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# Iteration of Rational Functions

*Complex Analytic Dynamical Systems*

With 35 Illustrations



Springer

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