geq 0,$$

where $N_\sigma(T, t)$ is the switching numbers of $\sigma(t)$ within the interval $[T, t)$, N_0 is the chatter bound, and the switching signal $\sigma(t)$ is called to have an average dwell-time τ_α.

Lemma 1 *[28] Assuming that $M \in \mathbb{R}^{N\times N}$ is an M-matrix, and there exists a positive vector $\xi = (\xi_1, ..., \xi_N)^T \in \mathbb{R}^N$ that satisfies $M^T\xi = 1_N$ and $\Lambda M + M^T\Lambda > 0$, where $\Lambda = diag\{\xi_1, ..., \xi_N\}$.*

Lemma 2 *[30] The Laplacian matrix $\mathcal{L}$ has a zero eigenvalue, and all other eigenvalues have positive real parts if and only if the the directed graph $\mathcal{G}$ contains a directed spanning tree.*

Lemma 3 *[31] Assuming that $A > 0 \in \mathbb{R}^{N\times N}$, then, the following inequality can be deduced:*

$$\lambda_{min}(A^{-1}B)x^TAx \leq x^TBx \leq \lambda_{max}(A^{-1}B)x^TAx, \tag{10}$$

where $\lambda_{min}(A^{-1}B)$ is the smallest eigenvalues and $\lambda_{max}(A^{-1}B)$ is the largest eigenvalues of $A^{-1}B$.

Lemma 4 *[15] For any given $\epsilon > 0$ and $x, y \in \mathbb{R}^n$, then, the following inequality holds:*

$$x^Ty + y^Tx \leq \epsilon x^Tx + \epsilon^{-1}y^Ty.$$

3 Main ruslts

3.1 Consensus of multi-agent systems with the leader containing a directed path to every follower

In this subsection, the consensus problem is discussed for systems (1) and (3) under switching topologies. Before moving on, based on Assumption 3, the following conclusions are presented.

Mark the leader as node 0, and a new topology graph $\hat{\mathcal{G}}^{\sigma(t)}$ can be constructed as follows:

$$\hat{\mathcal{L}}^{\sigma(t)} = \begin{pmatrix} 0 & \mathbf{0}_N^T \\ -d(t) & \bar{\mathcal{L}}^{\sigma(t)} \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

where $d(t) = (d_1(t), ..., d_N(t))^T$. According to Assumption 3, one can obtain that $\hat{\mathcal{G}}^{\sigma(t)}$ contains a spanning tree and takes node 0 as the root. Matrix $\bar{\mathcal{L}}^{\sigma(t)}$ satisfies the condition of Lemma 2, thus, $\bar{\mathcal{L}}^{\sigma(t)}$ is a M-matrix. Therefore, using Lemma 1, there exist positive vectors $\xi^i = [\xi_1^i, \xi_2^i, ..., \xi_N^i]^T$ such that

$$(\bar{\mathcal{L}}^i)^T\xi^i = \mathbf{1}_N \tag{11}$$

and

$$\bar{\Lambda}^i\bar{\mathcal{L}}^i + (\bar{\mathcal{L}}^i)^T\bar{\Lambda}^i > 0, \tag{12}$$

where $\bar{\Lambda}^i = diag\{\xi_1^i, \xi_2^i, ..., \xi_N^i\}$, for each $i = 1, ..., s$. The purpose of constructing $\hat{\mathcal{G}}^{\sigma(t)}$ is to show that $\bar{\mathcal{L}}^i$ is a M-matrix. Therefore, the multiple Lyapunov function mentioned later can be constructed by M-matrix theory.

Now, the following Theorem 1 is proposed, which illustrates the theoretical result of this section.

Theorem 1 *Suppose that Assumption 1–3 holds, the consensus problem of systems (1) and (3) can be solved if for any given scalars $\bar{m} > 0, \alpha > 0, \omega > 0, \epsilon > 0, \delta > 0$, and a matrix $R > 0$ satisfy the following inequalities holds:*

$$\Phi = \begin{bmatrix} \Phi_{11} & \sqrt{\epsilon}RG^T & \sqrt{\alpha}\omega R & \sqrt{\kappa\lambda_{\max}(R^{-1})}R\Gamma^T \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{13}$$

where $\Phi_{11} = AR + RA^T + \epsilon^{-1}\delta^2 FF^T + \alpha I - \frac{\bar{m}\nu}{\xi_{max}} BB^T + \gamma R$. $\xi_{max} = max\{\xi_j\}$, $\xi_{min} = min\{\xi_j\}$, $j \in \{1, ..., N\}$ *and* $\kappa = \xi_{max}/\xi_{min}$. $\nu = \lambda_{min}((\bar{\mathcal{L}}^{\sigma(t)})^T \bar{\Lambda}^{\sigma(t)} + \bar{\Lambda}^{\sigma(t)}\bar{\mathcal{L}}^{\sigma(t)})$. *The average dwell time satisfies* $\tau_\alpha > ln\kappa/\gamma$ *and* $K = B^T R^{-1}$ *is the feedback matrix.*

Proof Construct multiple Lyapunov functions for the system (7) as follows:

$$V(e(t), t) = e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})e(t), \tag{14}$$

where $\bar{\Lambda}^{\sigma(t)} \in \{\bar{\Lambda}^1, ..., \bar{\Lambda}^s\}$, and matrices $\bar{\Lambda}^i, i = 1, ..., s$, are given in (12), the matrix $R > 0$. Define the following infinitesimal operator $\mathcal{F}$(see, e.g., [15]):

$$\mathcal{F}V(e(t), t) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[\mathbb{E}\{V(e(t+\varepsilon) \mid e(t), t)\} - V(e(t), t)]. \tag{15}$$

For $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, it means that the topology graph $\mathcal{G}^{\sigma(t)}$ is fixed. Then, the time derivative of $V(t)$ can be calculated as follows:

$$dV(e(t), t) = \mathcal{F}V(e(t), t)dt + 2e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})\bar{\varrho}(e(t), t)d\omega(t). \tag{16}$$

Furthermore,

$$\begin{aligned}
&\mathcal{F}V(e(t), t)\\
&= e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (A^T(t)R^{-1} + R^{-1}A(t)))e(t) + 2\alpha e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})F(e(t), t)\\
&+ 2(\alpha(t) - \alpha)e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})F(e(t), t) - \bar{m}e^T(t)[((\bar{\mathcal{L}}^{\sigma(t)})^T \bar{\Lambda}^{\sigma(t)} \otimes K^T B^T R^{-1})\\
&+ \bar{\Lambda}^{\sigma(t)}\bar{\mathcal{L}}^{\sigma(t)} \otimes R^{-1}BK]e(t) + trace[\bar{\varrho}^T(e(t), t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})\bar{\varrho}(e(t), t)].
\end{aligned} \tag{17}$$

Based on Assumption 1, Assumption 2 and Lemma 4, one can obtain that

$$\begin{aligned}
&e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (A^T(t)R^{-1} + R^{-1}A(t)))e(t)\\
&= \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)((A + \delta(t)FH(t)G)^T R^{-1} + R^{-1}(A + \delta(t)FH(t)G))e_i(t)\\
&= \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)(A^T R^{-1} + R^{-1}A)e_i(t)\\
&+ \delta \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)(G^T H^T(t)F^T R^{-1} + R^{-1}FH(t)G)e_i(t)\\
&+ (\delta(t) - \delta) \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)(G^T H^T(t)F^T R^{-1} + R^{-1}FH(t)G)e_i(t)\\
&\leq \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)(A^T R^{-1} + R^{-1}A + \epsilon^{-1}\delta^2 R^{-1}FF^T R^{-1} + \epsilon G^T G)e_i(t)\\
&+ (\delta(t) - \delta) \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)(G^T H^T(t)F^T R^{-1} + R^{-1}FH(t)G)e_i(t)\\
&= e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (A^T R^{-1} + R^{-1}A + \epsilon^{-1}\delta^2 R^{-1}FF^T R^{-1} + \epsilon G^T G))e(t)\\
&+ (\delta(t) - \delta)e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes ((G^T H^T(t)F^T R^{-1} + R^{-1}FH(t)G)))e(t).
\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}
&2\alpha e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})F(e(t),t) \\
&= 2\alpha \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t) R^{-1}(f(x_i(t),t) - f(x_0(t),t)) \\
&\le \alpha \sum_{i=1}^{N} \xi_i{}^{\sigma(t)} e_i^T(t)((R^{-1})^2 + \omega^2 I)e_i(t) \\
&= \alpha e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes ((R^{-1})^2 + \omega^2 I))e(t),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&trace[\bar{\varrho}^T(e(t),t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})\bar{\varrho}(e(t),t)] \\
&\le \xi_{max}\lambda_{max}(R^{-1}) trace[\bar{\rho}^T(e(t),t)\bar{\rho}(e(t),t)] \\
&\le \xi_{max}\lambda_{max}(R^{-1}) e^T(t)(I_N \otimes \Gamma^T\Gamma)e(t) \\
&\le \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(R^{-1}) e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes \Gamma^T\Gamma)e(t).
\end{aligned} \tag{20}$$

Then, substitute (18), (19) and (20) into (17), one has

$$\begin{aligned}
&\mathbb{E}\{\mathcal{F}V(e(t),t)\} \\
&\le e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (A^T R^{-1} + R^{-1}A + \epsilon^{-1}\delta^2 R^{-1} F F^T R^{-1} + \epsilon G^T G))e(t) \\
&+ \alpha e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes ((R^{-1})^2 + \omega^2 I))e(t) - \bar{m} e^T(t)[((\bar{\mathcal{L}}^{\sigma(t)})^T \bar{\Lambda}^{\sigma(t)} \otimes K^T B^T R^{-1}) \\
&+ \bar{\Lambda}^{\sigma(t)}\bar{\mathcal{L}}^{\sigma(t)} \otimes R^{-1}BK]e(t) + \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(R^{-1}) e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes \Gamma^T\Gamma)e(t).
\end{aligned} \tag{21}$$

Define $K = B^T R^{-1}$ and $\phi(t) = (\phi_1^T(t), ..., \phi_N^T(t))^T$ with $\phi_i(t) = R^{-1}e_i(t)$. Obviously, $e(t) = (I_N \otimes R)\phi(t)$, then,

$$\begin{aligned}
&-\bar{m} e^T(t)[((\bar{\mathcal{L}}^{\sigma(t)})^T \bar{\Lambda}^{\sigma(t)} \otimes K^T B^T R^{-1}) + \bar{\Lambda}^{\sigma(t)}\bar{\mathcal{L}}^{\sigma(t)} \otimes R^{-1}BK]e(t) \\
&= -\bar{m} e^T(t)[((\bar{\mathcal{L}}^{\sigma(t)})^T \bar{\Lambda}^{\sigma(t)} + \bar{\Lambda}^{\sigma(t)}\bar{\mathcal{L}}^{\sigma(t)}) \otimes R^{-1}BB^T R^{-1}]e(t) \\
&\le -\bar{m}\nu e^T(t)(I_N \otimes R^{-1}BB^T R^{-1})e(t) \\
&\le -\frac{\bar{m}\nu}{\xi_{max}} e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1}BB^T R^{-1})e(t).
\end{aligned} \tag{22}$$

Thus, substitute (22) into (21), one gets that

$$\begin{aligned}
&\mathbb{E}\{\mathcal{F}V(e(t),t)\} \\
&\le \phi^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (AR + RA^T + \epsilon^{-1}\delta^2 F F^T + \epsilon R G^T G R))\phi(t) \\
&+ \alpha\phi^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes (I_n + \omega^2 R^2))\phi(t) - \frac{\bar{m}\nu}{\xi_{max}}\phi^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes BB^T)\phi(t) \\
&+ \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(R^{-1})\phi^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R\Gamma^T\Gamma R)\phi(t) \\
&= \phi^T(t)[\bar{\Lambda}^{\sigma(t)} \otimes (AR + RA^T + \epsilon^{-1}\delta^2 F F^T + \epsilon R G^T G R + \alpha I_n + \alpha\omega^2 R^2 \\
&- \frac{\bar{m}\nu}{\xi_{max}} BB^T + \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(R^{-1}) R\Gamma^T\Gamma R)]\phi(t).
\end{aligned} \tag{23}$$

Based on (13) and Schur complement theory, combined with $e(t) = (I_N \otimes R)\phi(t)$, one has

$$\begin{aligned}
&\mathbb{E}\{\mathcal{F}V(e(t),t)\}\\
&\leq -\gamma\phi^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R)\phi(t)\\
&= -\gamma e^T(t)(\bar{\Lambda}^{\sigma(t)} \otimes R^{-1})e(t) = -\gamma V(e(t),t).
\end{aligned} \tag{24}$$

Therefore, it can be concluded that the following inequality holds from (15)

$$\frac{d\mathbb{E}\{V(e(t),t)\}}{dt} \leq -\gamma V(e(t),t), \tag{25}$$

for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, implies that

$$V(t) < e^{-\gamma t}V(0), \quad \forall t \in [0, t_2). \tag{26}$$

Then, according to simple calculation, one gets

$$V(t_2) \leq \kappa V(t_2^-), \tag{27}$$

where $\kappa = \xi_{max}/\xi_{min}$ and $V(t_2^-) = lim_{t<t_2, t\to t_2}V(t)$. Let $t_i, i = 1, ..., N_\sigma(0,t)$ and $t_1 < t_2 < ... < t_{N_\sigma(0,t)}$, $\sigma(t)$ is the switching point of time interval $[0,t)$. Then, one has

$$\begin{aligned}
V(t) &< e^{-\gamma(t-t_{N_\sigma(0,t)})}V(t_{N_\sigma(0,t)}) \leq \kappa e^{-\gamma(t-t_{N_\sigma(0,t)})}V(t^-_{N_\sigma(0,t)})\\
&\leq \kappa^2 e^{-\gamma(t-t_{N_\sigma(0,t)})}e^{-\gamma(t_{N_\sigma(0,t)}-t_{N_\sigma(0,t)-1})}V(t_{N_\sigma(0,t)-1})\\
&...\\
&\leq \kappa^{N_\sigma(0,t)}e^{-\gamma t}V(0) = e^{-\gamma t+N_\sigma(0,t)ln\kappa}V(0).
\end{aligned} \tag{28}$$

According to Definition 3, one can obtain that $N_\sigma(0,t) \leq N_0 + \frac{t}{\tau_\alpha}$. Thus, the following inequality can be derived:

$$V(t) < e^{-\gamma t+(N_0+\frac{t}{\tau_\alpha})ln\kappa}V(0) = \kappa^{N_0}e^{-(\gamma-\frac{ln\kappa}{\tau_\alpha})t}V(0). \tag{29}$$

In view of the fact that $\tau_\alpha > ln\kappa/\gamma$, which implies that the consensus problem between all followers and leader can be solved. □

3.2 Consensus of multi-agent systems with the leader frequently containing a directed path to every follower

In this subsection, the problem of consensus is discussed for systems (1) and (3) for situations where some followers cannot receive the leader's information. The topology will be divided into two categories, one that satisfies Assumption 3 and the other that does not. Then, $\tilde{\mathcal{G}}^1$ and $\tilde{\mathcal{G}}^2$ are defined to represent two different types of communication topologies. Therefore, let $\tilde{\mathcal{G}}^1$ be the first one, the communication topology can be assumed switches between $\tilde{\mathcal{G}}^1$ and $\tilde{\mathcal{G}}^2$ in turn.

Obviously, $\tilde{\mathcal{G}}^1$ satisfies Assumption 3, the corresponding matrix $\tilde{\mathcal{L}}^1 = [\tilde{l}_{ij}^{\sigma(t)}]_{N\times N} = \tilde{\mathcal{L}} + diag\{d_1(t), ..., d_N(t)\}$ is a non-singular M-matrix. Thus, based on Lemma 1, there exist positive vectors $\xi = [\xi_1, ..., \xi_N]^T \in \mathbb{R}^N$ that satisfy the following equation:

$$(\tilde{\mathcal{L}}^1)^T\xi = \mathbf{1}_N \tag{30}$$

and

$$\tilde{\Lambda}\tilde{\mathcal{L}}^1 + (\tilde{\mathcal{L}}^1)^T\tilde{\Lambda} > 0, \tag{31}$$

where $\tilde{\Lambda} = diag\{\xi_1, ..., \xi_N\}$. It is worth noting that $\tilde{\Lambda}$ is varying because of the pinning links and switching inner interaction topology in $\tilde{\mathcal{G}}^1$.

Theorem 2 *Under Assumption 1-2, suppose that $\tilde{\mathcal{G}}^1$ satisfies Assumption 3, the consensus of systems (1) and (3) can be achieved if there exist $\alpha > 0, \bar{m} > 0, \omega > 0, \epsilon > 0, \delta > 0, \eta_1 > 0, \eta_2 > 0$, and matrix $Q > 0$ satisfy the following inequalities holds:*

$$\Psi_1 = \begin{bmatrix} \Psi_{111} & \sqrt{\epsilon}QG^T & \sqrt{\alpha}\omega Q & \sqrt{\kappa\lambda_{\max}(Q^{-1})}Q\Gamma^T \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{32}$$

$$\Psi_2 = \begin{bmatrix} \Psi_{211} & \sqrt{\epsilon}QG^T & \sqrt{\alpha}\omega Q & \sqrt{\kappa\lambda_{\max}(Q^{-1})}Q\Gamma^T \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{33}$$

where $\Psi_{111} = AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \alpha I - \frac{\bar{m}\vartheta}{\xi_{max}} BB^T + \eta_1 Q$, $\Psi_{211} = AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \alpha I - \bar{m}\varpi BB^T - \eta_2 Q$. $\xi_{max} = max\{\xi_j\}$, $\xi_{min} = min\{\xi_j\}$, $j \in \{1, ..., N\}$ and $\kappa = \xi_{max}/\xi_{min}$. $\vartheta = \lambda_{min}(\tilde{\Lambda}\tilde{\mathcal{L}}^1 + (\tilde{\mathcal{L}}^1)^T\tilde{\Lambda})$, $\varpi = \lambda_{min}(\tilde{\Lambda}^{-1}(\tilde{\Lambda}\tilde{\mathcal{L}}^2 + (\tilde{\mathcal{L}}^2)^T\tilde{\Lambda}))$. The dwell time τ_0 and τ_1 satisfy $\eta_1\tau_0 - 2\eta_2\tau_1 > 2ln\kappa$, and $K = B^T Q^{-1}$ is the feedback matrix to be designed.

Proof It is worth noting that $\tilde{\mathcal{G}}^1$ is made up of a finite number of different types of communication topologies and satisfies Assumption 1, which results in different types of matrix $\tilde{\Lambda}$ in inequality (31). For simplicity, $\tilde{\Lambda}$ is used to represent the matrix in inequality (31). Thus, the multiple Lyapunov function can be constructed for system (7)as follows:

$$V(e(t), t) = e^T(t)(\tilde{\Lambda} \otimes Q^{-1})e(t), \tag{34}$$

where the matrix $Q > 0$ is a solution of (32) and (33). For $t \in [t_{2k-1}, t_{2k}), k \in N$, We can know that the type of topology belongs to $\tilde{\mathcal{G}}^1$. Define $K = B^T Q^{-1}$ and $\psi(t) = (\psi_1^T(t), ..., \psi_N^T(t))^T$ with $\psi_i(t) = Q^{-1}e_i(t)$. So, one has $e(t) = (I_N \otimes Q)\psi(t)$. Then, by a similar analysis to Theorem 1, one has

$$\begin{aligned} &\mathbb{E}\{\mathcal{F}V(e(t), t)\} \\ &\quad \le \psi^T(t)[\tilde{\Lambda} \otimes (AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \epsilon QG^T GQ + \alpha I_n + \alpha\omega^2 Q^2 \\ &\quad + \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(Q^{-1})Q\Gamma^T\Gamma Q)]\psi(t) - \bar{m}e^T(t)[((\tilde{\mathcal{L}}^1)^T\tilde{\Lambda} \\ &\quad + \tilde{\Lambda}\tilde{\mathcal{L}}^1) \otimes Q^{-1}BB^T Q^{-1}]e(t). \\ &\quad \le \psi^T(t)[\tilde{\Lambda} \otimes (AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \epsilon QG^T GQ + \alpha I_n + \alpha\omega^2 Q^2 \\ &\quad - \frac{\bar{m}\vartheta}{\xi_{max}} BB^T + \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(Q^{-1})Q\Gamma^T\Gamma Q)]\psi(t). \end{aligned} \tag{35}$$

Based on (32) and Schur complement theory, combined with $e(t) = (I_N \otimes Q)\psi(t)$, gives

$$\begin{aligned} &\mathbb{E}\{\mathcal{F}V(e(t), t)\} \\ &\quad \le -\eta_1\psi^T(t)(\tilde{\Lambda} \otimes Q)\psi(t) \\ &\quad = -\eta_1 e^T(t)(\tilde{\Lambda} \otimes Q^{-1})e(t) = -\eta_1 V(e(t), t). \end{aligned} \tag{36}$$

Therefore, it can be concluded that the following inequality holds from (34)

$$\frac{d\mathbb{E}\{V(e(t),t)\}}{dt} \leq -\eta_1 V(e(t),t). \tag{37}$$

For $t \in [t_{2k}, t_{2k+1})$, $k \in N$, the type of topology is in $\tilde{\mathcal{G}}^2$, one gets

$$\begin{aligned}
&\mathbb{E}\{\mathcal{F}V(e(t),t)\} \\
&\leq \psi^T(t)[\tilde{\Lambda} \otimes (AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \epsilon QG^T GQ + \alpha I_n + \alpha\omega^2 Q^2 \\
&+ \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(Q^{-1})Q\Gamma^T\Gamma Q)]\psi(t) - \bar{m}e^T(t)[((\tilde{\mathcal{L}}^2)^T\tilde{\Lambda} \\
&+ \tilde{\Lambda}\tilde{\mathcal{L}}^2) \otimes Q^{-1}BB^TQ^{-1}]e(t).
\end{aligned} \tag{38}$$

Note that matrix $(\tilde{\mathcal{L}}^2)^T\tilde{\Lambda} + \tilde{\Lambda}\tilde{\mathcal{L}}^2$ may not be positive definite. However, from Lemma 3, the following inequality is established

$$\lambda_{min}(\tilde{\Lambda}^{-1}(\tilde{\Lambda}\tilde{\mathcal{L}}^2 + (\tilde{\mathcal{L}}^2)^T\tilde{\Lambda}))\psi^T\tilde{\Lambda}\psi \leq \psi^T(\tilde{\Lambda}\tilde{\mathcal{L}}^2 + (\tilde{\mathcal{L}}^2)^T\tilde{\Lambda})\psi.$$

For any given $\psi \in \mathbb{R}^n$. Let $\varpi = \lambda_{min}(\tilde{\Lambda}^{-1}(\tilde{\Lambda}\tilde{\mathcal{L}}^2 + (\tilde{\mathcal{L}}^2)^T\tilde{\Lambda}))$, we have

$$\begin{aligned}
&\mathbb{E}\{\mathcal{F}V(e(t),t)\} \\
&\leq \psi^T(t)[\tilde{\Lambda} \otimes (AQ + QA^T + \epsilon^{-1}\delta^2 FF^T + \epsilon QG^T GQ + \alpha I_n + \alpha\omega^2 Q^2 \\
&- \bar{m}\varpi BB^T + \frac{\xi_{max}}{\xi_{min}}\lambda_{max}(Q^{-1})Q\Gamma^T\Gamma Q)]\psi(t).
\end{aligned} \tag{39}$$

Similarly, according to (33), one can get

$$\frac{d\mathbb{E}\{V(e(t),t)\}}{dt} \leq \eta_2 V(e(t),t). \tag{40}$$

For $\tau_0 \leq t_{k+1} - t_k \leq \tau_1$, $k \in N$, combining (37) and (40),

$$V(t_2) \leq \kappa V(t_2^-) < \kappa e^{-\eta_1(t_2-t_1)}V(t_1) \leq e^{-\eta_1\tau_0 + ln\kappa}V(0) \leq e^{-\zeta\tau_0}V(0) \tag{41}$$

and

$$V(t_3) \leq \kappa V(t_3^-) < \kappa e^{\eta_2(t_3-t_2)}V(t_2) \leq e^{-\eta_1\tau_0+\eta_2\tau_1+2ln\kappa}V(0) \leq e^{-\zeta\tau_0}V(0), \tag{42}$$

where $\kappa = \xi_{max}/\xi_{min}$, $V(t_2^-) = lim_{t<t_2, t\to t_2}V(t)$ and $\zeta = \eta_1 - 2(\eta_2\tau_1)/\tau_0 - 2ln\kappa/\tau_0 > 0$. Furthermore, the following inequality can be introduced by recursion:

$$V(t_{2k+1}) \leq e^{-\zeta\tau_0}V(t_{2k-1}) < e^{-k\zeta\tau_0}V(0) \tag{43}$$

and

$$V(t_{2k}) \leq e^{-\zeta\tau_0}V(t_{2(k-1)}) < e^{-k\zeta\tau_0}V(0). \tag{44}$$

For any $t > t_3$, there exists $k \geq 2$ satisfies $t_{2k-1} \leq t < t_{2k}$ or $t_{2k} \leq t < t_{2k+1}$. Therefore, one concludes that

$$V(t) \leq e^{-k\zeta\tau_0}V(0) < e^{-k\zeta\tau_0\frac{t}{2k\tau_1}}V(0) = e^{-\frac{\zeta\tau_0}{2\tau_1}t}V(0), \quad t = t_{2k-1}, \tag{45}$$

$$\begin{aligned}
V(t) &< e^{-\eta_1(t-t_{2k-1})}V(t_{2k-1}) < e^{-(\eta_1(t-t_{2k-1})+(k-1)\zeta\tau_0)}V(0) \\
&< e^{-(k-1)\zeta\tau_0\frac{t}{2k\tau_1}}V(0) = e^{-\frac{(k-1)\zeta\tau_0}{2k\tau_1}t}V(0), \quad t \in (t_{2k-1}, t_{2k}),
\end{aligned} \tag{46}$$

$$V(t) \leq e^{-k\zeta\tau_0}V(0) < e^{-k\zeta\tau_0\frac{t}{2k\tau_1}}V(0) = e^{-\frac{\zeta\tau_0}{2\tau_1}t}V(0), \quad t = t_{2k}, \tag{47}$$

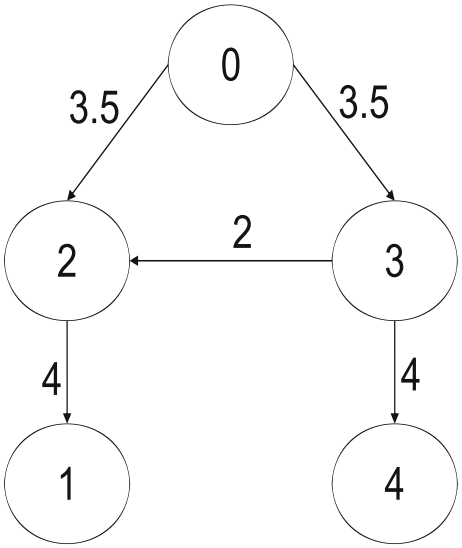

Fig. 1 Communication topology $\hat{\mathcal{G}}^1$

$$\begin{aligned} V(t) &< e^{\eta_2(t-t_{2k})}V(t_{2k}) < e^{\eta_2\tau_1}e^{-\eta_1\tau_0+\eta_2\tau_1+2ln\kappa}e^{-(k-1)\zeta\tau_0}V(0) \\ &< e^{-(k-1)\zeta\tau_0\frac{t}{2k\tau_1}}V(0) = e^{-\frac{(k-1)\zeta\tau_0}{2k\tau_1}t}V(0), \quad t \in (t_{2k}, t_{2k+1}). \end{aligned} \tag{48}$$

Finally, It follows from (45) to (48) that $e(t) \to 0$ as $t \to \infty$, which means that the consensus between all followers and leader is achieved. □

Remark 3 Our main innovation and contribution is to research the consensus problem under switching topology. Compared with some previous works, it is difficult to discuss the consensus problem when part of the topology does not include a directed spanning trees under switching topology, which can reduce the connection requirements of the topology structure. In the existing literature related to switching topology, such as references [25, 26] and [27], all the communication topologies are considered include a directed spanning tree, it is easier for followers to achieve consensus with the leader. In this paper, The topology will be divided into two categories, one that satisfies Assumption 3, which contains a directed spanning tree and the other that does not. In this case, the consensus between all followers and leader can still achieved. So, our results are more general.

4 Numerical simulations

In this section, two simulation examples are provided to verify the correctness of the proposed consensus criteria.

Example 1 Consider a multi-agent system consisting of four followers and one leader, whose dynamic behavior is described by (1) and (3). We take $\bar{m} = 2, \omega = 1, \alpha = \delta = 0.5, \epsilon = 5,$

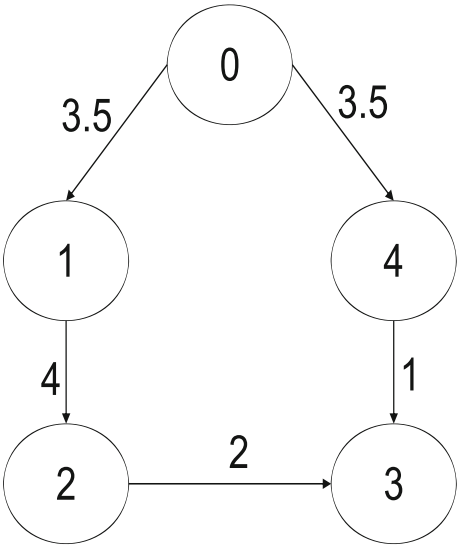

Fig. 2 Communication topology $\hat{\mathcal{G}}^2$

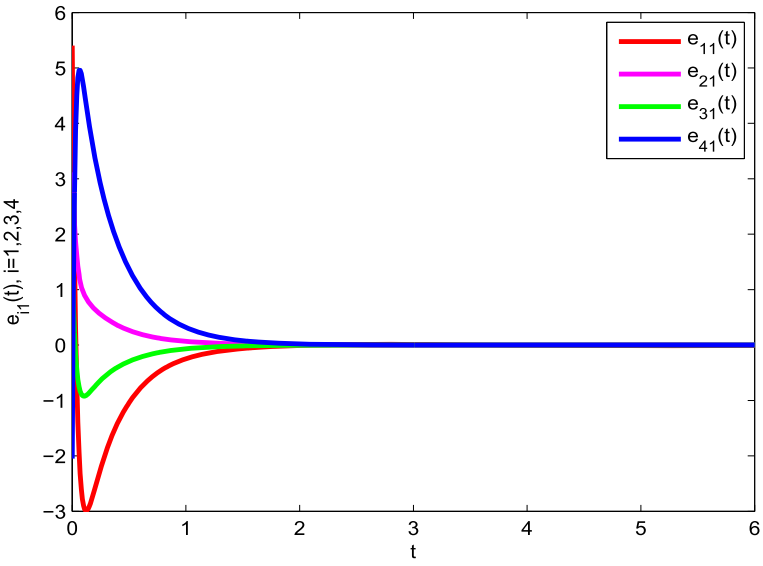

Fig. 3 Error signals of $e_{i1}(t),\ i = 1, 2, 3, 4$

and

$$A = \begin{pmatrix} -1.65 & -0.80 & -0.53 \\ 2.47 & -0.93 & 0.46 \\ -0.31 & -0.67 & -1.91 \end{pmatrix}, \quad B = \begin{pmatrix} -1.14 & 0.64 \\ 0.41 & -1.27 \\ 0.42 & 0.55 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.5 \end{pmatrix},$$

$$F = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, \quad G = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & -0.3 \end{pmatrix}, \quad H(t) = \begin{pmatrix} -sin(t) & 0 & 0 \\ 0 & cos(t) & 0 \\ 0 & 0 & sin(t) \end{pmatrix}.$$

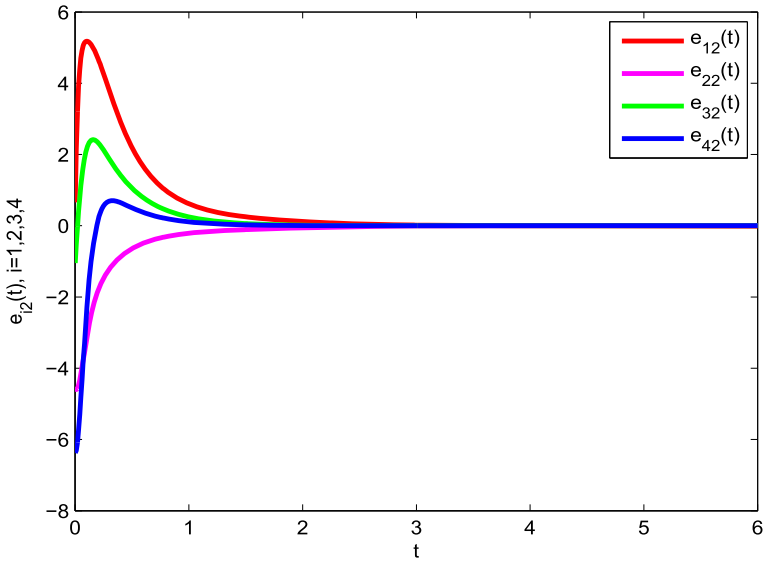

Fig. 4 Error signals of $e_{i2}(t),\ i = 1, 2, 3, 4$

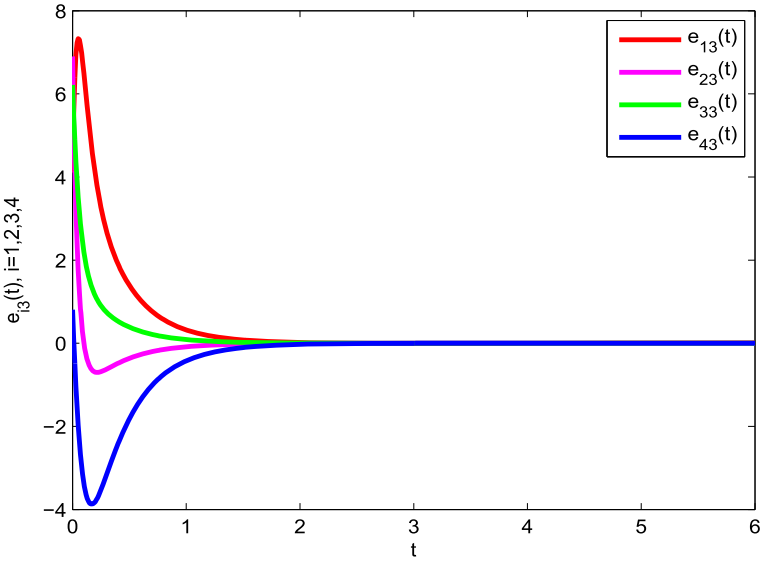

Fig. 5 Error signals of $e_{i3}(t),\ i = 1, 2, 3, 4$

The communication topology is shown in Figs. 1 and 2, both of which satisfy Assumption 3. Thus, one can obtain that

$$\bar{\mathcal{L}}^1 = \begin{pmatrix} 4 & -4 & 0 & 0 \\ 0 & 5.5 & -2 & 0 \\ 0 & 0 & 3.5 & 0 \\ 0 & 0 & -4 & 4 \end{pmatrix}, \bar{\mathcal{L}}^2 = \begin{pmatrix} 3.5 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 3.5 \end{pmatrix}.$$

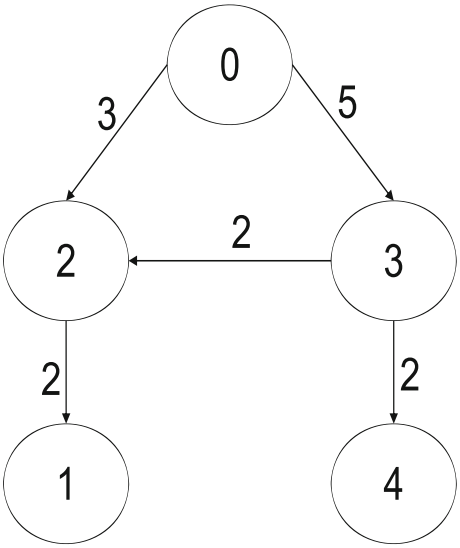

Fig. 6 Communication topology $\tilde{\mathcal{G}}^1$

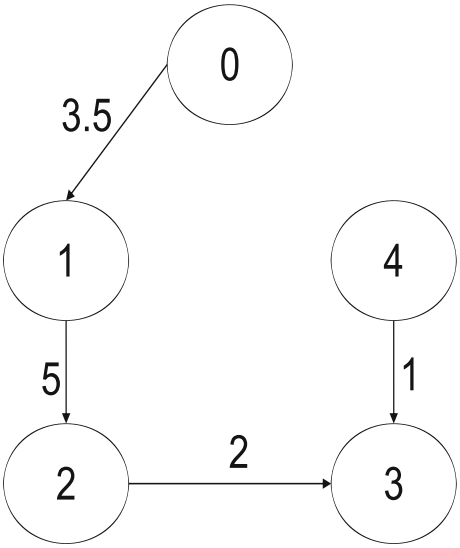

Fig. 7 Communication topology $\tilde{\mathcal{G}}^2$

Then, the parameters of the function are given as follows

$$f(x_i(t),t) = \begin{pmatrix} 2sin(t) + 0.2x_{i1}(t) - tanh(0.1x_{i1}(t)) \\ 0.2x_{i2}(t) - tanh(0.2x_{i2}(t)) \\ 0.2x_{i3}(t) \end{pmatrix},$$

$$\rho_i(x_i(t),t) = \begin{pmatrix} 0.2sin(t)x_{i1}(t) \\ 0.3sin(t)x_{i2}(t) \\ 0.4sin(t)x_{i3}(t) \end{pmatrix},$$

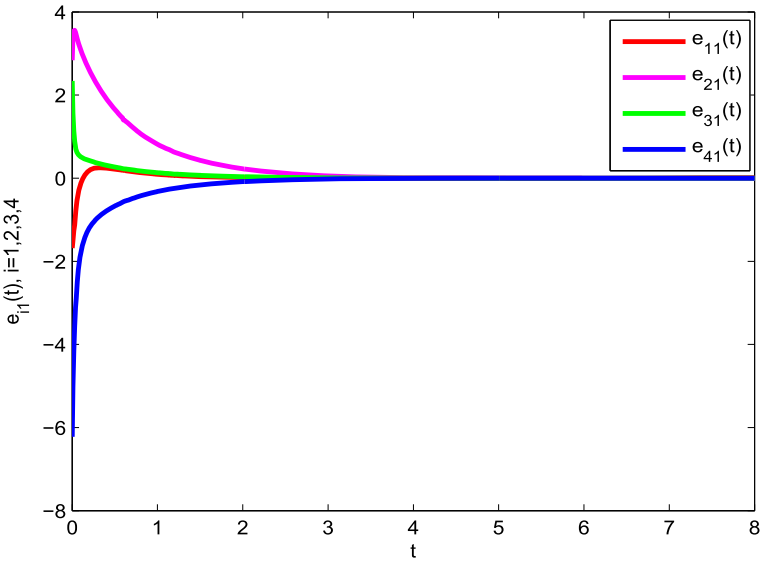

Fig. 8 Error signals of $e_{i1}(t),\ i = 1, 2, 3, 4$

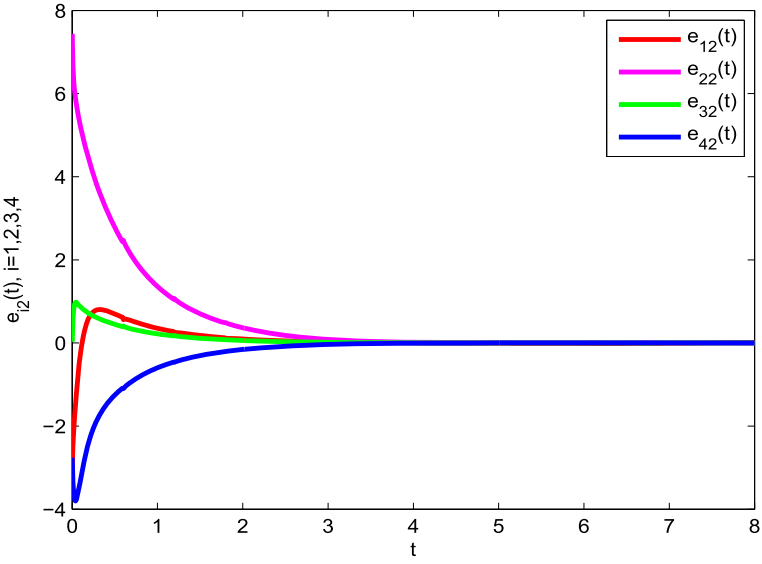

Fig. 9 Error signals of $e_{i2}(t),\ i = 1, 2, 3, 4$

where $i = 0, 1, 2, 3, 4$. Then, direct calculations give $\xi_{max} = 0.7792, \xi_{min} = 0.25$ and $\nu = 1.5026$. By selecting $\gamma = 1$ and solving this inequality (13) in Theorem 1, one has

$$K = \begin{pmatrix} -4.2022 & -1.7727 & -1.0433 \\ 1.2623 & -0.7945 & 1.7923 \end{pmatrix}.$$

Through analysis, when the dwell time is greater than 1.13 s, the consensus problem can be achieved. Suppose the topology switches back and forth between $\hat{\mathcal{G}}^1$ and $\hat{\mathcal{G}}^2$ at a speed

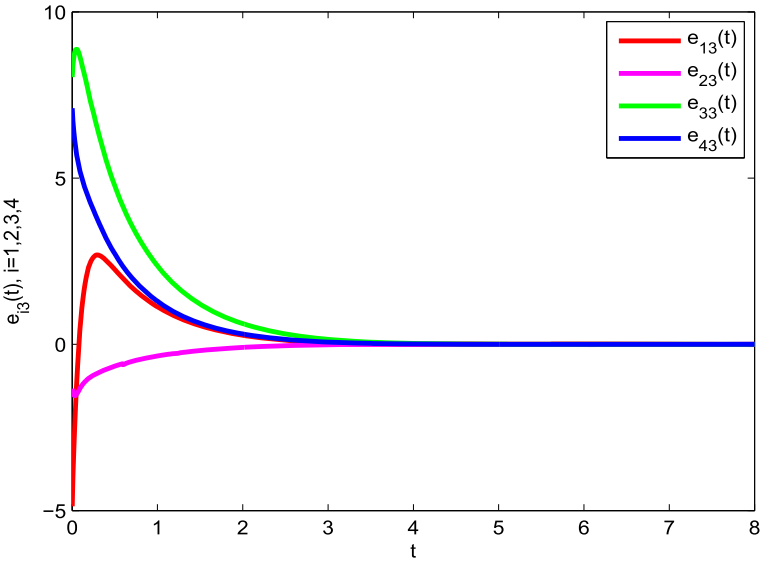

Fig. 10 Error signals of $e_{i3}(t),\ i = 1, 2, 3, 4$

of 1.2 s. The errors trajectories are shown in Figs. 3, 4 and 5, from which it means that the consensus between all followers and leader is achieved.

Example 2 The system is considered as in Example 1. The communication topology are shown in Figs. 6 and 7. Obviously, $\tilde{\mathcal{G}}^1$ satisfies Assumption 3 and graph $\tilde{\mathcal{G}}^2$ does not. Thus, one can obtain that

$$\tilde{\mathcal{L}}^1 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \quad \tilde{\mathcal{L}}^2 = \begin{pmatrix} 3.5 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, according to simple calculation, one has $\xi_{max} = 0.56$, $\xi_{min} = 0.4$, $\vartheta = 1.532$, $\varpi = -0.2054$. By choosing $\eta_1 = 2.2$, $\eta_2 = 0.6$ and solving the inequality (32) and (33) in Theorem 2, one has

$$K = \begin{pmatrix} -3.4729 & 0.9791 & 1.3327 \\ 2.3582 & -1.6464 & 1.5293 \end{pmatrix}.$$

Through the above analysis, Suppose the topology graph switches back and forth between $\tilde{\mathcal{G}}^1$ and $\tilde{\mathcal{G}}^2$ at a speed of 0.6 s, ie, $\tau_0 = \tau_1 = 0.6$. It is easy to get $\eta_1\tau_0 - 2\eta_2\tau_1 > 2ln\kappa$ with $\kappa = \xi_{max}/\xi_{min}$. The errors trajectories are given in Figs. 8, 9 and 10, from which it means that the consensus between all followers and leader is achieved.

5 Conclusion

The consensus problem has been discussed in this paper via uncertain pinning control under switching topologies. Firstly, the dynamic behavior of multi-agent systems with stochastic disturbances and randomly occurring nonlinearities is considered to reflect more realistic

situations. By defining the error system, the consensus problem can be converted into the stability problem to be discussed. Then, according to constructing an suitable multiple Lyapunov function, the consensus between all followers and leader can be achieved. Based on it, some sufficient conditions are established by using some inequality techniques and M-matrix theory. At the same time, the consensus can also be achieved that some followers cannot receive the leader's information under switching topologies. Finally, two examples are shown to verify the correctness of the proposed results. In addition, it remains a very interesting work to discuss the bipartite consensus problem via event-triggered control under switching topologies, which will continue to be discussed later.

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