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GAFA Geometric And Functional Analysis

CAT(0) SPACES OF HIGHER RANK I

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Abstract. A CAT(0) space has rank at least n if every geodesic lies in an n-flat. Ballmann's Higher Rank Rigidity Conjecture predicts that a CAT(0) space of rank at least 2 with a geometric group action is *rigid* – isometric to a Riemannian symmetric space, a Euclidean building, or splits as a metric product. This paper is the first in a series motivated by Ballmann's conjecture. Here we prove that a CAT(0) space of rank at least $n \geq 2$ is rigid if it contains a periodic n-flat and its Tits boundary has dimension $(n-1)$. This does not require a geometric group action. The result relies essentially on the study of flats which do not bound flat half-spaces – so-called *Morse flats*. We show that the Tits boundary $\partial_T F$ of a periodic Morse n-flat F contains a *regular point* – a point with a Tits-neighborhood entirely contained in $\partial_T F$. More precisely, we show that the set of singular points in $\partial_T F$ can be covered by finitely many round spheres of positive codimension.

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1 Introduction

1.1 Main results. A *Hadamard manifold* is a complete simply connected Riemannian manifold of non-positive sectional curvature. It has *higher rank* if every complete geodesic line is contained in an isometrically embedded Euclidean space of dimension at least two, a so-called *flat*. The structure of manifolds of non-positive

curvature and higher rank was clarified in the eighties, in the series of papers [BBE85, BBS85], culminating in Ballmann's celebrated Higher Rank Rigidity Theorem [Bal85, BS87, EH90]: If a higher rank Hadamard manifold admits quotients of finite volume, then it is either a Riemannian symmetric space or splits as a metric product. So in the presence of enough symmetry, higher rank implies special geometry. In particular, the theorem identifies higher rank as the exceptional case.

The modern theory of non-positive curvature focuses on metric spaces instead of manifolds, where the curvature condition is expressed by triangle comparison. The definition is due to Alexandrov, already in the early fifties, and later gained considerable momentum from Gromov's seminal paper [Gro87]. Nowadays, CAT(0) spaces – the synthetic versions of Hadamard manifolds – play an important role in mathematics beyond geometry and topology. They appear in many different fields such as geometric group theory, representation theory, arithmetic and optimization. For an account of the huge amount of literature on metric spaces with synthetic curvature bounds, see the bibliographies in [AKP19, Bal04, BH99]. The following conjecture, formulated by Ballmann and Buyalo in [BB08], predicts a generalization of the Higher Rank Rigidity Theorem from Hadamard manifolds to CAT(0) spaces. It is the main motivation for the present paper.

Conjecture (Higher Rank Rigidity). *Let X be a locally compact geodesically complete CAT(0) space. Suppose that Γ is a group of isometries of X with limit set $\Lambda(\Gamma) = \partial_\infty X$. Then* $\operatorname{diam}(\partial_T X) = \pi$ *implies that X is a Riemannian symmetric space or a Euclidean building of rank at least* 2*, or that X non-trivially splits as a metric product.*

The condition on the diameter expresses higher rank, as it is equivalent to saying that any complete geodesic bounds a flat half-plane, cf. Sect. 2.4. If a group Γ acts on a CAT(0) space X, then its *limit set* $\Lambda(\Gamma)$ is the set of accumulation points of a Γ-orbit in the ideal boundary $\partial_\infty X$. The main example of group actions on CAT(0) spaces with full limit set are *geometric actions*, meaning cocompact and properly discontinuous actions by isometries.

For singular spaces, Higher Rank Rigidity is wide open. It has only been established in the following short list of special cases:

- for a 2-dimensional simplicial complex with a piecewise smooth CAT(0) metric and a geometric group action [BB96];
- for a 3-dimensional simplicial complex with a piecewise Euclidean CAT(0) metric and a geometric group action [BB00];
- for a finite dimensional CAT(0) cube complex with a geometric group action [CS11];[1]

[1] For cube complexes, the statement of Higher Rank Rigidity takes the form of a splitting result, since neither irreducible symmetric spaces nor irreducible Euclidean buildings are cube complexes.

– for a locally compact and geodesically complete CAT(0) space such that its full isometry group does not fix a point at infinity and which supports a cocompact isometric action by an amenable locally compact group [CM15];
– for a locally compact and geodesically complete CAT(0) space with 1-dimensional Tits boundary which supports a geometric group action $\Gamma \curvearrowright X$ such that the induced action on the ideal boundary is not minimal, i.e. $\partial_\infty X$ contains a proper closed Γ-invariant subset [Ric19].

The paper [Ric19] in the last item relies heavily on [GS13] where the authors developed a general strategy to approach Ballmann's conjecture based on analyzing the dynamics of the induced action at infinity $\Gamma \curvearrowright \partial_\infty X$.

Note also that the interesting case in the next to last item concerns non-discrete groups since the only locally compact geodesically complete CAT(0) space which admits a cocompact isometric action by an amenable discrete group is the flat Euclidean space [AB98, Corollary C].

Theorem A. *Let X be a locally compact CAT(0) space whose Tits boundary has dimension $n-1 \geq 1$. Suppose that every geodesic in X lies in an n-flat. If X contains a periodic n-flat, then X is a Riemannian symmetric space or a Euclidean building, or X splits as a metric product.*

If the isometry group of X acts cocompactly, then the condition on the dimension of the Tits boundary simply means that X does not contain an $(n+1)$-flat [Kle99, Theorem C]. Thus, we obtain:

Corollary B. *Let X be a locally compact CAT(0) space whose isometry group acts cocompactly. Suppose that every geodesic in X lies in an n-flat but X does not admit $(n+1)$-flats. If X contains a periodic n-flat, then X is a Riemannian symmetric space or a Euclidean building, or X splits as a metric product.*

Another consequence of Theorem A is the following.

Corollary C. *Let X be a locally compact CAT(0) space whose Tits boundary has dimension $n-1 \geq 1$. Suppose that every geodesic in X lies in an n-flat. If X contains a periodic n-flat, then X has a unique decomposition into irreducible factors*

$$X = \mathbb{R}^j \times X_1 \times \cdots \times X_k \times Y_1 \times \cdots \times Y_l \times M_1 \times \cdots \times M_m$$

where X_a are rank 1 spaces, Y_b are irreducible Euclidean buildings of higher rank and M_c are irreducible Riemannian symmetric spaces of higher rank.

Note that the assumptions above imply that the factors X_a are not only rank 1 but have discrete Tits boundaries. Thus if X admits a cocompact group action, then every X_a is Gromov-hyperbolic.

Recall that in the smooth case, the higher rank property can be expressed by asking that every complete geodesic admits n linearly independent parallel Jacobi fields for some $n \geq 2$. In the presence of a geometric group action these Jacobi fields can

then be "integrated" to n-flats [Bal95, Section IV.4]. Hence, in the context of metric spaces, the assumption on the existence of n-flats in Theorem A is natural.[2] Recall that a flat is *periodic*, if its stabilizer in the isometry group of the surrounding space contains a subgroup acting geometrically on it. By [BH99, Flat Torus Theorem 7.1], the existence of a periodic n-flat in X is equivalent to the presence of a subgroup $\mathbb{Z}^n < \mathrm{Isom}(X)$. We point out that Theorem A does not assume any symmetries besides the existence of a single periodic n-flat. Also, if X is a rank n symmetric space or Euclidean building, and Γ acts cocompactly on X, then X contains a periodic n-flat [BB96, Theorem 8.9], [Mos73, Lemma 8.3], [PR72, Theorem 2.8]. We emphasize that some symmetry is required in order for the conclusion of Higher Rank Rigidity to hold true, since a geodesically complete CAT(0) space whose Tits boundary $\partial_T X$ has diameter π does not have to be a product or a Euclidean building [BB00].

We now discribe the main novelty of the present paper. Recall that a point in a metric space is called *regular*, if it has a neighborhood homeomorphic to an open set in a Euclidean space. The crucial ingredient behind Theorem A is a contribution to the following general conjecture which has been completely open.

Conjecture (Lytchak's Regular Point Conjecture). *Let X be a locally compact CAT(0) space with a geometric group action. Then $\partial_T X$ contains a regular point.*

Gromov first raised the question which CAT(1) spaces do contain regular points [Gro93, p.90]. A general CAT(1) space does not have to contain any regular points. On the other hand, locally compact and geodesically complete CAT(1) spaces always do, as shown by Lytchak-Nagano in [LN19], following ideas of Burago-Gromov-Perelman [BGP92]. However, the Tits boundary of a locally compact CAT(0) space with a geometric group action is typically not locally compact.

Recall that a flat in a CAT(0) space is called *Morse* if it does not bound a flat half-space. We prove:

Theorem D. [*Theorem 4.10*] *Let X be a locally compact CAT(0) space. Suppose that X contains a periodic Morse flat F. Then the complement in $\partial_T F$ of the set of regular points is a finite union of round spheres of positive codimension.*

In particular, the singular set in the Tits boundary of a periodic Morse $(n+1)$-flat is given by a union of finitely many hyperspheres and a closed set of finite $(n-2)$-dimensional Hausdorff measure. In this sense the Tits boundary of a periodic Morse flat resembles a Coxeter complex. This will be used in [Sta221] in order to control the geometry of certain submetries of the Tits boundary. Let us emphasize again that Theorem D does not assume any symmetry besides the periodic Morse flat, no geometric group action is required.

The present article is the first in a series motivated by the Higher Rank Rigidity Conjecture. The sequels all rely on the crucial new idea on how to find regular points

[2] See [OR22] for a definition of rank for CAT(0) spaces whose isometry groups satisfy the duality condition.

at infinity presented here. In [Sta221] we prove the following rigidity result which generalizes the Higher Rank Rigidity Theorem for Hadamard manifolds.

Theorem 1.1 ([Sta221, Main Theorem]). *Let X be a locally compact CAT(0) space with a geometric group action $\Gamma \curvearrowright X$. Suppose that there exists $n \geq 2$ such that every geodesic in X lies in an n-flat. If X contains a periodic Morse n-flat, then X is a Riemannian symmetric space or a Euclidean building or X non-trivially splits as a metric product.*

In [Sta222] we confirm Higher Rank Rigidity as well as the Closing Lemma for spaces without 3-flats. More precisely, we obtain

Theorem 1.2 ([Sta222, Theorems A and B]). *Let X be a locally compact geodesically complete CAT(0) space without 3-flats. Suppose X admits a geometric group action $\Gamma \curvearrowright X$. If X contains a complete geodesic which does not bound a flat half-plane, then it also contains a Γ-periodic geodesic which does not bound a flat half-plane. If every complete geodesic in X bounds a flat half-plane, then X is a Riemannian symmetric space, a Euclidean building or non-trivially splits as a metric product.*

2 Preliminaries

2.1 Metric spaces. We assume that the reader is familiar with the geometry of metric spaces with upper curvature bound, as references we mention [AKP19, Bal04, BH99, KL97]. For the required preliminaries on buildings we refer the reader to [KL97, Lee00].

Here we only want to agree on notation and collect some basic facts that will be used later.

Euclidean n-space with its flat metric will be denoted by $\mathbb{R}^n$. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ equipped with the induced metric will be referred to as a *round sphere*. Its intersection with a half-space $\mathbb{R}^{n-1} \times [0, \infty)$ is called a *round hemisphere*. We denote the distance between two points x and y in a metric space X by $|x, y|$. For $x \in X$ and $r > 0$, we denote by $B_r(x)$ and $\bar{B}_r(x)$ the open and closed r-ball around x, respectively. Similarly, $N_r(A)$ and $\bar{N}_r(A)$ denote the open and closed r-neighborhood of a subset $A \subset X$, respectively. A *geodesic* is an isometric embedding of an interval. It is called a *geodesic segment*, if it is compact. The *endpoints* of a geodesic segment c are denoted by ∂c.

A *branch point* is a point through which a certain geodesic can be extended in more than one way. More precisely, a point b is a branch point if there are geodesic segments c, c^+ and c^- such that $c \cup c^{\pm}$ is a geodesic and $c^+ \cap c^- = \{b\}$. In a *geodesic metric space* any pair of points is connected by a geodesic. Such a space is *geodesically complete* if every geodesic segment is contained in a complete local geodesic.

2.2 Spaces with an upper curvature bound. For any CAT(κ) space X, the angle between each pair of geodesics starting at the same point is well defined. Let x, y, z

be three points at pairwise distance less than $\frac{\pi}{\sqrt{\kappa}}$ in a CAT(κ) space X. Whenever $x \neq y$, the geodesic between x and y is unique and will be denoted by xy. For $y, z \neq x$, the angle at x between xy and xz will be denoted by $\angle_x(y,z)$. The *space of directions* or *link* at a point $x \in X$ will be denoted by $(\Sigma_x X, \angle)$, its elements are called *directions (at x)*.

The *dimension* $\dim(X)$ of a CAT(κ) space X refers to Kleiner's geometric dimension [Kle99]. It vanishes precisely when the space is discrete. In general, it is defined inductively:

$$\dim(X) = \sup_{x \in X} \{\dim(\Sigma_x X) + 1\}.$$

2.3 CAT(1) spaces. For two CAT(1) spaces Z_1 and Z_2 we denote by $Z_1 \circ Z_2$ their *spherical join*. It is a CAT(1) space of diameter π. A pair of points ξ^-, ξ^+ in a CAT(1) space Z are called *antipodes*, if their distance is at least π.

A subset $A \subset Z$ is called *spherical*, if it embeds isometrically into a round sphere. A *(spherical) n-lune of angle $\theta \in [0,\pi]$* in a CAT(1) space is a closed convex subset λ isometric to $S^{n-1} \circ [0,\theta]$. Bigons in CAT(1) spaces lead to spherical lunes [BB99, Lemma 2.5]. We will use the following more general result which follows immediately from [BB99, Lemma 2.5] and [Lyt05, Lemma 4.1].

LEMMA 2.1 (Lune Lemma). *Let ξ and $\hat{\xi}$ be antipodes at distance π in a CAT(1) space Z. If $\gamma^{\pm}$ are geodesics from ξ to $\hat{\xi}$, then $\gamma^- \cup \gamma^+$ is a spherical subset in Z. More precisely, if $\angle_\xi(\gamma^-,\gamma^+) \geq \pi$, then $\gamma^- \cup \gamma^+$ is a round 1-sphere in Z; and if $\angle_\xi(\gamma^-,\gamma^+) = \theta < \pi$, then $\gamma^- \cup \gamma^+$ bounds a spherical lune of angle θ. Similarly, let $\tau^{\pm} \subset Z$ be round n-hemispheres with $\partial\tau^- = \partial\tau^+$ and let $\zeta^{\pm} \in Z$ denote their centers. If $|\zeta^-,\zeta^+| \geq \pi$, then $\tau^- \cup \tau^+$ is a round n-sphere in Z; and if $|\zeta^-,\zeta^+| = \theta < \pi$, then $\tau^- \cup \tau^+$ bounds a spherical $(n+1)$-lune of angle θ.*

DEFINITION 2.2. Let Z be a CAT(1) space. We call a point $\xi \in Z$ *k-regular*, if it has a neighborhood homeomorphic to an open set in $\mathbb{R}^k$. The point ξ is called *k-spherical*, if there exists a radius $r > 0$ such that $B_r(\xi) \subset Z$ is isometric to an r-ball in S^k. In both definitions, if it's clear from the context, we neglect the quantification and speak of *regular* respectively *spherical* points.

LEMMA 2.3 ([BL06, Lemma 2.1]). *Let Z be a CAT(1) space of dimension n. If $\sigma \subset Z$ is a round n-sphere, then every point $\xi \in Z$ has an antipode in σ.*

This implies the following well-known criterion for geodesic completeness.

LEMMA 2.4. *Let Z be a CAT(1) space of dimension n. If every point $\xi \in Z$ is contained in a round n-sphere in Z, then Z is geodesically complete.*

Proof. If ξ is a point in Z contained in a round n-sphere $\sigma \subset Z$, then any local geodesic ending in the point ξ can be extended beyond ξ by a geodesic in σ, since any direction in $\Sigma_\xi Z$ has an antipode in $\Sigma_\xi \sigma \cong S^{n-1}$ by Lemma 2.3. □

A simple but useful property in CAT(1) spaces with a geodesically complete link is the following.

LEMMA 2.5. *Let Z be a CAT(1) space and $z \in Z$ a point where the link $\Sigma_z Z$ is geodesically complete. Then for every pair of directions $v, w \in \Sigma_z Z$ there exists an antipode $\hat{w} \in \Sigma_z Z$ of w, such that*

$$\angle_z(\hat{w}, v) = \pi - \angle_z(w, v).$$

Proof. If $\angle_z(v, w) = \pi$, then we can take $\hat{w} = v$. Otherwise, we extend the geodesic from w to v up to distance π. The endpoint $\hat{w}$ is then an antipode of w with the required property. □

The proof of Theorem A relies on the following rigidity result of Lytchak.

Theorem 2.6 ([Lyt05, Main Theorem]). *Let Z be a finite dimensional geodesically complete CAT(1) space. If Z has a proper closed subset A containing with each point all of its antipodes, then Z is a spherical join or a spherical building.*

2.4 CAT(0) spaces. The *ideal boundary* of a CAT(0) space X, equipped with the cone topology, is denoted by $\partial_\infty X$. If X is locally compact, then $\partial_\infty X$ is compact. The *Tits boundary* of X is denoted by $\partial_T X$, it is the ideal boundary equipped with the Tits metric $|\cdot,\cdot|_T$. Recall that the Tits metric is the intrinsic metric associated to the *Tits angle*.

If C is a closed convex subset, then it is CAT(0) with respect to the induced metric. In this case, C admits a 1-Lipschitz retraction $\pi_C : X \to C$. If C_1 and C_2 are closed convex subsets, then the distance function $d(\cdot, C_1)|_{C_2}$ is convex; and constant if and only if π_{C_1} restricts to an isometric embedding on C_2. We call C_1 and C_2 *parallel*, $C_1 \| C_2$, if and only if $d(\cdot, C_1)|_{C_2}$ and $d(\cdot, C_2)|_{C_1}$ are constant. Let $Y \subset X$ be a geodesically complete closed convex subset. Then we define the *parallel set* $P(Y)$ as the union of all closed convex subsets parallel to Y. The parallel set is closed, convex and splits canonically as a metric product

$$P(Y) \cong Y \times CS(Y)$$

where the *cross section* $CS(Y)$ is a closed convex subset [KL97, Sect. 2.3.3].

If X_1 and X_2 are CAT(0) spaces, then their metric product $X_1 \times X_2$ is again a CAT(0) space. We have $\partial_T(X_1 \times X_2) = \partial_T X_1 \circ \partial_T X_2$ and $\Sigma_{(x_1,x_2)}(X_1 \times X_2) = \Sigma_{x_1} X_1 \circ \Sigma_{x_2} X_2$. If X is a geodesically complete CAT(0) space, then any join decomposition of $\partial_T X$ is induced by a metric product decomposition of X [KL97, Proposition 2.3.7].

The Tits boundary of a closed convex subset $Y \subset X$ embeds canonically $\partial_T Y \subset \partial_T X$. If two closed convex subsets Y_1 and Y_2 intersect in X, then

$$\partial_T(Y_1 \cap Y_2) = \partial_T Y_1 \cap \partial_T Y_2.$$

For points $\xi \in \partial_\infty X$ and $x \in X$ we denote the *Busemann function centered at ξ based at x* by $b_{\xi,x}$. If $\rho : [0, \infty) \to X$ denotes the geodesic ray asymptotic to ξ and

with $\rho(0) = x$, then

$$b_{\xi,x}(y) = \lim_{t\to\infty}(|y, \rho(t)| - t).$$

It is a 1-Lipschitz convex function whose negative gradient at a point $y \in X$ is given by $\log_y(\xi)$. We denote the *horoball* centered at a point $\xi \in \partial_\infty X$ and based at the point $x \in X$ by

$$HB(\xi, x) := b^{-1}_{\xi,x}((-\infty, 0]).$$

It is a closed convex subset with

$$\partial_T HB(\xi, x) = \bar{B}_{\frac{\pi}{2}}(\xi) \subset \partial_T X.$$

A *n-flat* F in a CAT(0) space X is a closed convex subset isometric to $\mathbb{R}^n$. In particular, $\partial_T F \subset \partial_T X$ is a round $(n-1)$-sphere. On the other hand, if X is locally compact and $\sigma \subset \partial_T X$ is a round $(n-1)$-sphere, then either there exists an n-flat $F \subset X$ with $\partial_T F = \sigma$, or there exists a round n-hemisphere $\tau^+ \subset \partial_T X$ with $\sigma = \partial\tau^+$ [Lee00, Proposition 2.1]. Consequently, if $\partial_T X$ is $(n-1)$-dimensional, then any round $(n-1)$-sphere in $\partial_T X$ is the Tits boundary of some n-flat in X. Moreover, a round n-hemisphere $\tau^+ \subset \partial_T X$ bounds a flat $(n+1)$-half-space in X if and only if its boundary $\partial\tau^+$ bounds an n-flat in X. A flat is called *Morse*, if it does not bound a flat half-space.[3]

The following basic fact will be used repeatedly throughout the paper.

LEMMA 2.7 ([Lee00, Sublemma 2.3]). *Let F be a flat in a CAT(0) space and let $C \subset X$ be a closed convex subset so that $\partial_\infty F \subset \partial_\infty C$. Then C contains a flat F' parallel to F.*

A *flat (n-dimensional) half-space* $H \subset X$ is a closed convex subset isometric to a Euclidean half-space $\mathbb{R}^n_+$. Its boundary $\partial H \subset X$ is an $(n-1)$-flat and its Tits boundary is a round $(n-1)$-hemisphere $\partial_T H \subset \partial_T X$. Flat half-spaces will play a central role in our arguments later, and we agree to denote them by H and their boundaries by $\partial H = h$.

We define the *parallel set* of a round sphere $\sigma \subset \partial_T X$ as

$$P(\sigma) = P(F)$$

where F is a flat in X with $\partial_T F = \sigma$, if such a flat exists.

The proof of Theorem A relies on the following rigidity result of Leeb.

Theorem 2.8 ([Lee00, Main Theorem]). *Let X be a locally compact, geodesically complete CAT(0) space. If $\partial_T X$ is a connected thick irreducible spherical building, then X is either a symmetric space or a Euclidean building.*

[3] For a motivation of the term and further context see [HKS22, HKS23].

3 Recognizing a building at infinity

The aim of this section is to prove the following result.

Theorem 3.1. *Let X be a locally compact $CAT(0)$ space where every geodesic lies in an n-flat. Suppose that $\dim(\partial_T X) = n-1$. If $\partial_T X$ contains an open relatively compact subset U, then $\partial_T X$ is a spherical building.*

Note that the assumptions imply that X is geodesically complete since we can extend every geodesic segment inside a suitable n-flat to a complete geodesic line.

The result is similar to [BL06, Theorem 1.6] and builds on Theorem 2.6 in the same way. Note that if we would know that any pair of antipodes in $\partial_T X$ lies in a round $(n-1)$-sphere, then we could directly apply [BL06, Theorem 1.6].

Using the structure theory of locally compact and geodesically complete spaces with upper curvature bounds [LN19], one can deduce Theorem 3.1 from the following similar result which will suffice for our proof of Theorem A.

Theorem 3.2. *Let X be a locally compact $CAT(0)$ space where every geodesic lies in an n-flat. Suppose that $\dim(\partial_T X) = n-1$. If $\partial_T X$ contains an $(n-1)$-spherical point, then $\partial_T X$ is a spherical building.*

We need some preparation before we can provide the proof of Theorem 3.1.

Recall that a pair of antipodes in the Tits boundary of a CAT(0) space X does not have to bound a complete geodesic in X. Nevertheless, for locally compact geodesically complete spaces we have the following.

Lemma 3.3. *Let X be a locally compact, geodesically complete $CAT(0)$ space. Let ξ and $\hat{\xi}$ be antipodes in $\partial_T X$. Then there exists a sequence of complete geodesics c_k in X with $\partial_\infty c_k = \{\hat{\xi}, \xi_k\}$ and $\xi_k \to \xi$ in $\partial_\infty X$.*

Proof. If $|\xi, \hat{\xi}| > \pi$, then there is a complete geodesic c in X with $\partial_\infty c = \{\xi, \hat{\xi}\}$ [Lee00, Proposition 2.1]. Hence, we may assume $|\xi, \hat{\xi}| = \pi$. Fix a base point $o \in X$ and let $\rho : [0, \infty) \to X$ be a parametrization of the geodesic ray $o\xi$. Since X is locally compact and geodesically complete, all links in X are geodesically complete [LN19, Corollary 5.8, Corollary 5.9]. By Lemma 2.5, we can choose for each $k \in \mathbb{N}$ a complete geodesic c_k extending the geodesic ray $\rho(k)\hat{\xi}$ and such that

$$\angle_{\rho(k)}(\hat{\xi}, \xi) = \pi - \angle_{\rho(k)}(\xi, \xi_k)$$

where $\xi_k := c_k(+\infty)$. Now for every $p \in o\xi$ we have

$$\angle_p(\xi_k, \xi) \leq \angle_{\rho(k)}(\xi_k, \xi) = \pi - \angle_{\rho(k)}(\xi, \hat{\xi}) \to 0.$$

Since $\angle_{\rho(k)}(\xi, \hat{\xi}) \to |\xi, \hat{\xi}| = \pi$. As $p \in o\xi$ was arbitrary, this shows $\xi_k \to \xi$ in $\partial_\infty X$. Indeed, by local compactness, we may assume $\xi_k \to \xi'$ after passing to a subsequence. In particular, $\angle_x(\xi_k, \xi') \to 0$ for all $x \in X$. Now if $\xi \neq \xi'$, then there exists $q \in o\xi$ such

that $\angle_q(\xi,\xi') > 0$. But

$$\angle_q(\xi,\xi') \le \angle_q(\xi,\xi_k) + \angle_q(\xi_k,\xi') \to 0. \qquad \square$$

LEMMA 3.4. *Let X be a locally compact CAT(0) space. Suppose that every geodesic in X lies in an n-flat. If $\xi \in \partial_T X$ is an $(n-1)$-spherical point, then for every antipode $\hat{\xi}$ of ξ there exists a round $(n-1)$-sphere $\sigma \subset \partial_T X$ containing ξ and $\hat{\xi}$.*

Proof. Let $\xi \in \partial_T X$ be an $(n-1)$-spherical point, cf. Definition 2.2. Then there exists $s > 0$ such that $B_s(\xi)$ is contained in any $(n-1)$-sphere which contains ξ. If $\hat{\xi} \in \partial_T X$ is an antipode of ξ, then by Lemma 3.3, there exists a sequence of complete geodesics (c_k) in X with $\partial_\infty c_k = \{\xi, \hat{\xi}_k\}$ and $\hat{\xi}_k \to \hat{\xi}$. By assumption, there exists n-flats F_k with $c_k \subset F_k$. Hence, if $\eta \in \partial_T X$ is a point with $|\eta,\xi| = s$, then $|\eta,\hat{\xi}_k| = \pi - s$ for all $k \in \mathbb{N}$. Thus,

$$\pi = |\xi,\hat{\xi}| \le |\xi,\eta| + |\eta,\hat{\xi}| \le s + \liminf_{k\to\infty} |\eta,\hat{\xi}_k| = \pi,$$

by lower semi-continuity of the Tits metric. We obtain $|\eta,\hat{\xi}| = \pi - s$ for every point η at distance s from ξ. In particular, through every such point runs a geodesic from ξ to $\hat{\xi}$. Denote by $\sigma \subset \partial_T X$ the union of all these geodesics. It follows from Lemma 2.1 that σ is a round $(n-1)$-sphere. $\square$

PROPOSITION 3.5. *Let X be a locally compact CAT(0) space where every geodesic lies in an n-flat. Suppose* $\dim(\partial_T X) = n-1$. *Then the set $O \subset \partial_T X$ of $(n-1)$-spherical points is closed under taking antipodes.*

Proof. Let ξ be a $(n-1)$-spherical point in $\partial_T X$ and let $\hat{\xi}$ be an antipode. By Lemma 3.4, there exists a round $(n-1)$-sphere $\sigma \subset \partial_T X$ which contains ξ and $\hat{\xi}$. Since ξ is spherical, there exists $s > 0$ such that $B_s(\xi) \subset \sigma$. We claim that $B_s(\hat{\xi}) \subset \sigma$ holds as well. Let $\hat{\eta}$ be a point in $B_s(\hat{\xi})$. Since $\dim(\partial_T X) = n-1$, we can extend the geodesic $\hat{\eta}\hat{\xi}$, as in the proof of Lemma 2.4, up to an antipode η of $\hat{\eta}$ in σ. Then $|\eta,\xi| = |\hat{\eta},\hat{\xi}| < s$ and therefore η is spherical. By Lemma 3.4, we find a round $(n-1)$-sphere $\sigma' \subset \partial_T X$ which contains η and $\hat{\eta}$. Because $\eta \in B_s(\xi)$, we must have $B_s(\xi) \subset \sigma \cap \sigma'$. By construction, $\hat{\xi}$ also lies in $\sigma \cap \sigma'$. Now convexity of $\sigma \cap \sigma'$ implies $\sigma = \sigma'$ and therefore $\hat{\eta} \in \sigma$ as required. $\square$

LEMMA 3.6. *Let X be a locally compact CAT(0) space with* $\dim(\partial_T X) = n-1$ *and where every geodesic lies in an n-flat. Suppose X splits as a non-trivial metric product $X \cong X_1 \times X_2$. Then we have $n = n_1 + n_2$ where* $\dim(\partial_T X_j) = n_j - 1$, $j = 1,2$, *and every complete geodesic in X_j is contained in an n_j-flat.*

Proof. It is clear that the rank is additive. So let c_j be a complete geodesic in X_j and let $\tilde{c} = (c_1, c_2)$ be the diagonal geodesic in $c_1 \times c_2 \subset X_1 \times X_2$. If $\tilde{F}$ is an n-flat in X which contains $\tilde{c}$, then by [KL97, Lemma 2.3.8] there are flats $F_j \subset X_j$ such that $\tilde{F} \subset F_1 \times F_2$. In particular, $\dim(F_j) = n_j$ and $c_j \subset F_j$. $\square$

Proof of Theorem 3.2. By assumption, $\dim(\partial_T X) = n-1$ and every point ξ in $\partial_T X$ lies in a round $(n-1)$-sphere. Hence, Lemma 2.4 implies that $\partial_T X$ is geodesically complete.

We proceed by induction on the dimension of $\partial_T X$. If the $\dim(\partial_T X)$ is 0, then $\partial_T X$ is discrete and there is nothing to show. So let's assume $\dim(\partial_T X) \geq 1$. Recall that two antipodes ξ, $\hat{\xi}$ in $\partial_T X$ at distance $> \pi$ bound a complete geodesic in X [Lee00, Proposition 2.1]. Since every complete geodesic in X lies in an n-flat, we conclude that the diameter of $\partial_T X$ is π.

Suppose X splits as a non-trivial metric product $X \cong X_1 \times X_2$. Then by Lemma 3.6, the factors are subject to the induction hypothesis and we may assume that X is irreducible.

By assumption, the set of $(n-1)$-spherical points $O \subset \partial_T X$ is non-empty. If $O = \partial_T X$, then X is isometric to $\mathbb{R}^n$. So let us assume that O is a proper subset of $\partial_T X$. By Proposition 3.5, the set $\partial_T X \setminus O$ is a non-empty, closed proper subset which contains with every point also all its Tits antipodes. Since $\partial_T X$ is irreducible, Theorem 2.6 implies that it is a spherical building. □

Proof of Theorem 3.1. By Theorem 3.2, it is enough to find an $(n-1)$-spherical point in $\partial_T X$. If U is an open relatively compact subset of $\partial_T X$, then U contains an open subset U' homeomorphic to an n-manifold [LN19, Theorem 1.2]. Hence, for a point $\xi \in U'$ there exists an $\epsilon > 0$ such that $B_\epsilon(\xi) \subset U'$. If $\sigma \subset \partial_T X$ denotes a round $(n-1)$-sphere with $\xi \in \sigma$, then $\sigma \cap B_\epsilon(\xi) = B_\epsilon(\xi)$. Hence ξ is $(n-1)$-spherical. □

4 Rank n spaces with periodic n-flats

4.1 From branch points to orthogonal half-planes.

Definition 4.1. Let $F \subset X$ be a flat in a CAT(0) space. Then we call a flat half-space $H \subset X$ a *flat half-space orthogonal to* F, or, if the respective flat is clear from the context, an *orthogonal flat half-space* if

$$H \cap F = \partial H \quad \text{and} \quad \angle(H, F) \geq \frac{\pi}{2}$$

where the angle is measured in the cross section of the parallel set $P(\partial H)$. In case H is 2-dimensional, we adjust to *flat half-plane orthogonal to* F and *orthogonal flat half-plane*, respectively.

Definition 4.2. Let X be a locally compact CAT(0) space and $F \subset X$ a flat. Let $\mathrm{Isom}(X)$ denote the isometry group of X. We say that F is *periodic*, if its *stabilizer* $\mathrm{Stab}(F) := \{\gamma \in \mathrm{Isom}(X) |\ \gamma(F) = F\}$ contains a subgroup acting geometrically on F.

Lemma 4.3. *Let X be a locally compact CAT(0) space and let $F \subset X$ be a periodic n-flat with $n \geq 2$. Suppose that $F = F^- \times F^+$ is a metric product decomposition into flats $F^\pm$ of dimension $n^\pm$. Then $F^\pm$ is a periodic $n^\pm$-flat in the cross section $CS(F^\mp)$.*

Proof. By Bieberbach's theorem, the stabilizer of F contains a free abelian subgroup A of rank n which acts cocompactly on F by Euclidean translations. Since A preserves $\partial_T F$ pointwise, it also preserves $P(F^{\mp})$ and the splitting $P(F^{\mp}) \cong F^{\mp} \times CS(F^{\mp})$. Hence its projection to $CS(F^{\mp})$ contains a free abelian subgroup of rank $n^{\pm} = n - n^{\mp}$ which acts geometrically by Euclidean translations on $F^{\pm} \subset CS(F^{\mp})$. □

The following result implies that a branch point in the ideal boundary of a periodic flat leads to an orthogonal flat half-plane.

LEMMA 4.4 ([HK05, Lemma 2.3.1]). *Let X be a locally compact CAT(0) space. Let $F \subset X$ be a periodic flat. Let $\xi \in \partial_T F$ be a point such that there exists a sequence (ξ_k) in $\partial_T X \setminus \partial_T F$ which converges to ξ with respect to the Tits metric. Then there exists a flat half-plane $H \subset X$ asymptotic to ξ and orthogonal to F (cf. Definition 4.1).*

Proof. Since F is periodic, it is enough to find for every $R > 0$ a flat strip $S \cong \mathbb{R} \times [0, R]$ which is asymptotic to ξ and orthogonal to F. We fix a base point $o \in F$. For every $k \in \mathbb{N}$ we choose $x_k \in o\xi_k$ such that $|x_k, F| = R$. Denote by $\bar{x}_k \in F$ the nearest point to x_k. Choose isometries $\gamma_k \in \mathrm{Stab}(F)$ such that all points $\gamma_k(\bar{x}_k)$ lie in a fixed compact set. After passing to a subsequence, we obtain a limit geodesic c_∞ which is parallel to F and spans the required flats strip. More precisely, we have the convergence $\gamma_k(o\xi_k, x_k) \to (c_\infty, x_\infty)$ with respect to pointed Hausdorff topology. The complete geodesic c_∞ is asymptotic to ξ since $|\gamma_k \xi_k, \xi| = |\xi_k, \xi| \to 0$. Because $x_k o \subset \bar{N}_R(F)$, also c_∞ is contained in the R-neighborhood of F and has even distance R from F since $|x_\infty, F| = R$. Denote by $\bar{c}_\infty \subset F$ the complete geodesic which contains the point x_∞ and is asymptotic to ξ. Then c_∞ and $\bar{c}_\infty$ span the required flat strip by [KL97, Lemma 2.3.5]. □

COROLLARY 4.5. *Let X be a locally compact CAT(0) space. Let $\hat{F} \subset X$ be a periodic $(n+1)$-flat with $\partial_T \hat{F} = \hat{\sigma}$. Denote by $\lambda \subset \partial_T X$ a spherical $(k+1)$-lune with $k \leq n$ bounded by round k-hemispheres $\tau^{\pm}$. Suppose that $\lambda \cap \hat{\sigma} = \tau^-$. Then there exists a round $(k+1)$-hemisphere $\hat{\tau} \subset \partial_T X$ such that $\partial\hat{\tau} \subset \hat{\sigma}$ and $\tau^- \subset \partial\hat{\tau}$. Moreover, $\hat{\tau}$ is the ideal boundary of a flat half-space $\hat{H}$ orthogonal to $\hat{F}$.*

Proof. Set $\sigma = \partial\tau^{\pm}$ and write $\hat{\sigma} = \sigma \circ \sigma^{\perp}$. Let $\hat{F} = F \times F^{\perp}$ be a corresponding product splitting. We have $\lambda = \sigma \circ [0, \theta]$. By Lemma 4.3, $F^{\perp}$ is a periodic flat in the cross section $CS(F)$. Denote by ζ^- the center of τ^-. Then the cross section of λ provides a path in $\partial_T CS(F)$ which intersects $\partial_T F^{\perp}$ only in ζ^-. Thus, by Lemma 4.4, there exists an orthogonal flat half-plane $H \subset CS(F)$ for $F^{\perp}$ which is asymptotic to ζ^-. Hence, $\hat{H} := F \times H \subset P(F)$ and $\hat{\tau} := \partial_T \hat{H}$ define an orthogonal flat half-space and round $(k+1)$-hemisphere, respectively, as required. □

4.2 Orthogonal half-spaces from a sequence of lower dimensional ones. In this section we prove the following key result which allows us to understand the singular set in the Tits boundary of a periodic Morse flat. The responsible geometric prop-

erty behind the proof is flatness of intersecting parallel sets, similar as in [Sta15, Theorem 2].

PROPOSITION 4.6. *Let X be a locally compact $CAT(0)$ space. Let $\hat{F}$ be a periodic $\hat{n}$-flat in X with $\partial_T \hat{F} = \hat{\sigma}$. Suppose that there is a sequence of flat half-spaces (H_k) of dimension $d \leq n$, all orthogonal to $\hat{F}$, such that their boundary flats $h_k := \partial H_k$ are pairwise non-parallel. Then there exists a $(d+1)$-dimensional flat half-space $\hat{H}$ orthogonal to $\hat{F}$. Moreover, if the sequence of round spheres $\sigma_k = \partial_\infty h_k$ converges to a round sphere $\sigma \subset \hat{\sigma}$ and for every $k \in \mathbb{N}$ the convex hull of σ_k and σ is a round sphere of dimension $\hat{d}-1$, then we can arrange $\hat{H}$ to have dimension $\hat{d}+1$ and such that $\sigma \subset \partial_\infty \hat{H}$.*

Proof. Since $\hat{F}$ is periodic, we may assume that we have a convergent sequence of pairwise distinct pointed flat half-spaces $(H_k, p_k) \to (H_\infty, p_\infty)$, where $p_k \in F_k$. The limit H_∞ with boundary flat $h_\infty := \partial H_\infty$ is then still orthogonal to $\hat{F}$. Set $\partial_T H_k = \tau_k^+$, $\partial_T H_\infty = \tau_\infty^+$ and denote by ζ_k and ζ_∞ the respective centers. Because $p_\infty \in h_\infty$ and H_∞ is orthogonal to $\hat{F}$, the Busemann function $b_{\zeta_\infty, p_\infty}$ is non-negative on $\hat{F}$. Since the geodesic rays $p_k\zeta_k$ converge to $p_\infty\zeta_\infty$, we find a sequence $t_k \to \infty$ and points $x_k \in p_k\zeta_k$ such that $b_{\zeta_\infty,p_\infty}(x_k) \leq -t_k$. The geodesic ray $p_k\zeta_k$ lies in H_k and therefore x_k lies in $P(h_k)$. Thus $HB(\zeta_\infty, x_k) \cap P(h_k)$ is a non-empty closed convex set and we have

$$\partial_\infty(HB(\zeta_\infty, x_k) \cap P(h_k)) = \bar{B}_{\frac{\pi}{2}}(\zeta_\infty) \cap \partial_\infty P(h_k).$$

As $\hat{F} \subset P(h_k)$, we see $\partial_\infty h_\infty \subset \partial_\infty(HB(\zeta_\infty, x_k) \cap P(h_k))$. From Lemma 2.7 we infer that there is a $(d-1)$-flat $E_k \subset HB(\zeta_\infty, x_k) \cap P(h_k)$ with $\partial_\infty E_k = \partial_\infty h_\infty$. Since the flats h_k are pairwise non-parallel and $E_k \subset P(h_k)$, we see that there is a flat F_k in $P(h_k)$ of dimension at least d which contains E_k and such that $\partial_\infty h_k \subset \partial_\infty F_k$. Note that $b_{\zeta_\infty,p_\infty}$ is bounded above on E_k by $-t_k$ since $E_k \subset HB(\zeta_\infty, x_k)$. As $b_{\zeta_\infty,p_\infty}$ is non-negative on $\hat{F}$ we conclude $E_k \cap N_{t_k}(\hat{F}) = \emptyset$. Hence if $F_k' \subset \hat{F}$ denotes the closest parallel flat to F_k, then $|F_k, F_k'|_H \geq t_k$. In particular, F_k and F_k' bound a flat strip of dimension $\geq d+1$, orthogonal to $\hat{F}$ and of width at least t_k. Since $\hat{F}$ is periodic and $t_k \to \infty$, we obtain a flat half-space $\hat{H}$ orthogonal to $\hat{F}$ and of dimension at least $d+1$ as a limit.

To prove the finer statement, simply note that in the argument above the flat $E_k \subset P(h_k)$ spans a flat $F_k \subset P(h_k)$ of dimension $\hat{d}$ if and only if the convex hull of the spheres $\partial_\infty E_k = \partial_\infty h_\infty$ and $\partial_\infty h_k$ is a round sphere of dimension $\hat{d}-1$ in $\hat{\sigma}$. The rest of the argument stays the same. □

4.3 Tits boundary of a periodic Morse flat. In this final section we provide proofs of our main results.

DEFINITION 4.7. Let X be a locally compact CAT(0) space. Let $\hat{F} \subset X$ be a periodic flat with $\partial_T \hat{F} = \hat{\sigma}$. A round sphere $\sigma \subset \hat{\sigma}$ is called *singular*, if there exists a round

hemisphere $\tau^+ \subset \partial_T X$ with $\partial\tau^+ = \sigma$ and $\tau^+ \cap \hat{\sigma} = \sigma$ such that the union $\tau^+ \cup \tau^-$ is a round sphere for every round hemisphere $\tau^- \subset \hat{\sigma}$ with $\partial\tau^- = \sigma$. We call a singular sphere $\sigma \subset \hat{\sigma}$ *maximal*, if it is not contained in a singular sphere in $\hat{\sigma}$ of strictly larger dimension.

Note that if $\hat{F}$ is not Morse, then $\hat{\sigma}$ itself is singular.

COROLLARY 4.8. *Let X be a locally compact CAT(0) space. Let $\hat{F}$ be a periodic $(n+1)$-flat in X with $\partial_T \hat{F} = \hat{\sigma}$. Then the set of maximal singular spheres in $\hat{\sigma}$ is finite.*

Proof. Proposition 4.6 implies that a maximal singular sphere $\sigma \subset \hat{\sigma}$ has to be isolated in the following sense. If (σ_k) is a sequence of singular spheres which converges to a sphere $\sigma_\infty \subset \sigma$, then we must have $\sigma_k \subset \sigma$ for almost all $k \in \mathbb{N}$. This yields the claim. □

Recall that a point in $\hat{\sigma}$ is regular, if it has a Tits-neighborhood which is entirely contained in $\hat{\sigma}$. Otherwise we call it *singular*.

LEMMA 4.9. *Every singular point in $\hat{\sigma}$ is contained in a singular sphere.*

Proof. If $\xi \in \hat{\sigma}$ is a singular point, then there exists a sequence (ξ_k) in $\partial_T X \setminus \hat{\sigma}$ with $\xi_k \to \xi$. By Lemma 4.4, there exists a flat half-plane H orthogonal to $\hat{F}$ and with $\xi \in \partial_\infty H$. Set $\tau_1^+ := \partial_\infty H$. If $\partial\tau_1^+$ is not singular, then by the Lune Lemma (Lemma 2.1) we can find a half-circle $\tau_1^- \subset \hat{\sigma}$ with $\partial\tau_1^+ = \partial\tau_1^-$ such that τ_1^+ and τ_1^- span a 2-dimensional spherical lune λ_2. By Corollary 4.5, there exists a 2-dimensional round hemisphere $\tau_2^+ \subset \partial_T X$ with $\tau_2^+ \cap \hat{\sigma} = \partial\tau_2^+$ and $\tau_1^- \subset \partial\tau_2^+$, in particular $\xi \in \partial\tau_2^+$. If $\partial\tau_2^+$ is not singular, we repeat the argument to produce a 3-dimensional round hemisphere $\tau_3^+ \subset \partial_T X$ with $\xi \in \partial\tau_3^+$. After at most $(n-1)$ steps we end up with a singular sphere that contains the point ξ. □

Now we obtain the following refinement of Theorem D.

Theorem 4.10. *Let X be a locally compact CAT(0) space. Suppose that X contains a periodic Morse flat F. Then the complement in $\partial_T F$ of the set of regular points is a finite union of singular spheres of positive codimension.*

Proof. Immediate from Corollary 4.8 and Lemma 4.9. □

Proof of Theorem A. Let $F \subset X$ be a periodic n-flat. By Theorem D, $\sigma = \partial_T F$ contains a dense subset of spherical points. Then, by Theorem 3.2, $\partial_T X$ is a spherical join or a spherical building. Since X is geodesically complete and locally compact, the claim follows from Theorem 2.8 and [KL97, Proposition 2.3.7]. □

Proof of Corollary C. By Theorem A, Lemma 3.6 and Lemma 4.3, we can produce a product decomposition of X as claimed. Since every such decomposition induces a corresponding join decomposition of $\partial_T X$. The uniqueness statement follows from

the uniqueness of such join decompositions [Lyt05, Corollary 1.2] or alternatively from [FL08, Theorem 1.1]. □

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