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Action of automorphisms on irreducible characters of finite reductive groups of type A

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Abstract. Let G be a finite reductive group such that the derived subgroup of the underlying algebraic group is a product of quasi-simple groups of type A. In this paper, we give an explicit description of the action of automorphisms of G on the set of its irreducible complex characters. This generalizes a recent result of M. Cabanes and B. Späth [Equivariant character correspondences and inductive McKay condition for type A, *J. Reine Angew. Math.* **728** (2017), 153–194] and provides a useful tool for investigating the local sides of the local-global conjectures as one usually needs to deal with Levi subgroups. As an application we obtain a generalization of the stabilizer condition in the so-called inductive McKay condition [B. Späth, Inductive McKay condition in defining characteristic, *Bull. Lond. Math. Soc.* **44** (2012), no. 3, 426–438; Theorem 2.12] for irreducible characters of G. Moreover, a criterion is given to explicitly determine whether an irreducible character is a constituent of a given generalized Gelfand–Graev character of G.

1 Introduction

A finite reductive group is the fixed-point subgroup $G := \mathbf{G}^F$ of a connected reductive algebraic group $\mathbf{G}$ defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$, where $F\colon \mathbf{G} \to \mathbf{G}$ is the Frobenius map corresponding to this $\mathbb{F}_q$-structure. In recent years, many conjectures in representation theory of finite groups have been “reduced” to checking some new technical conditions about quasi-simple groups of Lie type. These new conditions to check involve analyzing the action of automorphisms of a quasi-simple Lie-type group on the set of its irreducible characters.

Question ([12, Problem 2.33]). *For G a quasi-simple group of Lie type, determine the action of* $\mathrm{Aut}(G)$ *on* $\mathrm{Irr}(G)$.

In this paper, considering a larger framework, we determine the action of automorphisms on irreducible characters of the finite reductive group G, where $\mathbf{G}$ is a reductive group whose derived subgroup is isomorphic to an (almost-direct)

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product of quasi-simple groups of type A. This generalizes an earlier result of Brunat and Himstedt [2, 3] concerning the action of automorphisms on the set of semisimple and regular characters, and the equivariant bijection presented in [5].

It has been shown by Lusztig that the irreducible characters of $G = \mathbf{G}^F$ can be partitioned into the so-called Lusztig geometrical (resp. rational) series, labelled by the semisimple $\mathbf{G}^*$-classes (resp. ${\mathbf{G}^*}^{F^*}$-classes) of ${\mathbf{G}^*}^{F^*}$, where $(\mathbf{G}^*, F^*)$ denotes a pair dual to $(\mathbf{G}, F)$. In this paper, we use the Kawanaka construction of generalized Gelfand–Graev characters (GGGCs) to separate the irreducible characters in geometric Lusztig series. Indeed, using a canonical regular closed embedding $i\colon \mathbf{G} \to \tilde{\mathbf{G}}$ into a connected-center group $\tilde{\mathbf{G}}$, it is shown that the various components of restrictions of irreducible characters of $\tilde{\mathbf{G}}$ to $\mathbf{G}$ can be distinguished by the G-classes of their unipotent support. We then study the action of outer automorphisms on parametrized irreducible characters by considering the induced action on the corresponding parameters.

To state our results, we need to explain some more notation. For irreducible characters of the connected-center group $\tilde{G}$, we will follow the parametrization given in [4, Theorem 3.1]. For an irreducible character $\chi_{\tilde{s},\tilde{\lambda}} \in \mathrm{Irr}(\tilde{G})$, we denote by $\mathcal{O}^*_{s,\lambda}$ the common wave front set of all irreducible constituents of $\mathrm{Res}^{\tilde{G}}_G \chi_{\tilde{s},\tilde{\lambda}}$; see for instance [18, Lemma 14.12]. The main goal of this paper is to show the following.

Theorem A (Theorem 3.6). *Assume that $G = \mathbf{G}^F$ is a finite reductive group whose derived subgroup is a product of quasi-simple groups of type A. Then, for any semisimple element $s \in G^*$, any unipotent character $\lambda \in \mathrm{Irr}(C^\circ_{G^*}(s))$, and any unipotent element $u \in \mathcal{O}^{*F}_{s,\lambda}$, one has*

$$\chi_{s,\lambda,\sigma(u)} = {}^{\sigma}\chi_{\sigma^*(s),\sigma^*(\lambda),u},$$

where $\sigma \in \mathrm{Aut}(\mathbf{G}, F)$ and $\sigma^* \in \mathrm{Aut}(G^*)$ is a dual automorphism.

We should remark that, in cases $G = \mathrm{SL}^\epsilon_n(q)$ with $\epsilon \in \{\pm\}$, the special linear $(\epsilon = +)$ or unitary group $(\epsilon = -)$, Theorem A can be seen to be equivalent to the result of [5, §8] on establishing an $\mathrm{Aut}(\mathrm{SL}^\epsilon_n(q))$-equivariant Jordan decomposition for $\mathrm{Irr}(\mathrm{SL}^\epsilon_n(q))$. Theorem A also provides a useful tool for investigating the so-called local-global conjectures as one usually needs to deal with irreducible characters of Levi subgroups. In the sequel, using Theorem A, we obtain a short proof of the stabilizer condition in the so-called inductive McKay condition, cf. [15, Theorem 2.12], for the irreducible characters of G. Denote by $\mathrm{Aut}(\mathbf{G}, F)$ the set of bijective morphisms of $\mathbf{G}$ which commute with the Frobenius map F. Let $D \subseteq \mathrm{Aut}(\mathbf{G}, F)$ be a submonoid whose image in $\mathrm{Aut}(G)$ is a subgroup. Identifying $D \to \mathrm{Aut}(\tilde{\mathbf{G}}, \tilde{F}) \to \mathrm{Aut}(\mathbf{G}, F)$ with its image in $\mathrm{Aut}(\tilde{G})$, we can form a semidirect product $\tilde{G}D = \tilde{G} \rtimes D$. This group acts naturally on the set $\mathrm{Irr}(G)$. We then have the following corollary which gives a generalization of the so-called stabilizer condition in the global side of the inductive McKay condition.

Corollary B (Theorem 3.8). *Assume that the G-conjugacy class of $u_0 \in \mathcal{O}^{*F}_{s,\lambda}$ is D-invariant and $\chi = \chi_{s,\lambda,u_0}$. Then*

$$(\tilde{G}D)_{\chi_0} = \tilde{G}_{\chi_0} D_{\chi_0}.$$

As a by-product of the main result, a criterion is obtained to determine whether an arbitrary irreducible character of G, which is parametrized by triple (s, λ, b), appears as a constituent of a given generalized Gelfand–Graev character (GGGC) Γ_a of G, which might be of independent interest since, as far as we are aware, it has not been written down explicitly so far. We should remark that Theorem C could be viewed as a generalized analogue form of [1, Proposition 15.13 and Corollaire 15.14] (see also [2, Theorem 3.1]) by using the known dual-group identifications; see for instance [1, Section 8, (8.4)]. In the sequel, we denote by $G_{[x]}$ the stabilizer of the G-conjugacy class $[x] = [x]_G$. Also, the common stabilizer of irreducible constituents of $\mathrm{Res}^{\tilde{G}}_{G} \chi_{\tilde{s},\tilde{\lambda}}$ is denoted by $\tilde{G}_{s,\lambda}$, see Section 3.1.

Theorem C (Theorem 4.3). *Let $s \in G^*$ be a semisimple element, $\lambda \in \mathrm{Irr}(C^{\circ}_{G^*}(s))$ a unipotent character, and $u \in \mathcal{O}^{*F}_{s,\lambda}$ a unipotent element. Then there exists a well-defined map $\varphi\colon \tilde{G}/\tilde{G}_{[u]} \to \tilde{G}/\tilde{G}_{s,\lambda}$ such that, for any $a \in \tilde{G}/\tilde{G}_{[u]}$ and $b \in \tilde{G}/\tilde{G}_{s,\lambda}$, we have $\langle \Gamma_a, \chi_{s,\lambda,b} \rangle \neq 0$ if and only if $\varphi(a) = b$.*

It is also worthwhile to mention that Theorem A may also be applied to a wide range of other questions concerning interactions between the structure of finite groups and the set of their character degrees, such as the Huppert conjecture [10] and its variations in which one needs to analyze the action of the automorphism group of a (quasi-)simple Lie-type group on a subset of its irreducible characters.

The rest of the paper is organized as follows. In Section 2, we introduce some basic facts and results about the generalized Gelfand–Graev characters and the centralizers of unipotent elements. Section 3 is at the heart of this paper. In Section 3, we use unipotent supports to separate irreducible characters in Lusztig series and obtain in this way a parametrization of $\mathrm{Irr}(G)$ which is shown to be equivariant with respect to the action of automorphisms; see Theorem 3.6. Moreover, some potential applications of Theorem 3.6 into other character theoretical problems are discussed in Section 3; see Theorem 3.8. Section 4 is devoted to an independent result. In this section, we give a criterion to explicitly determine whether an irreducible character is the constituent of a given GGGC of G (see Theorem 4.3).

1.1 Notation

We denote by $\operatorname{Res}_H^G \chi$ the restriction of a character χ of G to some subgroup $H \leq G$. Also, the induction of a character ψ of H to G is denoted by $\operatorname{Ind}_H^G \psi$. For $N \lhd G$ and $\chi \in \operatorname{Irr}(G)$, we denote by $\operatorname{Irr}(N|\chi)$ the set of irreducible constituents of the restriction $\operatorname{Res}_N^G \chi$. The stabilizer of $\psi \in \operatorname{Irr}(N)$ under the action of G on $\operatorname{Irr}(N)$ is denoted by G_ψ. Other notation is standard or will be defined where needed.

2 Main notions and background results

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\mathbb{K} = \overline{\mathbb{F}_p}$ of the finite field of prime order p and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure $G = \mathbf{G}^F$ on $\mathbf{G}$. Assuming p is a good prime for $\mathbf{G}$, a theory of generalized Gelfand–Graev characters (GGGCs) was developed by Kawanaka in [11]. These are certain characters Γ_u of G which are defined for any unipotent element $u \in G$. Note that $\Gamma_u = \Gamma_v$ whenever $u, v \in G$ are G-conjugate, so the GGGCs are naturally indexed by the unipotent conjugacy classes of G.

2.1 Unipotent supports and wave front sets

For the basic definitions of unipotent supports and wave front sets, we refer to [17]. Let $\rho \in \operatorname{Irr}(G)$ be an irreducible character and $\mathcal{O}$ an F-stable unipotent conjugacy class of $\mathbf{G}$. Geck [8, Theorem 1.4] and Taylor [17, Theorems 13.8 and 14.10] have shown that, whenever p is good for $\mathbf{G}$, any irreducible character ρ of G has a unique unipotent support $\mathcal{O}_\rho$ and a unique wave front set $\mathcal{O}_\rho^*$. Moreover, these turn out to be dual in the sense of Alvis–Curtis duality; see [17, Lemma 14.15].

From this point forward, we assume that $G = \mathbf{G}^F$ is a finite reductive group whose derived subgroup is a product of quasi-simple groups of type A. *We also fix a regular embedding in which $\tilde{\mathbf{G}}$ is the quotient of $\mathbf{G} \times \mathbf{T}$ by the closed normal subgroup $\{(z, z^{-1}) : z \in Z(\mathbf{G})\}$, where $\mathbf{T}$ is the F-stable maximal split torus of $\mathbf{G}$; see for instance* [9, §1.7].

2.2 Automorphisms of G

If $\sigma\colon \mathbf{G} \to \mathbf{G}$ is an automorphism of $\mathbf{G}$ stabilizing $\mathbf{T}$, then this extends to an automorphism $\tilde{\sigma}$ of $\mathbf{G} \times \mathbf{T}$ by setting $\tilde{\sigma}(g, t) = (\sigma(g), \sigma(t))$ and this restricts to an automorphism of $\tilde{\mathbf{G}}$ which we again denote by $\tilde{\sigma}$. If $\sigma \in \operatorname{Aut}(\mathbf{G}, F)$, then the restriction of σ to G will be an automorphism of G. Abusing the notation, we denote the corresponding automorphism of G by σ. The canonical regular closed embedding $i\colon \mathbf{G} \to \tilde{\mathbf{G}}$ then satisfies $\tilde{\sigma} \circ i = i \circ \sigma$. We have thus described an injective map $\mathrm{Aut}(\mathbf{G}, F) \mapsto \mathrm{Aut}(\tilde{\mathbf{G}}, \tilde{F})$ given by $\sigma \mapsto \tilde{\sigma}$, where $\tilde{F}$ denotes the corresponding Frobenius map of $\tilde{\mathbf{G}}$. Note that $\mathrm{Aut}(\mathbf{G}, F)$ is a monoid.

Let $\mathbf{G}^*$ and $\tilde{\mathbf{G}}^*$ denote dual groups of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ respectively. The embedding i gives rise to a surjective homomorphism $i\colon \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ between the dual groups. We also have dual Frobenius morphisms F^* on $\mathbf{G}^*$ and $\tilde{F}^*$ on $\tilde{\mathbf{G}}^*$ satisfying $F^* \circ i^* = i^* \circ \tilde{F}^*$.

Let $\mathcal{U}(\mathbf{G}) \subseteq \mathbf{G}$ be the set of unipotent elements of $\mathbf{G}$ and let

$$\mathcal{U}(G) = G \cap \mathcal{U}(\mathbf{G}).$$

Identifying $\mathbf{G}$ with its image $i(\mathbf{G}) \subseteq \tilde{\mathbf{G}}$, we have $\tilde{\mathbf{G}} = \mathbf{G}.Z(\tilde{\mathbf{G}})$. It follows that $\mathcal{U}(\mathbf{G}) = \mathcal{U}(\tilde{\mathbf{G}})$ and the orbits of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ on $\mathcal{U}(\mathbf{G})$, acting by conjugation, are the same. Given $x \in \mathbf{G}$, we denote by $A_{\mathbf{G}}(x)$ the component group of the centralizers $C_{\mathbf{G}}(x)/C^{\circ}_{\mathbf{G}}(x)$.

Lemma 2.1. *The centralizer of any unipotent element in $\tilde{\mathbf{G}}$ is connected.*

Proof. By [9, p. 62] and the fact that $Z(\tilde{\mathbf{G}})^{\circ} = Z(\tilde{\mathbf{G}})$, the surjective morphism

$$\pi_{\mathrm{ad}}\colon \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}/Z(\tilde{\mathbf{G}}) \to \mathrm{PGL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{PGL}_{n_s}(\mathbb{K})$$

is an adjoint quotient for some $n_1, \ldots, n_s \geq 1$. Let $\tilde{u} \in \tilde{\mathbf{G}}$ be a unipotent element and $u = u_1 \times \cdots \times u_s \in \mathrm{PGL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{PGL}_{n_s}(\mathbb{K})$ its image under the above adjoint map. Therefore, it follows from [16, Lemma 2.2] that

$$A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong A_{\mathrm{PGL}_{n_1}(\mathbb{K})}(u_1) \times \cdots \times A_{\mathrm{PGL}_{n_s}(\mathbb{K})}(u_s) = 1. \qquad \square$$

Lemma 2.2. *If $\mathcal{O} \subseteq \mathcal{U}(\mathbf{G})$ is an F-stable $\mathbf{G}$-orbit, then $\mathcal{O}^F \subseteq \mathcal{U}(G)$ is a single $\tilde{G}$-orbit.*

Proof. The statement follows by the Lang–Steinberg theorem since $C_{\tilde{\mathbf{G}}}(u)$ is connected. □

3 An equivariant character labelling

In this section, we study the action of automorphisms on irreducible characters of G. We separate the irreducible characters in Lusztig series by the conjugacy classes of rational unipotent elements in their unipotent support. Using this, we obtain a parametrization of irreducible characters which is shown to be equivariant under the action of automorphisms. We should remark that the action of automorphisms on semisimple and regular characters was already studied by Brunat and Himstedt in [3, Theorem 3.5] and [2, Proposition 3.3]. In the sequel, we generalize their approach by using the generalized Gelfand–Graev characters to determine the action of σ on an arbitrary irreducible character.

3.1 Action of automorphisms on arbitrary irreducible characters

A key idea to determine the action is to relate the action of outer automorphisms on irreducible characters to the action of the outer automorphisms on the corresponding GGGCs, which is explicitly described by the following lemma.

Lemma 3.1 ([18, Proposition 11.10]). *For any unipotent element $u \in G$ and any bijective morphism $\sigma\colon \mathbf{G} \to \mathbf{G}$ which commutes with F, we have ${}^{\sigma}\Gamma_u = \Gamma_{\sigma(u)}$.*

If $s \in G^*$ is a semisimple element, then there exists a semisimple element $\tilde{s} \in \tilde{G}^*$ such that $i^*(\tilde{s}) = s$. Since $\operatorname{Ker}(i^*)$ is connected, we have

$$i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^{\circ}_{\mathbf{G}^*}(s),$$

and inflating through i^*, we can identify the unipotent characters of $C_{\tilde{G}^*}(\tilde{s})$ and $C^{\circ}_{G^*}(s) = C^{\circ}_{\mathbf{G}^*}(s)^{F^*}$. Following the parametrization in [4, Theorem 3.1], let $\chi_{\tilde{s},\tilde{\lambda}}$ be the irreducible character of $\tilde{G}$ corresponding to the pair $(\tilde{s}, \tilde{\lambda})$, where $\tilde{s} \in \tilde{G}^*$ is a semisimple element and $\tilde{\lambda} \in \operatorname{Irr}(C_{\tilde{G}^*}(\tilde{s}))$ is a unipotent character. For any pair $(\tilde{s}, \tilde{\lambda})$, we let $\mathcal{E}(G, \tilde{s}, \tilde{\lambda})$ denote the irreducible constituents of the character $\operatorname{Res}^{\tilde{G}}_{G} \chi_{\tilde{s},\tilde{\lambda}}$.

Lemma 3.2. *Assume that $(\tilde{s}', \tilde{\lambda}')$, $(\tilde{s}, \tilde{\lambda})$ are two pairs such that $i^*(\tilde{s}') = s = i^*(\tilde{s})$ and $\tilde{\lambda}' = \lambda \circ i^* = \tilde{\lambda}$ for some $s \in G^*$ and the unipotent character $\lambda \in \operatorname{Irr}(C^{\circ}_{G^*}(s))$. Then $\mathcal{E}(G, \tilde{s}', \tilde{\lambda}') = \mathcal{E}(G, \tilde{s}, \tilde{\lambda})$.*

Proof. Clearly, $\tilde{s}' = \tilde{s}z$ for some $z \in \operatorname{Ker}(i^*)^{F^*} \leq Z(\tilde{G}^*)$. It follows from [6, Proposition 2.7 (i) and Theorem 7.1 (iii)] that $\chi_{\tilde{s}',\tilde{\lambda}'} = \chi_{\tilde{s},\tilde{\lambda}}\hat{z}$, where $\hat{z} \in \operatorname{Irr}(\tilde{G})$ is a linear character with G in its kernel; see [6, Proposition 2.6 (i)]. □

Therefore, for a pair (s, λ) with $s \in G$ a semisimple element and $\lambda \in \operatorname{Irr}(C^{\circ}_{G^*}(s))$ a unipotent character, we get a well-defined set $\mathcal{E}(G, s, \lambda) = \mathcal{E}(G, \tilde{s}, \lambda \circ i^*)$, where $\tilde{s} \in \tilde{G}^*$ satisfies $i^*(\tilde{s}) = s$.

Proposition 3.3. *If $\sigma \in \operatorname{Aut}(\mathbf{G}, F)$, then ${}^{\sigma}\mathcal{E}(G, \sigma^*(s), \sigma^*(\lambda)) = \mathcal{E}(G, s, \lambda)$, where $\sigma^* \in \operatorname{Aut}(\mathbf{G}^*, F^*)$ is dual to σ.*

Proof. Recall that, using the same arguments as in Section 2.2, the extended automorphism $\tilde\sigma$ of $\tilde{\mathbf{G}}$ can be chosen such that $\tilde\sigma \circ i = i \circ \sigma$. Let $\chi_{\tilde s,\tilde\lambda} \in \mathrm{Irr}(\tilde G)$ be such that $\tilde s \in \tilde G^*$ is a semisimple element and $\tilde\lambda \in \mathrm{Irr}(C_{\tilde G^*}(\tilde s))$, where $i^*(\tilde s) = s$ and $\tilde\lambda = \lambda \circ i^*$. Note that we have $\mathrm{Res}^{\tilde G}_G \chi_{\tilde s,\tilde\lambda} = \chi_{\tilde s,\tilde\lambda} \circ i$. Therefore,

$$\begin{aligned}{}^{\sigma}\mathrm{Res}^{\tilde G}_G \chi_{\sigma^*(\tilde s),\sigma^*(\tilde\lambda)} &= {}^{\sigma}(\chi_{\sigma^*(\tilde s),\sigma^*(\tilde\lambda)} \circ i)\\ &= \chi_{\sigma^*(\tilde s),\sigma^*(\tilde\lambda)} \circ i \circ \sigma^{-1} = (\chi_{\sigma^*(\tilde s),\sigma^*(\tilde\lambda)} \circ \tilde\sigma^{-1}) \circ i\\ &= \mathrm{Res}^{\tilde G}_G {}^{\tilde\sigma}\chi_{\sigma^*(\tilde s),\sigma^*(\tilde\lambda)},\end{aligned}$$

which leads to the desired assertion since we have ${}^{\tilde\sigma}\chi_{\tilde\sigma^*(\tilde s),\tilde\sigma^*(\tilde\lambda)} = \chi_{\tilde s,\tilde\lambda}$ by [4, Theorem 3.1]. □

Let $\mathcal{O}^*_\chi$ be the wave front set of $\chi \in \mathrm{Irr}(G)$. It is known that any two characters $\chi, \chi' \in \mathcal{E}(G, s, \lambda)$ have the same wave front set $\mathcal{O}^*_\chi = \mathcal{O}^*_{\chi'}$. Hence we get a well-defined $\mathbf{G}$-class $\mathcal{O}^*_{s,\lambda}$ equal to $\mathcal{O}^*_\chi$ for any $\chi \in \mathcal{E}(G, s, \lambda)$. The explicit computation of the map $(s, \lambda) \mapsto \mathcal{O}^*_{s,\lambda}$ is explained in [13, §13.3].

Given a $\chi \in \mathrm{Irr}(\tilde G)$, let $\tilde G_\chi \le \tilde G$ be the stabilizer of χ. As $\tilde G/G$ is abelian, $\tilde G_\chi$ is a normal subgroup of $\tilde G$. Hence, for any two characters $\chi, \chi' \in \mathcal{E}(G, s, \lambda)$, we have $\tilde G_\chi = \tilde G_{\chi'}$. We denote this common subgroup by $\tilde G_{s,\lambda}$. The following is due to Kawanaka in the case where $\mathbf{G} = \mathrm{SL}_n$; see [14, Theorem 2.10].

Lemma 3.4. *For any pair* (s, λ) *and any* $u \in \mathcal{O}^{*F}_{s,\lambda}$, *there exists a unique character* $\chi_{s,\lambda,u} \in \mathcal{E}(G, s, \lambda)$ *satisfying* $\langle \chi_{s,\lambda,u}, \Gamma_u \rangle = 1$.

Proof. Let $\tilde\chi \in \mathrm{Irr}(\tilde G)$ be a character such that $\mathrm{Res}^{\tilde G}_G \tilde\chi = \sum_{\chi \in \mathcal{E}(G,s,\lambda)} \chi$. By [17, Lemma 14.12], we have $\mathcal{O}^*_{s,\lambda}$ is the wave front set of $\tilde\chi$. Recall that, by the construction of GGGCs, we have $\tilde\Gamma_u = \mathrm{Ind}^{\tilde G}_G \Gamma_u$ for any $u \in \mathcal{U}(G)$. Now if $u \in \mathcal{O}^{*F}_{s,\lambda}$, then by [17, Corollary 15.7], we have

$$1 = \langle \tilde\chi, \tilde\Gamma_u \rangle = \langle \tilde\chi, \mathrm{Ind}^{\tilde G}_G \Gamma_u \rangle = \langle \mathrm{Res}^{\tilde G}_G \tilde\chi, \Gamma_u \rangle = \sum_{\chi \in \mathcal{E}(G,s,\lambda)} \langle \chi, \Gamma_u \rangle.$$

The statement now follows since each term in the right-hand side is a non-negative integer. □

By [7, Proposition 2.2], we have ${}^g\Gamma_u = \Gamma_{g^{-1}ug}$ for any $g \in \tilde G$ and $u \in \mathcal{U}(G)$. As $\langle \chi, \Gamma_u \rangle = \langle {}^g\chi, {}^g\Gamma_u \rangle$ and $\tilde G$ acts transitively on $\mathcal{E}(G, s, \lambda)$, it follows that the map $u \mapsto \chi_{s,\lambda,u}$ gives a surjection $\mathcal{O}^{*F}_{s,\lambda} \to \mathcal{E}(G, s, \lambda)$. The following addresses the uniqueness of this parametrization.

Lemma 3.5. *For any* $u, v \in \mathcal{O}^{*F}_{s,\lambda}$, *we have* $\chi_{s,\lambda,u} = \chi_{s,\lambda,v}$ *if and only if* u *and* v *are* $\tilde G_{s,\lambda}$*-conjugate.*

Proof. As $\mathcal{O}^{*F}_{s,\lambda}$ is a $\tilde{G}$-orbit, we have $u = {}^{g}v$ for some $g \in \tilde{G}$ and so $\Gamma_u = {}^{g}\Gamma_v$. In particular,

$$\chi_{s,\lambda,u} = {}^{g}\chi_{s,\lambda,v}$$

since $1 = \langle \chi_{s,\lambda,u}, \Gamma_u \rangle = \langle {}^{g}\chi_{s,\lambda,u}, {}^{g}\Gamma_u \rangle$. If $g \in \tilde{G}_{s,\lambda}$, then $\chi_{s,\lambda,u} = \chi_{s,\lambda,v}$ since $\tilde{G}_{s,\lambda}$ fixes every element of $\mathcal{E}(G, s, \lambda)$. Conversely, if $\chi_{s,\lambda,u} = \chi_{s,\lambda,v} = {}^{g}\chi_{s,\lambda,v}$, then $g \in \tilde{G}_{\chi_{s,\lambda,u}} = \tilde{G}_{s,\lambda}$. □

Theorem 3.6. *If* $\sigma \in \mathrm{Aut}(\mathbf{G}, F)$*, then*

$${}^{\sigma}\chi_{\sigma^*(s),\sigma^*(\lambda),u} = \chi_{s,\lambda,\sigma(u)}.$$

Proof. Let $\chi = \chi_{\sigma^*(s),\sigma^*(\lambda),u}$. By Proposition 3.3, we obtain that ${}^{\sigma}\chi \in \mathcal{E}(G, s, \lambda)$, and by [18, Proposition 10.11], we have ${}^{\sigma}\Gamma_u = \Gamma_{\sigma(u)}$. Now the statement follows since $1 = \langle \chi, \Gamma_u \rangle = \langle {}^{\sigma}\chi, {}^{\sigma}\Gamma_u \rangle$. □

3.2 Applications

In this subsection, we review some potential applications of Theorem 3.6. In [5], among other things, the authors investigated the global side of the so-called inductive McKay conditions for quasi-simple groups of type A. Indeed, they verified the so-called stabilizer condition as follows.

Theorem 3.7 ([5, Theorem 4.1]). *For any* $\tilde{\chi} \in \mathrm{Irr}(\mathrm{GL}_n(q))$*, there is a*

$$\chi_0 \in \mathrm{Irr}(\mathrm{SL}_n(q) | \tilde{\chi})$$

such that

$$(\mathrm{GL}_n(q) \times \langle F_p, \gamma \rangle)_{\chi_0} = \mathrm{GL}_n(q)_{\chi_0} \times (\langle F_p, \gamma \rangle)_{\chi_0}.$$

Using Theorem 3.6, we can precisely determine the character χ_0. Suppose that $D \subseteq \mathrm{Aut}(\mathbf{G}, F)$ is a submonoid whose image in $\mathrm{Aut}(G)$ is a subgroup. Identifying $D \to \mathrm{Aut}(\tilde{\mathbf{G}}, \tilde{F}) \to \mathrm{Aut}(\mathbf{G}, F)$ with its image in $\mathrm{Aut}(\tilde{G})$, we can form a semidirect product $\tilde{G}D = \tilde{G} \rtimes D$. This group acts naturally on the set $\mathrm{Irr}(G)$. We then have the following, which gives a generalization of the so-called stabilizer condition in the global side of inductive McKay condition.

Theorem 3.8. *Assume that the* G*-conjugacy class of* $u_0 \in \mathcal{O}^{*F}_{s,\lambda}$ *is* D*-invariant and* $\chi = \chi_{s,\lambda,u_0}$*. Then* $(\tilde{G}D)_{\chi_0} = \tilde{G}_{\chi_0} D_{\chi_0}$.

Proof. Assume that $g\sigma \in (\tilde{G}D)_{\chi_0}$ with $g \in \tilde{G}$ and $\sigma \in D$. As χ_0 is $g\sigma$-fixed, we must have

$$\mathcal{E}(G, s, \lambda) = {}^{g\sigma}\mathcal{E}(G, s, \lambda) = {}^{\sigma}\mathcal{E}(G, s, \lambda).$$

As $\sigma(u_0)$ and u_0 are G-conjugate, and hence $\tilde{G}_{s,\lambda}$-conjugate, we must also have

$$^{\sigma}\chi_{s,\lambda,u_0} = \chi_{s,\lambda,\sigma(u_0)} = \chi_{s,\lambda,u_0}.$$

Therefore, $\sigma \in D_{\chi_0}$, which implies that $g \in \tilde{G}_{\chi_0}$. □

4 Distribution of irreducible characters as constituents of GGGCs

The aim of this section is to investigate the distribution of irreducible characters of G as constituents of generalized Gelfand–Graev characters. If $u \in \mathcal{U}(G)$ is a unipotent element, then denote by $\tilde{G}_{[u]} \leq \tilde{G}$ the stabilizer of its G-conjugacy class $[u] = [u]_G$.

Lemma 4.1. *With the above notation, we have* $\tilde{G}_{[u]} \leq \tilde{G}_{s,\lambda}$ *provided* $u \in \mathcal{O}^{*F}_{s,\lambda}$.

Proof. If $g \in \tilde{G}_{[u]}$, then

$$1 = \langle \Gamma_u, \chi_{s,\lambda,u} \rangle = \langle {}^{g}\Gamma_u, {}^{g}\chi_{s,\lambda,u} \rangle = \langle \Gamma_u, {}^{g}\chi_{s,\lambda,u} \rangle.$$

But this implies that ${}^{g}\chi_{s,\lambda,u} = \chi_{s,\lambda,u}$ so that we get $g \in \tilde{G}_{s,\lambda}$. □

Now fix $u \in \mathcal{O}^{*F}_{s,\lambda}$. If $a \in \tilde{G}/\tilde{G}_{[u]}$, we set $\Gamma_a = \Gamma_{^g u}$, where $g \in \tilde{G}$ is an element such that $a = g\tilde{G}_{[u]}$. Similarly, if $b \in \tilde{G}/\tilde{G}_{s,\lambda}$, we then set $\chi_{s,\lambda,b} = \chi_{s,\lambda,{}^h u}$, where $h \in \tilde{G}$ is an element such that $b = h\tilde{G}_{s,\lambda}$. We should note that the fact that the character $\chi_{s,\lambda,b}$ is well-defined is deduced from Lemma 3.5 and Lemma 4.2 below.

Lemma 4.2. *For any* $g, h \in \tilde{G}$, *we have that* ${}^{g}u$ *and* ${}^{h}u$ *are* $\tilde{G}_{s,\lambda}$*-conjugate if and only if* $g\tilde{G}_{s,\lambda} = h\tilde{G}_{s,\lambda}$.

Proof. If ${}^{xg}u = {}^{h}u$ for some $x \in \tilde{G}_{s,\lambda}$, then $h^{-1}xg \in C_{\tilde{G}}(u) \leq \tilde{G}_{[u]}$. Therefore, Lemma 4.1 implies that $a = b$. The converse is clear. □

We are now in a position to state the main result of this section which concerns the distribution of an irreducible character as a constituent of a given GGGC.

Theorem 4.3. *We have a well-defined map* $\varphi\colon \tilde{G}/\tilde{G}_{[u]} \to \tilde{G}/\tilde{G}_{s,\lambda}$. *If* $a \in \tilde{G}/\tilde{G}_{[u]}$ *and* $b \in \tilde{G}/\tilde{G}_{s,\lambda}$, *then* $\langle \Gamma_a, \chi_{s,\lambda,b} \rangle \neq 0$ *if and only if* $\varphi(a) = b$.

Proof. We first note that φ is well-defined by Lemma 4.1. As above, let $g, h \in \tilde{G}$ such that $a = g\tilde{G}_{[u]}$ and $b = h\tilde{G}_{s,\lambda}$. We then have $\langle \Gamma_a, \chi_{s,\lambda,b} \rangle \neq 0$ if and only if $\langle \Gamma_{^g u}, \chi_{s,\lambda,{}^h u} \rangle = 1$, which is equivalent to $\chi_{s,\lambda,{}^g u} = \chi_{s,\lambda,{}^h u}$. By Lemma 3.5, this is equivalent to ${}^{g}u$ and ${}^{h}u$ are $\tilde{G}_{s,\lambda}$-conjugate. Now Lemma 4.2 implies that the latter statement is also equivalent to $\varphi(a) = b$. □

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