# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface to First Edition</td>
<td>v</td>
</tr>
<tr>
<td>Preface to Second Edition</td>
<td>vii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>ix</td>
</tr>
<tr>
<td>Introduction</td>
<td>xvii</td>
</tr>
<tr>
<td>Notes to the Reader</td>
<td>xxv</td>
</tr>
</tbody>
</table>

**CHAPTER 1**

Introduction to Lie Groups 1

1.1. Manifolds 2
   - Change of Coordinates 6
   - Maps Between Manifolds 7
   - The Maximal Rank Condition 7
   - Submanifolds 8
   - Regular Submanifolds 11
   - Implicit Submanifolds 11
   - Curves and Connectedness 12

1.2. Lie Groups 13
   - Lie Subgroups 17
   - Local Lie Groups 18
   - Local Transformation Groups 20
   - Orbits 22

1.3. Vector Fields 24
   - Flows 27
   - Action on Functions 30
   - Differentials 32
   - Lie Brackets 33
   - Tangent Spaces and Vectors Fields on Submanifolds 37
   - Frobenius' Theorem 38
1.4. Lie Algebras
One-Parameter Subgroups
Subalgebras
The Exponential Map
Lie Algebras of Local Lie Groups
Structure Constants
Commutator Tables
Infinitesimal Group Actions
1.5. Differential Forms
Pull-Back and Change of Coordinates
Interior Products
The Differential
The de Rham Complex
Lie Derivatives
Homotopy Operators
Integration and Stokes' Theorem
Notes
Exercises

CHAPTER 2
Symmetry Groups of Differential Equations
2.1. Symmetries of Algebraic Equations
Invariant Subsets
Invariant Functions
Infinitesimal Invariance
Local Invariance
Invariants and Functional Dependence
Methods for Constructing Invariants
2.2. Groups and Differential Equations
2.3. Prolongation
Systems of Differential Equations
Prolongation of Group Actions
Invariance of Differential Equations
Prolongation of Vector Fields
Infinitesimal Invariance
The Prolongation Formula
Total Derivatives
The General Prolongation Formula
Properties of Prolonged Vector Fields
Characteristics of Symmetries
2.4. Calculation of Symmetry Groups
2.5. Integration of Ordinary Differential Equations
First Order Equations
Higher Order Equations
Differential Invariants
Multi-parameter Symmetry Groups
Solvable Groups
Systems of Ordinary Differential Equations
Table of Contents

2.6. Nondegeneracy Conditions for Differential Equations 157
   Local Solvability 157
   Invariance Criteria 161
   The Cauchy–Kovalevskaya Theorem 162
   Characteristics 163
   Normal Systems 166
   Prolongation of Differential Equations 166
Notes 172
Exercises 176

CHAPTER 3
Group-Invariant Solutions 183
3.1. Construction of Group-Invariant Solutions 185
3.2. Examples of Group-Invariant Solutions 190
3.3. Classification of Group-Invariant Solutions 199
   The Adjoint Representation 199
   Classification of Subgroups and Subalgebras 203
   Classification of Group-Invariant Solutions 207
3.4. Quotient Manifolds 209
   Dimensional Analysis 214
3.5. Group-Invariant Prolongations and Reduction 217
   Extended Jet Bundles 218
   Differential Equations 222
   Group Actions 223
   The Invariant Jet Space 224
   Connection with the Quotient Manifold 225
   The Reduced Equation 227
   Local Coordinates 228
Notes 235
Exercises 238

CHAPTER 4
Symmetry Groups and Conservation Laws 242
4.1. The Calculus of Variations 243
   The Variational Derivative 244
   Null Lagrangians and Divergences 247
   Invariance of the Euler Operator 249
4.2. Variational Symmetries 252
   Infinitesimal Criterion of Invariance 253
   Symmetries of the Euler–Lagrange Equations 255
   Reduction of Order 257
4.3. Conservation Laws 261
   Trivial Conservation Laws 264
   Characteristics of Conservation Laws 266
4.4. Noether's Theorem 272
   Divergence Symmetries 278
Notes 281
Exercises 283
CHAPTER 5
Generalized Symmetries 286

5.1. Generalized Symmetries of Differential Equations 288
   Differential Functions 288
   Generalized Vector Fields 289
   Evolutionary Vector Fields 291
   Equivalence and Trivial Symmetries 292
   Computation of Generalized Symmetries 293
   Group Transformations 297
   Symmetries and Prolongations 300
   The Lie Bracket 301
   Evolution Equations 303

5.2. Recursion Operators, Master Symmetries and Formal Symmetries 304
   Fréchet Derivatives 307
   Lie Derivatives of Differential Operators 308
   Criteria for Recursion Operators 310
   The Korteweg–de Vries Equation 312
   Master Symmetries 315
   Pseudo-differential Operators 318
   Formal Symmetries 322

5.3. Generalized Symmetries and Conservation Laws 328
   Adjoins of Differential Operators 328
   Characteristics of Conservation Laws 330
   Variational Symmetries 331
   Group Transformations 333
   Noether’s Theorem 334
   Self-adjoint Linear Systems 336
   Action of Symmetries on Conservation Laws 341
   Abnormal Systems and Noether’s Second Theorem 342
   Formal Symmetries and Conservation Laws 346

5.4. The Variational Complex 350
   The D-Complex 351
   Vertical Forms 353
   Total Derivatives of Vertical Forms 355
   Functionals and Functional Forms 356
   The Variational Differential 361
   Higher Euler Operators 365
   The Total Homotopy Operator 368

Notes 374
Exercises 379

CHAPTER 6
Finite-Dimensional Hamiltonian Systems 389

6.1. Poisson Brackets 390
   Hamiltonian Vector Fields 392
   The Structure Functions 393
   The Lie–Poisson Structure 396
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2. Symplectic Structures and Foliations</td>
<td></td>
</tr>
<tr>
<td>The Correspondence Between One-Forms and Vector Fields</td>
<td>398</td>
</tr>
<tr>
<td>Rank of a Poisson Structure</td>
<td>399</td>
</tr>
<tr>
<td>Symplectic Manifolds</td>
<td>400</td>
</tr>
<tr>
<td>Maps Between Poisson Manifolds</td>
<td>401</td>
</tr>
<tr>
<td>Poisson Submanifolds</td>
<td>402</td>
</tr>
<tr>
<td>Darboux' Theorem</td>
<td>404</td>
</tr>
<tr>
<td>The Co-adjoint Representation</td>
<td>406</td>
</tr>
<tr>
<td>6.3. Symmetries, First Integrals and Reduction of Order</td>
<td></td>
</tr>
<tr>
<td>First Integrals</td>
<td>408</td>
</tr>
<tr>
<td>Hamiltonian Symmetry Groups</td>
<td>409</td>
</tr>
<tr>
<td>Reduction of Order in Hamiltonian Systems</td>
<td>412</td>
</tr>
<tr>
<td>Reduction Using Multi-parameter Groups</td>
<td>416</td>
</tr>
<tr>
<td>Hamiltonian Transformation Groups</td>
<td>418</td>
</tr>
<tr>
<td>The Momentum Map</td>
<td>420</td>
</tr>
<tr>
<td>Notes</td>
<td>427</td>
</tr>
<tr>
<td>Exercises</td>
<td>428</td>
</tr>
<tr>
<td>CHAPTER 7</td>
<td></td>
</tr>
<tr>
<td>Hamiltonian Methods for Evolution Equations</td>
<td></td>
</tr>
<tr>
<td>7.1. Poisson Brackets</td>
<td></td>
</tr>
<tr>
<td>The Jacobi Identity</td>
<td>434</td>
</tr>
<tr>
<td>Functional Multi-vectors</td>
<td>436</td>
</tr>
<tr>
<td>7.2. Symmetries and Conservation Laws</td>
<td></td>
</tr>
<tr>
<td>Distinguished Functionals</td>
<td>446</td>
</tr>
<tr>
<td>Lie Brackets</td>
<td>446</td>
</tr>
<tr>
<td>Conservation Laws</td>
<td>447</td>
</tr>
<tr>
<td>7.3. Bi-Hamiltonian Systems</td>
<td></td>
</tr>
<tr>
<td>Recursion Operators</td>
<td>452</td>
</tr>
<tr>
<td>Notes</td>
<td>461</td>
</tr>
<tr>
<td>Exercises</td>
<td>463</td>
</tr>
<tr>
<td>References</td>
<td>467</td>
</tr>
<tr>
<td>Symbol Index</td>
<td>489</td>
</tr>
<tr>
<td>Author Index</td>
<td>497</td>
</tr>
<tr>
<td>Subject Index</td>
<td>501</td>
</tr>
</tbody>
</table>