Research Article

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Well-posedness of Cauchy problem of fractional drift diffusion system in non-critical spaces with power-law nonlinearity

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Abstract: In this article, we consider the global and local well-posedness of the mild solutions to the Cauchy problem of fractional drift diffusion system with higher-order nonlinearity. The main difficulty comes from the higher-order nonlinearity. Instead of the convention that people always focus on the properties of the solution in critical spaces, here we are interested in non-critical spaces such as supercritical Sobolev spaces and subcritical Lebesgue spaces. For the initial data in these non-critical spaces, using the properties of fractional heat semigroup and the classical Hardy-Littlewood-Sobolev inequality, we obtain the existence and uniqueness of the mild solution, together with the decaying rate estimates in terms of time variable.

Keywords: drift-diffusion system, Keller-Segel equations, global solutions, fractional Laplacian, higher-order nonlinearity, multi-linear operator

MSC 2020: 35K45, 35K55, 35Q60, 35B40

1 Introduction

In this article, we consider the well-posedness to the Cauchy problem of fractional drift diffusion system with higher-order nonlinearity

$$\begin{cases} \partial_t v + \Lambda^\alpha v = -\nabla\cdot(v^m\nabla\phi), & t > 0,\ x \in \mathbb{R}^n, \\ \partial_t w + \Lambda^\alpha w = \nabla\cdot(w^m\nabla\phi), & t > 0,\ x \in \mathbb{R}^n, \\ \Delta\phi = v - w, & t > 0, x \in \mathbb{R}^n, \\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m \geq 1$ is an integer, $v(x,t)$ and $w(x,t)$ are the densities of negatively and positively charged particles and $\phi(x,t)$ is the electric potential determined by the Poisson equation $\Delta\phi = v - w$. $\Lambda = \sqrt{-\Delta}$ is the Calderón-Zygmund operator [1]. The difficulties mainly come from higher-order nonlinear couplings.

In the physical literature, such fractal anomalous diffusions have been recently enthusiastically embraced by a slew of investigators in the context of hydrodynamics, acoustics, trapping effects in surface diffusion, statistical mechanics, relaxation phenomena, and biology. For instance, astrophysics is a source of mean-field

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models of gravitationally attracting particles going back to the famous Chandrasekhar equation for the equilibrium of radiating stars [2,3]. Another source of related models is mathematical biology where chemotaxis (haptotaxis, angiogenesis, etc.) phenomena for populations of either cells or (micro)organisms are described by various modifications of the Keller-Segel systems [4–15].

In our previous study [16], we considered the global existence, regularizing decay rate, and asymptotic behavior of mild solutions to the Cauchy problem of fractional drift diffusion system [17–23] with power-law nonlinearity, we only studied the problem in the critical Besov spaces, and we obtained the global well-posedness since the critical index provides the minimal regularity for the initial data to ensure the existence of the mild solutions. But in this article, we are interested in the Cauchy problem of the drift-diffusion equation in non-critical spaces such as supercritical Sobolev spaces and subcritical Lebesgue spaces. Using the properties of fractional heat semigroup and the classical estimates of fractional heat kernel, we first proved the global-in-time existence and uniqueness of the mild solutions in the frame of mixed time-space Besov space with multi-linear continuous mappings. Then, we showed the asymptotic behavior and regularizing-decay rate estimates of the solution to equations with power-law nonlinearity by the method of multi-linear operator and the classical Hardy-Littlewood-Sobolev inequality.

But now we are interested in the well-posedness of solution to the Cauchy problem (1.1) with the initial data in some non-critical spaces, such as local solutions in supercritical Sobolev spaces $H_p^s(\mathbb{R}^n)$ and global solutions in subcritical Lebesgue spaces $L^p(\mathbb{R}^n)$.

By the fundamental solution of Laplacian

$$\Phi_n(x) = \begin{cases} -\dfrac{1}{2}|x|, & n = 1, \\ -\dfrac{1}{2\pi}\ln|x|, & n = 2, \\ \dfrac{1}{n(n-2)\omega(n)|x|^{n-2}}, & n \geq 3, \end{cases} \tag{1.2}$$

where $\omega(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, the electric potential ϕ can be expressed by the convolution

$$\phi = (-\Delta)^{-1}(w - v) = \Phi_n * (w - v) = \int_{\mathbb{R}^n} \Phi_n(x - y)(w - v)(y)\mathrm{d}y.$$

$\Lambda = \sqrt{-\Delta}$ is the Calderón-Zygmund operator, and the fractional Laplacian $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ with $1 < \alpha < 2n$ is a non-local fractional differential operator defined as

$$\Lambda^\alpha v(x) = \mathcal{F}^{-1}\,|\xi|^\alpha \mathcal{F}v(\xi),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse [1].

In probabilistic terms, replacing the Laplacian Δ by its fractional power $-\Lambda^\alpha = -(-\Delta)^{\frac{\alpha}{2}}$, it leads to interesting and largely open questions of extensions of results for Brownian motion-driven stochastic equations to those driven by Lévy α-stable flights [24].

An important technical difficulty is that the densities of the semigroups generated by $-\Lambda^\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ do not decay rapidly in $x \in \mathbb{R}^n$ as is the case of the heat semigroup $S(t) = e^{t\Delta}$ $(\alpha = 2)$, the Gauss-Weierstrass kernel $K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^2})$ decays exponentially, while the densities $\mathcal{F}^{-1}(e^{-t|\xi|^\alpha})(0 < \alpha < 2)$ of non-Gaussian Lévy α-stable semigroups $S_\alpha(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}}$ have only an algebraic decay rate $|x|^{-n-\alpha}$. For the fractional operator and the fractional heat kernel, we can also refer [25–31].

By the fractional heat semigroup $e^{-t\Lambda^\alpha}$ and the well-known Duhamel principle, we rewrite System (1.1) as a system of integral equations

$$\begin{cases} v(t) = e^{-t\Lambda^\alpha}v_0 + B(v, \ldots, v, w), \\ w(t) = e^{-t\Lambda^\alpha}w_0 + B(w, \ldots, w, v), \end{cases} \tag{1.3}$$

$$B(\underbrace{v, \ldots, v}_{m}, w) = \int_0^t e^{-(t-\tau)\Lambda^\alpha}\nabla\cdot(v^m\nabla(-\Delta)^{-1}(w - v))(\tau)\mathrm{d}\tau. \tag{1.4}$$

A solution of (1.3) and (1.4) is called a mild solution of the Cauchy problem (1.1).

As usual, the fractional Sobolev spaces and their homogeneous versions are defined by

$$H_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda_s f \in L^p\}, \ \dot{H}_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \dot{\Lambda}_s f \in L^p\},$$

where Λ_s and $\dot{\Lambda}_s$ are the operators with symbols $\Lambda_s(\xi) = (1 + |\xi|^2)^{s/2}$ and $\dot{\Lambda}_s = |\xi|^s$ (these spaces are sometimes also denoted $L^{p,s}(\mathbb{R}^n)$, see [32]).

If $(v(x, t), w(x, t))$ is a solution of the Cauchy problem (1.1), then for any $\lambda > 0$,

$$(v_\lambda(x, t), w_\lambda(x, t)) = \left(\lambda^{\frac{\alpha}{m}} v(\lambda x, \lambda^\alpha t), \lambda^{\frac{\alpha}{m}} w(\lambda x, \lambda^\alpha t)\right)$$

is also a solution of the Cauchy problem (1.1) with the initial data

$$(v_\lambda(x, 0), w_\lambda(x, 0)) = \left(\lambda^{\frac{\alpha}{m}} v_0(\lambda x), \lambda^{\frac{\alpha}{m}} w_0(\lambda x)\right).$$

$(v_\lambda(x, t), w_\lambda(x, t))$ is called the self-similar solution. We can verify that $H_p^{s_c}(\mathbb{R}^n)$ ($s_c = \frac{n}{p} - \frac{\alpha}{m}$) is a critical space for the Cauchy problem (1.1), i.e., the self-similar initial data $(v_0(x), w_0(x))$ are invariant under the norm $\|\cdot\|_{H_p^{s_c}}$, thus the solution $(v_\lambda(x, t), w_\lambda(x, t))$ of System (1.1) too. This means that the index s_c provides the minimal regularity for the initial data to ensure the well-posedness of the Cauchy problem (1.1).

To obtain the local existence of the solutions in supercritical Sobolev spaces, we need the following two assumptions on s:

(H1) There exists $m > 1$ such that

$$s \leq \frac{mn}{p(m+1)} - \frac{1}{m+1}.$$

(H2) There exists $m > 1$ such that

$$\frac{mn}{p(m+1)} - \frac{1}{m+1} < s < \min\left\{\frac{mn}{p(m+1)} + 1, \frac{n}{p} - 1\right\}.$$

To avoid technical problems, we will assume that

$$s \geq \frac{n}{p} - \frac{n+1}{m+1} \tag{1.5}$$

and that

$$s \geq 0. \tag{1.6}$$

We want to solve the Cauchy problem (1.1) in supercritical spaces $H_p^s(\mathbb{R}^n)$; the main idea is to counterbalance the loss of smoothness coming from the nonlinear terms by the smoothing effects of the heat kernel. To measure the loss of smoothness on the $H_p^s(\mathbb{R}^n)$ scale coming from the composition by $F(v, w) = v^m\nabla(-\Delta)^{-1}(v - w)$, we give the following theorem.

Theorem 1.1. *Let $p \in (0, \infty)$ and s satisfy that*

$$\max\left\{0, \frac{n}{p} - \frac{n+1}{m+1}\right\} < s < \frac{n}{p} - 1. \tag{1.7}$$

Denote $s_m = (m+1)s - \frac{mn}{p} + 1$. If **(H1)** *or* **(H2)** *is fulfilled, then for all $(v, w) \in H_p^s(\mathbb{R}^n)$, $F(v, w) \in H_p^{s_m}(\mathbb{R}^n)$. Furthermore, there exists a constant C independent of (v, w) such that*

$$\|F(v,w)\|_{H_p^{s_m}} \le C\|v\|_{H_p^s}^m \|v-w\|_{H_p^s}.$$

Remark 1.1.

(1) Note that the condition $s \le \frac{mn}{p(m+1)} - \frac{1}{m+1}$ in assumption **(H1)** is equivalent to $s_m \le 0$. In the same way, the condition $\frac{mn}{p(m+1)} - \frac{1}{m+1} < s < \min\left\{\frac{mn}{p(m+1)} + 1, \frac{n}{p} - 1\right\}$ in assumption **(H2)** is equivalent to $0 < s_m < m$.

(2) The hypothesis $s > \max\left\{0, \frac{n}{p} - \frac{n+1}{m+1}\right\}$ ensures that $F(v,w)$ is well defined as an element of tempered distribution space.

(3) The value of s_m given by Theorem 1.1 is optimal. To see this, we have just to consider the example of $v(x) = w(x) = \psi(x)x^{-\beta}$, where ψ is a truncation function near 0, and β is an arbitrary nonnegative constant.

Using the nonlinear estimates given by Theorem 1.1 and the fixed point theorem, we prove the following result about the local Cauchy problem in supercritical Sobolev spaces $H_p^s(\mathbb{R}^n)$.

Theorem 1.2. (Local existence in H_p^s) *Let* $p \in (1,\infty)$. *Assume that* (1.5) *and* (1.6) *hold, and* **(H1)** *or* **(H2)** *is fulfilled.*

(1) *If the initial data* $(v_0, w_0) \in H_p^s(\mathbb{R}^n)$ *with* $s > s_c = \frac{n}{p} - \frac{\alpha}{m}$, *there exists a unique solution* $(V(t,x), W(t,x))$ *of* (1.1) *in* $C([0,T_m), H_p^s)$ *with* $T_m \ge \frac{C}{8}\|(v_0,w_0)\|_{H_p^s}^{-\nu}$ *with* $\nu = \frac{s-s_c}{\alpha}$. *If* $T_m < +\infty$, *then*

$$\lim_{t\to T_m} \|(V(t,x), W(t,x))\|_{H_p^s} = +\infty.$$

(2) *Furthermore, we have the following smoothing effect*: $B(V,W) \in C([0,T_m), H_p^{s+\theta})$ *for all* $\theta < m(s - s_c)$ *if* $s < n/p - 1$.

(3) *Let* $(V,W) \in C([0,T_1), H_p^s)$ *and* $(\tilde{V}, \tilde{W}) \in C([0,T_2), H_p^s)$ *be two solutions of* (1.1) *for the respective initial data* (v_0,w_0) *and* $(\tilde{v}_0, \tilde{w}_0)$. *Then, for all* $T < \min\{T_1, T_2\}$,

$$\|(V-\tilde{V}, W-\tilde{W})\|_{C([0,T),H_p^s)} \le C(T)(\|v_0-\tilde{v}_0\|_{H_p^s}^{1/2} + \|w_0-\tilde{w}_0\|_{H_p^s}^{1/2}).$$

Because of the definition of s_c, we see that $L^p(\mathbb{R}^n)$ is supercritical for the Cauchy problem (1.1) if and only if $p > p_c$, where p_c is defined as $p_c = \frac{mn}{\alpha}$, coming from $s_c = 0$.

Next, we give the local (respectively, global) well-posedness of the solution of the Cauchy problem (1.1) with the initial data in supercritical (respectively, critical) spaces. The similar results have been proved by Weissler [33] for nonlinear heat equations, by Kato [34] for the Navier-Stokes equations, and by Giga and Sawada [35] for the general problems.

Theorem 1.3. (Global and local existence in L^p) (1) (Existence) *Let*

$$\bar{p}_c = \max\left\{1, p_c, \frac{(m+1)n}{n+1}\right\}.$$

Suppose $(v_0,w_0) \in L^r$ *for a fixed* $r > \bar{p}_c$ *or* $r = p_c > 1$. *Then, there exist a constant* $T_0 > 0$ *and a solution* $(V(t,x), W(t,x))$ *of* (1.1) *on* $[0,T_0)$ *such that*

$$(t^\sigma V(t), t^\sigma W(t)) \in BC([0,T_0), L^p), \tag{1.8}$$

with $r \le p < n$, $0 \le \sigma < \frac{1}{m+1}$, *or* $n \le p < \infty$, $\sigma m + \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{n}\right) < 1$, *where* $\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$. *And*

$$(t^\sigma \|W(t)\|_{L^p}, t^\sigma \|W(t)\|_{L^p}) \to 0, \quad \text{as } t \to 0, \tag{1.9}$$

with $r < p < n$, $0 < \sigma < \frac{1}{m+1}$ *or* $n \le p < \infty$, $\sigma m + \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{n}\right) < 1$.

(2) (*Estimate for* T_0) *If* $r > \bar{p}_c$ *and* $\beta(r) = \frac{mn}{\alpha r}$, *we have*

$$T_0 \geq C\|(v_0, w_0)\|_{L^r}^{-\frac{m}{1-\beta(r)}}. \tag{1.10}$$

(3) (*Global existence for small initial data*) *If there exists a positive constant* ε *small enough such that* $\|(v_0, w_0)\|_{L^{p_c}} < \varepsilon$, *then* $T_0 = \infty$ *if* $p_c > 1$ *and we have*

$$\|(V(t), W(t))\|_{L^p} \leq Ct^{-\sigma}, \quad \forall 0 < t < \infty, \tag{1.11}$$

with C *independent of* t, *provided that* $p \geq p_c$.

(4) (*Uniqueness*) *Solutions of* (1.1) *satisfying* (1.8) *and* (1.9) *for some* $0 < \sigma < \frac{1}{m+1}$, $p > \frac{(m+1)n}{n+1}$, *and* $\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$ *are unique. If* $r > p_c$, σ *may equal zero and* (1.9) *is not necessary to guarantee the uniqueness. In particular if* $r > \bar{p}_c$, *solutions are unique in* $BC([0, T_0), L^r)$, *provided that* $r > m+1$.

We can also prove a slight improvement result if we consider the initial data in critical space $L^{p_c}(\mathbb{R}^n)$.

Corollary 1.1. *Assume that* $\alpha > m$ *and* $p_c > 1$. *Let* $\gamma(q) = \frac{n}{\alpha}(\frac{1}{p_c} - \frac{1}{q})$. *If there exists a constant* A *such that* $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n)$ *with* $\|(v_0, w_0)\|_{L^{p_c}} \leq A$, *then there is a unique global solution* $(V(t,x), W(t,x))$ *of* (1.1) *such that for all* $q \in [p_c, \infty)$,

$$\|(V(t,\cdot), W(t,\cdot))\|_{L^q} \leq Ct^{-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}). \tag{1.12}$$

Then, using Corollary 1.1, we will consider the case of the initial data with arbitrary high norm in subcritical spaces $L^p(\mathbb{R}^n)$ and small norm in critical spaces $L^{p_c}(\mathbb{R}^n)$.

Proposition 1.1. *Let* $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *with* $p \leq p_c$ *and assume that* $\|(v_0, w_0)\|_{L^{p_c}} \leq A$. *Let us consider* $(V(t,x), W(t,x))$ *the global solution of* (1.1) *given by Theorem* 1.3. *Then,*

$$(V(t,x), W(t,x)) \in BC(\mathbb{R}^+, L^p) \cap BC(\mathbb{R}^+, L^{p_c}), \tag{1.13}$$

$$\|(V(t,\cdot), W(t,\cdot))\|_{L^r} \leq Ct^{-\frac{n}{\alpha}(\frac{1}{p}-\frac{1}{r})}(\|v_0\|_{L^p} + \|w_0\|_{L^p}), \tag{1.14}$$

for all $r \geq p$ *and* $t > 0$.

Using Proposition 1.1, we will prove the following result on the global Cauchy problem with the initial data in supercritical spaces $H_p^s(\mathbb{R}^n)$.

Theorem 1.4. *Assume that* $p_c > 1$ *and* $p \in (p_c m^{-1}, p_c]$.

(1) *There exists a constant* A' *such that, for all* $(v_0, w_0) \in H_p^{s_c}(\mathbb{R}^n)$ *with* $\|(v_0, w_0)\|_{H_p^{s_c}} \leq A'$, *there is a unique global solution* $(V(t,x), W(t,x))$ *of* (1.1) *in* $C([0, +\infty), H_p^{s_c})$, *which satisfies* (1.8)–(1.9). *Furthermore,* $(V(t,x), W(t,x))$ *satisfies* (1.13) *and* (1.14).

(2) *Let* $(v_0, w_0) \in H_p^s(\mathbb{R}^n)$ *with* $s > s_c$. If $\|(v_0, w_0)\|_{H_p^{s_c}} \leq A'$, *then for the local solution* $(V(t,x), W(t,x))$ *of* (1.1) *given by Theorem* 1.3, *we have* $t^{(s-s_c)/\alpha}(V, W) \in BC(\mathbb{R}^+, H_p^s)$ *and satisfy* (1.13) *and* (1.14).

Remark 1.2. There is no restriction on the size of $\|(v_0, w_0)\|_{H_p^s}$ in Part (2) of Theorem 1.4: we just assume that the initial data are small enough in the critical spaces $H_p^{s_c}(\mathbb{R}^n)$.

The outline of the rest of the article is as follows. In Section 2, we give several useful lemmas such as the classical Hardy-Littlewood-Sobolev inequality and $L^p - L^q$ and $H_p^{s+\theta} - H_p^s$ estimates for the semigroup operator $e^{-t\Lambda^\alpha}$. In Section 3, we establish the local existence and uniqueness of mild solution in supercritical spaces $H_p^s(\mathbb{R}^n)$, together with the smooth effects and continuous dependence with respect to the initial data. In Section 4, we discuss the local and global Cauchy problem of the mild solution in supercritical spaces $L^r(\mathbb{R}^n)$. In Section 5, we prove the improvement result Corollary 1.1 and consider the global Cauchy problem with the initial data in $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $H_p^{s_c}(\mathbb{R}^n) \cap H_p^s(\mathbb{R}^n)$, respectively. Finally, we prove Theorem 1.1 in Appendix with tedious calculation.

2 Preliminaries

For the Laplacian operator Δ and the Calderón-Zygmund operator $\Lambda = \sqrt{-\Delta}$, we have the following classical Hardy-Littlewood-Sobolev inequality.

Lemma 2.1. [36,37] *Let* $1 < p < n$, *and the nonlocal operator* $(-\Delta)^{-\frac{1}{2}}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^{\frac{np}{n-p}}(\mathbb{R}^n)$, *i.e.*, $\forall f \in L^p(\mathbb{R}^n)$,

$$\|(-\Delta)^{-\frac{1}{2}} f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\nabla(-\Delta)^{-1} f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)}.$$

For the fractional power operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ and the semigroup operator $e^{-t\Lambda^\alpha}$, we first consider the Cauchy problem for the homogeneous linear fractional heat equation:

$$\begin{cases} \partial_t u + \Lambda^\alpha u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

By the Fourier transform, the solution can be written as

$$u(t,x) = e^{-t\Lambda^\alpha} u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha} \mathcal{F} u_0(\xi)) = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha}) * u_0(x) = K_t(x) * u_0(x),$$

where the fractional heat kernel

$$K_t(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t|\xi|^\alpha} \mathrm{d}\xi = t^{-\frac{n}{\alpha}} K(xt^{-\frac{1}{\alpha}}),$$

the function $K(x) \in L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ denotes the space of functions $f \in C(\mathbb{R}^n)$ satisfying that $\lim_{|x|\to\infty} f(x) = 0$.

For the semigroup operator $e^{-t\Lambda^\alpha}$, we have $L^p - L^q$ and $H_p^{s+\theta} - H_p^s$ estimates [38].

Lemma 2.2. (a) *Let* $1 \leq p \leq q \leq \infty$, $\alpha > 0$, *and there exists* C *such that*

$$\|e^{-t\Lambda^\alpha} f\|_{L^q} \leq C(n,\alpha) t^{-\frac{n}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n).$$

(b) *Let* $1 \leq p \leq q \leq \infty$, $\alpha > 0$, *and* $\gamma > 0$. *There exists* C *such that*

$$\|\Lambda^\gamma e^{-t\Lambda^\alpha} f\|_{L^q} \leq C(n,\alpha) t^{-\frac{\gamma}{\alpha}-\frac{n}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n).$$

(c) *Let* $\theta \geq 0$, $\alpha > 0$, *and* $t \in (0,T]$. *There exists* C *such that*

$$\|e^{-t\Lambda^\alpha} f\|_{H_p^{s+\theta}} \leq C(T) t^{-\frac{\theta}{\alpha}} \|f\|_{H_p^s}, \quad \forall f \in H_p^s(\mathbb{R}).$$

(d) *Let* $\theta \geq 0$, $\alpha > 0$, *and* $t \in (0,T]$. *There exists* C *such that*

$$\|e^{-t\Lambda^\alpha} f\|_{\dot{H}_p^{s+\theta}} \leq C(T) t^{-\frac{\theta}{\alpha}} \|f\|_{\dot{H}_p^s}, \quad \forall f \in H_p^s(\mathbb{R}).$$

Lemma 2.3. *Let* $p \in (1,+\infty)$, $\theta \in \mathbb{R}$, *and* $s \in \mathbb{R}$. *Then,*

$$\|f\|_{H_p^s}^2 \leq C\|f\|_{H_p^{s+\theta}} \|f\|_{H_p^{s-\theta}}, \quad \forall f \in H_p^{s+\theta}(\mathbb{R}).$$

3 Local Cauchy problem in supercritical space $H_p^s(\mathbb{R}^n)$

In this section, we prove Theorem 1.2. We first prove the existence of solution in Section 3.1 and then uniqueness in Section 3.2. In Section 3.3, we study the smoothing effects for (1.1), and in Section 3.4, we consider the continuous dependence of the solutions with respect to the initial dada.

3.1 Existence

First, we assume that all conditions in Theorem 1.1 hold true and the initial data (v_0, w_0) belong to subcritical space $H_p^s(\mathbb{R}^n)$. In the sequel, C will denote a positive constant that may be changed from one line to another. To simplify the notations, we have

$$B(v, w)(t, x) = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla\cdot(v^m \nabla(-\Delta)^{-1}(v - w))(\tau)\mathrm{d}\tau.$$

We introduce the exponent $\tilde{p}$ given by $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s}{n}$, and by (1.8) and (1.9). Since $s > s_c$, we have $\tilde{p} \geq \frac{n(m+1)}{n+1}$ and $\tilde{p} > p_c$. We define the spaces $Y = C([0, T), H_p^s)$ and $X = C([0, T), L^{\tilde{p}})$. Hence, by the Sobolev embedding theorem, $Y \hookrightarrow X$. Now, let us consider the sequence of functions:

$$\begin{cases} v^0 = e^{-t\Lambda^\alpha} v_0(x), & v^{j+1} = v^0 + B(v^j, w^j), \\ w^0 = e^{-t\Lambda^\alpha} w_0(x), & w^{j+1} = w^0 + B(w^j, v^j). \end{cases} \tag{3.1}$$

First, we are going to prove that $\{(v^j, w^j)\}$ converges strongly in X to a limit (V, W), which verifies (1.7) (this proof follows closely Giga's proof but we detail it for the reader's convenience), and second, using the new estimates given by Theorem 1.1, we will show that (V, W) belongs also to Y. By Part (1) of Lemma 2.2, we have

$$\|v^0, w^0\|_X \leq \|v_0, w_0\|_{L^{\tilde{p}}} \leq C\|v_0, w_0\|_{H_p^s}. \tag{3.2}$$

For (v, w) and $(\tilde{v}, \tilde{w})$ in X,

$$B(v, w) - B(\tilde{v}, \tilde{w}) = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla\cdot(v^m \nabla(-\Delta)^{-1}(v - w) - \tilde{v}^m \nabla(-\Delta)^{-1}(\tilde{v} - \tilde{w}))(\tau)\mathrm{d}\tau = B_1 + B_2, \tag{3.3}$$

where

$$B_1 = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla\cdot((v^m - \tilde{v}^m)\nabla(-\Delta)^{-1}(v - w))(\tau)\mathrm{d}\tau,$$
$$B_2 = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla\cdot(\tilde{v}^m \nabla(-\Delta)^{-1}(v - \tilde{v} - (w - \tilde{w})))(\tau)\mathrm{d}\tau.$$

Since we are working in the whole Euclidian space $\mathbb{R}^n$, the operators $e^{t\Lambda^\alpha}$ and ∇ are some Fourier multipliers, and so we have

$$e^{t\Lambda^\alpha}\nabla = \nabla e^{t\Lambda^\alpha} = e^{t\Lambda^\alpha/2}\nabla e^{t\Lambda^\alpha/2}.$$

Furthermore, by Lemmas 2.2 and 2.3 and Hölder's inequality, we obtain

$$\begin{aligned}\|B_1\|_{L^{\tilde p}} &\le \int_0^t \|e^{-(t-\tau)\Lambda^\alpha}\nabla\cdot((v^m-\tilde v^m)\nabla(-\Delta)^{-1}(v-w))(\tau)\|_{L^{\tilde p}}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{-\beta}\|(v^m-\tilde v^m)\nabla(-\Delta)^{-1}(v-w)(\tau)\|_{L^q}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{-\beta}\|(v-\tilde v)(|v|^{m-1}+|\tilde v|^{m-1})\nabla(-\Delta)^{-1}(v-w)(\tau)\|_{L^q}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{-\beta}\|v-\tilde v\|_{L^{\tilde p}}(\|v\|_{L^{\tilde p}}^{m-1}+\|\tilde v\|_{L^{\tilde p}}^{m-1})\|\nabla(-\Delta)^{-1}(v-w)(\tau)\|_{L^{\frac{n\tilde p}{n-\tilde p}}}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{-\beta}\|v-\tilde v\|_{L^{\tilde p}}(\|v\|_{L^{\tilde p}}^{m-1}+\|\tilde v\|_{L^{\tilde p}}^{m-1})\|v-w\|_{L^{\frac{n\tilde p}{n-\tilde p}}}\mathrm{d}\tau,\end{aligned}$$

where $\frac{1}{q}=\frac{m}{\tilde p}+\frac{n-\tilde p}{n\tilde p}$ and by $s>s_c$, $\beta=\frac{1}{\alpha}+\frac{n}{\alpha}\left(\frac{1}{q}-\frac{1}{\tilde p}\right)=\frac{mn}{\alpha\tilde p}<1$. Then,

$$\|B_1\|_X \le CT^{1-\beta}\|v-\tilde v\|_X(\|v\|_X^{m-1}+\|\tilde v\|_X^{m-1})\|v-w\|_X. \tag{3.4}$$

Similarly, we have

$$\|B_2\|_X \le CT^{1-\beta}\,\|\tilde v\|_X^m\|v-\tilde v-(w-\tilde w)\|_X. \tag{3.5}$$

Combining (3.3)–(3.5), we have

$$\|B(v,w)-B(\tilde v,\tilde w)\|_X \le CT^{1-\beta}\{\|v-\tilde v\|_X(\|v\|_X^{m-1}+\|\tilde v\|_X^{m-1})\|v-w\|_X+\|\tilde v\|_X^m\|v-\tilde v-(w-\tilde w)\|_X\}. \tag{3.6}$$

Due to $B(0,0)=0$ and (3.2) and (3.6), we have

$$\begin{cases}\|v^{j+1}\|_X \le \|v_0\|_{H_p^s}+CT^{1-\beta}\|v^j\|_X^m\|v^j-w^j\|_X,\\ \|v^{j+1}-v^j\|_X \le CT^{1-\beta}\{\|v^j-v^{j-1}\|_X(\|v^j\|_X^{m-1}+\|v^{j-1}\|_X^{m-1})\|v^j-w^j\|_X+\|v^{j-1}\|_X^m\|v^j-v^{j-1}-(w^j-w^{j-1})\|_X\}.\end{cases}$$

Similarly,

$$\begin{cases}\|w^{j+1}\|_X \le \|w_0\|_{H_p^s}+CT^{1-\beta}\|w^j\|_X^m\|v^j-w^j\|_X,\\ \|w^{j+1}-w^j\|_X \le CT^{1-\beta}\{\|w^j-w^{j-1}\|_X(\|w^j\|_X^{m-1}+\|w^{j-1}\|_X^{m-1})\|v^j-w^j\|_X+\|w^{j-1}\|_X^m\|v^j-v^{j-1}-(w^j-w^{j-1})\|_X\}.\end{cases}$$

Then, a standard fixed point argument shows that, for

$$T<\frac{C}{4}\|(v_0,w_0)\|_{H_p^s}^{-\nu^{-1}}\left(\nu=\frac{s-s_c}{\alpha}\right), \tag{3.7}$$

the sequence $\{(v^j,w^j)\}$ converges strongly in X to limit (V,W), which obviously solves (1.1) since $\tilde p\ge\frac{n(m+1)}{n+1}$ by $\tilde p\ge\frac{n(m+1)}{n+1}$ and $\tilde p>p_c$.

Now, we must prove that the limit solution belongs also to Y. Let $(v,w)\in Y$; using Theorem 1.1, we have

$$\begin{aligned}\|B(v,w)\|_{H_p^s} &\le \int_0^t \|e^{-(t-\tau)\Lambda^\alpha}\nabla\cdot(v^m\nabla(-\Delta)^{-1}(v-w))(\tau)\|_{H_p^s}\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{1}{\alpha}(s-s_m)}\|v^m\nabla(-\Delta)^{-1}(v-w)(\tau)\|_{H_p^s}\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-\beta}\|v\|_{H_p^s}^m\|v-w\|_{H_p^s}\mathrm{d}\tau,\end{aligned}$$

where $\beta=\frac{1}{\alpha}+\frac{1}{\alpha}(s-s_m)=\frac{mn}{\alpha\tilde p}$, then $\|B(v,w)\|_Y\le CT^{1-\beta}\|v\|_Y^m\|v-w\|_Y$, and so, by (3.2),

$$\|v^{j+1}\|_Y \le \|v_0\|_{H_p^s}+CT^{1-\beta}\|v^j\|_Y^m\|v^j-w^j\|_Y. \tag{3.8}$$

Similarly, we have

$$\|w^{j+1}\|_Y \le \|w_0\|_{H_p^s} + CT^{1-\beta}\|w^j\|_Y^m \|v^j - w^j\|_Y. \tag{3.9}$$

If T satisfies (3.7), thanks to (3.8)–(3.9), we see that $\|(v^j, w^j)\|_Y$ remains bounded; thus, we can extract a subsequence (v^{j_k}, w^{j_k}), which converges weakly-* to a limit $(\tilde{V}, \tilde{W}) \in Y$. Now, the (v^{j_k}, w^{j_k}) converges to (V, W) and converges to $(\tilde{V}, \tilde{W})$ in Y and so (V, W) agrees with $(\tilde{V}, \tilde{W})$. Thus, we have proved the existence of a solution in $C([0, T), H_p^s)$.

The estimate for T_m comes from (3.7), which gives $T_m \ge \frac{C}{8}\|(v_0, w_0)\|_{H_p^s}^{-\nu}$ with $\nu = \frac{s - s_c}{\alpha}$.

If $T_m < +\infty$, this explicit lower bound obviously allows us to show the blow-up in H_p^s norm (one can prove the blow-up in $L^{\tilde{p}}(\mathbb{R}^n)$ when it holds in H_p^s).

3.2 Uniqueness

Let $(V, W) \in Y$ and $(\tilde{V}, \tilde{W}) \in Y$ be two solutions for the same initial data (v_0, w_0) and $T < \min\{T_m(V, W), T_m(\tilde{V}, \tilde{W})\}$. Then, since (V, W) and $(\tilde{V}, \tilde{W})$ solve (1.1), respectively, we have $\|V - \tilde{V}\|_X = \|B(V, W) - B(\tilde{V}, \tilde{W})\|_X$. Denote

$$M = \sup_{t \in [0,T]} \{\|(V, W)\|_{L^{\tilde{p}}}, \|(\tilde{V}, \tilde{W})\|_{L^{\tilde{p}}}\},$$

by (3.6), we obtain

$$\begin{aligned} \|V - \tilde{V}\|_X &\le CT^{1-\beta}M^m(\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X), \\ \|W - \tilde{W}\|_X &\le CT^{1-\beta}M^m(\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X). \end{aligned}$$

So,

$$\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X \le CT^{1-\beta}M^m(\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X).$$

For T small enough, we have

$$\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X \le \frac{1}{2}(\|V - \tilde{V}\|_X + \|W - \tilde{W}\|_X),$$

and so $(V, W) = (\tilde{V}, \tilde{W})$ on $[0, T]$. To conclude, we just have to prove that $T_m(V, W) = T_m(\tilde{V}, \tilde{W})$; otherwise, we can assume that $T_m(V, W) < T_m(\tilde{V}, \tilde{W})$. By iteration, we can obtain that $(V, W) = (\tilde{V}, \tilde{W})$ on $[0, T_m(V, W)]$, then

$$+\infty = \lim_{t \to T_m(V,W)} \|(V(t, \cdot), W(t, \cdot))\|_X = \lim_{t \to T_m(V,W)} \|(\tilde{V}(t, \cdot), \tilde{W}(t, \cdot))\|_X < \infty,$$

which is an apparent contradiction. This ends the proof of uniqueness.

3.3 Smoothing effects

Let (V, W) be a solution of (1.1). Using Lemma 2.2, we easily obtain that

$$\|V(t) - e^{-t\Lambda}v_0\|_{H_p^{s+\theta}} \le C\int_0^t (t - \tau)^{-\frac{1}{\alpha} - \frac{1}{\alpha}(s+\theta-s_m)} \|V\|_{H_p^s}^m \|V - W\|_{H_p^s} \mathrm{d}\tau,$$

and so, for all $\theta < \alpha(1 - \beta) = m(s - s_c)$, we obtain

$$\|V(t) - e^{-t\Lambda}v_0\|_{H_p^{s+\theta}} \le CT^{1-\beta-\frac{\theta}{\alpha}}\|V\|_Y^m \|V - W\|_Y.$$

Similarly,

$$\|W(t) - e^{-t\Lambda}w_0\|_{H_p^{s+\theta}} \le CT^{1-\beta-\frac{\theta}{\alpha}}\|W\|_Y^m\|V - W\|_Y.$$

Then, we prove the smoothing effects.

3.4 Continuous dependence with respect to the initial data

Let (V, W) and $(\tilde{V}, \tilde{W})$ be two solutions of (1.1) for the initial data (v_0, w_0) and $(\tilde{v}_0, \tilde{w}_0)$, respectively. Let

$$T < \min\{T_m(V, W), T_m(\tilde{V}, \tilde{W})\}, M = \sup_{t\in[0,T]}\{\|(V, W)\|_{H_p^s}, \|(\tilde{V}, \tilde{W})\|_{H_p^s}\}.$$

Assume that $s_m \le 0$, i.e., $s \le \frac{mn}{p(m+1)} - \frac{1}{m+1}$, one obtains

$$\|V - \tilde{V}\|_{H_p^s} \le \|v_0 - \tilde{v}_0\|_{H_p^s} + C\int_0^t (t-\tau)^{-\beta}\|V^m\nabla(-\Delta)^{-1}(V - W) - \tilde{V}^m\nabla(-\Delta)^{-1}(\tilde{V} - \tilde{W})\|_{H_p^{s_m}}\mathrm{d}\tau.$$

Since $s_m \le 0$ and $q = \frac{np}{n - s_m p}$, we can use the Sobolev embedding $L^q(\mathbb{R}^n) \hookrightarrow H_p^{s_m}(\mathbb{R}^n)$, which leads to

$$\|V - \tilde{V}\|_{H_p^s} \le \|v_0 - \tilde{v}_0\|_{H_p^s} + C\int_0^t (t-\tau)^{-\beta}\|V^m\nabla(-\Delta)^{-1}(V - W) - \tilde{V}^m\nabla(-\Delta)^{-1}(\tilde{V} - \tilde{W})\|_{L^q}\mathrm{d}\tau.$$

By $L^p - L^q$ estimate, we have

$$\begin{aligned}
&\|V^m\nabla(-\Delta)^{-1}(V - W) - \tilde{V}^m\nabla(-\Delta)^{-1}(\tilde{V} - \tilde{W})\|_{L^q}\\
&\le \|(V^m - \tilde{V}^m)\nabla(-\Delta)^{-1}(V - W)\|_{L^q} + \|\tilde{V}^m\nabla(-\Delta)^{-1}(V - \tilde{V} - (W - \tilde{W}))\|_{L^q}\\
&\le \|(V - \tilde{V})(|V|^{m-1} + |\tilde{V}|^{m-1})\nabla(-\Delta)^{-1}(V - W)\|_{L^q} + \|\tilde{V}\|_{L^{\tilde{p}}}^m\|V - \tilde{V} - (W - \tilde{W})\|_{L^{\tilde{p}}}\\
&\le \|V - \tilde{V}\|_{L^{\tilde{p}}}(\|V\|_{L^{\tilde{p}}}^{m-1} + \|\tilde{V}\|_{L^{\tilde{p}}}^{m-1})\|V - W\|_{L^{\tilde{p}}} + \|\tilde{V}\|_{L^{\tilde{p}}}^m(\|V - \tilde{V}\|_{L^{\tilde{p}}} + \|W - \tilde{W}\|_{L^{\tilde{p}}})\\
&\le \|V - \tilde{V}\|_{H_p^s}(\|V\|_{H_p^s}^{m-1} + \|\tilde{V}\|_{H_p^s}^{m-1})\|V - W\|_{H_p^s} + \|\tilde{V}\|_{H_p^s}^m(\|V - \tilde{V}\|_{H_p^s} + \|W - \tilde{W}\|_{H_p^s}).
\end{aligned}$$

Thus,

$$\|V - \tilde{V}\|_Y \le \|v_0 - \tilde{v}_0\|_{H_p^s} + CT^{1-\beta}M^m(\|V - \tilde{V}\|_Y + \|W - \tilde{W}\|_Y).$$

Similarly,

$$\|V - \tilde{V}\|_Y + \|W - \tilde{W}\|_Y \le \|v_0 - \tilde{v}_0\|_{H_p^s} + \|w_0 - \tilde{w}_0\|_{H_p^s} + 2CT^{1-\beta}M^m(\|V - \tilde{V}\|_Y + \|W - \tilde{W}\|_Y).$$

Taking T small enough such that $2CT^{1-\beta}M^m \le 1/2$, we obtain that

$$\|V - \tilde{V}\|_Y + \|W - \tilde{W}\|_Y \le 2(\|v_0 - \tilde{v}_0\|_{H_p^s} + \|w_0 - \tilde{w}_0\|_{H_p^s}). \tag{3.10}$$

To conclude, we have to relax our assumption on s. Since (V, W) and $(\tilde{V}, \tilde{W})$ are the solutions of (1.1),

$$\|V - \tilde{V}\|_Y \le \|v_0 - \tilde{v}_0\|_{H_p^s} + \|B(V, W) - B(\tilde{V}, \tilde{W})\|_Y.$$

By Lemma 2.3, we see that

$$\|B(V, W) - B(\tilde{V}, \tilde{W})\|_{H_p^s} \le (\|B(V, W)\|_{H_p^{s+\theta}} + \|B(\tilde{V}, \tilde{W})\|_{H_p^{s+\theta}})^{1/2}\|B(V, W) - B(\tilde{V}, \tilde{W})\|_{H_p^{s-\theta}}^{1/2}.$$

One can choose $\theta < m(s - s_c)$ such that

$$s_c < s - \theta < \frac{mn}{p(m+1)} - \frac{1}{m+1}. \tag{3.11}$$

Using the smoothing effects, the first term in the left-hand side of the last inequality is bounded by

$$\{CT^{1-\beta-\theta/\alpha}(\|V\|_Y + \|W\|_Y)\}^{1/2} \le C(T)M^{(m+1)/2}.$$

Using (3.10) and (3.11), we bound the second term by

$$\|v_0 - \tilde{v}_0\|^{1/2}_{H^{s-\theta}_p} + \|V - \tilde{V}\|_{H^{s-\theta}_p} \le C(\|v_0 - \tilde{v}_0\|^{1/2}_{H^{s-\theta}_p} + \|w - w_0\|^{1/2}_{H^{s-\theta}_p}) \le C(\|v_0 - \tilde{v}_0\|^{1/2}_Y + \|w - w_0\|^{1/2}_Y).$$

Combining these two inequalities, we obtain that

$$\|V - \tilde{V}\|_Y \le C(T)(\|v_0 - \tilde{v}_0\|^{1/2}_Y + \|w - w_0\|^{1/2}_Y).$$

Thus,

$$\|V - \tilde{V}\|_Y + \|W - \tilde{W}\|_Y \le C(T)(\|v_0 - \tilde{v}_0\|^{1/2}_Y + \|w - w_0\|^{1/2}_Y).$$

The proof of Part (3) is complete.

4 Cauchy problem in supercritical space $L^r(\mathbb{R}^n)$

In this section, we prove Theorem 1.3, i.e., we describe the local and global Cauchy problem in supercritical spaces $L^r(\mathbb{R}^n)$.

We begin with estimate for $e^{-(t-\tau)\Lambda^\alpha}\nabla\cdot(v^m\nabla(-\Delta)^{-1}(v - w))(\tau)$.

$$\begin{aligned}
&\|e^{-(t-\tau)\Lambda^\alpha}\nabla\cdot((v^m\nabla(-\Delta)^{-1}(v - w)) - (\tilde{v}^m\nabla(-\Delta)^{-1}(\tilde{v} - \tilde{w})))(\tau)\|_{L^s} \\
&\le C(t-\tau)^{-\frac{1}{\alpha}-\frac{1}{\alpha}(\frac{1}{h}-\frac{1}{s})}\{\|(v^m - \tilde{v}^m)\nabla(-\Delta)^{-1}(v - w)\|_h + \|\tilde{v}^m\nabla(-\Delta)^{-1}(v - \tilde{v} - (w - \tilde{w}))\|_{L^h}\} \qquad (4.1)\\
&\le Ct^{-(\beta(p)-\delta)}\{\|v - \tilde{v}\|_{L^p}(\|v\|^{m-1}_{L^p} + \|\tilde{v}\|^{m-1}_{L^p})\|v - w\|_{L^p} + \|\tilde{v}\|^m_{L^p}\|v - \tilde{v} - (w - \tilde{w})\|_{L^p}\},
\end{aligned}$$

where $\frac{1}{h} = \frac{m}{p} + \frac{1}{p} - \frac{1}{n}$, $\beta(p) = \frac{mn}{\alpha p}$, and $\delta = \frac{n}{\alpha}\left(\frac{1}{s} - \frac{1}{p}\right)$, (v, w) and $(\tilde{v}, \tilde{w})$ belong to L^p, provided that $p > \frac{n(m+1)}{n+1}$ and $s \ge h$.

Next for $T_0 > 0$, we derive *a priori* estimates for

$$K_j = K_j(T_0) = \sup_{0<t\le T_0} t^\sigma\|(v^j, w^j)\|_{L^p}, \quad \forall j \ge 0,$$

where (v^j, w^j) is defined in (3.1) and σ and p satisfy that

$$\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right), 0 \le \sigma < \frac{1}{m+1}, p > \frac{n(m+1)}{n+1}, p \ge r, p \ne p_c. \qquad (4.2)$$

Remark 4.1. It is easy to find a nonempty set of numbers σ, p meeting (4.2). Indeed, the definition of p_c shows that $\frac{m+1}{p_c} = \frac{1}{p_c} + \frac{\alpha}{n}$, which gives $\frac{n}{\alpha}(\frac{1}{r} - \frac{n+1}{n(m+1)}) < \frac{1}{m+1}$ for $r = p_c > 1$. If $r > p_c$, this is obvious. This shows that there exist σ and p satisfying (4.2).

To estimate K_j, let us recall the scheme

$$v^{j+1} = v^0 + B(v^j, w^j), w^{j+1} = w^0 + B(w^j, v^j), j \in \mathbb{N}. \qquad (4.3)$$

Applying (4.1) with $(v, w) = (v^j, w^j)$ and $(\tilde{v}, \tilde{w}) = (0, 0)$ to the second term $B(v^j, w^j)$, we obtain

$$t^\sigma\|B(v^j, w^j)\|_{L^p} \le Ct^\sigma\int_0^t (t-\tau)^{-\beta(p)}\|v^j\|^m_{L^p}\|v^j - w^j\|_{L^p}(\tau)\mathrm{d}\tau,$$

where $p > \frac{n(m+1)}{n+1}$ is used. This gives an iterative estimate

$$K_{j+1} \le K_0 + CBK_j^{m+1}T_0^{1-\beta(r)}, \quad \forall j \in \mathbb{N}, \qquad (4.4)$$

where $B = \int_0^1 (1-\tau)^{-\beta(p)}\tau^{\sigma(m+1)}d\tau$. Since $\beta(r) = \beta(p) + \sigma m$, the assumptions $\sigma < \frac{1}{m+1}$ and $p > p_c'$ ensure the convergence of B since $\beta(p_c) = 1$.

For a technical reason, we use a less sharp estimate essentially same as (4.4):

$$K_{j+1} \le K_0 + 2CBK_j^{m+1}T_0^{1-\beta(r)}, \quad \forall j \in \mathbb{N}. \tag{4.5}$$

If K_0 or T_0 satisfies that

$$K_0^m T_0^{1-\beta(r)} \le \frac{1}{2^{2+m}CB}, \tag{4.6}$$

due to (4.5), an elementary calculation shows that

$$K_j < 2K_0, \quad \forall j \in \mathbb{N} \quad \text{and} \quad 2CBK_j^m T_0^{1-\beta(r)} < \frac{1}{2}, \quad \forall j \in \mathbb{N} \cup \{0\}. \tag{4.7}$$

We thus have *a priori* estimate for K_j under Condition (4.6).

We next study what conditions on T_0 and (v_0, w_0) can guarantee (4.6). First, we prove that for $\sigma > 0$,

$$t^\sigma \|e^{-t\Lambda^\alpha}(v_0, w_0)\|_{L^p} \to 0 \quad \text{as } t \to 0. \tag{4.8}$$

Since $C_c(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ is dense in L^r, there is a sequence $\{v_0^i\}$ in $C_c(\mathbb{R}^n)$ such that $v_0^i \to v_0$ in L^r. Then,

$$t^\sigma \|e^{-t\Lambda^\alpha}v_0\|_{L^p} \le t^\sigma \|e^{-t\Lambda^\alpha}(v_0 - v_0^i)\|_{L^p} + t^\sigma \|e^{-t\Lambda^\alpha}v_0^i\|_{L^p} \le C\|v_0 - v_0^i\|_{L^r} + t^\sigma \|e^{-t\Lambda^\alpha}v_0^i\|_{L^p},$$

where the constant C is independent of i and t. Since $v_0^i \in L^p$, we have

$$\|e^{-t\Lambda^\alpha}v_0^i\|_{L^p} \le C\|v_0^i\|_{L^p}.$$

Thus, $t^\sigma \|e^{-t\Lambda^\alpha}v_0^i\|_{L^p} \to 0$ as $t \to 0$ since $\sigma > 0$. A similar result is valid for w_0, and we then have (4.8), which particularly implies that for $\sigma > 0$,

$$K_0 = \sup_{0<t\le T_0} t^\sigma \|(v^0, w^0)\|_{L^p} = \sup_{0<t\le T_0} t^\sigma \|e^{-t\Lambda^\alpha}(v_0, w_0)\|_{L^p} \to 0, \quad \text{as } T_0 \to 0. \tag{4.9}$$

If $r > p_c'$ (consequently $\beta(r) < 1$), then Condition (1.10) ensures (4.6) since $K_0/\|(v_0, w_0)\|_{L^r}$ is bounded independent of (v_0, w_0) and T_0. In the case $r = p_c$, (4.9) shows that for small T_0, we have (4.6) for every $T_0 > 0$. Moreover, if $\beta(p_c) = 1$, (4.6) does not contain T_0 explicitly, so K_j is bounded on $(0, T)$ even if $T = \infty$. Therefore, (4.6) holds under the assumptions on (v_0, w_0) and T_0 in (1)–(3) of Theorem 1.3.

To see the existence, it remains to prove the convergence of $\{(v^j, w^j)\}$ as $j \to \infty$. Actually, we shall first prove that $t^\sigma(v^j, w^j)$ converges in $BC([0, T_0), L^p)$ provided that p and σ satisfy (4.2). Note that (4.7) implies that $t^\sigma(v^j, w^j) \in BC([0, T_0), L^p)$. Moreover, from (4.6), (4.7), and (4.9), it follows that if $\sigma > 0$, $t^\sigma(v^j, w^j)$ tends to zero as $t \to 0$. To show the convergence, we consider the successive difference of (v^j, w^j) constructed by (4.3):

$$v^{j+1} - v^j = B(v^j, w^j) - B(v^{j-1}, w^{j-1}),$$
$$w^{j+1} - w^j = B(w^j, v^j) - B(w^{j-1}, v^{j-1}).$$

Just like deriving (4.4), applying (4.1) with $s = p$, we have

$$t^\sigma \|v^{j+1} - v^j\|_{L^p} + t^\sigma \|w^{j+1} - w^j\|_{L^p} \le 2CBT_0^{1-\beta(r)}K_j^m \cdot \left\{ \sup_{0<\tau\le T_0} \tau^\sigma \|v^j - v^{j-1}\|_{L^p} + \sup_{0<\tau\le T_0} \tau^\sigma \|w^j - w^{j-1}\|_{L^p} \right\},$$

where $p > p_c$ is used. Since $2CBT_0^{1-\beta(r)}K_j^m < 1/2 < 1$, this shows that there is a pair of functions (V, W) such that $\lim_{j\to\infty} t^\sigma(v^j, w^j) = t^\sigma(V, W)$ in $BC([0, T_0), L^p)$, which solves

$$V = v^0 + B(V, W) \quad \text{and} \quad W = w^0 + B(W, V).$$

Also, if $\sigma > 0$, $t^\sigma \|(V, W)\|_{L^p}$ tends to zero as $t \to 0$ since each $t^\sigma(v^j, w^j)$ have the same property.

To complete the proof of (1.8) and (1.9), we have to relax the condition on p. Let $p' \in [r, p]$ and $\sigma' = \frac{n}{\alpha}(\frac{1}{r} - \frac{1}{p'})$. We shall prove that $t^{\sigma'}(v^j, w^j)$ converges in $BC([0, T_0), L^{p'})$ and that $t^{\sigma'}\|(V, W)\|_{L^{p'}}$ tends to zero as $t \to 0$ if $\sigma' > 0$. Applying (4.1) with $s = p'$ yields

$$\begin{aligned} t^{\sigma'}\|v^{j+1} - v^j\|_{L^{p'}} + t^{\sigma'}\|w^{j+1} - w^j\|_{L^{p'}} \le 2C'T_0^{1-\beta(r)}K_j^m \\ \cdot \left\{ \sup_{0<\tau\le T_0} \tau^{\sigma}\|v^j - v^{j-1}\|_{L^p} + \sup_{0<\tau\le T_0} \tau^{\sigma}\|w^j - w^{j-1}\|_{L^p} \right\}, \end{aligned} \tag{4.10}$$

with a different constant C'. Since $t^{\sigma'}(v^j, w^j)$ converges in $BC([0, T_0), L^p)$, this implies

$$t^{\sigma'}(v^j, w^j) \to t^{\sigma'}(V, W) \quad \text{in } BC([0, T_0), L^{p'}).$$

Thus, we have proved (1.8). Since $t^{\sigma}\|(v^j, w^j)\|_{L^p}$ and $t^{\sigma'}\|(v^0, w^0)\|_{L^{p'}}$ both tend to zero as $t \to 0$ by (4.8), (4.10) implies (1.9). The asymptotic behavior (1.11) comes from (4.5) if p and σ satisfy (4.2). For general p in Theorem 1.3, it is not difficult to see that (1.11) also holds by using (4.8) and (4.10). Thus, we have proved (1)–(3) of Theorem 1.3.

It remains to prove the uniqueness (4). Let (V, W) and $(\tilde{V}, \tilde{W})$ be two solutions of (1.1) satisfying assumptions of (4). We may assume that (V, W) and $(\tilde{V}, \tilde{W})$ satisfy (1.8)–(1.9) for same σ and p such that $0 < \sigma < \frac{1}{m+1}$, $p > \frac{n(m+1)}{n+1}$, and $\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$. If $r > p_c'$, σ may equal zero and we assume (1.8) only. In fact, if (V, W) also satisfy (1.8)–(1.9) for every σ' and p' such that $0 \le \sigma' < \sigma$, $\sigma' = \frac{n}{\alpha}(\frac{1}{r} - \frac{1}{p'})$. To see this, we just use the estimate

$$t^{\sigma'}\|V\|_{L^{p'}} \le t^{\sigma'}\|v^0\|_{L^{p'}} + CT_0^{1-\beta(r)}\left(\sup_{0<\tau\le T_0} \tau^{\sigma}\|V\|_{L^p} \right)^m \left(\sup_{0<\tau\le T_0} \tau^{\sigma}\|V - W\|_{L^p} \right),$$

which is proved similar to (4.10).

We first consider the case $r > p_c'$. Let K be a constant such that

$$t^{\sigma}\|(V, W)\|_{L^p}, \quad t^{\sigma}\|(\tilde{V}, \tilde{W})\|_{L^p} \le K, \quad 0 \le t \le T_0,$$

where σ and p satisfy $0 \le \sigma < \frac{1}{m+1}$, $p > \frac{n(m+1)}{n+1}$, and $\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$. Since

$$V - \tilde{V} = B(V, W) - B(\tilde{V}, \tilde{W}), \; W - \tilde{W} = B(W, V) - B(\tilde{W}, \tilde{V}),$$

by definition, we have the estimate for $0 \le \tau \le t_0$,

$$t^{\sigma}\|V - \tilde{V}\|_{L^p} + t^{\sigma}\|W - \tilde{W}\|_{L^p} \le 2Ct_0^{1-\beta(r)}K^m \sup_{0\le\tau\le t_0} \tau^{\sigma}(\|V - \tilde{V}\|_{L^p} + \|W - \tilde{W}\|_{L^p}), \tag{4.11}$$

which follows from (4.1) similar to (4.10). Since $r > p_c'$, we can take t_0 small so that $2Ct_0^{1-\beta(r)}K^m < 1$. This implies that $V = \tilde{V}$ on $[0, t_0)$. Since (V, W), $(\tilde{V}, \tilde{W}) \in BC([t_0, T_0), L^p)$. The aforementioned argument with initial data $V(t_0) = \tilde{V}(t_0)$ and $W(t_0) = \tilde{W}(t_0)$ shows that $(V, W) = (\tilde{V}, \tilde{W})$ on $[t_0, t_0 + t_0)$. By iterating, we have $(V, W) = (\tilde{V}, \tilde{W})$ on $[0, T_0)$.

It remains to discuss the case $r = p_c$. Let $K(t_0)$ be a constant such that

$$t^{\sigma}\|(V, W)\|_{L^p}, \quad t^{\sigma}\|(\tilde{V}, \tilde{W})\|_{L^p} \le K(t_0), \quad 0 \le t \le t_0,$$

where $0 \le \sigma < \frac{1}{m+1}$, $p > \frac{n(m+1)}{n+1}$, and $\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right)$. Here, by (1.9), $K(t_0)$ tends to zero as $t_0 \to 0$. Instead of (4.11), for $0 \le \tau \le t_0$, we have

$$t^{\sigma}\|V - \tilde{V}\|_{L^p} + t^{\sigma}\|W - \tilde{W}\|_{L^p} \le 2CK(t_0)^m \sup_{0\le\tau\le t_0} \tau^{\sigma}(\|V - \tilde{V}\|_{L^p} + \|W - \tilde{W}\|_{L^p}),$$

since $\beta(p_c) = 1$. Take $t_0 > 0$ small so that $2CK(t_0)^m < 1$. As is seen in the preceding paragraph, we have $(V, W) = (\tilde{V}, \tilde{W})$ on $[0, t_0)$. By iterating as earlier, we have $(V, W) = (\tilde{V}, \tilde{W})$ on $[0, T_0)$. This concludes the proof of uniqueness.

Finally, we prove that for $p > n$, the mild solution of the Cauchy problem (1.1) satisfies the estimates (1.8)–(1.9), too. Of course, we first derive *a priori* estimate for

$$K_j = K_j(T_0) = \sup_{0<t\le T_0} t^{\sigma}\|(v^j, w^j)\|_{L^p}, j \ge 0, T_0 > 0,$$

the argument is similar as earlier, here σ and p satisfy that

$$\sigma = \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{p}\right), \quad \sigma m + \frac{n}{\alpha}\left(\frac{1}{r} - \frac{1}{n}\right) < 1, \ m \le p, n \le p.$$

Thus, we obtain the existence of the solution. Then, we relax the index $p \ge m$ as earlier. Combining with the uniqueness of solution, we finish the proof of Theorem 1.3.

5 Global Cauchy problem in subcritical space $L^q(\mathbb{R}^n)$

In this section, we prove Theorem 1.4. We study the global Cauchy problem for small initial data in $L^{p_c}(\mathbb{R}^n)$. First, in Section 5.1, we study the case of initial data that belong only to $L^{p_c}(\mathbb{R}^n)$ and we prove Corollary 1.1. In Section 5.2, we study the global Cauchy problem for initial data in $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ when $L^p(\mathbb{R}^n)$ is subcritical for (1.1) and we prove Proposition 1.1. Finally, in Section 5.3, we consider initial data in $H_p^s(\mathbb{R}^n)$ space, and then, we prove Theorem 1.4.

5.1 Initial data in $L^{p_c}(\mathbb{R}^n)$

Let us consider $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n)$. In Theorem 1.3, we have proved that there exists a non-negative absolute constant A such that if $\|(v_0, w_0)\|_{L^{p_c}} \le A$, then there exists a unique global solution $(V, W) \in BC(\mathbb{R}^+, L^q)$ to Problem (1.1), which satisfies

$$t \mapsto t^{\gamma(q)}\|(V(t,\cdot), W(t,\cdot))\|_{L^q} \in BC(\mathbb{R}^+), \tag{5.1}$$

for all q and $\gamma(q)$ such that $p_c \le q < n$ and $0 \le \gamma(q) < \frac{1}{m+1}$, and

$$\lim_{t\to 0^+} t^{\gamma(q)}\|(V(t,\cdot), W(t,\cdot))\|_{L^q} = 0, \tag{5.2}$$

for all q and $\gamma(q)$ such that

$$p_c < q < n, q > \frac{(m+1)n}{n+1} \quad \text{and} \quad 0 < \gamma(q) < \frac{1}{m+1}. \tag{5.3}$$

First, we are going to prove that for $p_c \le q < n$ and $0 < \gamma(q) < \frac{1}{m+1}$,

$$\|(V(t,\cdot), W(t,\cdot))\|_{L^q} \le Ct^{-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}), \tag{5.4}$$

which is a little more precise than the estimate $\|(V, W)\|_{L^q} \le Ct^{-\gamma(q)}$.

Second, we are going to relax the restriction $\gamma(q) < \frac{1}{m+1}$ in this estimate. Indeed, when $\alpha < m(m+1)$, it is easy to check that $\gamma(q) < \frac{1}{m+1}$ is fulfilled for all $q \in [p_c, n)$ and so, the asymptotic estimate (5.4) too. On the contrary, when $\alpha \ge m(m+1)$, one must assume that $q \in \left[p_c, \left(\frac{1}{p_c} - \frac{\alpha}{n(m+1)}\right)^{-1}\right)$ to be sure that $\gamma(q) < \frac{1}{m+1}$ holds. So, when $\alpha < m(m+1)$, the asymptotic estimates are proved only for q in the range $\left[p_c, \left(\frac{1}{p_c} - \frac{\alpha}{n(m+1)}\right)^{-1}\right)$ and we want to show that they hold for all exponents $q \in [p_c, n)$.

To prove Corollary 1.1, let us come back to the proof of Theorem 1.3. In the critical case (when $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n)$), to prove the existence of solutions for (1.1), one introduce, for $p_c < q < n$, $q > \frac{(m+1)n}{n+1}$ and $0 < \gamma(q) < \frac{1}{m+1}$, the Banach spaces

$$X_q = \{f(t,x) : t \mapsto t^{\gamma(q)}\|f(t,x)\|_{L^q} \in BC(\mathbb{R}^+)\},$$
$$Y = \{f(t,x) : t \mapsto \|f(t,x)\|_{L^{p_c}} \in BC(\mathbb{R}^+)\}.$$

Then, for the sequence $\{(v^j, w^j)\}$ defined in (3.1), we have the estimates

$$\begin{cases} \|v^{j+1}\|_{X_q} \le C_1\|v^0\|_{X_q} + C_2\|v^j\|_{X_q}^m\|v^j - w^j\|_{X_q}, \\ \|w^{j+1}\|_{X_q} \le C_1\|w^0\|_{X_q} + C_2\|w^j\|_{X_q}^m\|v^j - w^j\|_{X_q}, \end{cases} \tag{5.5}$$

where $\|f(t,x)\|_{X_q} = \sup_{t>0}\|f(t,x)\|_{L^q}$. Then, when $\|(v_0, w_0)\|_{L^{p_c}} \le A$, using (5.5) and (5.2)–(5.3), one can prove that the $\{(v^j, w^j)\}$ converge in X_q to $(V(t,x), W(t,x))$ the unique solution of (1.1) such that (5.2) and (5.3) are fulfilled. Furthermore, to prove that $(V(t,x), W(t,x))$ also belong to $BC(\mathbb{R}^+, L^{p_c})$, one can easily check that

$$\|B(V,W)\|_Y \le C\|V\|_{X_q}^m\|V - W\|_{X_q}$$

as soon as $p_c < q < n$, $q > \frac{(m+1)n}{n+1}$, and $0 < \gamma(q) < \frac{1}{m+1}$.

Now, let us come back to the proof of Corollary 1.1. By (5.5), it is obvious that the sequence $\{(v^j, w^j)\}$ stay in the ball $B(0, 2R)$ for the X_q topology as soon as $C_2(2C_1R)^{m+1} \le C_1R$ with $R = \max\{\|v^0\|_{X_q}, \|w^0\|_{X_q}\}$, which holds for

$$\max\{\|v^0\|_{X_q}, \|w^0\|_{X_q}\} \le \frac{1}{2C_1}\left(\frac{1}{2C_2}\right)^{1/m}.$$

Now, by Lemma 2.2, we have

$$\|v^0\|_{X_q} \le C\|v_0\|_{L^{p_c}}, \ \ \|w^0\|_{X_q} \le C\|w_0\|_{L^{p_c}},$$

and for

$$r \coloneqq \max\{\|v_0\|_{L^{p_c}}, \|w_0\|_{L^{p_c}}\} \le \frac{1}{2CC_1}\left(\frac{1}{2C_2}\right)^{1/m},$$

there exists a global solution $(v(t,x), w(t,x))$ of (1.1), which belongs to the ball

$$B(0, 2C_1R) \subset B(0, 2CC_1r),$$

for the X_q topology. Thus, the proof of Corollary 1.1 is completed for the exponent q such that $p_c < q < n$, $q > \frac{(m+1)n}{n+1}$, and $0 < \gamma(q) < \frac{1}{m+1}$. If we can deal with the special case of $L^{p_c}(\mathbb{R}^n)$ norm, we finish the proof completely.

Now, we are going to prove that the asymptotic estimates

$$\|(V(t,\cdot), W(t,\cdot))\|_{L^q} \le Ct^{-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})$$

hold also when $\gamma(q) \ge \frac{1}{m+1}$. First, for (v_0, w_0) such that $\|(v_0, w_0)\|_{L^{p_c}} \le A$, let us consider $(v(t,x), w(t,x))$ the solution of (1.1). Let us consider q_0 an exponent such that $q_0 > p_c$ and $\gamma(q_0) \ge \frac{1}{m+1}$ (such a q_0 always exists since $p_c > 1$: see the remark after Theorem 1.3). Next, let us consider the sequence q_i defined by

$$\frac{n}{\alpha}\left(\frac{1}{q_i} - \frac{1}{q_{i+1}}\right) = \delta_i < \frac{1}{m+1}, \tag{5.6}$$

and note that $\{q_i\}$ is increasing and there exists q_k such that $\frac{n}{\alpha q_k} < \frac{1}{m+1}$.

Let us define

$$I(q_i, q_{i+1}) = \int_0^1 (1-\tau)^{-\frac{mn}{\alpha q_{i+1}}}\tau^{-(m+1)\delta_i}\mathrm{d}\tau.$$

Then, by (5.6), for all $i \ge 0$, $I(q_i, q_{i+1}) < +\infty$. Now, we pick $t_0 > 0$ and we consider (V', W') the solution of

$$\begin{cases} V'(t,x) = e^{t\Lambda^\alpha}V_0' + B(V', W'), \\ W'(t,x) = e^{t\Lambda^\alpha}W_0' + B(W', V'), \\ V'(0,x) = V_0'(x) = V(t_0, x), \\ W'(0,x) = W_0'(x) = W(t_0, x). \end{cases} \tag{5.7}$$

Similar to the previous steps and $\gamma(q_0) \ge \frac{1}{m+1}$, we have $(V_0', W_0') \in L^{p_c}(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n)$ with

$$\|(V_0', W_0')\|_{L^{q_0}} \le Ct^{-\gamma(q_0)}(\|V_0\|_{L^{p_c}} + \|W_0\|_{L^{p_c}}). \tag{5.8}$$

Lemma 5.1. *Let $T = T(t_0)$ such that*

$$(2C)^{m+1}T^{1-p_c/q_0}I(q_0, q_1)(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}}) < 1, \tag{5.9}$$

then for all $t \in [0, T)$, we have

$$\|(V'(t,\cdot), W'(t,\cdot))\|_{L^{q_1}} \le Ct^{\delta_0}(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}}). \tag{5.10}$$

Proof. Indeed, since $(V_0'(x), W_0'(x)) \in L^{q_0}(\mathbb{R}^n)$ with $q_0 > p_c$, following the proof of Theorem 1.3, we see that the sequence

$$v^0 = e^{-t\Lambda^\alpha}V_0',\ v^{j+1} = v^0 + B(v^j, w^j),$$
$$w^0 = e^{-t\Lambda^\alpha}W_0',\ w^{j+1} = w^0 + B(w^j, v^j).$$

converges strongly to $(\dot{V}, \dot{W})$ in $C([0,T), L^{q_0}(\mathbb{R}^n))$. By Lemma 2.2, (v^0, w^0) obviously satisfies (5.10) for all $t > 0$. Now, if v^j satisfies (5.10), then

$$\begin{aligned} \|v^{j+1}(t)\|_{L^{q_1}} &\le C\|V_0'\|_{L^{q_0}} + C\int_0^t (t-\tau)^{\frac{mn}{\alpha q_1}}\|v^j\|_{L^{q_1}}^m \|v^j - w^j\|_{L^{q_1}} \mathrm{d}\tau \\ &\le C\|V_0'\|_{L^{q_0}} + C2^{m+1}C^{m+1}(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}})^{m+1}\int_0^t (t-\tau)^{-\frac{mn}{\alpha q_1}}\tau^{-(m+1)\delta_0}\mathrm{d}\tau \end{aligned}$$

for all $t \in (0, T)$, and so

$$\begin{aligned} \|v^{j+1}(t)\|_{L^{q_1}} &\le 2Ct^{-\delta_0}(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}}) \\ &\quad \cdot \left\{\frac{1}{2} + 2^m C^{m+1}(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}})^m I(q_0, q_1)T^{1-\frac{p_c}{q_0}}\right\}, \quad \forall t \in (0, T). \end{aligned}$$

Hence, if T satisfies (5.9),

$$\|(v^{j+1}(t,\cdot), w^{j+1}(t,\cdot))\|_{L^{q_1}} \le 2Ct^{\delta_0}(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}}).$$

So, by introduction, (5.10) holds for all $j \in \mathbb{N}$, and thus, Lemma 5.1 is proved. □

Using the uniqueness result in the supercritical case and (5.7), we see that

$$V'(t,x) = V(t+t_0, x),\ \ W'(t,x) = W(t+t_0, x),$$

due to Lemma 5.1, for each $t \in [0, T(t_0))$ and $T(t_0)$ satisfies (5.9), we have

$$\|(V(t+t_0,\cdot), W(t+t_0,\cdot))\|_{L^{q_1}} \le 2Ct^{-\delta}(\|V(t_0)\|_{L^{q_0}} + \|W(t_0)\|_{L^{q_0}}).$$

Now, we claim that there exists an absolute constant A' such that, when $\|(v_0, w_0)\|_{L^{p_c}} \le A'$, one can always take $T(t_0) = \frac{1}{2}$ in the previous inequality. Indeed, by Lemma 5.1, we have only to make sure that

$$(2C)^{m+1}\left(\frac{t_0}{2}\right)^{1-p_c/q_0} I(q_0, q_1)(\|V_0'\|_{L^{q_0}} + \|W_0'\|_{L^{q_0}})^m < 1,$$

combining with (5.8) and following by

$$(2C)^{m+1}\left(\frac{t_0}{2}\right)^{1-p_c/q_0-m\gamma(q_0)} I(q_0, q_1)(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m < 1,$$

and $1 - \frac{p_c}{q_0} - m\gamma(q_0) = 0$, it is sufficient to make sure that

$$(2C)^{m+1} I(q_0, q_1)(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m < 1.$$

Thus, when (v_0, w_0) is small enough in $L^{p_c}(\mathbb{R}^n)$, (5.9) holds for each $t_0 > 0$, and so

$$\left\|\left(V\left(\frac{3t_0}{2}\right), W\left(\frac{3t_0}{2}\right)\right)\right\|_{L^{q_1}} \leq 4Ct_0^{\delta_0} t_0^{-\gamma(q_0)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}),$$

and since t_0 is arbitrary, then for all $t > 0$, we have

$$\|(V(t), W(t))\|_{L^{q_1}} \leq 4Ct^{-\frac{n}{\alpha}\left(\frac{1}{p_c}-\frac{1}{q_1}\right)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}).$$

Now, since $I(q_i, q_{i+1}) < \infty$ and the required estimate for q_1 defined by (5.6) is proved, we have just to iterate this proof to obtain the required estimate in $L^{q_2}(\mathbb{R}^n)$-norm. Thus, for each q_i, the proof follows by induction. Now, if $q \in (q_i, q_{i+1})$, we obtain the result by interpolation. Thus, we have proved that the global solution $(V(t,x), W(t,x))$ of (1.1) satisfies that

$$\|(V(t), W(t))\|_{L^{q_1}} \leq Ct^{-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}), \quad \forall q \in [p_c, n).$$

For the case $p \in [n, +\infty)$, the proof is similar as earlier, we omit the detail.

5.2 Initial data in $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$

Let $p < p_c$. For the initial data $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $\|(v_0, w_0)\|_{L^{p_c}} \leq A$, $(V(t,x), W(t,x))$ is the mild solution of (1.1), which belongs to $BC(\mathbb{R}^+, L^{p_c})$ and satisfies Estimates (5.1)–(5.3). Using the slight improvement about the decay of the $L^q(\mathbb{R}^n)$ norms (Estimates (1.12) of Corollary 1.1) that we previously proved, we are first going to show that the solution belongs to $L^p(\mathbb{R}^n)$ for all t (Step 1), then we will prove that (V, W) belongs to $BC(\mathbb{R}^n, L^p(\mathbb{R}^n))$ (Step 2) and next, (V, W) satisfies the asymptotic estimates (1.12).

Step 1. For $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and we want to prove that, for any $T > 0$,

$$\|(V(t), W(t))\|_{L^p} \leq C(T), \quad \forall t \in [0, T]. \tag{5.11}$$

First, assume that

$$\max\left\{1, \frac{p_c}{m}\right\} \leq p < p_c. \tag{5.12}$$

Since $(V(t), W(t))$ is a solution for (1.1), for all $T > 0$ and $t \in [0, T]$,

$$\begin{aligned}\|V(t)\|_{L^p} &\leq C\|v_0\|_{L^p} + \|B(V, W)\|_{L^p}\\ &\leq C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{n}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}\|V^m\nabla(-\Delta)^{-1}(V-W)\|_{L^r}\,\mathrm{d}\tau\\ &\leq C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\frac{n}{\alpha q}}\|V\|_{L^{pm}}^m\|V-W\|_{L^q}\,\mathrm{d}\tau,\end{aligned}$$

where $\frac{1}{r} = \frac{m}{pm} + \frac{1}{q} - \frac{1}{n}$ and $p_c \leq q < n$. If we choose q such that $q \approx n$, by (5.12), $pm > p_c$, using Estimate (1.12) of Corollary 1.1, we obtain that

$$\|V(t)\|_{L^p} \le C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\frac{n}{\alpha q}}\tau^{-m\gamma(pm)-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^{m+1}\mathrm{d}\tau,$$
$$\le C\|v_0\|_{L^p} + C(T)(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^{m+1},$$

where $p_c \le q < n$, for all $\frac{p_c}{m} \le p < p_c$, we have $0 < m\gamma(pm) + \gamma(q) < 1 - \frac{n}{\alpha q} < 1$.

Similarly,

$$\|W(t)\|_{L^p} \le C\|v_0\|_{L^p} + C(T)(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^{m+1}.$$

Thus, we obtain Estimate (1.14) for p satisfying (5.12), and if $p_c < m$, the proof is complete.

Assume now that

$$\max\left\{1, \frac{p_c}{m^2}\right\} \le p < \frac{p_c}{m}. \tag{5.13}$$

First, if $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then $(v_0, w_0) \in L^q(\mathbb{R}^n)$ for all $[p_c/m, p_c)$, and then, by the previous result, $\|(v(t), w(t))\|_{L^q} \le C(T, v_0, w_0)$ for all q in the range $[p_c/m, p_c)$.

Second, since $(v(t,x), w(t,x))$ is a solution of (1.1),

$$\begin{aligned}\|V(t)\|_{L^p} &\le C\|v_0\|_{L^p} + \|B(V,W)\|_{L^p}\\ &\le C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{n}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}\|V^m\nabla(-\Delta)^{-1}(V-W)\|_{L^r}\mathrm{d}\tau\\ &\le C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\frac{n}{\alpha q}}\|V\|_{L^{pm}}^m\|V-W\|_{L^q}\mathrm{d}\tau,\end{aligned}$$

where $\frac{1}{r} = \frac{m}{pm} + \frac{1}{q} - \frac{1}{n}$ and $p_c \le q < n$. Choosing q such that $q \approx n$, by (5.13), $pm \in [\frac{p_c}{m}, p_c)$. Hence, by the previous result, we can use the bound $\|(V(t,x), W(t,x))\|_{L^{pm}} \le C(T, v_0, w_0)$, which leads to

$$\|V(t), W(t)\|_{L^p} \le C\|v_0, w_0\|_{L^p} + C(T, v_0, w_0).$$

Thus, Estimate (5.11) holds for all p in the range $[\max\{1, p_c m^{-2}\}, p_c)$.

Finally, for $p \in [p_c m^{-n-1}, p_c m^{-n})$, the proof of (5.11) follows easily by induction.

Step 2. In Step 1, we have proved that $(V(t,x), W(t,x))$ is the solution of (1.1) that belongs to $L^p(\mathbb{R}^n)$ for all $t \ge 0$ when (v_0, w_0) belongs to $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ and when (v_0, w_0) is small enough in $L^{p_c}(\mathbb{R}^n)$. Now, we are going to prove that $(V(t,x), W(t,x))$ belongs to $BC(\mathbb{R}^+, L^p(\mathbb{R}^n))$. Let us consider $T > 0$ and t in $[0, T]$. First, since (V, W) is a mild solution of (1.1), by Lemma 2.2,

$$\begin{aligned}\|V(t)\|_{L^p} &\le C\|v_0\|_{L^p} + \|B(V,W)\|_{L^p}\\ &\le C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\|V^m\nabla(-\Delta)^{-1}(V-W)\|_{L^q}\mathrm{d}\tau,\end{aligned}$$

where q is any exponent in $[1, p)$ which will be fixed later, and where $\xi(q)$ is defined by

$$\xi(q) = \frac{1}{\alpha} + \frac{n}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right). \tag{5.14}$$

Using Hölder's inequality, we obtain

$$\|V(t)\|_{L^p} \le C\|v_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\|V\|_{L^{qq_1}}\|V\|_{L^{qq_2(m-1)}}^{m-1}\|V-W\|_{L^{q_3}}\mathrm{d}\tau,$$

where $\frac{1}{q} = \frac{1}{qq_1} + \frac{1}{qq_2(m+1)} + \frac{1}{q_3} - \frac{1}{n}$ and $p_c \le q_3 < n$. Furthermore, we choose q_1 such that $qq_1 = p$ to obtain

$$\|V(t)\|_{L^p} \le C\|v_0\|_{L^p} + C\|V\|_{L^\infty([0,\infty),L^p)}\int_0^t (t-\tau)^{-\xi(q)}\|V\|^{m-1}_{L^{qq_2(m-1)}}\|V-W\|_{L^{q_3}}\mathrm{d}\tau.$$

Now, if we choose q such that $q \approx p$ with $q < p$ and choose q_3 such that $q_3 = n$ with $q_3 < n$ then, since $qq_1 = p$, i.e., $q_1 \approx 1$ and q_3 is large enough. Hence, it follows that $qq_2(m+1) \ge p_c$. Next, for $qq_2(m+1) \ge p_c$, by Corollary 1.1 and the $L^{qq_2(m-1)}(\mathbb{R}^n)$-norm, we have

$$\|V(t)\|_{L^{qq_2(m-1)}} \le Ct^{-\gamma(qq_2(m-1))}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}),$$

and by $L^{q_3}(\mathbb{R}^n)$-norm, we obtain

$$\|V-W\|_{L^{q_3}} \le Ct^{-\gamma(q_3)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}}).$$

Thus,

$$\|V(t)\|_{L^p} \le C\|v_0\|_{L^p} + C\|V\|_{L^\infty([0,T],L^p)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m\int_0^t (t-\tau)^{-\xi(q)}\tau^{-\theta(q)}\mathrm{d}\tau,$$

where

$$\theta(q) = (m-1)\gamma(qq_2(m-1)) + \gamma(q_3) = 1 - \frac{n}{\alpha}\left(\frac{1}{qq_2} + \frac{1}{q_3}\right) \approx 1 - \frac{1}{\alpha}, \tag{5.15}$$

since $q_3 \approx n$ and q_2 is large enough, when we choose $q \approx p$. One can easily check that $0 < \xi(q) < 1$ and $0 < \theta(q) < 1$ (since $\alpha > m \ge 1$). Furthermore, $\xi(q) + \theta(q) = 1$, and so,

$$\|V(t)\|_{L^p} \le C\|v_0\|_{L^p} + C\|V\|_{L^\infty([0,T],L^p)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m. \tag{5.16}$$

Now, if $\|(v_0, w_0)\|_{L^{p_c}}$ is small enough, then $C(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m \le \frac{1}{2}$, then, by (5.16) and $\|V\|_{L^\infty([0,T],L^p)} < \infty$ for all $T > 0$, we have

$$\|V\|_{L^\infty([0,T],L^p)} \le \frac{\|v_0\|_{L^p}}{1 - C(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m} \le 2\|v_0\|_{L^p}.$$

To conclude, we have just to remark that the right-hand side of this estimate does not depend of T. Thus, we have proved that (V, W), the mild solution of (1.1) that belongs to $BC(\mathbb{R}^+, L^p(\mathbb{R}^n))$.

Step 3. Now, we have to prove the $L^r(\mathbb{R}^n)$ Estimate (1.14) of Proposition 1.1. They hold obviously for the term $(e^{-t\Lambda^\alpha}v_0, e^{-t\Lambda^\alpha}w_0)$ by Lemma 2.2; hence, we just deal with the nonlinear term $B(V, W)$ and $B(W, V)$. First, let us suppose that

$$\delta(r) = \frac{n}{\alpha}\left(\frac{1}{p} - \frac{1}{r}\right) < 1 - \frac{1}{\alpha}. \tag{5.17}$$

Then,

$$\begin{aligned}\|B(V,W)\|_{L^r} &\le C\int_0^t (t-\tau)^{-\delta(r)}\left\| e^{-\frac{1}{2}(t-\tau)\Lambda^\alpha}\nabla\cdot(V^m\nabla(-\Delta)^{-1}(V-W))\right\|_{L^p}\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\|V\|_{L^{qq_1}}\|V\|^{m-1}_{L^{qq_2(m-1)}}\|V-W\|_{L^{q_3}}\mathrm{d}\tau,\end{aligned}$$

where $q \in [1,p)$, $p_c \le q_3 < n$, $\frac{1}{q} = \frac{1}{qq_1} + \frac{1}{qq_2(m+1)} + \frac{1}{q_3} - \frac{1}{n}$, and $\xi(q)$ is given by (5.14). Now, taking $qq_1 = p$ with $q \approx p$, then $q_1 \approx 1$, taking $q_3 = n$ with $p_c \le q_3 < n$ and $qq_2(m+1) \ge p_c$, and so using Corollary 1.1, we obtain

$$\|B(V,W)\|_{L^r} \le C(\sup_{t\in\mathbb{R}^+}\|V\|_{L^p})(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m\int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\tau^{-\theta(q)}\mathrm{d}\tau,$$

where $\theta(q)$ is given by (5.15). If (5.17) holds, then one can choose q, q_1, and q_2 such that $0 < \theta(q) < 1$, $0 < \delta(r) + \xi(q) < 1$, and $\xi(q) + \theta(q) = 1$, and so

$$\|B(V,W)\|_{L^r} \le Ct^{-\delta(r)}\left(\sup_{t\in\mathbb{R}^+}\|V\|_{L^p}\right)(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})^m.$$

Then, since $\|v\|_{L^p} \le C\|v_0\|_{L^p}$ (by Step 2), we have $\|B(V,W)\|_{L^r} \le Ct^{-\delta(r)}\|v_0\|_{L^p}$.

Now, if (5.17) is not fulfilled, we build a sequence $\{r_i\}$ defined by

$$r_0 = p,\ \frac{n}{\alpha}\left(\frac{1}{r_i} - \frac{1}{r_{i+1}}\right) = \delta_i < \max\left\{1 - \frac{1}{\alpha}, m+1\right\}.$$

Now, if $p < r_1 < r_2 < p_c$, since (V,W) is bounded in $L^p \cap L^{p_c}$, then (V,W) is also bounded in L^r for all r in $[p, p_c]$ and for each $t \ge 0$.

Let $t_0 > 0$ and (V', W') be the solution of (5.7). We have already proved that

$$(V_0', W_0') \in L^{p_c}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n),$$

with $\|V_0'\|_{L^{r_1}} \le Ct_0^{-\delta(r_1)}\|v_0\|_{L^p}$,$\|W_0'\|_{L^{r_1}} \le Ct_0^{-\delta(r_1)}\|w_0\|_{L^p}$, and furthermore, $(V'(t,\cdot), W'(t,\cdot))$ is bounded in $L^{p_c}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n)$. So we just have to iterate the previous proof to estimate $V'(t_0,x) = V(2t_0,x)$ and $W'(t_0,x) = W(2t_0,x)$ in $L^{r_2}(\mathbb{R}^n)$ norm with respect to $V_0'(x) = V(t_0,x)$ and $W_0'(x) = W(t_0,x)$ in $L^{r_1}(\mathbb{R}^n)$ norm to obtain the required estimate and we can do this until $r_i \le p_c$.

Now, let us denote by N the first index such that $r_N = p_c$. We have proved that

$$\|V'(t)\|_{L^r_N} \le Ct^{\delta(r_N)}\|v_0\|_{L^p} \quad\text{and}\quad \|W'(t)\|_{L^r_N} \le Ct^{\delta(r_N)}\|w_0\|_{L^p}, \tag{5.18}$$

where (V', W') is the mild solution of (5.7) with the initial data replaced by

$$V_0'(x) = V((N-1)t_0, x) \quad\text{and}\quad W_0'(x) = W((N-1)t_0, x).$$

Then, we can choose r_{N+1} such that $p_c < r_{N+1} < n$. Followed by Corollary 1.1, we have

$$\begin{aligned}\|V'(t)\|_{L^{r_{N+1}}} &\le Ct^{\gamma(r_{N+1})}(\|V_0'\|_{L^{p_c}} + \|W_0'\|_{L^{p_c}}),\\ \|W'(t)\|_{L^{r_{N+1}}} &\le Ct^{\gamma(r_{N+1})}(\|V_0'\|_{L^{p_c}} + \|W_0'\|_{L^{p_c}}),\end{aligned} \tag{5.19}$$

where $\gamma(r_{N+1})$ is defined in Corollary 1.1 and $(V'(t), W'(t))$ is the mild solution of (5.7) with the initial data replaced by $V_0'(x) = V(Nt_0, x)$ and $W_0'(x) = W(Nt_0, x)$. Then, combining (5.18)–(5.19), we have

$$\begin{aligned}\|V(t_0(N+1))\|_{L^{r_{N+1}}} &\le Ct_0^{\delta(r_{N+1})}(\|v_0\|_{L^p} + \|w_0\|_{L^p}),\\ \|W(t_0(N+1))\|_{L^{r_{N+1}}} &\le Ct_0^{\delta(r_{N+1})}(\|v_0\|_{L^p} + \|w_0\|_{L^p}).\end{aligned}$$

This ends the proof of Proposition 1.1 since $t_0 > 0$ is arbitrary.

5.3 Initial data in $H_p^s(\mathbb{R}^n)$

Let us consider an initial data (v_0, w_0) such that $\|(v_0,w_0)\|_{H_p^{sc}} \le A'$. Then, by the Sobolev embedding theorem, $(v_0, w_0) \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and if A' is small enough, then $\|(v_0,w_0)\|_{L^{p_c}} \le A$. So, according to Proposition 1.1, there exists a unique global solution $(V(t,x), W(t,x))$ of (1.1), and this solution satisfies (1.13) and (1.14). Hence, to prove that (V,W) belongs to $BC(\mathbb{R}^+, H_p^{s_c})$, we have only to check that (V,W) remains bounded in the homogeneous space $\dot{H}_p^{s_c}(\mathbb{R}^n)$, thanks to the following well-known inequality:

$$\|f\|_{H_p^s} \le C(\|f\|_{L^p} + \|f\|_{\dot{H}_p^s}), \quad \forall s \ge 0.$$

Now, since (V,W) is a solution of (1.1),

$$
\begin{aligned}
\|V(t)\|_{\dot{H}_p^{s_c}} &\le C\|v_0\|_{H_p^{s_c}} + \|B(V,W)\|_{\dot{H}_p^{s_c}} \\
&\le C\|v_0\|_{H_p^{s_c}} + C\int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{s_c}{\alpha}} \left\| e^{-\frac{1}{2}(t-\tau)\Lambda^\alpha} V^m \nabla(-\Delta)^{-1}(V-W)\right\|_{L^p} \mathrm{d}\tau \\
&\le C\|v_0\|_{H_p^{s_c}} + C\int_0^t (t-\tau)^{-\frac{1+s_c}{\alpha}-\frac{n}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|V^m \nabla(-\Delta)^{-1}(V-W)\|_{L^r} \mathrm{d}\tau \\
&\le C\|v_0\|_{H_p^{s_c}} + C\int_0^t (t-\tau)^{\lambda(q)} \|V\|_{L^q}^m \|V-W\|_{L^{q_1}} \mathrm{d}\tau,
\end{aligned}
$$

where

$$\lambda(q) = \frac{1+s_c}{\alpha} + \frac{n}{\alpha}\left(\frac{m}{q} + \frac{1}{q_1} - \frac{1}{n} - \frac{1}{p}\right),$$

with $q \in (p_c, pm]$ and $q_1 \in [p_c, n)$.

Since $p > p_c m^{-1}$, one can easily check that $s_c < \alpha\left(1 - \frac{1}{m}\right) < \alpha - 1$ ($m < \alpha$). Taking $q_1 \approx n$ with $q_1 < n$ and $q \approx pm$, one can always choose q such that $0 < \lambda(q) < 1$. Then, for this choice of q, we obtain

$$
\begin{aligned}
\|V(t)\|_{\dot{H}_p^{s_c}} &\le C\|v_0\|_{H_p^{s_c}} + C\left(\sup_{t\in\mathbb{R}^+} t^{\gamma(q)}\|V\|_{L^q}\right)^m \\
&\quad \cdot \left(\sup_{t\in\mathbb{R}^+} t^{\gamma(q_1)}\|V-W\|_{L^{q_1}}\right) \int_0^t (t-\tau)^{\lambda(q)}\tau^{-m\gamma(q)-\gamma(q_1)}\mathrm{d}\tau \\
&\le C\|v_0\|_{H_p^{s_c}} + C\left(\sup_{t\in\mathbb{R}^+} t^{\gamma(q)}\|V\|_{L^q}\right)^m \left(\sup_{t\in\mathbb{R}^+} t^{\gamma(q_1)}\|V-W\|_{L^{q_1}}\right),
\end{aligned}
$$

since $\gamma(q) + m\gamma(q) + \gamma(q_1) = 1$ and $m\gamma(q) + \gamma(q_1) \in (0,1)$. But by Corollary 1.1, we know that $t^{\gamma(q)}\|V\|_{L^q}$ remains bounded for all $t > 0$ and so (V, W) belongs to $BC(\mathbb{R}^n, \dot{H}_p^{s_c})$. Thus, we have proved that (V, W) belongs to $BC(\mathbb{R}^+, H_p^{s_c})$. This ends the proof of Part (1).

Now, let $(v_0, w_0) \in H_p^s(\mathbb{R})$ such that $\|(v_0, w_0)\|_{H_p^{s_c}} \le A'$. Then, according to Part (1) of Theorem 1.2 and to Part (1) of Theorem 1.4, there exists a unique solution of (1.1) in $C([0, T_m), H_p^s) \cap BC(\mathbb{R}^+, H_p^{s_c})$ and so, to prove Part (2), we must show that blow-up in $H_p^s(\mathbb{R}^n)$ norm cannot occur when $t \ge T_m$. But, like the proof of Part (1), one can easily show that for $T > 0$, $t \in [T_m, T_m + T]$,

$$\|V(t) - e^{-t\Lambda^\alpha}v_0\|_{H_p^{s_c+\theta}} \le C\int_0^t (t-\tau)^{-\eta(\theta)}\|V\|_{p_c}^m \|V-W\|_{L^q}\mathrm{d}\tau, \tag{5.20}$$

where $\eta(\theta) = \frac{1+s_c+\theta}{\alpha} + \frac{n}{\alpha}(\frac{m}{p_c})$, $0 < \theta < \frac{\alpha}{m} - 1$, and $q \approx n$ with $q < n$. By $H_p^{s_c}(\mathbb{R}^n) \hookrightarrow L^{p_c}(\mathbb{R}^n)$ and (5.4), we obtain that the right-hand side of (5.20) is bounded by

$$
\begin{aligned}
&C\int_0^t (t-\tau)^{-\eta(\theta)}\|V\|_{H_p^{s_c}}^m \tau^{-\gamma(q)}(\|v_0\|_{L^{p_c}} + \|w_0\|_{L^{p_c}})\mathrm{d}\tau \\
&\le C\left(\sup_{t\in\mathbb{R}^n}\|V\|_{H_p^{s_c}}^m\right)^m \int_0^t (t-\tau)^{-\eta(\theta)}\tau^{-\gamma(q)}\mathrm{d}\tau \\
&\le Ct^{-\frac{\theta}{\alpha}}\left(\sup_{t\in\mathbb{R}^n}\|V\|_{H_p^{s_c}}^m\right)^m,
\end{aligned}
$$

since $0 < \eta(\theta) < 1$, $0 < \gamma(q) < 1$, and $\eta(\theta) + \gamma(q) = 1 + \frac{\theta}{\alpha}$, where $\gamma(q)$ is given by Corollary 1.1. Hence, if blow-up holds in $H_p^{s_c+\theta}(\mathbb{R}^n)$, it holds in $H_p^{s_c}(\mathbb{R}^n)$ norm, this contradicts Part (1). Now since $s > s_c$ is arbitrary, we have just to iterate like the proof of Part (2) of Theorem 1.2 with $\theta < m(s - s_c)$ replaced by $\theta = m(s - s_c)$.

6 Proof of Theorem 1.1

In this section, we prove Theorem 1.1, the main part is the following nonlinear estimate:

$$\|F(v,w)\|_{H_p^{s_m}} \leq C\|v\|_{H_p^s}^m \|v-w\|_{H_p^s}$$

that we used in a crucial way in the proof of Theorem 1.2 (our result about local existence and uniqueness for the Cauchy problem (1.1)). First, we are going to consider the case **(H1)** (i.e., when $s_m \leq 0$), Then, after recalling a few results about Littlewood-Paley analysis, we will prove Theorem 1.1 when the assumption **(H2)** is fulfilled ($0 < s_m < m$).

6.1 Case $s_m \leq 0$

Here, we suppose that $\max\left\{0, \frac{n}{p} - \frac{n+1}{m+1}\right\} < s$ and that **(H1)** is fulfilled, i.e. $s_m < 0$. Now, consider $(v,w) \in H_p^s(\mathbb{R}^n)$. Then, by the Sobolev embedding theorem ($s_m \leq 0$ and $p \in (1,+\infty)$), we have

$$L^q(\mathbb{R}^n) \hookrightarrow H_p^{s_m}(\mathbb{R}^n), \quad q = \frac{np}{n - s_m p}. \tag{6.1}$$

Together with Hölder's inequality and Lemma 2.1, for all $0 < s < \frac{n}{p} - 1$ and $r = \frac{np}{n-sp}$,

$$\begin{aligned} \|F(v,w)\|_{H_p^{s_m}} &= \|v^m \nabla(-\Delta^{-1}(v-w))\|_{H_p^{s_m}} \\ &\leq C\|v^m \nabla(-\Delta^{-1}(v-w))\|_{L^q} \\ &\leq C\|v\|_{L^r}^m \|\nabla(-\Delta^{-1}(v-w))\|_{L^{\frac{nr}{n-r}}} \\ &\leq C\|v\|_{L^r}^m \|v-w\|_{L^r}. \end{aligned} \tag{6.2}$$

Now, by the Sobolev embedding theorem ($s > 0$ and $p \in (1,+\infty)$), we have

$$H_p^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n),$$

Thus, with the estimate (6.2), we obtain

$$\|F(v,w)\|_{H_p^{s_m}} \leq C\|v\|_{H_p^s}^m \|v-w\|_{H_p^s}.$$

This implies our claim.

6.2 Littlewood-Paley analysis

Let us first recall the Littlewood-Paley dyadic decomposition for a tempered distribution. Let φ_0 be a non-negative radial test function such that $\widehat{\varphi_0}(\xi) = 1$ for $|\xi| \leq 3/4$ and such that $\widehat{\varphi_0}(\xi) = 0$ for $|\xi| \geq 1$. Let $\varphi_j(x) = 2^{nj}\varphi_0(2^j x)$, i.e., $\widehat{\varphi_j}(x) = \widehat{\varphi_0}(2^{-j}x)$, and let us consider the partial sum operators S_j associated with the φ_j and defined by

$$S_j(f)(x) = \varphi_j \star f(x).$$

Now, define $\psi_0(x) = \varphi_0$ and $\psi_j(x) = \varphi_{j+1}(x) - \varphi_j(x)$ and, in the same way as previously, consider the operators Δ_j defined by

$$\Delta_j(f)(x) = S_{j+1}(f)(x) - S_j(f)(x) = \psi_j \star f(x).$$

Thus,

$$f = \lim_{j\to\infty} S_j(f) = \Delta_0 + \sum_{j=0}^{\infty} \Delta_j(f). \tag{6.3}$$

More precisely, one can prove the following result [39].

Proposition 6.1. *The convergence in* (6.3) *occurs in* $H_p^s(\mathbb{R}^n)$ *for all* p *in* $(1, +\infty)$ *and for all* s *in* $\mathbb{R}$. *Furthermore,*

$$\|f\|_{H_p^s} \sim \|\Delta_0(f)\|_{L^p} + \left\| \left(\sum_{j=0}^{\infty} 4^{js} \, |\Delta_j(f)|^2 \right)^{1/2} \right\|_{L^p}, \quad \forall f \in H_p^s(\mathbb{R}^n).$$

Now, we give some classical lemmas which will be of great use in the sequel.

Lemma 6.1. [40] (Bernstein's inequalities) *Let* $p \in [1, \infty]$.

(1) *If* f *has its spectrum in the ball* $B(0, r)$, *then there exists a constant* C *independent of* f *and* r *such that*

$$\|\Lambda_s f\|_{L^p} \le C r^s \|f\|_{L^p}, \quad \forall s > 0.$$

(2) *If* f *has its spectrum in the ring* $C(0, Ar, Br) = \{\xi : Ar \le |\xi| \le Br\}$, *then there exists some constants* C_1 and C_2 *independent of* f *and* r *such that*

$$C_1 r^s \|f\|_{L^p} \le \|\Lambda_s f\|_{L^p} \le C_2 r^s \|f\|_{L^p}, \quad \forall s > 0.$$

For the behavior of $S_j(u)$ and $\Delta_j(u)$ in $L^\infty(\mathbb{R}^n)$-norm when $u \in H_p^s(\mathbb{R}^n)$, we give the following.

Lemma 6.2. [40] *Let* $s_n = s - \frac{n}{p}$.

(1) *For all* $s \in \mathbb{R}$, $\|\Delta_k(u)\|_{L^\infty} \le C 2^{-ks_n} \|u\|_{H_p^s}$;

(2) *If* $s < n/p$, *then* $\|S_k(u)\|_{L^\infty} \le C 2^{-ks_n} \|u\|_{H_p^s}$.

Lemma 6.3. [32] *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of functions in* $\mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\operatorname{supp}(\hat{f}_k) \subset B(0, C2^k).$$

Then, there exists a constant C *such that*

$$\left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_j(f_k)|^2 \right)^{1/2} \right\|_{L^p} \le C \left\| \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p}.$$

6.3 Paracomposition formula

To prove Theorem 1.1, we use the paracomposition technique (see [32,40–42]), which generalizes the paraproduct technique introduced by Bony. We rewrite $F(v, w)$ as the series

$$\begin{aligned} F(v, w) &= F(S_0(v), S_0(w)) + (F(S_1(v), S_1(w)) - F(S_0(v), S_0(w))) \\ &\quad + \cdots + (F(S_{k+1}(v), S_{k+1}(w)) - F(S_k(v), S_k(w))) + \cdots \\ &= F(S_0(v), S_0(w)) + \sum_{k=0}^{\infty} S_{k+1}(v)^m \nabla(-\Delta)^{-1} \Delta_k(v - w) \\ &\quad + \sum_{k=0}^{\infty} (S_{k+1}(v)^m - S_k(v)^m) \nabla(-\Delta)^{-1} S_k(v - w) \\ &= F(S_0(v), S_0(w)) + \sum_{k=0}^{\infty} S_{k+1}(v)^m \nabla(-\Delta)^{-1} \Delta_k(v - w) + \sum_{k=0}^{\infty} \Delta_k(v) m_k(v, w), \end{aligned}$$

where

$$m_k(v,w) = m\int_0^1 (S_k(v) + t\Delta_k(v))^{m-1}\nabla(-\Delta)^{-1}S_k(v-w)\mathrm{d}t.$$

Thus, we have

$$\begin{aligned}
\|m_k(v,w)\|_{L^\infty} &\le m\int_0^1 \|S_k(v) + t\Delta_k(v)\|_{L^\infty}^{m-1}\|S_{k+1}(\nabla(-\Delta)^{-1}S_k(v-w))\|\mathrm{d}t\\
&\le C2^{-k\left(s-\frac{n}{p}\right)(m-1)}\|v\|_{H_p^s}^{m-1}2^{-k(s+1-\frac{n}{p})}\|\nabla(-\Delta)^{-1}(v-w)\|_{H_p^{s+1}}\\
&\le C2^{-k\left(s-\frac{n}{p}\right)(m-1)}\|v\|_{H_p^s}^{m-1}2^{-k(s+1-\frac{n}{p})}\|v-w\|_{H_p^s}.
\end{aligned}\tag{6.4}$$

To relocate the $m_k(u)$ spectrums, we introduce a second Littlewood Paley's partition of unity

$$\widehat{\varphi_{-1}}\left(\frac{\xi}{A2^k}\right) + \sum_{p=0}^{\infty}\hat{\psi}\left(\frac{\xi}{A2^{k+p}}\right) = 1,$$

and $m_k(v,w) = m_{k,-1}(v,w) + \sum_{p=0}^{\infty} m_{k,p}(v,w)$, where

$$\begin{cases}
m_{k,-1}(v,w) = \mathcal{F}^{-1}\left(\widehat{\varphi_{-1}}\left(\frac{\xi}{A2^k}\right)\right)\star m_k(v,w),\\
m_{k,p}(v,w) = \mathcal{F}\left(\hat{\psi}\left(\frac{\xi}{A2^{k+p}}\right)\right)\star m_k(v,w).
\end{cases}$$

Thus,

$$\sum_{k=0}^{\infty}\Delta_k(v)m_k(v,w) = \sum_{k=0}^{\infty}\Delta_k(v)m_{k,-1}(v,w) + \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\Delta_k(v)m_{k,p}(v,w),$$

and we want to prove that each of these terms belongs to $H_p^{s_m}(\mathbb{R}^n)$, where $s_m > 0$.

We now describe the fact that the series $\sum_{k=0}^{\infty}\Delta_k(v)m_{k,-1}(v,w)$ belongs to $H_p^{s_m}$. We first give the following lemma.

Lemma 6.4. *Under (H2), we have*

$$\|m_{k,-1}(v,w)\|_{L^\infty} \le C2^{k\left(s-\frac{n}{p}\right)(m-1)}\|v\|_{H_p^s}^{m-1}2^{-k(s+1-\frac{n}{p})}\|v-w\|_{H_p^s},\quad \forall k\in\mathbb{N}.$$

Proof. Since φ_0 be a non-negative radial test function, we have

$$\begin{aligned}
\|m_{k,-1}(v,w)\|_{L^\infty} &\le \left\|\mathcal{F}^{-1}\left(\widehat{\varphi_{-1}}\left(\frac{\xi}{A2^k}\right)\right)\star m_k(v,w)\right\|_{L^\infty}\\
&\le \left\|\mathcal{F}^{-1}\left(\widehat{\varphi_{-1}}\left(\frac{\xi}{A2^k}\right)\right)\right\|_{L^1}\|m_k(v,w)\|_{L^\infty}\\
&\le C\|m_k(v,w)\|_{L^\infty}\\
&\le C2^{-k\left(s-\frac{n}{p}\right)(m-1)}\|v\|_{H_p^s}^{m-1}2^{-k(s+1-\frac{n}{p})}\|v-w\|_{H_p^s}.
\end{aligned}$$

Together with (6.4) and Lemma 6.2, Lemma 6.4 is concluded. □

To prove that the series belongs to $H_p^{s_m}(\mathbb{R}^n)$, it is then sufficient to show that the function

$$\sigma(x) = \left(\sum_{j=0}^{\infty} 4^{js_m} \left|\Delta_j\left(\sum_{k=0}^{\infty} \Delta_k(v) m_{k,-1}(v,w)\right)\right|^2\right)^{1/2} \in L^p.$$

By construction, the $m_{k,-1}(v,w)$ spectrums are in the ball $B(0, A2^k)$ and the $\Delta_k(v)$ spectrums are in the rings $C(0, 2^{-1}A2^k, 2A2^k)$. Taking $A = 100$ (for instance), then the composition $m_{k,-1}(v,w)\Delta_k(v)$ spectrums are in some extended balls $B(0, A'2^k)$ and so, there exists an integer N such that $\Delta_j(\Delta_k(v)m_{k,-1}(v,w))$ are disjointed. So,

$$\begin{aligned}\left|\Delta_j\left(\sum_{k=0}^{\infty} \Delta_k(v) m_{k,-1}(v,w)\right)\right|^2 &= \left|\Delta_j\left(\sum_{k=j-N}^{\infty} \Delta_k(v) m_{k,-1}(v,w)\right)\right|^2 \\ &\le C4^{-js_m} \sum_{k=j-N}^{\infty} 4^{ks_m} |\Delta_j(\Delta_k(v)m_{k,-1}(v,w))|^2,\end{aligned}$$

by Cauchy-Schwartz inequality applied to the sequences

$$\{2^{-ks_m} I_{k \ge j-N}\} \quad \text{and} \quad \{2^{ks_m}\Delta_j(\Delta_k(v)m_{k,-1}(v,w))I_{k\ge j-N}\}$$

(note that $s_m > 0$ is needed). Then, by definition of $\sigma(x)$, we obtain

$$\sigma(x) \le C\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_j(2^{ks_m}\Delta_k(v)m_{k,-1}(v,w))|^2\right)^{1/2},$$

and Lemma 6.3 applied to the sequence $\{2^{-ks_m} I_{k\ge j-N}\}$ leads to

$$\|\sigma(x)\|_{L^p} \le C\left\|\left(\sum_{k=0}^{\infty} 4^{ks_m} |\Delta_k(v)m_{k,-1}(v,w)|^2\right)^{1/2}\right\|_{L^p}.$$

Now, using Lemma 6.4,

$$\begin{aligned}|\Delta_k(v)m_{k,-1}(v,w)|^2 &\le |\Delta_k(v)|^2 \|\Delta_k(v)m_{k,-1}(v,w)\|_{L^\infty}^2 \\ &\le C4^{-k\left(s-\frac{n}{p}\right)(m-1)} \|v\|_{H_p^s}^{2(m-1)} 4^{-k\left(s+1-\frac{n}{p}\right)} \|v-w\|_{H_p^s}^2 |\Delta_k(v)|^2,\end{aligned}$$

and so,

$$\|\sigma(x)\|_{L^p} \le C\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s} \left\|\left(\sum_{k=0}^{\infty} 4^{ks} |\Delta_k(v)|^2\right)^{1/2}\right\|_{L^p} \le C\|v\|_{H_p^s}^{m}\|v-w\|_{H_p^s}.$$

Thus, the series belongs to $H_p^{s_m}(\mathbb{R}^n)$ and its norm is bounded by $C\|v\|_{H_p^s}^m\|v-w\|_{H_p^s}$.

Then, we describe the fact that the series $\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\Delta_k(v)m_{k,p}(v,w)$ belongs to $H_p^{s_m}(\mathbb{R}^n)$.

For the fixed $p \ge 0$, we define

$$\ell_p(x) = \sum_{p=0}^{\infty} \Delta_k(v)m_{k,p}(v,w).$$

Taking the constant A large enough, one can check that the $\Delta_k(v)m_{k,p}(v,w)$ spectrums are in some rings $\{\xi : C_1 2^{p+k} \le |\xi| \le C_2 2^{p+k}\}$. So, there exists an integer K (which does not depend on p) such that those rings are K to K disjointed. So, we can use the Littlewood-Paley analysis on the K partial sums $\ell_p^r(x)$ defined by

$$\ell_p^r(x) = \sum_{k=r \bmod (K)} \Delta_k(v)m_{k,p}(v,w), \;\; r \in \{0, \dots, K-1\},$$

and, by Proposition 6.1, we know that for all $r \in \{0, \dots, K-1\}$,

$$\|\ell_p^r\|_{H_p^{s_m}} \le C \left\| \left(\sum_{k=r \mod(K)} 4^{(k+p)s_m} |\Delta_k(v) m_{k,p}(v,w)|^2 \right)^{1/2} \right\|_{L^p}.$$

Lemma 6.5. *Under* **(H2)**, *we have*

$$\|m_{k,p}(v,w)\|_{L^\infty} \le C 2^{-k\left(s-\frac{n}{p}\right)m-k} 2^{-pm} \|v\|_{H_p^s}^{m-1} \|v-w\|_{H_p^s}.$$

Proof. Let us define $m = N + \nu$, where $N = [m]$ and $\nu \in [0,1)$, and

$$P_k^t(v)(x) = S_k(v)(x) + t\Delta_k(v)(x).$$

By Lemma 6.1 applied with $p = \infty$,

$$\|m_{k,p}(v,w)\|_{L^\infty} \le C 2^{-(k+p)m} \|m_k(v,w)\|_{C^m}, \tag{6.5}$$

where $C^\theta(\mathbb{R}^n)$ denotes the Hölder space of order θ endowed with the norm:

$$\begin{aligned} \|h\|_{C^\theta} &= \|h\|_{L^\infty} + \cdots + \|D^\theta h\|_{L^\infty}, \quad \text{if } \theta \in \mathbb{N}, \\ \|h\|_{C^\theta} &= \|h\|_{C^N} + \sup_{|x-y|<1} \frac{|D^N h(x) - D^N(y)|}{|x-y|^\nu}, \quad \text{if } \theta \notin \mathbb{N}. \end{aligned}$$

So, by (6.5),

$$\begin{aligned} \|m_{k,p}(v,w)\|_{L^\infty} \le C 2^{-(k+p)m} \Big(&\|m_k(v,w)\|_{L^\infty} + \cdots + \|D^N m_k(v,w)\|_{L^\infty} \\ &+ \sup_{|x-y|<1} \frac{|D^N m_k(v,w)(x) - D^N m_k(v,w)(y)|}{|x-y|^\nu} \Big). \end{aligned} \tag{6.6}$$

The bound of $\|m_k(v,w)\|_{L^\infty}$ is easy to establish: we have just to argue as in the proof of Lemma 6.4 to obtain

$$\|m_k(v,w)\|_{L^\infty} \le C 2^{-k\left(s-\frac{n}{p}\right)(m-1)} \|v\|_{H_p^s}^{m-1} 2^{-k(s+1-\frac{n}{p})} \|v-w\|_{H_p^s}. \tag{6.7}$$

Next, we must bound $\|D^j m_k(v,w)\|_{L^\infty}$ for $j \in \{1, \dots, N\}$. Let γ be multi-index such that $\gamma = \gamma_1 + \cdots + \gamma_m$ with total length $|\gamma| = |\gamma_1| + \cdots + |\gamma_m| = j$, then,

$$\partial^\gamma m_k(v,w)(x) = \sum_{\gamma'+\eta=\gamma} \int_0^1 \sum_{q=1}^{|\gamma'|} \sum_{\gamma_1+\cdots+\gamma_q=\gamma'} D^{q+1}(P_k^t(x))^m \partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x) \partial^\eta (\nabla(-\Delta)^{-1} S_k(v-w))(x) \mathrm{d}t.$$

By Lemmas 6.4 and 6.5,

$$\|D^{q+1}(P_k^t(x))^m\|_{L^\infty} \le C \|P_k^t(x)\|_{L^\infty}^{m-q-1} \le C 2^{-k\left(s-\frac{n}{p}\right)(m-q-1)} \|v\|_{H_p^s}^{m-q-1}, \tag{6.8}$$

$$\|\partial^{\gamma_i} P_k^t(x)\|_{L^\infty} \le C 2^{|\gamma_i|k} \|P_k^t(x)\|_{L^\infty} \le C 2^{|\gamma_i|k} 2^{-k\left(s-\frac{n}{p}\right)} \|v\|_{H_p^s}, \tag{6.9}$$

and furthermore,

$$\begin{aligned} \|\partial^\eta (\nabla(-\Delta)^{-1} S_k(v-w))\|_{L^\infty} &\le C 2^{|\eta|k} \|S_k(\nabla(-\Delta)^{-1}(v-w))\|_{L^\infty} \\ &\le C 2^{|\eta|k} 2^{-k(s+1-\frac{n}{p})} \|\nabla(-\Delta)^{-1}(v-w)\|_{H_p^{s+1}} \\ &\le C 2^{|\eta|k} 2^{-k(s+1-\frac{n}{p})} \|v-w\|_{H_p^s}. \end{aligned} \tag{6.10}$$

Thus, for $j \in \{1, \dots, N\}$,

$$\|D^j m_k(v,w)\|_{L^\infty} \le C2^{jk}2^{-k\left(s-\frac{n}{p}\right)m-k}\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s}. \tag{6.11}$$

To conclude, we must estimate

$$\sup_{|x-y|<1}\frac{|D^N m_k(v,w)(x)-D^N m_k(v,w)(y)|}{|x-y|^\nu} =: D(x,y).$$

Let γ be a multi-index of length N. Then,

$$\partial^\gamma m_k(v,w)(x)-\partial^\gamma m_k(v,w)(y) =: I(x,y)+J(x,y)+K(x,y),$$

where

$$\begin{aligned} I(x,y) = \sum_{\gamma'+\eta=\gamma}\int_0^1\sum_{q=1}^{|\gamma'|}\sum_{\gamma_1+\dots+\gamma_q=\gamma'} &[D^{q+1}(P_k^t(x))^m - D^{q+1}(P_k^t(y))^m] \\ &\cdot\left(\prod_{i=1}^q \partial^{\gamma_i}P_k^t(x)\right)\partial^\eta(\nabla(-\Delta)^{-1}S_k(v-w))(x)\mathrm{d}t, \end{aligned} \tag{6.12}$$

$$\begin{aligned} J(x,y) = \sum_{\gamma'+\eta=\gamma}\int_0^1\sum_{q=1}^{|\gamma'|}\sum_{\gamma_1+\dots+\gamma_q=\gamma'} & D^{q+1}(P_k^t(y))^m\sum_{j=1}^q\left(\prod_{i>j}\partial^{\gamma_i}P_k^t(x)\right)(\partial^{\gamma_j}P_k^t(x)-\partial^{\gamma_j}P_k^t(y)) \\ &\times\left(\prod_{i<j}\partial^{\gamma_i}P_k^t(y)\right)\partial^\eta(\nabla(-\Delta)^{-1}S_k(v-w))(x)\mathrm{d}t, \end{aligned} \tag{6.13}$$

$$\begin{aligned} K(x,y) = \sum_{\gamma'+\eta=\gamma}\int_0^1\sum_{q=1}^{|\gamma'|}\sum_{\gamma_1+\dots+\gamma_q=\gamma'} & D^{q+1}(P_k^t(y))^m\left(\prod_{i=1}^q\partial^{\gamma_i}P_k^t(y)\right) \\ &\cdot\partial^\eta\nabla(-\Delta)^{-1}(S_k(v-w)(x)-S_k(v-w)(y))\mathrm{d}t. \end{aligned} \tag{6.14}$$

By definition of $C^s(\mathbb{R})$ and Lemma 6.2, we have

$$\begin{aligned} &\sup_{|x-y|<1}\frac{|D^{q+1}(P_k^t(x))^m - D^{q+1}(P_k^t(y))^m|}{|x-y|^\nu} \\ &\le \sup_{|x-y|<1}\frac{C|(P_k^t(x))^{m-q-1}-(P_k^t(y))^{m-q-1}|}{|x-y|^\nu} \\ &\le C\sup_{|x-y|<1}\frac{|P_k^t(x)-P_k^t(y)|}{|x-y|^\nu}(\|P_k^t(x)\|_{L^\infty}^{m-q-2}+\|P_k^t(y)\|_{L^\infty}^{m-q-2}) \\ &\le C\|P_k^t\|_{C^\nu}\|P_k^t\|_{L^\infty}^{m-q-2} \le C2^{k\nu}2^{-k\left(s-\frac{n}{p}\right)(m-q-1)}\|v\|_{H_p^s}^{m-q-1}, \end{aligned} \tag{6.15}$$

and

$$\sup_{|x-y|<1}\frac{|\partial^{\gamma_j}(P_k^t(v)(x)-P_k^t(v)(y))|}{|x-y|^\nu} \le \|P_k^t(v)\|_{C^{|\gamma_j|+\nu}} \le C2^{k(|\gamma_j|+\nu)}\|P_k^t(v)\|_{L^\infty} \le C2^{k(|\gamma_j|+\nu)}2^{-k\left(s-\frac{n}{p}\right)}\|v\|_{H_p^s}. \tag{6.16}$$

Similarly, we have

$$\sup_{|x-y|<1}\frac{|\partial^\eta\nabla(-\Delta)^{-1}(S_k(v-w)(x)-S_k(v-w)(y))|}{|x-y|^\nu} \le C2^{k(|\gamma_\eta|+\nu)}2^{-k(s+1-\frac{n}{p})}\|v-w\|_{H_p^s}. \tag{6.17}$$

Then, by (6.8)–(6.10) and (6.12)–(6.17), we have

$$\begin{aligned}D(x,y) &= \sup_{|x-y|<1} \frac{I(x,y)+J(x,y)+K(x,y)}{|x-y|^{\nu}} \\ &\le C2^{km}2^{-k\left(s-\frac{n}{p}\right)m-k}\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s}.\end{aligned} \tag{6.18}$$

Now, by (6.6), (6.7), (6.11), and (6.18), we have

$$\begin{aligned}\|m_{k,p}(v,w)\|_{L^\infty} &\le C2^{-(k+p)m}\left(\sum_{j=0}^{N} 2^{kj}2^{-k\left(s-\frac{n}{p}\right)m-k} + 2^{km}2^{-k\left(s-\frac{n}{p}\right)m-k}\right)\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s} \\ &\le C2^{-pm}2^{-k\left(s-\frac{n}{p}\right)m-k}\left(\sum_{j=0}^{N}2^{kj-mk}+1\right)\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s} \\ &\le C2^{-pm}2^{-k\left(s-\frac{n}{p}\right)m-k}\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s}.\end{aligned}$$

This ends the proof of Lemma 6.5. □

Then, by Lemma 6.5,

$$\begin{aligned}\|\ell_p^r\|_{H_p^{s_m}} &\le C\left\|\left(\sum_{k=r \bmod (K)} 4^{(k+p)s_m}\,|\Delta_k(v)|^2\|m_{k,p}(v,w)\|_{L^\infty}^2\right)^{1/2}\right\|_{L^p} \\ &\le C\left\|\left(\sum_{k=r \bmod (K)} 4^{ks_m}\,|\Delta_k(v)|^2 4^{-k\left(s-\frac{n}{p}\right)m-k}\right)^{1/2}\right\|_{L^p} 2^{-pm}2^{ps_m}\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s} \\ &\le C2^{p(s_m-m)}\|v\|_{H_p^s}^{m-1}\|v-w\|_{H_p^s}\left\|\left(\sum_{k=r \bmod (K)} 4^{ks}\,|\Delta_k(v)|^2\right)^{1/2}\right\|_{L^p} \\ &\le C2^{p(s_m-m)}\|v\|_{H_p^s}^{m}\|v-w\|_{H_p^s}.\end{aligned}$$

Thus, for $s_m < m$, the K series $\{\ell_p^r\}_{p\in\mathbb{N}}$ are uniformly convergent in $H_p^{s_m}(\mathbb{R}^n)$ and

$$\sum_{r\in\{0,\ldots,K-1\}} \|\ell_p^r\|_{H_p^{s_m}} \le C\|v\|_{H_p^s}^{m}\|v-w\|_{H_p^s}.$$

Thus, the series belongs to $H_p^{s_m}(\mathbb{R}^n)$ and its norm is bounded by $C\|v\|_{H_p^s}^{m}\|v-w\|_{H_p^s}$.

To end the proof of Theorem 1.1, we need to prove that $\sum_{k=0}^{\infty} S_{k+1}(v)^m\nabla(-\Delta)^{-1}\Delta_k(v-w)$ and $F(S_0(v),S_0(w))$ belong to $H_p^{s_m}(\mathbb{R}^n)$; using the same methods as in Section 6.3, we can obtain that they are bounded by $C\|v\|_{H_p^s}^{m}\|v-w\|_{H_p^s}$.

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