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Besicovitch Almost Anti-periodic Solution of Octonion-Valued Cohen–Grossberg Neural Networks with Delays on Time Scales

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Abstract

In this work, we put forward a concept of Besicovitch almost anti-periodic functions on time scales, which is new even when time scale $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. Based on this, we use the fixed point theorem and analytical techniques to obtain the existence, uniqueness and global exponential stability of Besicovitch almost anti-periodic solutions for a class of octonion-valued Cohen–Grossberg neural networks with time delays on time scales. Finally, the validity of the results is verified by a numerical example.

Keywords Besicovitch almost anti-periodicity · Time scale · Octonion-valued Cohen–Grossberg neural network · Global exponential stability

1 Introduction

It is well known that anti-periodicity is a special kind of periodicity, and anti-periodic oscillation is one of the important qualitative properties of differential equations and neural network systems described by differential equations [1–6]. Recently, M. Kostić put forward a concept of almost anti-periodic functions in the sense of Bohr and Stepanov [7]. Like periodicity and almost periodicity, almost anti-periodicity is also a generalization of anti-periodicity. At the same time, almost anti-periodic functions are a special case of almost periodic functions. That is to say, almost anti-periodic oscillations more accurately describe the motional law of matters than almost periodic oscillations. Consequently, almost anti-periodic oscillation is also one of the important dynamics of differential equations and neural networks [8–11]. Furthermore, we know that Besicovitch almost (B_p-almost) periodicity is a very general almost periodicity. In a certain sense, it includes Bohr almost periodicity, Stepanov almost periodicity and Weyl almost periodicity [12–14]. Therefore, it is very natural and reasonable to

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introduce and study B_p-almost anti-periodicity. Meanwhile, it is meaningful and challenging to study the B_p-almost anti-periodic oscillation for neural networks.

On the one hand, with the continuous progress of science and technology, and the continuous development of neural network theory, more and more massive multi-dimensional data need to be processed by neural networks. As we know, using ordinary real-valued neural networks to process massive multidimensional data requires more connection weight functions and occupies more storage space in the processor. However, using complex-valued [15–20], quaternion-valued [21–25], Clifford-valued [26–32], and octonion-valued neural networks [33–37] to process the same data only requires less connection weight functions and less storage space. Therefore, the theoretical and applied research on the algebraic valued neural networks mentioned above has received increasing attention. It is worth mentioning that the research on octonion-valued neural networks has just started. Since the multiplication of octonion algebra is neither commutative nor associative, little is known about the dynamics of octonion-valued neural networks. And most of the existing results have been obtained by decomposing the octonion-valued system under consideration into real-valued or complex-valued systems [38–45]. Therefore, it is necessary to explore direct methods to study the various dynamics of such neural networks.

On the other hand, the neural network models described by difference equations play an important role in theory and practical application. Thanks to the time scale theory [46, 47], the research on neural network models described by dynamic equations on time scales can unify the research on neural network models described by differential equations and difference equations respectively.

Inspired by the above observation and considering that, in the past decades, various types of Cohen–Grossberg neural network (CGNN) [48–53] have been successfully applied in many fields, the main purpose of this work is to introduce a concept of B_p-almost anti-periodic functions on time scales, and study the existence and stability of B_p-almost anti-periodic solutions of a class of octonion-valued CGNNs on time scales.

The main contributions of this paper are as follows:

1. We propose a concept of B_p-almost anti-periodic function on time scales, which is new even for functions defined on the set of real numbers or the set of integers.
2. This is the first paper to study the existence of B_p-almost anti-periodic solutions of neural networks. Our results are new to neural networks described by differential equations or difference equations.
3. When the neural networks we consider degenerate into real-valued, complex-valued and quaternion-valued networks, our results are also new.
4. The results and methods in this paper can be used to study the existence of B_p-almost anti-periodic solutions to other types of dynamic equations.
5. Our research method is a direct approach, which does not decompose the octonion-valued system under consideration into real-valued, complex-valued, or quaternion-valued systems.

The rest of this paper is organized as follows: In Sect. 2, we recall some relevant definitions and give a description of the model. In Sect. 3, we propose a definition of B_p-almost anti-periodic functions on time scales. In Sect. 4, we study the existence and global exponential stability of B_p-almost anti-periodic solutions for a class of CGNNs on time scales. In Sect. 5, we present an example to show the validity of our conclusion. In Sect. 6, we provide a brief conclusion.

2 Preliminaries and Model Description

The algebra of octonions is defined as

$$\mathbb{O} = \left\{ x = \sum_{l=0}^{7} x^l e_l : x^l \in \mathbb{R}, l = 0, 1, \ldots, 7 \right\},$$

where $e_l, l = 0, 1, \cdots, 7$ are the octonion units and they satisfy the following multiplication table:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

From the table, one can easily check that the multiplication of octonions is neither commutative nor associative. For each $x \in \mathbb{O}$, we define its conjugate as $\bar{x} = x^0 e_0 - \sum_{l=1}^{7} x^l e_l$ and its norm as $\|x\|_{\mathbb{O}} = \sqrt{x\bar{x}}$. For $y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{O}^n$, we can define $\|y\|_{\mathbb{O}^n} = \max_{l \in I[n]}\{\|y_l\|_{\mathbb{O}}\}$. For more information about the octonion algebra, we refer to [54].

Let $\mathbb{T}$ be a time scale. For a function $y = \sum_{l=0}^{7} y^l e_l : \mathbb{T} \to \mathbb{O}$, where $y^l : \mathbb{T} \to \mathbb{R}, l = 0, 1, \ldots, 7$ are delta differentiable, the delta derivative of y is defined by $y^\Delta(t) = \sum_{l=0}^{7}(y^l)^\Delta(t)e_l$. For a function $z = \sum_{l=0}^{7} z^l e_l : \mathbb{T} \to \mathbb{O}$, where $z^l : \mathbb{T} \to \mathbb{R}, l = 0, 1, \ldots, 7$ are rd-continuous, the delta integral of z from a to b is given by $\int_a^b z(s)\Delta s = \sum_{l=0}^{7}(\int_a^b z^l(s)\Delta s)e_l$.

Throughout this paper, we use $\mathcal{R}^+$ to denote the set of all positive regressive functions from $\mathbb{T}$ to $\mathbb{R}$. We denote by $L^p_{loc}(\mathbb{T}, \mathbb{O}^n)$ the set of all locally p-th Δ-integrable functions from $\mathbb{T}$ to $\mathbb{O}^n$ and by $L^\infty(\mathbb{T}, \mathbb{O}^n)$ the set of all strongly Δ-measurable and essentially bounded functions from $\mathbb{T}$ to $\mathbb{O}^n$, respectively. Obviously, $L^\infty(\mathbb{T}, \mathbb{O}^n)$ is a Banach space with the norm $\|f\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{T}} \|f(t)\|_{\mathbb{O}^n}$ for $f \in L^\infty(\mathbb{T}, \mathbb{O}^n)$.

Lemma 2.1 [55] *If $a, b \in C_{rd}\left(\mathbb{T}, \mathbb{R}^+\right)$ with $-a, -b \in \mathcal{R}^+$ and $t, s \in \mathbb{T}, \xi \in \Pi$, then $e_{-a}(t+\xi, \sigma(s+\xi)) - e_{-b}(t, \sigma(s)) = \int_{\sigma(s)}^{t} e_{-b}(t, \sigma(\theta))(b(\theta) - a(\theta+\xi))e_{-a}(\theta+\xi, \sigma(s+\xi))\Delta\theta$.*

Definition 2.1 [55] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \text{for all } t \in \mathbb{T}\} \neq \{0\}.$$

Obviously, if $\mathbb{T}$ is an almost periodic time scale, then the graininess function μ of $\mathbb{T}$ satisfies $\mu^+ := \sup\limits_{t\in\mathbb{T}} \mu(t) < +\infty$. From now on, unless otherwise specified, the time scale $\mathbb{T}$ refers to the almost periodic time scale.

The model we consider in this paper is the following octonion-valued CGNNs with delays on time scale $\mathbb{T}$:

$$x_i^\Delta(t) = -\alpha_i(x_i(t))\left[\beta_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\theta_j)) - I_i(t)\right], \quad (1)$$

where $i \in I[n] := \{1, 2, \ldots, n\}$, $x_i(t) \in \mathbb{O}$ is the state of the ith unit at time t, $\alpha_i(\cdot) : \mathbb{O} \to \mathbb{O}$ is the amplification function, $\beta_i(\cdot) : \mathbb{O} \to \mathbb{O}$ is the behaved function, and $f_i(\cdot), g_i(\cdot) : \mathbb{O} \to \mathbb{O}$ are the activation functions, $a_{ij}, b_{ij} \in \mathbb{O}$ present the connection weight, the discretely delayed connection weight, respectively, $\theta_j \in \Pi^+$ denotes the time delay, $I_i(t) \in \mathbb{O}$ is the external input at time t.

To make model (1) meaningful, we stipulate that for $a, b, c \in \mathbb{O}$, $abc = (ab)c$.

The initial values of (1) are

$$x_i(s) = \varphi_i(s) \in L^\infty([-\varrho, 0]_{\mathbb{T}}, \mathbb{O}), \quad i \in I[n], \quad \varrho = \max_{j \in I[n]}\{\theta_j\}.$$

3 B_p-Almost Anti-periodic Functions on Time Scales

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space and $f \in L^p_{loc}(\mathbb{T}, \mathbb{E})$, where $p \geq 1$, we define the Besicovitch seminorm by

$$\|f\|_{B^p} := \limsup_{T \to +\infty} \left[\frac{1}{2T}\int_{-T}^{T} \|f(t)\|^p \Delta t\right]^{\frac{1}{p}}.$$

Denote by $B^p_b(\mathbb{T}, \mathbb{E})$ the space of all B^p-bounded functions.

Let $L^\infty(\mathbb{T}, \mathbb{E})$ be the collection of functions $x : \mathbb{T} \to \mathbb{E}$ that are measurable and essentially bounded, then it is a Banach space with the norm $\|x\|_\infty = \operatorname{ess\,sup}_{t \in \mathbb{T}} \|x(t)\|$ for $x \in L^\infty(\mathbb{T}, \mathbb{E})$.

Definition 3.1 A function $f \in B^p_b(\mathbb{T}, \mathbb{E})$ is called B_p-almost periodic if for each $\epsilon > 0$ the set

$$E_{bpap}(f, \epsilon) = \{\tau \in \Pi : \|f(\cdot + \tau) - f(\cdot)\|_{B^p} < \epsilon\}$$

is relatively dense. We will denote by $B^pAP(\mathbb{T}, \mathbb{E})$ the space of all such functions.

Definition 3.2 A function $f \in B^p_b(\mathbb{T}, \mathbb{E})$ is called B_p-almost anti-periodic if for each $\epsilon > 0$ the set

$$\vartheta_{bpap}(f, \epsilon) = \{\tau \in \Pi : \|f(\cdot + \tau) + f(\cdot)\|_{B^p} < \epsilon\}$$

is relatively dense. We will denote by $B^pANP(\mathbb{T}, \mathbb{E})$ the set of all such functions.

Example 3.1 If $\mu(t) < \frac{1}{2}$ for any $t \in \mathbb{T}$, for $p \geq 1$, we can easily check that the following functions are not periodic, anti-periodic or almost anti-periodic but B_p-almost anti-periodic,

$$(1)\ \ f(t) = \begin{cases} \cos(\pi t) + \cos(\sqrt{2}\pi t) + e_2(t, 0) + \frac{(-\sigma(t)-t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t \in (-\infty, 0]_{\mathbb{T}}; \\ \cos(\pi t) + \cos(\sqrt{2}\pi t) + e_{-2}(t, 0) + \frac{(\sigma(t)+t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t \in (0, +\infty)_{\mathbb{T}}. \end{cases}$$

$$(2)\ \ g(t) = \begin{cases} \sin(\pi t) + \sin(\sqrt{2}\pi t) + e_2(t, 0) + \frac{(-\sigma(t)-t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t \in (-\infty, 0]_{\mathbb{T}}; \\ \sin(\pi t) + \sin(\sqrt{2}\pi t) + e_{-2}(t, 0) + \frac{(\sigma(t)+t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t \in (0, +\infty)_{\mathbb{T}}. \end{cases}$$

Remark 3.1 From Sect. 3.1, 3.2 and 3.1, one can see that the space of B_p-almost anti-periodic functions is a proper subspace of the space of B_p-almost anti-periodic functions.

By Sect. 2.1 and the Fubini theorem on time scales, one can easily show that

Lemma 3.1 *If* $f \in L^{\infty}(\mathbb{T}, \mathbb{E})$, $a \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *is bounded with* $-a \in \mathcal{R}^+$ *(the set of all positive regressive functions),* $a^+ = \sup_{t\in\mathbb{T}} a(t)$, $a^- = \inf_{t\in\mathbb{T}} a(t) > 0$, $a^+\mu^+ < 1$ *and* $\tau \in \Pi^+$. *Then*

$$\left\| \int_{-\infty}^{\cdot} e_{-a}(\cdot, \sigma(s)) f(s)\Delta s \right\|_{B^p} \leq \frac{1}{a^-(1-a^+\mu^+)^{\frac{1}{p}}} \left(1 + \frac{1}{1-e^{-a^-\tau}}\right)^{\frac{1}{p}} \|f\|_{B^p}.$$

Definition 3.3 Function $f = (f_1, f_2, \cdots, f_n) : \mathbb{T} \to \mathbb{O}^n$ is called B_p-almost anti-periodic if for every $i \in I[n]$, f_i is B_p-almost anti-periodic.

In the sequel, we will use the following symbols:

$$a_{ij}^* = \|a_{ij}\|_{\mathbb{O}}, \quad b_{ij}^* = \|b_{ij}\|_{\mathbb{O}}, \quad c_{ij}^* = \|c_{ij}\|_{\mathbb{O}}, \quad \mu^+ = \sup_{t\in\mathbb{T}} \mu(t), \quad I_i^+ = \|I_i\|_{\infty}$$

and make the following assumptions:

(H$_1$) For $i \in I[n]$, there exist constants $L_i^{\alpha} \geq 0$ such that for any $x, y \in \mathbb{O}$,

$$\|\alpha_i(x) \pm \alpha_i(y)\|_{\mathbb{O}} \leq L_i^{\alpha}\|x \pm y\|_{\mathbb{O}}.$$

(H$_2$) For $i \in I[n]$, there exist positive constants $\gamma_i > 0$ with $\gamma_i\mu^+ < 1$, $\gamma_i \in \mathcal{R}^+$ and constants $L_i^{\Gamma} \geq 0$ such that functions $\Gamma_i : \mathbb{O} \to \mathbb{O}$ defined by $\Gamma_i(u) := \gamma_i u - \alpha_i(u)\beta_i(u)$ for any $u \in \mathbb{O}$ satisfying, for any $x, y \in \mathbb{O}$,

$$\|\Gamma_i(x) \pm \Gamma_i(y)\|_{\mathbb{O}} \leq L_i^{\Gamma}\|x \pm y\|_{\mathbb{O}}.$$

(H$_3$) For $j \in I[n]$, functions $f_j, g_j \in C(\mathbb{O}, \mathbb{O})$ and there exist positive constants $L_j^f, L_j^g > 0$ such that, for any $x, y \in \mathbb{O}$,

$$\|f_j(x) \pm f_j(y)\|_{\mathbb{O}} \leq L_j^f\|x \pm y\|_{\mathbb{O}}, \quad \|g_j(x) \pm g_j(y)\|_{\mathbb{O}} \leq L_j^g\|x \pm y\|_{\mathbb{O}}.$$

(H$_4$) Function $I = (I_1, I_2, \cdots, I_n) \in B^pANP(\mathbb{T}, \mathbb{O}^n) \cap L^{\infty}(\mathbb{T}, \mathbb{O}^n)$.

(H$_5$) There exist positive constants $r > 0$ and $h \in \Pi^+$ such that $\max_{i\in I[n]}\{\rho_i H_i\} < 1$, where

$$\Theta_i = \frac{1}{(1-\mu^+\gamma_i)^{\frac{1}{p}}}\left(1 + \frac{1}{1-e^{-\gamma_i h}}\right)^{\frac{1}{p}} (> 1),$$

$$\rho_i = \frac{1}{\gamma_i}\left[L_i^{\Gamma} + 2rL_i^{\alpha}\sum_{j=1}^{n}\left(a_{ij}^*L_j^f + b_{ij}^*L_j^g\right) + L_i^{\alpha}I_i^+\right].$$

4 B_p-Almost Anti-periodic Solutions

4.1 Existence

By (H$_2$), system (1) can be written as

$$\begin{aligned} x_i^{\Delta}(t) = &- \gamma_i x_i(t) + \Gamma_i(x_i(t)) + \alpha_i(x_i(t)) \\ &\times \left(\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\theta_j)) + I_i(t)\right), \ i \in I[n]. \end{aligned} \tag{2}$$

Definition 4.1 Under assumptions (H_1)-(H_4), if $x = (x_1, x_2, \dots, x_n)^T \in L^\infty(\mathbb{T}, \mathbb{O}^n)$ is a solution of the following equation

$$x_i(t) = \int_{-\infty}^{t} e_{-\gamma_i}(t, \sigma(s))\Bigg[\Gamma_i(x_i(s)) + \alpha_i(x_i(s))\Bigg(\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\theta_j)) + I_i(s)\Bigg)\Bigg]\Delta s,\ \ i \in I[n]. \tag{3}$$

then x is said to be a mild solution of system (1).

By Sect. 2.1 and conditions (H_1)-(H_3), one can easily get that

Remark 4.1 (1^0) If there exists a positive constant $a > 0$ with $-a \in \mathcal{R}^+$, for any $\tau \in \Pi$ and $t \geq \sigma(s)$, $e_{-a}(t+\tau, \sigma(s+\tau)) = e_{-a}(t, \sigma(s))$;
(2^0) obviously, $\alpha_i(0) = \Gamma_i(0) = f_j(0) = g_j(0) = 0$.

Theorem 4.1 *Assume that* (H_1)-(H_5) *are fulfilled. Then system* (1) *has a unique* B_p*-almost anti-periodic mild solution in* $\mathbb{Y} = \{\varphi : \varphi \in L^\infty(\mathbb{T}, \mathbb{O}^n), \|\varphi\|_\infty \leq r\}$.

Proof Noting that $(L^\infty(\mathbb{T}, \mathbb{O}^n), \|\cdot\|_\infty)$ is a Banach space and $\mathbb{Y}$ is a closed convex subset of $L^\infty(\mathbb{T}, \mathbb{O}^n)$. Consider an operator $T : \mathbb{Y} \to L^\infty(\mathbb{T}, \mathbb{O}^n)$ defined by

$$(T\phi)(t) = ((T\phi)_1(t), (T\phi)_2(t), \dots, (T\phi)_n(t))^T,$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in \mathbb{Y}$, $t \in \mathbb{T}$ and

$$(T\phi)_i(t) = \int_{-\infty}^{t} e_{-\gamma_i}(t, \sigma(s))\Bigg[\Gamma_i(\phi_i(s)) + \alpha_i(\phi_i(s))\Bigg(\sum_{j=1}^{n} a_{ij} f_j(\phi_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(\phi_j(s-\theta_j)) + I_i(s)\Bigg)\Bigg]\Delta s,\ \ i \in I[n].$$

Step 1: We will show that $T(\mathbb{Y}) \subset \mathbb{Y}$. In view of (H_1)-(H_5) and Sect. 4.1, for any $\phi \in \mathbb{Y}$, one gets

$$\begin{aligned}
\|(T\phi)\|_\infty = \max_{i \in I[n]} \Bigg\{ \operatorname*{ess\,sup}_{t \in \mathbb{T}} \Bigg\| \int_{-\infty}^{t} e_{-\gamma_i}(t, \sigma(s))\Bigg[\Gamma_i(\phi_i(s)) - \Gamma_i(0) + (\alpha_i(\phi_i(s)) - \alpha_i(0)) \\
\times \Bigg(\sum_{j=1}^{n} a_{ij}(f_j(\phi_j(s)) - f_j(0)) + \sum_{j=1}^{n} b_{ij}(g_j(\phi_j(s-\theta_j)) - g_j(0))\Bigg) \\
+ (\alpha_i(\phi_i(s)) - \alpha_i(0)) I_i(s)\Bigg]\Delta s \Bigg\|_{\mathbb{O}} \Bigg\} \\
\leq \max_{i \in I[n]} \Bigg\{ \int_{-\infty}^{t} e_{-\gamma_i}(t, \sigma(s))\Bigg[L_i^\Gamma \|\phi_i\|_\infty + L_i^\alpha \|\phi_i\|_\infty \Bigg(\sum_{j=1}^{n} a_{ij}^* L_j^f \|\phi_j\|_\infty \\
+ \sum_{j=1}^{n} b_{ij}^* L_j^g \|\phi_j\|_\infty\Bigg) + L_i^\alpha \|\phi_i\|_\infty I_i^+\Bigg]\Delta s \Bigg\} \\
\leq \max_{i \in I[n]} \Bigg\{ \frac{1}{\gamma_i}\Bigg[L_i^\Gamma + L_i^\alpha \|\phi_i\|_\infty \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) + L_i^\alpha I_i^+\Bigg] \|\phi\|_\infty \Bigg\}
\end{aligned}$$

$$\leq \max_{i\in I[n]}\left\{\frac{1}{\gamma_i}\left[L_i^{\Gamma}+rL_i^{\alpha}\sum_{j=1}^{n}\left(a_{ij}^{*}L_j^{f}+b_{ij}^{*}L_j^{g}\right)+L_i^{\alpha}I_i^{+}\right]r\right\}\leq r.$$

That is, $T(\mathbb{Y})\subset\mathbb{Y}$.

Step 2: we will prove that T has a fixed point in $\mathbb{Y}$. For any $\varphi,\psi\in\mathbb{Y}$, by (H$_1$)-(H$_5$) and Sect. 4.1, we infer that

$$\begin{aligned}
&\|(T\varphi)-(T\psi)\|_{\infty}\\
&=\max_{i\in I[n]}\left\{\operatorname*{ess\,sup}_{t\in\mathbb{T}}\left\|\int_{-\infty}^{t}e_{-\gamma_i}(t,\sigma(s))\left[\Gamma_i(\varphi_i(s))-\Gamma_i(\psi_i(s))+\alpha_i(\varphi_i(s))\right.\right.\right.\\
&\quad\times\left(\sum_{j=1}^{n}a_{ij}(f_j(\varphi_j(s))-f_j(\psi_j(s)))+\sum_{j=1}^{n}b_{ij}(g_j(\varphi_j(s-\theta_j))-g_j(\psi_j(s-\theta_j)))\right)\\
&\quad\left.\left.\left.+(\alpha_i(\varphi_i(s))-\alpha_i(\psi_i(s)))\left(\sum_{j=1}^{n}a_{ij}f_j(\psi_j(s))+\sum_{j=1}^{n}b_{ij}g_j(\psi_j(s-\theta_j))+I_i(s)\right)\right]\Delta s\right\|_{\mathbb{O}}\right\}\\
&\leq\max_{i\in I[n]}\left\{\int_{-\infty}^{t}e_{-\gamma_i}(t,\sigma(s))\left[L_i^{\Gamma}\|\varphi_i-\psi_i\|_{\infty}+rL_i^{\alpha}\left(\sum_{j=1}^{n}a_{ij}^{*}L_j^{f}\|\varphi_j-\psi_j\|_{\infty}\right.\right.\right.\\
&\quad\left.\left.\left.+\sum_{j=1}^{n}b_{ij}^{*}L_j^{g}\|\varphi_j-\psi_j\|_{\infty}\right)+L_i^{\alpha}\|\varphi_i-\psi_i\|_{\infty}\left(r\sum_{j=1}^{n}a_{ij}^{*}L_j^{f}+r\sum_{j=1}^{n}b_{ij}^{*}L_j^{g}+I_i^{+}\right)\right]\Delta s\right\}\\
&\leq\max_{i\in I[n]}\left\{\frac{1}{\gamma_i}\left[L_i^{\Gamma}+2rL_i^{\alpha}\sum_{j=1}^{n}\left(a_{ij}^{*}L_j^{f}+b_{ij}^{*}L_j^{g}\right)+L_i^{\alpha}I_i^{+}\right]\|\varphi-\psi\|_{\infty}\right\}\\
&\leq\max_{i\in I[n]}\{\rho_i\}\|\varphi-\psi\|_{\infty},
\end{aligned}$$

which combined with (H$_5$) implies that T is a contraction mapping. As a consequence, T has a unique fixed point $\hat{x}=(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$ in $\mathbb{Y}$. Hence, $\hat{x}$ is a unique essentially bounded solution of (1).

Step 3: we will show that the $\hat{x}$ is B_p-almost anti-periodic. For $i\in I[n]$, we can rewrite (3) as

$$\begin{aligned}
\hat{x}_i(t)=\int_{-\infty}^{t}e_{-\gamma_i}(t,\sigma(s))\left\{\Gamma_i(\hat{x}_i(s))+\alpha_i(\hat{x}_i(s))\left[\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s))\right.\right.\\
\left.\left.+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s-\theta_j))+I_i(s)\right]\right\}\Delta s. \qquad (4)
\end{aligned}$$

By (H$_4$), for any $\epsilon>0$, there exists a $\tau_i(\epsilon)\in\vartheta_{bpap}(I_i,\epsilon)$ such that

$$\|I_i(\cdot+\tau_i)+I_i(\cdot)\|_{B^p}<\epsilon,\ \ i\in I[n]. \qquad (5)$$

On one hand, by Lemma 4.1, Sects. 4.1 and 2.1, we can obtain that

$$\|\hat{x}_i(\cdot+\tau_i)+\hat{x}_i(\cdot)\|_{B^p}=\left\|\int_{-\infty}^{\cdot}e_{-\gamma_i}(\cdot,\sigma(s))F_i(s)\Delta s\right\|_{B^p},\ \ i\in I[n], \qquad (6)$$

where

$$F_i(s)=\Gamma_i(\hat{x}_i(s+\tau_i))+\Gamma_i(\hat{x}_i(s))+\alpha_i(\hat{x}_i(s+\tau_i))$$

$$\times\left(\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s+\tau_i))+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s+\tau_i-\theta_j))+I_i(s+\tau_i)\right)$$
$$+\alpha_i(\hat{x}_i(s))\left[\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s))+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s-\theta_j))+I_i(s)\right].$$

According to $(H_1)-(H_5)$, (5), Sect. 3.1 and Sect. 4.1, for $h\in\Pi^+$ and $i\in I[n]$, we deduce that

$$\begin{aligned}
&\left\|\int_{-\infty}^{\cdot}e_{-\gamma_i}(\cdot,\sigma(s))F_i(s)\Delta s\right\|_{B^p}\\
\leq&\frac{1}{\gamma_i(1-\mu^+\gamma_i)^{\frac{1}{p}}}\left(1+\frac{1}{1-e^{-\gamma_i h}}\right)^{\frac{1}{p}}\Big\|\Gamma_i(\hat{x}_i(s+\tau_i))+\Gamma_i(\hat{x}_i(s))+\alpha_i(\hat{x}_i(s+\tau_i))\\
&\times\left(\sum_{j=1}^{n}a_{ij}(f_j(\hat{x}_j(s+\tau_i))+f_j(\hat{x}_j(s)))+\sum_{j=1}^{n}b_{ij}(g_j(\hat{x}_j(s+\tau_i-\theta_j))+g_j(\hat{x}_j(s-\theta_j)))\right.\\
&\left.+I_i(s+\tau_i)+I_i(s)\right)+(\alpha_i(\hat{x}_i(s))-\alpha_i(\hat{x}_i(s+\tau_i)))\\
&\times\left(\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s))+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s-\theta_j))+I_i(s)\right)\Big\|_{B^p}\\
\leq&\frac{1}{\gamma_i}\Theta_i\Big[L_i^{\Gamma}\|\hat{x}_i(\cdot+\tau_i)+\hat{x}_i(\cdot)\|_{B^p}+rL_i^{\alpha}\Big(\sum_{j=1}^{n}a_{ij}^*L_j^f\|\hat{x}_j(\cdot+\tau_i)+\hat{x}_j(\cdot)\|_{B^p}\\
&+\sum_{j=1}^{n}b_{ij}^*L_j^g\|\hat{x}_j(\cdot+\tau_i)+\hat{x}_j(\cdot)\|_{B^p}+\|I_i(\cdot+\tau_i)+I_i(\cdot)\|_{B^p}\Big)\\
&+L_i^{\alpha}\|\hat{x}_i(\cdot+\tau_i)-\hat{x}_i(\cdot)\|_{B^p}\Big(r\sum_{j=1}^{n}a_{ij}^*L_j^f+r\sum_{j=1}^{n}b_{ij}^*L_j^g+I_i^+\Big)\Big]\\
\leq&\frac{1}{\gamma_i}\Theta_i\Big[L_i^{\Gamma}+rL_i^{\alpha}\sum_{j=1}^{n}\big(a_{ij}^*L_j^f+b_{ij}^*L_j^g\big)\Big]\|\hat{x}_i(\cdot+\tau_i)+\hat{x}_i(\cdot)\|_{B^p}\\
&+\frac{1}{\gamma_i}\Theta_i\Big[rL_i^{\alpha}\sum_{j=1}^{n}\big(a_{ij}^*L_j^f+b_{ij}^*L_j^g\big)+L_i^{\alpha}I_i^+\Big]\|\hat{x}_i(\cdot+\tau_i)-\hat{x}_i(\cdot)\|_{B^p}+\frac{rL_i^{\alpha}}{\gamma_i}\Theta_i\epsilon.
\end{aligned}\tag{7}$$

On the other hand, by (4) and Sect. 2.1, we can get that

$$\|\hat{x}_i(\cdot+\tau_i)-\hat{x}_i(\cdot)\|_{B^p}=\left\|\int_{-\infty}^{\cdot}e_{-\gamma_i}(\cdot,\sigma(s))G_i(s)\Delta s\right\|_{B^p},\tag{8}$$

where

$$\begin{aligned}
G_i(s)=&\Gamma_i(\hat{x}_i(s+\tau_i))-\Gamma_i(\hat{x}_i(s))+\alpha_i(\hat{x}_i(s+\tau_i))\\
&\times\left(\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s+\tau_i))+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s+\tau_i-\theta_j))+I_i(s+\tau_i)\right)\\
&-\alpha_i(\hat{x}_i(s))\left(\sum_{j=1}^{n}a_{ij}f_j(\hat{x}_j(s))+\sum_{j=1}^{n}b_{ij}g_j(\hat{x}_j(s-\theta_j))+I_i(s)\right).
\end{aligned}$$

By (H$_1$)-(H$_5$), (5), Sect. 3.1 and Sect. 4.1, for $h \in \Pi^+$ and $i \in I[n]$, one can easily get that

$$\begin{aligned}
&\left\| \int_{-\infty}^{\cdot} e_{-\gamma_i}(\cdot, \sigma(s)) G_i(s) \Delta s \right\|_{B^p} \\
&\leq \frac{1}{\gamma_i (1 - \mu^+ \gamma_i)^{\frac{1}{p}}} \left(1 + \frac{1}{1 - e^{-\gamma_i h}}\right)^{\frac{1}{p}} \Bigg[L_i^{\Gamma} \|\hat{x}_i(\cdot + \tau_i) - \hat{x}_i(\cdot)\|_{B^p} \\
&\quad + r L_i^{\alpha} \Bigg(\sum_{j=1}^{n} a_{ij}^* L_j^f \|\hat{x}_j(\cdot + \tau_i) + \hat{x}_j(\cdot)\|_{B^p} + \sum_{j=1}^{n} b_{ij}^* L_j^g \|\hat{x}_j(\cdot + \tau_i) + \hat{x}_j(\cdot)\|_{B^p} \\
&\quad + \|I_i(\cdot + \tau_i) + I_i(\cdot)\|_{B^p} \Bigg) + L_i^{\alpha} \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} \Bigg(r \sum_{j=1}^{n} a_{ij}^* L_j^f + r \sum_{j=1}^{n} b_{ij}^* L_j^g + I_i^+ \Bigg) \Bigg] \\
&\leq \frac{1}{\gamma_i} \Theta_i \Bigg[2 r L_i^{\alpha} \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) + L_i^{\alpha} I_i^+ \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} \\
&\quad + L_i^{\Gamma} \|\hat{x}_i(\cdot + \tau_i) - \hat{x}_i(\cdot)\|_{B^p} + r L_i^{\alpha} \epsilon \Bigg]. \qquad (9)
\end{aligned}$$

Since $\Theta_i > 1$, by (8) and (9), one can infer that

$$\begin{aligned}
\|\hat{x}_i(\cdot + \tau_i) - \hat{x}_i(\cdot)\|_{B^p} &\leq \left[1 - \frac{L_i^{\Gamma} \Theta_i}{\gamma_i}\right]^{-1} \frac{\Theta_i}{\gamma_i} [(\gamma_i \rho_i - L_i^{\Gamma}) \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + r L_i^{\alpha} \epsilon] \\
&= \frac{\Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} [(\gamma_i \rho_i - L_i^{\Gamma}) \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + r L_i^{\alpha} \epsilon] \\
&= \frac{\gamma_i \rho_i \Theta_i - L_i^{\Gamma} H_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \epsilon \\
&\leq \frac{\gamma_i - L_i^{\Gamma} H_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \epsilon \\
&= \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \epsilon. \qquad (10)
\end{aligned}$$

Consequently, it follows from (6), (7), (10) that

$$\begin{aligned}
&\|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} \\
&\quad \leq \frac{\Theta_i}{\gamma_i} \left[L_i^{\Gamma} + r L_i^{\alpha} \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) \right] \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} \\
&\quad + \frac{\Theta_i}{\gamma_i} \left[r L_i^{\alpha} \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) + L_i^{\alpha} I_i^+ \right] \|\hat{x}_i(\cdot + \tau_i) - \hat{x}_i(\cdot)\|_{B^p} + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i} \epsilon \\
&\leq \Theta_i \rho_i \|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} + \frac{\Theta_i}{\gamma_i} \left[r L_i^{\alpha} \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) + L_i^{\alpha} I_i^+ \right] \frac{r L_i^{\alpha} \Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} \epsilon + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i} \epsilon,
\end{aligned}$$

hence,

$$\|\hat{x}_i(\cdot + \tau_i) + \hat{x}_i(\cdot)\|_{B^p} < \frac{\frac{\Theta_i}{\gamma_i} \left[r L_i^{\alpha} \sum_{j=1}^{n} \left(a_{ij}^* L_j^f + b_{ij}^* L_j^g\right) + L_i^{\alpha} I_i^+ \right] \frac{r L_i^{\alpha} \Theta_i}{\gamma_i - L_i^{\Gamma} \Theta_i} + \frac{r L_i^{\alpha} \Theta_i}{\gamma_i}}{1 - \Theta_i \rho_i} \epsilon, \quad i \in I[n],$$

which implies that $\hat{x}$ is B_p-almost anti-periodic. This ends the proof.

Remark 4.2 Due to the fact that the set composed of almost anti-periodic functions does not have a linear structure, $B^p ANP(\mathbb{T}, \mathbb{E})$ is not a Banach space. Our method of proving Theorem 4.1 is different from the usual method of proving the existence of almost periodic solutions. And for dynamic equations on time scales, our method of proving Theorem 4.1 is different from existing methods.

4.2 Global Exponential Stability

Theorem 4.2 *Under the assumptions of Sect. 4.1. Let x be the B_p-almost anti-periodic solution of system (1) with the initial value φ and y be an arbitrary solution of system (1) with the initial value ψ, respectively. Then there exist positive constants λ with $\ominus\lambda \in \mathcal{R}^+$ and M such that*

$$\|x(t) - y(t)\|_{\mathbb{O}^n} \leq M e_{\ominus\lambda}(t, 0)\|\varphi - \psi\|_\tau, \quad t \in (0, +\infty)_{\mathbb{T}},$$

where $\|\varphi-\psi\|_\tau = \sup_{s\in[-\varrho,0]_{\mathbb{T}}} \|\varphi(s)-\psi(s)\|_{\mathbb{O}^n}$, i.e., the B_p-almost anti-periodic mild solution x of system (1) is globally exponentially stable.

Proof Let $z(t) = y(t) - x(t)$, for $i \in I[n]$, by (2), we have

$$z_i^\Delta(t) = y_i^\Delta(t) - x_i^\Delta(t) = -\gamma_i z_i(t) + F_i(t, y_i, x_i), \quad i \in I[n], \tag{11}$$

where

$$\begin{aligned} F_i(t, y_i, x_i) =& \Gamma_i y_i(t)) - \Gamma_i(x_i(t)) + (\alpha_i(y_i(t)) - \alpha_i(x_i(t)))\Big[\sum_{j=1}^n a_{ij} f_j(y_j(t)) \\ &+ \sum_{j=1}^n b_{ij} g_j(y_j(t-\theta_j)) + I_i(t)\Big] + \alpha_i(x_i(t))\Big[\sum_{j=1}^n a_{ij}(f_j(y_j(t)) - f_j(x_j(t))) \\ &+ \sum_{j=1}^n b_{ij}(g_j(y_j(t-\theta_j)) - g_j(x_j(t-\theta_j)))\Big]. \end{aligned}$$

Based on (11), one can have that

$$z_i(t) = z_i(0)e_{-\gamma_i}(t, 0) + \int_0^t e_{-\gamma_i}(t, \sigma(s))F_i(s, y_i, x_i)\Delta s, \;\; i \in I[n]. \tag{12}$$

For $i \in I[n]$, set

$$N_i(\lambda) = \gamma_i \ominus \lambda - \Xi_i(\lambda),$$

where

$$\Xi_i(\lambda) = (1 + \mu^+\lambda)\Big[L_i^\Gamma + rL_i^\alpha \sum_{j=1}^n (a_{ij}^* L_j^f + b_{ij}^* L_j^g e^{\lambda\theta_j}) + rL_i^\alpha \zeta_i + L_i^\alpha I_i^+\Big].$$

Then, we have $N_i(\lambda) \to -\infty$ as $\lambda \to +\infty$, and $N_i(0) > 0$ by (H_5). Hence, by the continuity of $N_i(\lambda)$, one can take a number λ with $0 < \lambda < \min_{i\in I[n]}\{\gamma_i\}$ such that $N_i(\lambda) > 0, i \in I[n]$, which implies that $\Xi_i(\lambda) < \gamma_i \ominus \lambda$. From ($H_5$), we have

$$M := \min_{i\in I[n]} \Big\{\gamma_i[\Xi_i(0)]^{-1}\Big\} > 1,$$

thus,

$$\frac{1}{M} \leq \frac{\Xi_i(0)}{\gamma_i} < \frac{\Xi_i(\lambda)}{\gamma_i \ominus \lambda} < 1.$$

For any $\epsilon > 0$, it is obvious that

$$\|z(t)\|_{\mathbb{O}^n} < M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(t, 0), \ \ t \in [-\varrho, 0]_{\mathbb{T}}.$$

Furthermore, we claim that

$$\|z(t)\|_{\mathbb{O}^n} < M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(t, 0), \ \ t \in [0, +\infty)_{\mathbb{T}}. \tag{13}$$

If not, then there is a positive constant $\hat{t} \in \mathbb{T}$ such that

$$\|z(\hat{t})\|_{\mathbb{O}^n} = M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(\hat{t}, 0) \tag{14}$$

and

$$\|z(t)\|_{\mathbb{O}^n} < M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(t, 0), \ \ t \in [0, \hat{t})_{\mathbb{T}}.$$

Hence, from (12), one has

$$\begin{aligned}
\|z_i(\hat{t})\|_{\mathbb{O}} <&(\|\psi - \varphi\|_\tau + \epsilon)e_{-\gamma_i}(\hat{t}, 0) + M(\|\psi - \varphi\|_\tau + \epsilon)\int_0^{\hat{t}} e_{-\gamma_i}(\hat{t}, \sigma(s))\bigg(L_i^\Gamma e_{\ominus\lambda}(s, 0) \\
&+ L_i^\alpha e_{\ominus\lambda}(s, 0)\bigg[r\sum_{j=1}^n a_{ij}^* L_j^f + r\sum_{j=1}^n b_{ij}^* L_j^g + I_i^+\bigg] \\
&+ rL_i^\alpha\bigg(\sum_{j=1}^n a_{ij}^* L_j^f e_{\ominus\lambda}(s, 0) + \sum_{j=1}^n b_{ij}^* L_j^g e_{\ominus\lambda}(s - \theta_j, 0)\bigg)\bigg)\Delta s \\
\leq& M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(\hat{t}, 0)\bigg\{\frac{e_{-\gamma_i \oplus\lambda}(\hat{t}, 0)}{M} + \int_0^{\hat{t}} e_{-\gamma_i\oplus\lambda}(\hat{t}, \sigma(s))\bigg[L_i^\Gamma e_\lambda(\sigma(s), s) \\
&+ L_i^\alpha e_\lambda(\sigma(s), s)\bigg(r\sum_{j=1}^n a_{ij}^* L_j^f + r\sum_{j=1}^n b_{ij}^* L_j^g + I_i^+\bigg) \\
&+ rL_i^\alpha\bigg(\sum_{j=1}^n a_{ij}^* L_j^f e_\lambda(\sigma(s), s) + \sum_{j=1}^n b_{ij}^* L_j^g e_\lambda(\sigma(s), s - \theta_j)\bigg)\bigg]\Delta s\bigg\} \\
\leq& M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(\hat{t}, 0)\bigg\{\frac{e_{-\gamma_i \oplus\lambda}(\hat{t}, 0)}{M} + \frac{1 + \mu^+\lambda}{\gamma_i \ominus \lambda}(1 - e_{-\gamma_i\oplus\lambda}(\hat{t}, 0)) \\
&\times \bigg[L_i^\Gamma + rL_i^\alpha\sum_{j=1}^n(a_{ij}^* L_j^f + b_{ij}^* L_j^g) + L_i^\alpha I_i^+ + rL_i^\alpha\sum_{j=1}^n(a_{ij}^* L_j^f + b_{ij}^* L_j^g e^{\lambda\theta_j})\bigg]\bigg\} \\
<& M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(\hat{t}, 0).
\end{aligned}$$

Consequently,

$$\|z(\hat{t})\|_{\mathbb{O}^n} < M(\|\psi - \varphi\|_\tau + \epsilon)e_{\ominus\lambda}(\hat{t}, 0),$$

which contradicts (14), and so the claim is true. Letting $\epsilon \to 0^+$, (13) yields

$$\|z(t)\|_{\mathbb{O}^n} \leq M\|\psi - \varphi\|_\tau e_{\ominus\lambda}(t, 0).$$

This completes the proof.

5 A Numerical Example

Example 5.1 In network (1), let $n=2$, $r=20$, for $i, j=1,2$ and $x\in\mathbb{O}$, take

$$\begin{aligned}
f_1(x)=&0.03\sin(x^0)e_0+0.01\tanh(x^1)e_1+0.03\tanh(x^2)e_2+0.02\sin(x^3)e_3\\
&+0.001\sin(x^4)e_4+0.003\sin(x^5)e_5+0.002\sin(x^6)e_6+0.03\arctan(x^7)e_7,\\
f_2(x)=&0.01\sin(x^0)e_0+0.025\sin(x^1)e_1+0.025\tanh(x^2)e_2+0.025\sin(x^3)e_3\\
&+0.03\arctan(x^4)e_4+0.005\sin(x^5)e_5+0.04\tanh(x^6)e_6+0.001\sin(x^7)e_7,\\
g_1(x)=&0.003\sin(x^0)e_0+0.001\tanh(x^1)e_1+0.006\sin(x^2)e_2+0.006\sin(x^3)e_3\\
&+0.02\arctan(x^4)e_4+0.001\sin(x^5)e_5+0.02\tanh(x^6)e_6+0.003\sin(x^7)e_7,\\
g_2(x)=&0.03\sin(x^0)e_0+0.02\sin(x^1)e_1+0.02\sin(x^2)e_2+0.01\sin(x^3)e_3\\
&+0.03\sin(x^4)e_4+0.02\sin(x^5)e_5+0.01\sin(x^6)e_6+0.02\sin(x^7)e_7,\\
I_1(t)=&0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_0+0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_1\\
&+0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_2+0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_3\\
&+\begin{cases}0.01\cos(\pi t)+0.01\cos(\sqrt{2}\pi t)+\frac{0.01(-\sigma(t)-t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t\in(-\infty,0]_{\mathbb{T}}\\ 0.01\cos(\pi t)+0.01\cos(\sqrt{2}\pi t)+\frac{0.01(\sigma(t)+t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t\in(0,+\infty)_{\mathbb{T}}\end{cases}e_4\\
&+0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_5+0.01(\cos(\pi t)+\cos(\sqrt{2}\pi t))e_6\\
&+\begin{cases}0.01\cos(\pi t)+0.01\cos(\sqrt{2}\pi t)+0.01e_2(t,0), & t\in(-\infty,0]_{\mathbb{T}}\\ 0.01\cos(\pi t)+0.01\cos(\sqrt{2}\pi t)+0.01e_{-2}(t,0), & t\in(0,+\infty)_{\mathbb{T}}\end{cases}e_7,\\
I_2(t)=&0.01(\sin(t)+\sin(\sqrt{3}t))e_0+0.01(\sin(t)+\sin(\sqrt{3}t))e_1\\
&+\begin{cases}0.01\sin(t)+0.01\sin(\sqrt{3}t)+\frac{0.01(-\sigma(t)-t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t\in(-\infty,0]_{\mathbb{T}}\\ 0.01\sin(t)+0.01\sin(\sqrt{3}t)+\frac{0.01(\sigma(t)+t)^{\frac{1}{p}}}{(1+\sigma^2(t))^{\frac{1}{p}}(1+t^2)^{\frac{1}{p}}}, & t\in(0,+\infty)_{\mathbb{T}}\end{cases}e_2\\
&+0.01(\sin(t)+\sin(\sqrt{3}t))e_3+0.01(\sin(t)+\sin(\sqrt{3}t))e_4\\
&+\begin{cases}0.01\sin(t)+0.01\sin(\sqrt{3}t)+0.01e_2(t,0), & t\in(-\infty,0]_{\mathbb{T}}\\ 0.01\sin(t)+0.01\sin(\sqrt{3}t)+0.01e_{-2}(t,0), & t\in(0,+\infty)_{\mathbb{T}}\end{cases}e_5\\
&+0.01(\sin(t)+\sin(\sqrt{3}t))e_6+0.01(\sin(t)+\sin(\sqrt{3}t))e_7,\\
a_{11}=&-0.1\sin(\sqrt{2})+0.1\cos(\sqrt{3})e_1,\quad a_{12}=0.1\cos(\sqrt{2})+0.1\sin(\sqrt{5})e_2,\\
a_{21}=&0.1\cos(\sqrt{2})+0.1\sin(\sqrt{2})e_3,\quad a_{22}=-0.05\sin(\sqrt{2})+0.05\cos(\sqrt{5})e_4,\\
b_{11}=&-0.05\cos(\sqrt{2})+0.05\sin(\sqrt{3})e_5,\quad b_{12}=0.01\sin(\sqrt{2})+0.01\cos(\sqrt{3})e_6,\\
b_{21}=&0.04\sin(\sqrt{2})+0.04\cos(\sqrt{5})e_7,\quad b_{22}=0.05\cos(1)+0.05\sin(\sqrt{2})e_3+0.05\cos(\sqrt{2})e_6.
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$, let

$$\theta_i=6,\quad h=0,\quad p=1,\quad \alpha_i(x)=x,\quad \beta_i(x)=18+\frac{1}{1+\overline{x}x}.$$

If $\mathbb{T}=\mathbb{Z}$, let

$$\theta_i=4,\quad h=0,\quad p=4,\quad \alpha_i(x)=1,\quad \beta_i(x)=x.$$

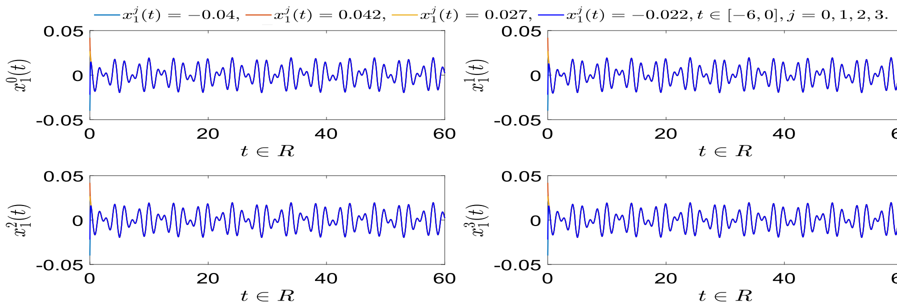

Fig. 1 $\mathbb{T}=\mathbb{R}$, states x_1^0, x_1^1, x_1^2 and x_1^3 of system (1) with different initial values

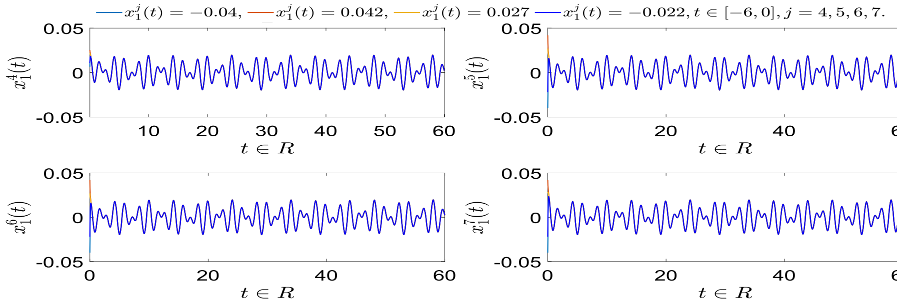

Fig. 2 $\mathbb{T}=\mathbb{R}$, states x_1^4, x_1^5, x_1^6 and x_1^7 of system (1) with different initial values

By simple calculations, we have for $i=1,2$,

$$a_{11}^*=0.1\sin(\sqrt{2}),\quad a_{12}^*=0.1\sin(\sqrt{5}),\quad a_{21}^*=0.1\sin(\sqrt{2}),\quad a_{22}^*=0.05\sin(\sqrt{2}),$$
$$b_{11}^*=0.05\sin(\sqrt{3}),\quad b_{12}^*=0.01\sin(\sqrt{2}),\quad b_{21}^*=0.04\sin(\sqrt{2}),\quad b_{22}^*=0.05\sin(\sqrt{2}),$$
$$L_1^f=0.03,\quad L_2^f=0.04,\quad L_1^g=0.02,\quad L_2^g=0.03,\quad L_1^h=0.04,\quad L_2^h=0.02,\quad I_i^+\le 0.065.$$

For $\mathbb{T}=\mathbb{R}$, $i\in I[n]$, we have $\gamma_i=18$, $\Gamma_i(x)=-\frac{x}{1+\bar{x}x}$,

$$L_i^\alpha=3,\quad L_i^\Gamma=1.25,\quad \zeta_i<0.0149,\quad \rho_i<0.162,\quad \Theta_i=2,\quad \max_i \rho_i H_i<0.324<1.$$

For $\mathbb{T}=\mathbb{Z}$, $i\in I[n]$, we have $\gamma_i=0.8$, $\Gamma_i(x)=-0.2x$,

$$L_i^\alpha=0,\quad L_i^\Gamma=0.2,\quad \zeta_i<0.0233,\quad \rho_i<0.25,\quad \Theta_i<2,\quad \max_i \rho_i H_i<0.5<1.$$

So, $(\mathrm{H}_1)-(\mathrm{H}_5)$ are fulfilled. By Sect. 4.2, system (1) possesses a unique B_p-almost anti-periodic mild solution, which is globally exponential stable (see Figs. 1, 2, 3, 4, 5, 6, 7 and 8).

Remark 5.1 There are no known results that can yield the results of Example 5.1.

6 Conclusions

In this paper, we have proposed a concept of B_p-almost anti-periodic functions on time scales, and obtained the existence and global exponential stability of B_p-almost anti-periodic mild

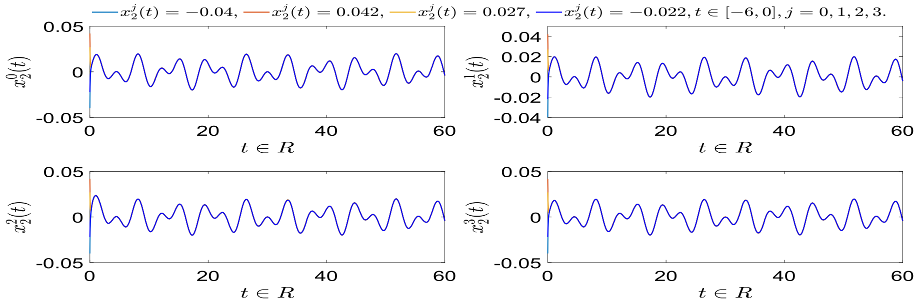

Fig. 3 $\mathbb{T} = \mathbb{R}$, states x_2^0, x_2^1, x_2^2 and x_2^3 of system (1) with different initial values

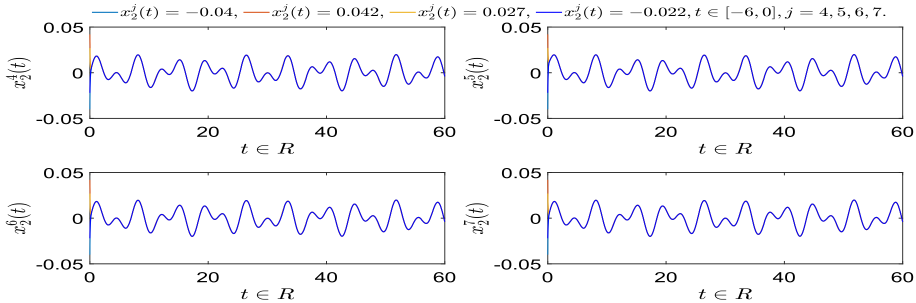

Fig. 4 $\mathbb{T} = \mathbb{R}$, states x_2^4, x_2^5, x_2^6 and x_2^7 of system (1) with different initial values

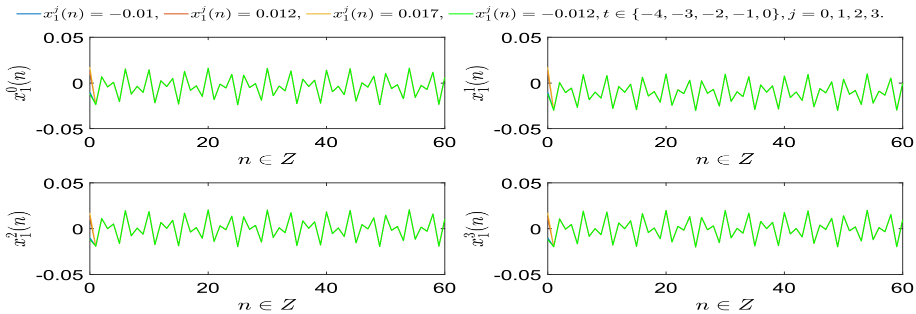

Fig. 5 $\mathbb{T} = \mathbb{Z}$, states x_1^0, x_1^1, x_1^2 and x_1^3 of system (1) with different initial values

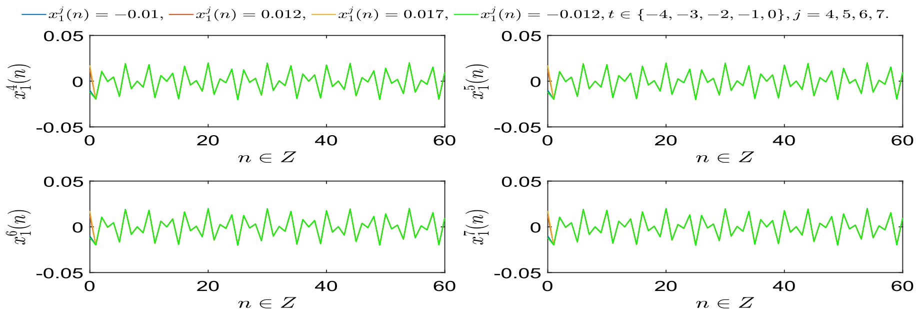

Fig. 6 $\mathbb{T} = \mathbb{Z}$, states x_1^4, x_1^5, x_1^6 and x_1^7 of system (1) with different initial values

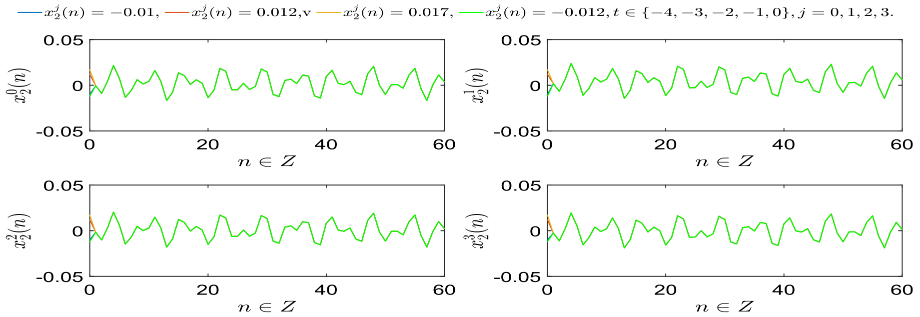

Fig. 7 $\mathbb{T} = \mathbb{Z}$, states x_2^0, x_2^1, x_2^2 and x_2^3 of system (1) with different initial values

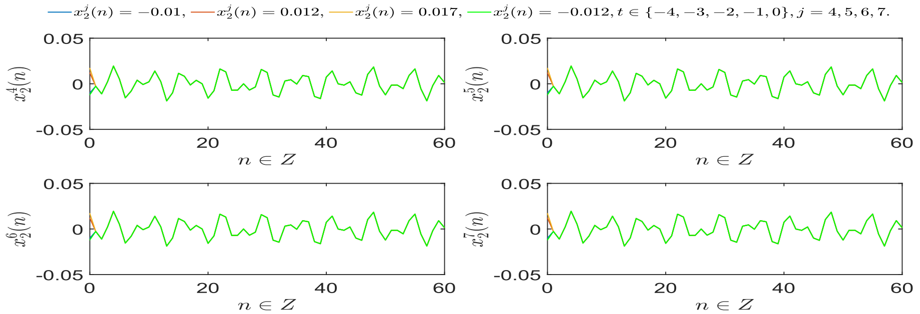

Fig. 8 $\mathbb{T} = \mathbb{Z}$, states x_2^4, x_2^5, x_2^6 and x_2^7 of system (1) with different initial values

solutions for a class of Cohen–Grossberg neural networks by using the fixed point theorem, inequality techniques and the method of contradiction. The concepts and research methods proposed in this paper can be used to study the existence of almost anti-periodic solutions to other types of neural network models and dynamic equations on time scales.

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