Research Article

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Hyponormality on a weighted Bergman space of an annulus with a general harmonic symbol

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Abstract: In this work we consider the hyponormality of Toeplitz operators on the Bergman space of the annulus with a logarithmic weight. We give necessary conditions when the symbol is of the form $\varphi + \overline{\psi}$ where both φ and ψ are analytic on the annulus $\{z \in \mathbb{C};\ 1/2 < |z| < 1\}$ and of the form $\sum_{n\geq 1} a_n z^n + b_n \frac{1}{z^n}$. We also show a sufficient condition for hyponormality on the Bergman space of the unit disk with a square Logarithmic weight.

Keywords: hyponormality; Toeplitz operators; positive matrices; analytic functions; weighted Bergman space; annulus

MSC 2020: 47B35; 47B20; 15B48; 30Bxx

1 Introduction

A bounded operator T on a Hilbert space is hyponormal if $T^*T - TT^*$ is positive. Hyponormality of Toeplitz operators on the Hardy space was considered by Cowen [1,2]. The first work on hyponormality on the Bergman space can be found in [3]. A general necessary condition, in the case of a harmonic symbol, is shown. An improvement of the necessary condition, which consists of a local version of it, uses function theory and is due to Ahern and Cuckovic [4]. Their method was recently extended to weighted Bergman spaces [5]. An improvement of the necessary condition in a special case is due Curto and Cuckovic [6]. Many partial results can be found in the literature, we cite for example results due to Phukon [7], Hwang [8,9]. Some of the results on hyponormality on the Bergman space of the unit disk, in the case of non-harmonic symbols, are due to Fleeman, and Liaw [10] and Simanek [11]. Sufficient conditions for hyponormality when the analytic part of the symbol is a monomial are given in [3]. Most of the other results on hyponormality on the Bergman space deal mostly with the case of specific symbols and use matrix computations. Some of these results can be found in [9]. Recent results on hyponormality on the Bergman space of an annulus can be found in [12,13]. In this work we generalize the necessary condition for hyponormality of Toeplitz operators $B_{\varphi+\overline{\psi}}$, on the Bergman space of a annulus with a logarithmic weight [13], to the case where both φ and ψ are of the form $\sum_{n\geq 1} a_n z^n + \sum_{n\geq 1} b_n \frac{1}{z^n}$. We begin with definitions and notations. Set $C_{1/2} = \{z \in \mathbb{C}; 1/2 < |z| < 1\}$. The space L^2_μ is the space of measurable functions f on $C_{1/2}$ such that $\int_{C_{1/2}} |f|^2 \mathrm{d}\mu(z) < \infty$ where $\mathrm{d}\mu(z) = \frac{8}{\pi(3-2\ln 2)} r|\log r| \mathrm{d}r\mathrm{d}\theta$. The subspace of L^2_μ consisting of analytic functions is denoted by A^2_μ. If f is analytic on $C_{1/2}$, we have $f = \sum_{-\infty}^{-1} c_n z^n + \sum_0^\infty c_n z^n$ and

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$$\|f\|^2 = \frac{1}{3-2\ln 2}\sum_{0}^{\infty}\frac{2^{2n+2}-1-(2n+2)\ln 2}{2^{2n}(n+1)^2}|c_n|^2 + \frac{8(\ln 2)^2}{3-2\ln 2}|c_{-1}|^2$$
$$+\frac{4}{3-2\ln 2}\sum_{2}^{\infty}\frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2}+1}{(n-1)^2}|c_{-n}|^2.$$

The space A^2_μ has the following orthonormal basis:

$$\{e_n;\ n\geq 0\}\cup\{e_{-1}\}\cup\{e_{-n};\ n\geq 2\} = \left\{\sqrt{3-2\ln 2}\frac{2^n(n+1)}{\sqrt{2^{2n+2}-1-(2n+2)\ln 2}}z^n,\ n\geq 0\right\}$$
$$\cup\left\{\frac{\sqrt{3-2\ln 2}}{2\sqrt{2}\ln 2}\frac{1}{z}\right\}$$
$$\cup\left\{\frac{\sqrt{3-2\ln 2}}{2}\frac{n-1}{\sqrt{2^{2n-1}(n-1)\ln 2 - 2^{2n-2}+1}}\frac{1}{z^n}, n\geq 2\right\}.$$

For h bounded measurable on $C_{1/2}$ we define the Toeplitz operator B_h by $B_h(f) = P(hf)$ where P is the orthogonal projection of L^2_μ onto A^2_μ. We also define Hankel operators by $H_h(f) = (I-P)(hf)$. We start with some properties of Toeplitz operators in general. These properties are easy to prove and the proof is omitted.

2 General properties of Toeplitz and Hankel operators

Lemma 2.1. *Let φ and ψ be bounded measurable on $C_{1/2}$. The following holds:*

(a) $B_{\varphi+\psi} = B_\varphi + B_\psi$.
(b) $B^*_\varphi = B_{\overline{\varphi}}$.
(c) $B_\varphi B_\psi = B_{\varphi\psi}$ *if ψ is analytic or φ is conjugate analytic.*
(d) $B^*_\varphi B_\varphi - B_\varphi B^*_\varphi = H^*_{\overline{\varphi}} H_\varphi$ *if φ is analytic.*

As in the case of the Bergman space of the unit disk, the following proposition gives equivalent forms of hyponormality. The Douglas lemma [14] is used to show (c) implies (d).

Proposition 2.2. *For φ_1 and φ_2 bounded analytic on $C_{1/2}$ the following statements are equivalents:*

(i) $B_{\varphi_1+\overline{\varphi_2}}$ *is hyponormal.*
(ii) $B_{\overline{\varphi_2}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_2}} \leq B_{\overline{\varphi_1}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_1}}$.
(iii) $H^*_{\overline{\varphi_2}}H_{\varphi_2} \leq H^*_{\overline{\varphi_1}}H_{\varphi_1}$.
(iv) $H_{\overline{\varphi_2}} = KH_{\overline{\varphi_1}}$ *where K is a bounded operator of norm less than or equal 1.*

Computations involving the projection are given in the next lemma.

Lemma 2.3. [15] *The orthogonal projection of L^2_μ onto A^2_μ satisfies the following properties:*

(1) $P(z^m\overline{z^n}) = \frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2}\frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2}-1-(2(m-n)+2)\ln 2}z^{m-n}$, $m\geq n\geq 0$.
(2) $P(z^m\overline{z^n}) = \frac{1}{4}\frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2}\frac{((n-m-1)^2}{2^{2(n-m)-1}((n-m)-1)\ln 2-2^{2(n-m)-2}+1}\frac{1}{z^{n-m}}$, $n-m\geq 2, m\geq 0$.
(3) $P(z^m\overline{z^{m+1}}) = \frac{1}{8(\ln 2)^2}\frac{2^{2m+2}-1-(2m+2)\ln 2}{2^{2m}(m+1)^2}\frac{1}{z}$, $m\geq 0$.
(4) $P(\frac{1}{z^m}\overline{z^n}) = \frac{2^{2m-1}(m-1)\ln 2-2^{2m-2}+1}{(m-1)^2}\frac{(m+n-1)^2}{2^{2(m+n)-1}(m+n-1)\ln 2-2^{2(m+n)-2}+1}\frac{1}{z^{m+n}}$, $m\geq 2, n\geq 0$.
(5) $P(\frac{1}{z}\overline{z^n}) = 2(\ln 2)^2\frac{n^2}{2^{2n+1}n\ln 2-2^{2n}+1}\frac{1}{z^{n+1}}$, $n\geq 1$.
(6) $P(\frac{1}{\overline{z^n}}z^n) = \frac{2^{2n+2}-1-(2n+2)\ln 2}{2^{2n}(n+1)^2}\frac{2^{2(m+n)}(m+n+1)^2}{2^{2(m+n)+2}-1-(2(m+n)+2)\ln 2}z^{m+n}$, $n\geq 0, m\geq 0$.

(7) $P(\frac{1}{\overline{z^m}}\frac{1}{z^n}) = 4\frac{2^{2n-1}(n-1)\ln 2-2^{2n-2}+1}{(n-1)^2}\frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2}-1-(2(m-n)+2)\ln 2}z^{m-n}$, $m \geq n$, $n \geq 2$.

(8) $P(\frac{1}{\overline{z^m}}\frac{1}{z}) = 2(\ln 2)^2\frac{2^{2m}m^2}{2^{2m}-1-2m\ln 2}z^{m-1}$, $m \geq 1$.

(9) $P(\frac{1}{\overline{z^m}}\frac{1}{z^n}) = \frac{2^{2n-1}(n-1)\ln 2-2^{2n-2}+1}{(n-1)^2}\frac{(n-m-1)^2}{2^{2(n-m)-1}(n-m-1)\ln 2-2^{2(n-m)-2}+1}\frac{1}{z^{n-m}}$, $n-m \geq 2$, $m \geq 1$.

(10) $P(\frac{1}{\overline{z^m}}\frac{1}{z^{m+1}}) = \frac{1}{2(\ln 2)^2}\frac{m^2}{2^{2m+1}m\ln 2-2^{2m}+1}\frac{1}{z}$, $m \geq 1$.

3 The necessary condition

Set $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 = \sum_{n\geq 1}a_n z^n$, $\varphi_2 = \sum_{n\geq 1}b_n\frac{1}{z^n}$ and $\psi = \psi_1 + \psi_2$, with $\psi_1 = \sum_{n\geq 1}c_n z^n$, $\psi_2 = \sum_{n\geq 1}d_n\frac{1}{z^n}$. We have the following identity

$$B_{\overline{\varphi}}B_{\varphi} - B_{\varphi}B_{\overline{\varphi}} = B_{\overline{\varphi_1}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_1}} + B_{\overline{\varphi_2}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_2}} + B_{\overline{\varphi_1}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_1}} + B_{\overline{\varphi_2}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_2}} \tag{1}$$

The following lemmas were proven in [13,15].

Lemma 3.1. *Let $\varphi_1 = \sum_{n\geq 1}a_n z^n$ be bounded and analytic on $C_{1/2}$. For k and $l \geq 1$ we have*

$$\begin{aligned}
&\langle B_{\overline{\varphi_1}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_1}}(e_k), e_l\rangle \\
&= \sum_{n+k-l\geq 1, n\geq 1}\overline{a_{n+k-l}}a_n\frac{2^k(k+1)2^l(l+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}\frac{(2^{2(n+k)+2}-1-(2(n+k)+2)\ln 2)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}2^{2(n+k)}(n+k+1)^2} \\
&- \sum_{n+k-l\geq 1, l\geq n\geq 1}\overline{a_{n+k-l}}a_n\frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)}\frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}2^{2(l-n)}(l-n+1)^2}{(2^{2(l-n)+2}-1-(2(l-n)+2)\ln 2)} \\
&- \frac{1}{8(\ln 2)^2}\overline{a_{k+1}}a_{l+1}\frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)}\frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)} \\
&- \frac{1}{4}\sum_{n\geq l+2}\overline{a_{n+k-l}}a_n\frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)}\frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}(n-l-1)^2}{(2^{2(n-l)-1}(n-l-1)\ln 2-2^{2(n-l)-2}+1)}.
\end{aligned}$$

Lemma 3.2. *Let $\varphi_2 = \sum_1^\infty b_n\frac{1}{z^n}$ be bounded and analytic on $C_{1/2}$. For k and $l \geq 1$ we have*

$$\begin{aligned}
&\langle B_{\overline{\varphi_2}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_2}}(e_k), e_l\rangle \\
&= \sum_{k\geq n\geq 1, n+l-k\geq 1}\frac{\overline{b_{n+l-k}}b_n(2^k(k+1)2^l(l+1)(2^{2(k-n)+2}-1-(2(k-n)+2)\ln 2)}{\sqrt{[2^{2(k+2)}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{2(k-n)(k-n+1)^2}} \\
&+ 8(\ln 2)^2\overline{b_{l+1}}b_{k+1}\frac{2^k(k+1)}{\sqrt{2^{2(k+2)}-1-(2k+2)\ln 2}}\frac{2^l(l+1)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} \\
&+ 4\sum_{n\geq k+2}\frac{\overline{b_{n+l-k}}b_n2^k(k+1)2^l(l+1)[2^{2(k-n)-1}(k-n-1)\ln 2-2^{2(k-n)-2}+1]}{\sqrt{[2^{2(k+2)}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}(k-n-1)^2} \\
&- \sum_{n\geq 1, n+l-k\geq 1}\frac{\overline{b_{n+l-k}}b_n\sqrt{[2^{2(k+2)}-1-(2k+2)\ln 2][2^{2l+2}-1-(2l+2)\ln 2]}2^{2(n+l)}(n+l+1)^2}{2^k(k+1)2^l(l+1)[2^{2(n+l)+2}-1-(2(n+l)+2)\ln 2]}.
\end{aligned}$$

Denote by $(\lambda_{k,l})$ the matrix of $B_{\overline{\varphi_1}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_1}}$ in the orthonormal basis $\{e_n,\ n\in\mathbb{Z}\}$, and by $(\gamma_{k,l})$ the matrix of $B_{\overline{\varphi_2}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_2}}$. On the Hardy space of the unit disk H^2, denote by T_ω the Toeplitz operator on H^2 with symbol ω, and denote by $(\theta_{k,l})$, $(\chi_{k,l})$ respectively the matrices of the (possibly unbounded) Toeplitz operators

$T_{2|\varphi_1'|^2}$ and $T_{2|\widetilde{\varphi_2}'|^2}$, where $\widetilde{\varphi_2} = \sum_{n\geq 1}\overline{b_n}z^n$. Note some of our results in [13] are in terms of matrices $m_{l,l+p}$, while here for our purposes we express convergence results in terms of $m_{l+p,p}$. We need the following lemma.

Lemma 3.3. *Assume that $\varphi_1' \in H^2$. Then we have $\lim_{l\to\infty} l^2\lambda_{l+p,l} = \theta_{s+p,s}$*

Proof. We have

$$\begin{aligned}
\lambda_{l+p,l} &= \sum_{n\geq p+1} \overline{a_{n-p}}a_n \frac{2^l(l+1)2^{l+p}(l+p+1)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} \frac{(2^{2(n+l)+2}-1-(2(n+l)+2)\ln 2)}{\sqrt{2^{2(l+p)+2}-1-(2(l+p)+2)\ln 2}2^{2(n+l)}(n+l+1)^2} \\
&- \sum_{l+p\geq n\geq p+1} \overline{a_{n-p}}a_n \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)2^{l+p}(l+p+1)} \\
&\times \frac{\sqrt{2^{2(l+p)+2}-1-(2(l+p)+2)\ln 2}2^{2(l+p-n)}(l+p-n+1)^2}{(2^{2(l+p-n)+2}-1-(2(l+p-n)+2)\ln 2)} \\
&- \frac{1}{8(\ln 2)^2}\overline{a_{l+1}}a_{l+p+1} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)} \frac{\sqrt{2^{2(l+p)+2}-1-(2(l+p)+2)\ln 2}}{2^{l+p}(l+p+1)} \\
&- \frac{1}{4}\sum_{n\geq l+p+2} \overline{a_{n-p}}a_n \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)2^{l+p}(l+p+1)} \frac{\sqrt{2^{2(l+p)+2}-1-(2(l+p)+2)\ln 2}(n-l-p-1)^2}{(2^{2(n-l-p)-1}(n-l-p-1)\ln 2-2^{2(n-l-p)-2}+1)}.
\end{aligned}$$

With obvious notations write

$$\lambda_{l+p,l} = \sum_{p+1\leq n\leq l+p} \overline{a_{n-p}}a_n R_{n,l,p} + \overline{a_{l+1}}a_{l+p+1}S_{p,l} + \sum_{n\geq l+p+2} \overline{a_{n-p}}a_n T_{n,p,l}.$$

An involved, but elementary computation, shows that

$$\lim_{l\to\infty} l^2 R_{n,l,p} = 2n(n-p),$$

and that

$$l^2|R_{n,l,p}| \leq Cn(n-p), \quad p+1\leq n\leq l+p, \quad C \text{ is a constant.}$$

Writing $\sum_{p+1\leq n\leq l+p}\overline{a_{n-p}}a_n l^2 R_{n,l,p}$ as an integral with respect to the counting measure and applying the dominated convergence theorem we see, since $\sum_{n\geq 1} n^2|a_n|^2$ is convergent, that

$$\lim_{l\to\infty} \sum_{p+1\leq n\leq l+p} \overline{a_{n-p}}a_n l^2 R_{n,l,p} = 2\sum_{n\geq p+1} n(n-p)\overline{a_{n-p}}a_n.$$

Clearly

$$\lim_{l\to\infty} l^2\overline{a_{l+1}}a_{l+p+1}S_{p,l} = 0.$$

Similarly we show

$$\lim_{l\to\infty} l^2 \sum_{n\geq l+p+2} \overline{a_{n-p}}a_n T_{n,p,l} = 0.$$

We see that $2\sum_{n\geq p+1} n(n-p)\overline{a_{n-p}}a_n = \theta_{s+p,s}$, where $(\theta_{i,j})$ is the matrix of the Hardy space Toeplitz operator $T_{2|\varphi_1'|^2}$. □

The following lemma can be shown using the same method and the proof is omitted.

Lemma 3.4. *Assume $\widetilde{\varphi_2}' \in H^2$. Then we have $\lim_{l\to\infty} l^2\gamma_{l+p,l} = \chi_{s+p,s}$.*

Let us compute the matrix of $B_{\overline{\varphi_1}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_1}}$ in the orthonormal basis $\{e_n,\ n \in \mathbb{Z}\}$. Set $Q_i = \frac{2^i(i+1)}{\sqrt{2^{2i+2}-1-(2i+2)\ln 2}}$.

Lemma 3.5. *The matrix of $B_{\overline{\varphi_1}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_1}}$ is given by $(\zeta_{l,k})$ where*

$$\zeta_{l,k} = Q_k Q_l \sum_{1\le n\le k-l-1} \frac{1}{Q^2_{n+l}}\overline{a_n} b_{k-n-l} - \frac{1}{Q_k Q_l}\sum_{1\le n\le k-l-1} \overline{a_n} b_{k-n-l} Q^2_{k-n}.$$

Proof. We have for $l, k \ge 1$

$$\begin{aligned}\left\langle B_{\overline{\varphi_1}}B_{\varphi_2}e_k, e_l\right\rangle &= Q_k Q_l(3-2\ln 2)\sum_{m,n\ge 1}\overline{a_n}b_m\left\langle z^{k-m}, z^{n+l}\right\rangle \\ &= Q_k Q_l \sum_{1\le n\le k-l-1}\frac{1}{Q^2_{n+l}}\overline{a_n}b_{k-n-l}\end{aligned}$$

and $\left\langle B_{\overline{\varphi_1}}B_{\varphi_2}e_k, e_l\right\rangle = 0$ if $k-l\le 1$. Similarly we have

$$\begin{aligned}\left\langle B_{\varphi_2}B_{\overline{\varphi_1}}e_k, e_l\right\rangle &= \sum_{m,l\ge 1}\overline{a_n}b_m\left\langle P(\overline{z}^n e_k), P\Big(\frac{1}{\overline{z}^m}e_l\Big)\right\rangle \\ &= \sum_{m,l\ge 1}\overline{a_n}b_m(3-2\ln 2)Q_kQ_l\left\langle P(\overline{z}^n z^k), P\Big(\frac{1}{\overline{z}^m}z^l\Big)\right\rangle \\ &= \sum_{1\le n\le k-l-1}\overline{a_n}b_{k-n-l}(3-2\ln 2)Q_kQ_l\left\langle \frac{Q^2_{k-n}}{Q^2_k}z^{k-n}, \frac{Q^2_{k-n}}{Q^2_l}z^{k-n}\right\rangle \\ &= \frac{1}{Q_kQ_l}\sum_{1\le n\le k-l-1}\overline{a_n}b_{k-n-l}Q^2_{k-n}\end{aligned}$$

and $\left\langle B_{\varphi_2}B_{\overline{\varphi_1}}e_k, e_l\right\rangle = 0$ if $k-l\le 1$. □

Notice that $(\zeta_{l,k})_{l,k\ge 2}$ is upper triangular. Since $B_{\overline{\varphi_2}}B_{\varphi_1} - B_{\varphi_1}B_{\overline{\varphi_2}}$ is the adjoint of $B_{\overline{\varphi_1}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_1}}$ we deduce the following lemma.

Lemma 3.6. *The matrix of $B_{\overline{\varphi_1}}B_{\varphi_2} - B_{\varphi_2}B_{\overline{\varphi_1}}$ is given by*

$$\eta_{l,k} = Q_kQ_l\sum_{1\le n\le l-k-1}\frac{1}{Q^2_{n+k}}a_n\overline{b_{l-n-k}} - \frac{1}{Q_kQ_l}\sum_{1\le n\le l-k-1}a_n\overline{b_{l-n-k}}Q^2_{l-n},\quad k,l\ge 2.$$

Let us rewrite the expression of $\eta_{l,k}$.

$$\begin{aligned}\eta_{l+p,l} &= Q_lQ_{l+p}\sum_{1\le n\le p-1}\frac{1}{Q^2_{n+l}}a_n\overline{b_{p-n}} - \frac{1}{Q_lQ_{l+p}}\sum_{1\le n\le p-1}a_n\overline{b_{p-n}}Q^2_{l+p-n} \\ &= \sum_{1\le n\le p-1}a_n\overline{b_{p-n}}\left(\frac{Q_lQ_{l+p}}{Q^2_{n+l}} - \frac{Q^2_{l+p-n}}{Q_lQ_{l+p}}\right).\end{aligned}$$

We get an asymptotic expression of $\eta_{l+p,l}$.

Lemma 3.7. $\lim_{l\to\infty} l^2\eta_{l+p,l} = -2\sum_{1\le n\le p-1}n(p-n)a_n\overline{b_{p-n}}$.

The matrix $(\sigma_{i,j})$ given by $\sigma_{i+p,i} = -2\sum_{1\le n\le p-1}n(p-n)a_n\overline{b_{p-n}}$ and $\sigma_{i,i+p} = \overline{\sigma_{i+p,i}}$ is the matrix of the Hardy space Toeplitz operator with symbol $-4\ \mathrm{Re}\Big(\varphi_1'\widetilde{\varphi_2}' - (a_1\overline{b_1} + 2(a_2\overline{b_1} + a_1\overline{b_2})z)\Big)$, where $\widetilde{\varphi_2}(z) = \sum_{n\ge 1}\overline{b_n}z^n$. Let

$(\Phi_{i,j})$ denote the matrix of $B_{\overline{\varphi}}B_{\varphi} - B_{\varphi}B_{\overline{\varphi}}$ in the basis $\{e_n,\ n \in \mathbb{Z}\}$. From (1) using Lemmas 3.3, 3.4, and 3.7 we deduce the following

Lemma 3.8. *Under the assumptions $\varphi_1', \widetilde{\varphi_2}' \in H^2$, we have*

$$\lim_{l\to\infty} l^2\Phi_{l+p,l} = \theta_{s+p,s} + \chi_{s+p,s} + \sigma_{s+p,s}.$$

We have our first main result where we denote by $(\Psi_{i,j})$ the matrix of $B_{\overline{\psi}}B_{\psi} - B_{\psi}B_{\overline{\psi}}$.

Theorem 3.9. *Assume φ_1, φ_2 are analytic and bounded, with $\varphi_1', \widetilde{\varphi_2}' \in H^2$ and that $B_{\varphi+\overline{\psi}}$ is hyponormal. Then $\psi_1', \widetilde{\psi_2}' \in H^2$ and we have*

$$\begin{aligned}|\psi_1'|^2 + |\widetilde{\psi_2}'|^2 - 2\ \mathrm{Re}\Big(\psi_1'\widetilde{\psi_2}' - (c_1\overline{d_1} + 2(c_2\overline{d_1} + c_1\overline{d_2})z)\Big)\\ \le |\varphi_1'|^2 + |\widetilde{\varphi_2}'|^2 - 2\ \mathrm{Re}\Big(\varphi_1'\widetilde{\varphi_2}' - (a_1\overline{b_1} + 2(a_2\overline{b_1} + a_1\overline{b_2})z)\Big)\end{aligned}$$

a.e on the unit circle.

Proof. Hyponormality implies

$$i^2\Psi_{i,i} \le i^2\Phi_{i,i} = i^2(\lambda_{i,i} + \gamma_{i,i}).$$

Thus

$$\liminf_{i\to\infty} i^2\Psi_{i,i} \le \liminf_{i\to\infty} i^2\Phi_{i,i}.$$

From Lemmas 3.3 and 3.4 we have

$$\liminf_{i\to\infty} i^2\Phi_{i,i} = \sum_1^\infty 2n^2\big(|a_n|^2 + |b_n|^2\big) < \infty$$

since $\varphi_1', \widetilde{\varphi_2}' \in H^2$. Write $\Psi_{i,i} = C_{i,i} + D_{i,i}$, where $(C_{i,j})$ and $(D_{i,j})$, respectively, denote the matrices of $B_{\overline{\psi_1}}B_{\psi_1} - B_{\psi_1}B_{\overline{\psi_1}}$ and $B_{\overline{\psi_2}}B_{\psi_2} - B_{\psi_2}B_{\overline{\psi_2}}$. Using Lemma 3.2 we have with obvious notations

$$C_{i,i} = \sum_{1\le n\le i} E_{i,n}|c_n|^2 + F_i|c_{i+1}|^2 + \sum_{n\ge i+2} G_{i,n}|c_n|^2$$

where, for example, we have

$$\begin{aligned}E_{i,n} = \frac{2^{2i}(i+1)^2}{2^{2i+2} - 1 - (2i+2)\ln 2}\,\frac{(2^{2(n+i)+2} - 1 - (2(n+i)+2)\ln 2)}{2^{2(n+i)}(n+i+1)^2}\\ - \frac{2^{2i+2} - 1 - (2i+2)\ln 2}{2^{2i}(i+1)^2}\,\frac{2^{2(i-n)}(i-n+1)^2}{(2^{2(i-n)+2} - 1 - (2(i-n)+2)\ln 2)}.\end{aligned}$$

We see that $E_{i,n}$ is positive, since it is the ith diagonal term of $B_{\overline{z}^n}B_{z^n} - B_{z^n}B_{\overline{z}^n}$. Clearly F_i and $G_{i,n}$ are positive. A computation shows that

$$\lim_{i\to\infty} i^2E_{i,n} = 2n^2.$$

Writing $\sum_{1\le n\le i} i^2E_{i,n}|c_n|^2$ as an integral with respect to the counting measure and applying Fatou's lemma we obtain

$$\liminf_{i\to\infty} i^2C_{i,i} = \sum_{n\ge1} 2n^2|c_n|^2 \le \sum_1^\infty 2n^2\big(|a_n|^2 + |b_n|^2\big) < \infty.$$

We can deduce that $i^2\sum_{n\ge i+2} G_{i,n}|c_n|^2 \to 0$ as $i \to \infty$ in a similar way, and it is clear that $\lim_{i\to\infty} i^2F_i|c_{i+1}|^2 = 0$. Thus $\psi_1' \in H^2$. We similarly verify that $\widetilde{\psi_2}' \in H^2$. We get that $\lim_{i\to\infty} i^2\Psi_{i+p,i} = \varrho_{s+p,s}$ where $(\varrho_{i,j})$ is the matrix

of the Hardy space Toeplitz operator with symbol $2|\psi_1'|^2 + 2|\widetilde{\psi_2}'|^2 - 4\mathrm{Re}\Big(\psi_1'\widetilde{\psi_2}' - (c_1\overline{d_1} + 2(c_2\overline{d_1} + c_1\overline{d_2})z)\Big)$. Finally, by a property of Toeplitz forms [16, Theorem (c), p. 19], the inequality $H^*_{\overline{\psi}}H_{\psi} \leq H^*_{\overline{\varphi}}H_{\varphi}$ implies

$$|\psi_1'|^2 + |\widetilde{\psi_2}'|^2 - 2\ \mathrm{Re}\Big(\psi_1'\widetilde{\psi_2}' - (c_1\overline{d_1} + 2(c_2\overline{d_1} + c_1\overline{d_2})z\Big)$$
$$\leq |\varphi_1'|^2 + |\widetilde{\varphi_2}'|^2 - 2\ \mathrm{Re}\Big(\varphi_1'\widetilde{\varphi_2}' - (a_1\overline{b_1} + 2(a_2\overline{b_1} + a_1\overline{b_2})z\Big).$$

□

Corollary 3.10. *If φ_1, φ_2, ψ_1, ψ_2 are analytic and bounded, with φ_1', $\widetilde{\varphi_2}'$ in H^2 and satisfy $c_1\overline{d_1} = c_2\overline{d_1} + c_1\overline{d_2} = a_1\overline{b_1} = a_2\overline{b_1} + a_1\overline{b_2} = 0$. Then if $B_{\varphi+\overline{\psi}}$ is hyponormal we have ψ_1', $\widetilde{\psi_2}'$ are in H^2 and $|\psi_1' - \widetilde{\psi_2}'| \leq |\varphi_1' - \widetilde{\varphi_2}'|$ a.e on the unit circle.*

4 The sufficient condition

In the second part of this work we show a sufficient condition for hyponormality of Toeplitz operators on the Bergman space of the unit disk, with a square logarithmic weight, in the case the symbol is of the form $z^m + \overline{g}$, where g is a polynomial of degree m. We start with some definitions and notations. We consider the Hilbert space of measurable functions on the unit disk D such that $\int_D |f(z)|^2 d\nu(z) < \infty$, where $\mathrm{d}\nu(z) = \frac{2}{\pi}(\log|z|)^2 \mathrm{d}A(z)$, and $dA(z)$ is the Lebesgue measure. When f is analytic on D, we have $f = \sum a_n z^n$ and $\|f\|^2 = \int_D |f(z)|^2 \mathrm{d}\mu(z) = \sum \frac{1}{(n+1)(n+2)^2}|a_n|^2$. Denote by $L^2_{a,\nu}$ the closed subspace of such functions. Its orthonormal basis is given by $\{\sqrt{n+1}(n+2)z^n,\ n \geq 0\}$. Toeplitz operators are defined by $B_{\varphi}(f) = P(f\varphi)$, where φ is a bounded and measurable function on the disk, f is in $L^2_{a,\nu}$, and P is the orthogonal projection of $L^2(D, d\nu)$ on $L^2_{a,\nu}$. Hankel operators are defined by $H_{\varphi}(f) = (I - P)(f\varphi)$. Basic properties of Toeplitz operators and Hankel operators on $L^2_{a,\nu}$ are identical to the ones on A^2_{μ} listed in the first part, and to the case of the unweighted Bergman space of the unit disk [17,18]. We need the following computational lemma, where $m \geq 1$.

Lemma 4.1. *The matrix of $H^*_{\overline{z^m}}H_{\overline{z^m}}$ in the orthonormal basis $\Big\{\sqrt{i+1}(i+2)z^i,\ i \geq 0\Big\}$ is diagonal and is given by:*

$$d_i = \begin{cases} \dfrac{(i+1)(i+2)^2}{(i+1+m)(i+2+m)^2}, & i < m, \\ \dfrac{(i+1)(i+2)^2}{(i+1+m)(i+2+m)^2} - \dfrac{(i+1-m)(i+2-m)^2}{(i+1)(i+2)^2}, & i \geq m. \end{cases}$$

This leads to the following proposition.

Proposition 4.2. *Let m, n be two integers with $n < m$. The operator $B_{z^m+\lambda\overline{z}^n}$ is hyponormal on $L^2_{a,\nu}$ if and only if $|\lambda| \leq \sqrt{\frac{n+1}{m+1}}\frac{n+2}{m+2}$.*

Proof. Hyponormality is equivalent to three inequalities

$$|\lambda|^2 \frac{(i+1)(i+2)^2}{(i+1+n)(i+2+n)^2} \leq \frac{(i+1)(i+2)^2}{(i+1+m)(i+2+m)^2},\ i \leq n-1, \tag{2}$$

$$|\lambda|^2 \frac{(i+1)(i+2)^2}{(i+1+n)(i+2+n)^2} - \frac{(i+1-n)(i+2-n)^2}{(1+1)(i+2)^2} \leq \frac{(i+1)(i+2)^2}{(i+1+m)(i+2+m)^2},\ n \leq i \leq m-1, \tag{3}$$

$$|\lambda|^2 \frac{(i+1)(i+2)^2}{(i+1+n)(i+2+n)^2} - \frac{(i+1-n)(i+2-n)^2}{(1+1)(i+2)^2} \leq \frac{(i+1)(i+2)^2}{(i+1+m)(i+2+m)^2} - \frac{(i+1-m)(i+2-m)^2}{(1+1)(i+2)^2}, i \geq m. \tag{4}$$

Inequality (2) is equivalent to

$$|\lambda| \leq \min\left\{ \sqrt{\frac{i+1+n}{i+1+m}}\frac{i+2+n}{i+2+m},\ i \leq n-1 \right\}.$$

Clearly $\sqrt{\frac{i+1+n}{i+1+m}}\frac{i+2+n}{i+2+m}$ increases with i. Thus (2) is equivalent to

$$|\lambda| \leq \sqrt{\frac{n+1}{m+1}}\frac{n+2}{m+2}.$$

Inequality (3) is equivalent to

$$|\lambda|^2 \leq \min\left\{ \frac{i+1+n}{i+1+m}\frac{(i+2+n)^2}{(i+2+m)^2}\frac{(i+1)^2(i+2)^4}{(i+1)^2(i+2)^4-((i+1)^2-n^2)((i+2)^2-n^2)^2},\ n \leq i \leq m-1 \right\}. \quad (5)$$

It is easy to see that $\frac{i+1+n}{i+1+m}\frac{(i+2+n)^2}{(i+2+m)^2}$ increases with i. The expression $1-\frac{n^2}{(i+1)^2}$ increases with i, and so does $(1-\frac{n^2}{(i+2)^2})^2$. Thus

$$1-\left(1-\frac{n^2}{(i+1)^2}\right)\left(1-\frac{n^2}{(i+2)^2}\right)^2$$

is decreasing and its inverse $\frac{(i+1)^2(i+2)^4}{(i+1)^2(i+2)^4-((i+1)^2-n^2)((i+2)^2-n^2)^2}$ is increasing and the minimum in (5) is assumed at $i=n$. We conclude that inequality (3) is equivalent to

$$|\lambda| \leq \sqrt{\frac{2n+1}{n+m+1}}\frac{(2n+2)}{(n+m+2)}\frac{(n+1)(n+2)^2}{\sqrt{(n+1)^2(n+2)^4-((n+1)^2-n^2)((n+2)^2-n^2)^2}}.$$

Clearly the right hand side of the last inequality is greater than $\sqrt{\frac{n+1}{m+1}}\frac{n+2}{m+2}$. So if inequality (2) is satisfied so is (3). Inequality (4) is equivalent to

$$|\lambda|^2 \leq \inf\left\{ \frac{i+1+n}{i+1+m}\frac{(i+2+n)^2}{(i+2+m)^2}\frac{(i+1)^2(i+2)^4-((i+1)^2-m^2)((i+2)^2-m^2)^2}{(i+1)^2(i+2)^4-((i+1)^2-n^2)((i+2)^2-n^2)^2},\ i \geq m \right\}.$$

Since $m>n$ we have

$$\frac{(i+1)^2(i+2)^4-((i+1)^2-m^2)((i+2)^2-m^2)^2}{(i+1)^2(i+2)^4-((i+1)^2-n^2)((i+2)^2-n^2)^2} > 1.$$

As mentioned before $\frac{i+1+n}{i+1+m}\frac{(i+2+n)^2}{(i+2+m)^2}$ increases with i, and thus if inequality (2) is satisfied, so is (4). Thus hyponormality of $B_{z^m+\lambda\overline{z}^n}$ is equivalent to $|\lambda| \leq \sqrt{\frac{n+1}{m+1}}\frac{n+2}{m+2}$. □

Notice the result holds also if $m=n$. It is not difficult to see that if $B_{z^m+\overline{g_1}}$ and $B_{z^m+\overline{g_2}}$ are hyponormal on $L^2_{a,v}$, and a and b are two complex numbers such that $|a|+|b| \leq 1$, then $B_{z^m+a\overline{g_1}+b\overline{g_2}}$ is hyponormal. We state our second main result.

Theorem 4.3. *Assume $\lambda_1,\ldots,\lambda_m$ are complex numbers such that $\sum_{1\leq n\leq m}|\lambda_n| \leq 1$, and set $g=\sum_{1\leq n\leq m}\lambda_n\sqrt{\frac{n+1}{m+1}}\frac{n+2}{m+2}z^n$. Then we have $B_{\overline{g}}B_g - B_gB_{\overline{g}} \leq B_{\overline{z}^m}B_{z^m} - B_{z^m}B_{\overline{z}^m}$.*

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