On Representations of Affine Hecke Algebras of Type $B$
## CONTENTS

Introduction 4  
Acknowledgments 6  
Notation 7  
1. The Algebras 9  
2. The Main Tools – Mackey Theorem and Duality 28  
3. Formal Characters 36  
4. Crystal operators 46  
5. Facts About Representations of the Affine Hecke Algebra of Type A 55  
6. Classification For Subcategories 58  
7. Eigenvalues 1 and $-1$ 61  
Appendix A. German Abstract 82  
References 88
Introduction

Affine Hecke algebras form an intriguing field of study as there are many interactions with fascinating areas and questions in representation theory. For example, the category of finite-dimensional complex representations of an affine Hecke algebra is equivalent to a certain subcategory of representations of a $p$-adic algebraic group. Certain finite-dimensional quotients of affine Hecke algebras, e.g. Ariki-Koike algebras, also play important roles in questions arising in the representation theory of finite groups of Lie type.

In type $A$, Grojnowski [8] proved that the sum over $n$ of the Grothendieck groups of the affine Hecke algebras $\mathcal{H}_n^A$ of type $A$ corresponding, in some sense, to $GL_n$ is isomorphic to the dual of the positive part of the Kostant $\mathbb{Z}$-form $U_+^+$ of the universal enveloping algebra of $\hat{\mathfrak{sl}}_l$, where $l$ is the multiplicative order of the deformation parameter involved in the definition of $\mathcal{H}_n^A$. Certain crystal operators on the irreducible modules give partial branching rules and it is shown that the irreducible modules correspond to the crystal basis of $U_+^+$, thus closely relating the representation theory of affine Hecke algebras to that of Kac-Moody algebras and quantum groups.

In this thesis, we apply Grojnowski’s methods to affine Hecke algebras of type $B$.

In the first chapter we give an overview of some different ways in which affine Hecke algebras arise and explain correlations of different definitions of an affine Hecke algebra. Then we define the affine Hecke algebra $\mathcal{H}_n$, which is the main interest of study, along with the subalgebra $\mathcal{H}_n^R$, which we will investigate in Chapter 6. We apply Clifford theory to clarify the representation theoretic relationship between the two algebras. The Chapter concludes with some computational lemmas that will be of use later on.

The second through fourth chapters closely follow [3] and informal lecture notes [13] that are now part of [14]. The second chapter provides an affine version of the Mackey Theorem, yielding a filtration instead of a decomposition, and investigates the effect of dualizing with respect to a certain antiautomorphism $\tau$, resulting in a relation between the induction and coinduction functors.
The third chapter introduces the concept of formal characters, which are the main tool in understanding finite-dimensional irreducible modules for the affine Hecke algebras of type $A$, but as it turns out in Chapter 4, they only serve their purpose in type $B$ when we restrict ourselves to a subcategory with additional conditions on the eigenvalues of certain algebra elements on the modules. In this subcategory we obtain a similar result as in type $A$, namely that all irreducible modules can be obtained as the irreducible cosocles of modules induced from certain parabolic subalgebras and that the restriction of an $\mathcal{H}_n$-module to $\mathcal{H}_{n-1}$ has a multiplicity-free socle. The fourth chapter also introduces the so-called crystal operators as well as the crystal graph which are analogs of the operators on module categories in type $A$ which actually link the representation theory of the affine Hecke algebra of type $A$ to the theory of crystal bases.

The fifth chapter gives a summary of results in type $A$, especially the combinatorial description of the crystal graph.

In Chapter 6 we consider subcategories where the irreducible modules are obtained by induction from irreducibles in type $A$, which gives us explicit formulae for the action of the crystal operators in those cases. The last chapter then deals with the cases where the formal character arguments used in type $A$ do not apply. We give a condition under which the results from Chapter 4 can be recovered even in those cases and then proceed to giving partial results and examples in the other cases.
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Notation

$W_n^{\text{fin}}$ finite Weyl group of type $B_n$ 10
$R_n$ weight lattice of $SO_n$ or $GL_n$ 11
$P_n$ weight lattice of $GO_n$ 11
$\mathcal{P}_n = F[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]$ Laurent polynomials 12
$\mathcal{H}_n^{\text{fin}}$ finite Hecke algebra of type $B$ 12
$\mathcal{H}_n = \mathcal{H}_n^P$ affine Hecke algebra of type $B$ using $P_n$ 12
$\mathcal{H}_n^R$ affine Hecke algebra of type $B$ using $R_n$ 13
$\mathcal{R}_n = F[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ Laurent polynomials 13
$A\text{-mod}^{\text{fd}}$ finite-dimensional modules 13
$K(A\text{-mod}^{\text{fd}})$ Grothendieck group 14
$\sigma$ automorphism $X_0 \mapsto -X_0$ 19
$T_{k,l}$ $T_k \cdots T_l$ 21
$T_{k,0,l}$ $T_k T_{k-1} \cdots T_0 T_1 \cdots T_l$ 21
$s_{k,l}$ $s_k \cdots s_l$ 21
$s_{k,0,l}$ $s_k s_{k-1} \cdots s_0 s_1 \cdots s_l$ 21
$<$ Bruhat order on $W_n^{\text{fin}}$ 25
$W_I^{\text{fin}}$ parabolic subgroup of $W_n^{\text{fin}}$ 28
$(m_1, \ldots, m_l)$ special case of $I$ 28
$\mathcal{H}_I$ parabolic subalgebra of $\mathcal{H}_n$ 28
$\text{ind}_I^J, \text{res}_I^J$ $\text{ind}_{\mathcal{H}_I}^{\mathcal{H}_J}, \text{res}_{\mathcal{H}_I}^{\mathcal{H}_J}$ 29
$D_I, D_I^{-1}$ distinguished left/right coset representatives 29
$D_{I,J}$ distinguished double coset representatives 29
$dM$ twisted module 33
$\tau$ antiautomorphism, identity on generators 34
$(\underline{a}) = (a_0, \ldots, a_n)$ one-dimensional module for $\mathcal{P}_n$ 36
$M[\underline{a}]$ submodule: only composition factors $(\underline{a})$ 36
$\text{ch}$ formal characters 37
$\chi_\alpha, \chi_\gamma$ central characters 40
$\mathcal{H}_n[\gamma]\text{-mod}^{\text{fd}}$ block corresponding to $\chi_\gamma$ 41
$L^A(\alpha^{(m)})$ Kato module in type $A$ 41
$\Delta_a, \Delta_a^{(m)}$ functor picking generalised $a$ eigenspaces 46
$\epsilon_a(M)$ length of longest $a$-tail in $\text{ch} M$ 47
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_a, c_a$</td>
<td>$a$-induction, $a$-restriction</td>
<td>49</td>
</tr>
<tr>
<td>$\tilde{f}_a, \tilde{c}_a$</td>
<td>crystal operators</td>
<td>49</td>
</tr>
<tr>
<td>$L(a)$</td>
<td>labeling via crystal paths</td>
<td>51</td>
</tr>
<tr>
<td>$\text{Rep}\mathcal{H}_n$</td>
<td>no eigenvalues $\pm 1$</td>
<td>51</td>
</tr>
<tr>
<td>$\text{Rep}_\lambda\mathcal{H}_n^A$</td>
<td>only eigenvalues in $I_{\lambda}^+$</td>
<td>55</td>
</tr>
<tr>
<td>$M_{\Gamma}$</td>
<td>irreducible in type $A$ labeled by multisegment $\Gamma$</td>
<td>55</td>
</tr>
<tr>
<td>$\tilde{c}_a^A, \tilde{f}_a^A$</td>
<td>crystal operators in type $A$</td>
<td>56</td>
</tr>
<tr>
<td>$\tilde{c}_a^A, \tilde{f}_a^A$</td>
<td>crystal operators in type $A$ acting on first position</td>
<td>56</td>
</tr>
<tr>
<td>$\epsilon_{\lambda}^*(M)$</td>
<td>$\max{k \mid \tilde{c}_a^A M \neq 0}$</td>
<td>56</td>
</tr>
<tr>
<td>$\text{Rep}_\lambda\mathcal{H}_n$</td>
<td>only eigenvalues in $I_{\lambda}$</td>
<td>58</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>antiautomorphism $d \circ \tau$</td>
<td>58</td>
</tr>
</tbody>
</table>
1. The Algebras

An affine Hecke algebra in general is the deformation of a (possibly extended) affine Weyl group. Affine Weyl groups arise naturally in representation theory of objects of affine (tame) type in the same ways in which finite Weyl groups arise when dealing with objects classified by Dynkin diagrams. For example, finite Weyl groups are associated to finite-dimensional simple Lie algebras, and analogously, affine Weyl groups are associated to Kac-Moody Lie algebras of affine type. The affine Weyl group in generators and relations can be read off from the Euclidian diagram classifying the affine type in the same way in which any Coxeter group can be read off from the corresponding Coxeter graph: we take a generator for every vertex of the graph and on the set \( \{ s_i \mid i \in I \} \) of generators we impose relations

\[
\begin{align*}
{s_i}^2 &= 1 \\
(s_i s_j)^{m_{i,j} + 2} &= 1
\end{align*}
\]

for all \( i, j \in I \), where \( m_{i,j} \) denotes the number of edges between vertex \( i \) and vertex \( j \). For explicit computations though, the more convenient presentation is the one as semidirect product of the finite Weyl group of the corresponding diagram of finite type with a specific weight lattice. This can be found in [11], Proposition 6.5.

A finite Weyl group \( W_0 \) may also be obtained from a connected reductive group \( G \) over \( \mathbb{C} \) with a maximal torus \( T \) as the quotient of the normalizer of \( T \) in \( G \) by \( T \), \( W_0 = N_G(T)/T \). This acts on the character group \( X = \text{Hom}(T, \mathbb{C}^*) \), so we may form the semidirect product \( W = W_0 \ltimes X \). A group \( W \) arising in this way is called an extended affine Weyl group, the actual affine Weyl group coming from one particular group \( G \) depending on the type of \( W_0 \), compare [19], Chapter 2. Different extended affine Weyl groups with the same underlying finite Weyl group are called isogenous.

A deformation of a Coxeter group is obtained by altering the eigenvalues of the generators from 1 and \(-1\) to some deformation parameter and the negative of its inverse. Here any two generators must be deformed with the same parameter if they are conjugate in the group.
Since now a generator and its inverse no longer coincide we have to decide how to distribute inverses in the mixed relations and we do this by placing inverses on the second half of the generators. Thus, the deformation of a group algebra of the above Coxeter group would be an associative algebra over some field $F$ containing a set of deformation parameters $\{q_i \mid i \in I\}$ with generators $\{T_i \mid i \in I\}$ and relations

\[(T_i - q_i)(T_i + q_i^{-1}) = 0\]
\[(T_j T_i)^{\frac{m_{i,j}+1}{2}} = (T_i T_j)^{\frac{m_{i,j}+1}{2}} \quad \text{if } m_{i,j} \text{ is even}\]
\[(T_j T_i)^{\frac{m_{i,j}+1}{2}} T_i = (T_i T_j)^{\frac{m_{i,j}+1}{2}} T_j \quad \text{if } m_{i,j} \text{ is odd}.\]

We now consider type $B_n^{(1)} = B\tilde{D}_n$, labeled by the Euclidian diagram

\[
\begin{array}{cccccccccccc}
& & & & & & & & & & \omega \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & 0 \\
0 & 1 & 2 & \cdots & n-2 & n-1
\end{array}
\]

which is obtained from the Dynkin diagram of type $B_n$ by adding the vertex $\omega$ and give an overview of the results given in [11]. Here $m_{i,i+1} = m_{i+1,i} = 1$ for $i = 1, \ldots, n - 1$ and $m_{n-2,\omega} = m_{\omega,n-2} = 1$. Furthermore $m_{0,1} = m_{1,0} = 2$, and all other $m_{i,j} = 0$. The affine Weyl group $W_n$ has generators $\{s_0, \ldots, s_{n-1}, s_\omega\}$ and relations

\[s_i^2 = 1 \quad \text{for } i \in \{0, 1, \ldots, n - 1, \omega\}\]
\[s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0\]
\[s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } 1 \leq i \leq n - 2\]
\[s_{n-2} s_\omega s_{n-2} = s_\omega s_{n-2} s_\omega\]

and all other pairs of generators commute. The subgroup

\[W_n^{\text{fin}} := \langle s_0, \ldots, s_{n-1} \rangle\]

is the finite Weyl group of type $B_n$, which is isomorphic to the wreath product of a cyclic group of order two and $S_n$, the symmetric group on $n$ letters. In [11], §6.7, the element

\[t := s_\omega s_{n-2} s_{n-3} \cdots s_1 s_0 s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 s_0 s_1 \cdots s_{n-3} s_{n-2},\]
which is the product of the new generator $s_\omega$ with the reflection at the highest positive root $\beta = s_{n-2} \cdots s_1 s_0 s_1 \cdots s_{n-3} s_{n-2} \alpha_{n-1}$, where $\alpha_{n-1}$ denotes the simple root corresponding to the node $n-1$ in the Euclidian diagram above, is introduced.

This element is of infinite order and, under the action of $W_n^{\text{fin}}$ generates the normal subgroup called the translation lattice. In type $B_n^{(1)}$, this turns out to be isomorphic to the weight lattice $\mathcal{R}_n$ of the special orthogonal group $SO_n$ which is also the root lattice of type $B_n$. The root system has a realization in the Euclidian space $\mathbb{R}^n$, where all roots are given by $\{e_i, \pm e_i \pm e_j \mid 1 \leq i, j \leq n\}$ for an orthonormal standard basis $\{e_i \mid 1 \leq i \leq n\}$ of $\mathbb{R}^n$. A basis for the root system is given by the set $\{e_1, e_{i+1} - e_i \mid 1 \leq i \leq n - 1\}$, cf. [10], Section 12.1. Multiplicatively written, the root lattice is the free abelian group generated by $X_1, \ldots, X_n$, on which the subgroup

$$\langle s_1, \ldots, s_{n-1} \rangle \cong S_n$$

acts by the natural permutation representation and the generator $s_0$ inverts $X_1$, i.e. $X_1 s_0 = s_0 X_1^{-1}$, and commutes with all other $X_i$. Here, $X_i$ corresponds to the root $e_i$ in the additive root system. In this presentation as semidirect product, the element $t$ corresponds to the product $X_{n-1} X_n$, see [11], §6.7. All $s_i$ but $s_0$ are conjugate in $W_n$ as they are joined by single edges, giving $s_i = (s_{i+1} s_i) s_{i+1} (s_i s_{i+1})$ for $1 \leq i \leq n-2$ and $s_\omega = (s_{n-2} s_\omega) s_{n-2} (s_\omega s_{n-2})$, so we have to choose the same deformation parameter $q$ for all generators $s_1, \ldots, s_{n-1}, s_\omega$, while we will, in full generality, have a different deformation parameter $p$ for the generator corresponding to the node 0.

We will in general use a larger lattice, namely the weight lattice $P_n$ of the general orthogonal group $GO_n$, the group of $n \times n$-matrices $A$ with respect to an orthonormal basis leaving the orthogonal form invariant up to a scalar, i.e. satisfying $A^T A = \lambda E$ for some scalar $\lambda$. This lattice has an extra generator $X_0$ which commutes with the symmetric group part of $W_n^{\text{fin}}$ and satisfies the relation $X_0 s_0 = s_0 X_0 X_1$. The reason for using this lattice is vaguely speaking – we will see this in more detail in the part about Clifford theory at the end of this Chapter – that a
larger lattice helps to distinguish representations by only looking at the action of the lattice.

The isomorphism between the two different presentations of affine Weyl groups – the Coxeter presentation and the presentation as semidirect product of the finite Weyl group with a translation lattice – carries over to affine Hecke algebras. Thus, fixing an algebraically closed field $F$ of characteristic not equal to two containing deformation parameters $p$ and $q$ which are not roots of unity, we arrive at defining $\mathcal{H}_n := \mathcal{H}'_n$ to be the associative $F$-algebra on generators

$$X_0^{\pm 1}, \ldots, X_n^{\pm 1}, T_0, \ldots, T_{n-1},$$

where the $T_i$ generate a finite Hecke algebra $\mathcal{H}_{n}^{\text{fin}}$ of type $B_n$ with relations

\begin{align}
(1) \quad & (T_0 - p)(T_0 + p^{-1}) = 0 \\
(2) \quad & (T_i - q)(T_i + q^{-1}) = 0 \quad \text{for } i \geq 1 \\
(3) \quad & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{for } i \geq 1 \\
(4) \quad & T_iT_j = T_jT_i \quad \text{for } |i - j| > 1 \\
(5) \quad & T_1T_0T_1T_0 = T_0T_1T_0T_1,
\end{align}

the $X_i^{\pm 1}$ generate a Laurent polynomial ring $\mathcal{P}_n$ and those two subalgebras are subject to the mixed relations

\begin{align}
(6) \quad & T_0X_0T_0 = X_0X_1 \\
(7) \quad & T_iX_j = X_jT_i \quad \text{for } j \neq i, i + 1 \\
(8) \quad & T_iX_iT_i = X_{i+1} \quad \text{for } i \geq 1.
\end{align}

For a reduced expression $w = s_{i_1} \cdots s_{i_k}$ of an element $w \in W_n^{\text{fin}}$, we define $T_w := T_{i_1} \cdots T_{i_k}$. This does not depend on the choice of reduced expression and is therefore well-defined.

In [15] Lusztig proves a general result on bases of affine Hecke algebras in the case where $p$ and $q$ are distinct powers of the same deformation parameter $v_0$, but the proof doesn’t rely on this and carries over to the general case, see [19]. In our case this result gives the following two bases for $\mathcal{H}_n$:
Combining this with the well-known basis for $\mathcal{H}_n$ (e.g. [1], Theorem 3.10), $\mathcal{H}_n$ has -- defining the so called Jucys-Murphy elements $L_i := T_{i-1}T_{i-2} \cdots T_1 T_0 T_1 \cdots T_{i-2} T_{i-1}$ -- the following bases:

\[
\left\{ X_0^{c_0}X_1^{c_1} \cdots X_n^{c_n} L_1^{\alpha_1} \cdots L_n^{\alpha_n} T_w \bigg| \begin{array}{c}
(c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}, \\
(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n, \\
w \in S_n \end{array} \right\},
\]

and

\[
\left\{ L_1^{\alpha_1} \cdots L_n^{\alpha_n} T_w X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n} \bigg| \begin{array}{c}
(c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}, \\
(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n, \\
w \in S_n \end{array} \right\}.
\]

The deformation of the affine Weyl group using the root system of type $B$ is naturally a subalgebra of $\mathcal{H}_n$ generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ and $T_0, \ldots, T_{n-1}$ which we denote by $\mathcal{H}_n^R$. Here we need an additional relation

\[
X_1 T_0 = T_0 X_1^{-1} + (p - p^{-1})(X_1 + 1)
\]

which, in $\mathcal{H}_n$, can be derived from relation (6) since

\[
X_1 T_0 = X_0^{-1} T_0 X_0 T_0 T_0
= (p - p^{-1}) X_0^{-1} T_0 X_0 T_0 + X_0^{-1} T_0 X_0
= (p - p^{-1}) X_1 + X_0^{-1} T_0^{-1} X_0 + (p - p^{-1})
= (p - p^{-1})(X_1 + 1) + T_0 X_0^{-1} X_1^{-1} X_0
= T_0 X_1^{-1} + (p - p^{-1})(X_1 + 1).
\]

The commutative subalgebra generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ will be denoted by $\mathcal{R}_n$. All modules in consideration will be left modules that are finite-dimensional over $F$ and the category of such modules for an algebra $A$ will be denoted by $A\text{-mod}^{fd}$. It is well-known that all irreducible representations of $\mathcal{H}_n$ are finite-dimensional, since, by Frobenius reciprocity, we can obtain every irreducible module by taking an irreducible
module of $\mathcal{P}_n$, inducing it to $\mathcal{H}_n$ and taking a quotient thereof. By Hilbert’s Nullstellensatz, all irreducibles for $\mathcal{P}_n$ are one-dimensional, whence every irreducible module for $\mathcal{H}_n$ is a quotient of a module of dimension $2^n n!$, the index of $\mathcal{P}_n$ in $\mathcal{H}_n$.

The Grothendieck group of the category $A$-mod$^{\text{id}}$ will be denoted by $K(A$-mod$^{\text{id}}$). If we have an automorphism $\psi$ of $A$, we will, for any $M \in A$-mod$^{\text{id}}$ denote by $M^\psi$ the module obtained from $M$ by twisting the action with $\psi$. This is equal to $M$ as an abelian group but the operation of $A$ is now via the new multiplication $\odot$ defined by $a \odot m = \psi(a)m$ for $a \in A, m \in M$. The smallest integer $k$ such that $M^{\psi^k} \cong M$ as an $A$-module will be called the order of $\psi$ on $M$ whereas the order of $\psi$ (without specification of a module) will denote the order of $\psi$ on $A$.

We will use Clifford theory to move between modules for both algebras $\mathcal{H}_R^n$ and $\mathcal{H}_P^n$. This idea to explore the interplay between different affine Hecke algebras of the same isogeny class is originally due to Xi [19] and has been worked out in detail by Ram and Ramagge [16]. In fact, Clifford theory works in a more general setting, which has been studied by Dade in [5].

**Lemma 1.1.** Let $n$ be a natural number, $K$ an algebraically closed field of characteristic $p \geq 0$ with $p \nmid n$. Let $A$ be a $K$-algebra and let $B$ be a subalgebra of $A$, such that $A$ is free as a $B$-module on basis \{x$^s | 0 \leq s \leq n - 1$\} for an invertible element $x$ in $A$, and $\mathbb{Z}/n\mathbb{Z}$-graded, i.e. $Bx^sBx^t = Bx^{s+t}$. Let $\sigma$ be an automorphism of $A$ with $\sigma|_B = \text{id}_B$. Let $\psi : a \mapsto x^{-1}ax$ be conjugation with $x$, so $\psi(B) = B$. Then the order $d$ of $\psi$ on $N$ divides $n$ and for $k := n/d$ we have

$$\text{res}^A_B M = \bigoplus_{j=0}^{d-1} N^{\psi^j}$$

and

$$\text{ind}^A_B N = \bigoplus_{j=0}^{k-1} M_j$$

for irreducible and pairwise non-isomorphic modules $M_j$. All $\sigma$-conjugates of $M$ occur as some $M_j$ in this decomposition, so in particular, the order of $\sigma$ on $M_j$ is less or equal to $k$ for all $0 \leq j \leq k - 1$. 
Proof. Since \( x^n \in B \), \( N^{\psi^n} \cong N \) for any \( B \)-module \( N \). Now let \( d \) be the smallest natural number such that \( N^{\psi^d} \cong N \) and assume \( d \) does not divide \( n \). Then for \( n = qd + r \), \( N^{\psi^d \psi^r} \cong N \), i.e. \( N^{\psi^r} \cong N \), but \( r < d \), a contradiction. So, indeed \( d \) does divide \( n \).

Let \( f : N \to N^{\psi^d} \) be an isomorphism and note that then \( f^j : N \to N^{\psi^{dj}} \) is also an isomorphism. In particular, since \( x^n \in B \), \( f^k : N \to N^{\psi^{dk}} \) is a scalar multiple of multiplication with \( x^n \), so by normalizing, we can assume that \( f^k \) is in fact multiplication with \( x^{-n} \).

Now take any irreducible \( B \)-submodule \( N \) of \( M \) and consider \( \text{ind}^B_B N \), where we set \( B' := \bigoplus_{0 \leq j \leq k} Bx^j \).

Claim 1: \( \text{ind}^B_B N \) is a completely reducible \( B' \)-module, decomposing into a direct sum of \( k \) non-isomorphic \( B' \)-modules \( L_i, i = 0, \ldots, k - 1 \), where each \( L_i \) is isomorphic to \( N \) as \( B \)-module.

As a \( B \)-module
\[
\text{ind}^B_B N \cong \bigoplus_{j=0}^{k-1} x^j \otimes N.
\]

Let \( \zeta \) be a primitive \( k \)-th root of unity and define \( L_i \) to be the subspace of \( \text{ind}^B_B N \) consisting of all elements
\[
a_i := \sum_{j=0}^{k-1} \zeta^{ji} x^j \otimes f^j(a)
\]
where \( a \) runs through \( N \). It is straightforward to check that \( b \cdot a_i = (ba)_i \), so \( L_i \) is a \( B \)-submodule of \( \text{ind}^B_B N \). \( L_i \) is also a \( B' \)-submodule of \( \text{ind}^B_B N \).

To see this, it suffices to show \( x^d L_i \subseteq L_i \) since
\[
B' = \bigoplus_{j=0}^{k-1} Bx^j
\]
as a \( B \)-module. But for \( a \) in \( N \),
\[
x^d a_i = \sum_{j=0}^{k-1} \zeta^{ji} x^{(j+1)d} \otimes f^j(a)
= \zeta^{-i} \sum_{j=0}^{k-1} \zeta^{(j+1)i} x^{(j+1)d} \otimes f^{j+1} f^{-1}(a)
= \zeta^{-i} (f^{-1}(a))_i,
\]
which is again an element of \( L_i \).

Now let \( 0 \leq i, l \leq k - 1 \) and suppose \( i \neq l \) but \( L_i \cong L_l \), i.e. there exists a \( B' \)-module isomorphism \( g : L_i \rightarrow L_l \). Since \( \text{res}^{B'}_B L_i \) is isomorphic to \( \text{res}^{B'}_B L_l \) via the isomorphism \( \tilde{g} : a_i \mapsto a_l \) and this is the only isomorphism up to a scalar by Schur’s Lemma and the irreducibility of \( \text{res}^{B'}_B L_l \cong N \). Since, if \( g \) is an isomorphism, \( \lambda g \) is, for any \( \lambda \in K \), also an isomorphism, we can choose \( g \) to coincide with the map \( \tilde{g} \). But then

\[
g(x^d a_i) = \zeta^{-i} g((f^{-1}(a)), i)
= \zeta^{-i}(f^{-1}(a))_i
\]

but also

\[
g(x^d a_i) = x^d g(a_i)
= x^d a_l
= \zeta^{-l}(f^{-1}(a))_l
\]

whence we conclude that \( i = l \), contrary to our assumption. This proves Claim 1.

Now let \( L \) be one of the irreducible \( B' \)-submodules of \( \text{ind}^{B'}_B N \) and consider \( \text{ind}^A_{B'} L \). The set \( \{x^j \mid 0 \leq j \leq d - 1\} \) forms a basis of \( A \) as a \( \mathbb{Z}/d\mathbb{Z} \)-graded \( B' \)-module. The automorphism \( \psi \) leaves \( B' \) invariant as it leaves \( B \) invariant and fixes \( x^j \) for \( 0 \leq j \leq k - 1 \). Therefore we can twist any \( B' \)-module with \( \psi \) and again obtain an \( B' \)-module. Since \( N^{\psi j} \not\cong N \) for \( j < d \), we also have \( x^j \otimes L \cong L^{\psi j} \not\cong L \) for \( j < d \).

**Claim 2:** \( \text{ind}^A_{B'} L \) is an irreducible \( A \)-module.

We have

\[
\text{Ann}_{B'} x^j \otimes L \not\supset \bigcap_{0 \leq j \leq d - 1, j \neq i} \text{Ann}_{B'} x^j \otimes L.
\]

Otherwise an inclusion of the annihilators which are the kernels of the representations

\[
\rho_{\neq i} : B' \rightarrow \text{End}_K( \bigoplus_{0 \leq j \leq d - 1, j \neq i} x^j \otimes L)
\]

and

\[
\rho_i : B' \rightarrow \text{End}_K(x^i \otimes L)
\]
would give rise to a projection in the opposite direction on the side of the images, which cannot happen since the $x^j \otimes L$ are pairwise non-isomorphic irreducible $B'$-modules.

Let $0 \neq a = \sum_{j=0}^{d-1} x^j \otimes a_j$ with $a_j \in L$ be an element of $\text{ind}^A_B L$. Suppose $a_i \neq 0$. Then the left ideal $\text{Ann}_{B'} x^i \otimes a_i$ contains the maximal two-sided ideal $\text{Ann}_{B'} x^i \otimes L$. If it also contained $\bigcap_{0 \leq j \leq d-1, j \neq i} \text{Ann}_{B'} x^j \otimes L$, it would contain the two-sided ideal $\text{Ann}_{B'} x^i \otimes L + \bigcap_{0 \leq j \leq d-1, j \neq i} \text{Ann}_{B'} x^j \otimes L$, which, by the above, is strictly larger than $\text{Ann}_{B'} x^i \otimes L$ and thus equals $B'$ by the maximality of $\text{Ann}_{B'} x^i \otimes L$.

Therefore we can find a non-zero element such that

$$ya = yx^i \otimes a_i = x^i \otimes y'a_i \neq 0,$$

where $y' = x^{-i}yx^i$ and therefore we have an element $0 \neq b_i := y'a_i \in L$ such that $x^i \otimes b_i \in B'a$. Thus, $x^i \otimes L$ is a $B'$-submodule of $B'a$ and $x^j \otimes L$ is contained in $B'a$ for all $j = 0, \ldots, d-1$. This proves Claim 2.

Now, what is left for us to prove is that the $M_i := \text{ind}^A_B L_i$ are also pairwise non-isomorphic and that all conjugates of $M$ by distinct powers of $\sigma$ occur in the decomposition.

Frobenius reciprocity gives

$$\text{Hom}_A(M_i, M_l) \cong \text{Hom}_{B'}(L_i, \bigoplus_{j=0}^{d-1} x^j \otimes L_l)$$

$$\cong \text{Hom}_{B'}(L_i, L_l) \oplus \bigoplus_{j=1}^{d-1} \text{Hom}_{B'}(L_i, x^j \otimes L_l).$$

$\text{Hom}_{B'}(L_i, L_l) = 0$ for $i \neq l$, according to Claim 1, while

$$\bigoplus_{j=1}^{d-1} \text{Hom}_{B'}(L_i, x^j \otimes L_l) \hookrightarrow \bigoplus_{j=1}^{d-1} \text{Hom}_B(\text{res}^B_{B'} L_i, \text{res}^B_{B'} x^j \otimes L_l)$$

but the latter is just $\bigoplus_{j=1}^{d-1} \text{Hom}_B(N, N^{\psi^j})$ which is zero by hypothesis. So the $M_i$ are pairwise non-isomorphic.

Now we have $\text{ind}^A_B N \cong \bigoplus_{i=0}^{k-1} M_i$ and $\text{res}^A_B M_i \cong \bigoplus_{j=0}^{d-1} x^j \otimes N$. In particular, this implies the asserted decomposition of $M$, since $M$ is isomorphic
to one of the $M_i$ by Frobenius reciprocity. As $\sigma$ fixes $B$ pointwise, $X^{\sigma^l} = X$ for all $B$-modules $X$ and all $l \in \mathbb{Z}$. So

$$K \cong \text{Hom}_B(N, \text{res}_{B_i}^A M_i)$$
$$\cong \text{Hom}_B(N, \text{res}_{B_i}^A M_i^{\sigma^l})$$
$$\cong \text{Hom}_A(\bigoplus_{j=0}^{k-1} M_j, M_i^{\sigma^l})$$

gives an inclusion of $\{M_i^{\sigma^l} \mid l \in \mathbb{Z}\}$ into $\{M_j \mid 0 \leq j \leq k - 1\}$ for a fixed $i$. Therefore the order of $\sigma$ on $M_i$ is at most $k$ for any $i$.

Now, in order to apply this to our situation, observe that $X_0^2 \prod_{i=1}^n X_i$ is central in $\mathcal{H}_n$. To verify this, note that every generator $T_i$ commutes with all but $X_i$ and $X_{i+1}$. But, for $i \geq 1$, relation (7)

$$T_i X_i X_{i+1} = X_i T_i^{-1} X_{i+1} = X_i X_{i+1} T_i.$$  

Similarly, $X_0^2 X_1 T_0 = X_0 X_1 T_0^{-1} X_0 X_1 = T_0 X_0^2 X_1$, by relation (6) so $X_0^2 \prod_{i=1}^n X_i$ is indeed central and therefore acts as a scalar $\mu_M$ on an irreducible module $M \in \mathcal{H}_n$-mod. Thus, the action of $\mathcal{H}_n$ on an irreducible module $M \in \mathcal{H}_n$-mod factors over the quotient algebra $\mathcal{H}_n^{\mu M} := \mathcal{H}_n / (X_0^2 \prod_{i=1}^n X_i - \mu_M)$.

**Lemma 1.2.** $\mathcal{H}_n^{\mu M}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{H}_n^R$-module with basis $\{1, X_0\}$.

**Proof.** We first show that $\mathcal{H}_n^R$ is contained in $\mathcal{H}_n^{\mu M}$. Since $X_0^2 \prod_{i=1}^n X_i$ is central the ideal generated by $X_0^2 \prod_{i=1}^n X_i - \mu_M$ is already contained in the right ideal $(X_0^2 \prod_{i=1}^n X_i - \mu_M)\mathcal{H}_n$ which is generated over $F$ by

$$(X_0^2 \prod_{i=1}^n X_i - \mu_M) X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n} L_1^{\alpha_1} \cdots L_n^{\alpha_n} T_w$$

$$= (X_0^{c_0+2} X_1^{c_1+1} \cdots X_n^{c_n+1} - \mu_M X_0^{c_0} X_1^{c_1} \cdots X_n^{c_n}) L_1^{\alpha_1} \cdots L_n^{\alpha_n} T_w$$

for $(c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}, (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$ and $w \in S_n$.

No finite linear combination of those can have degree 0 in $X_0$, therefore the intersection of the ideal generated by $(X_0^2 \prod_{i=1}^n X_i - \mu_M)$ with $\mathcal{H}_n^R$ is zero and we can view $\mathcal{H}_n^R$ as a subalgebra of $\mathcal{H}_n^{\mu M}$.  

As \( X_0 \) commutes with all generators \( X_j \) and all generators \( T_j \) except \( T_0 \) and

\[
T_0 X_0 = T_0^{-1} X_0 + (p - p^{-1}) X_0
= X_0 T_0 X_1^{-1} + (p - p^{-1}) X_0
= X_0 (T_0 X_1^{-1} + (p - p^{-1}))
\]

we see that \( \mathcal{H}_n^R X_0 \subseteq X_0 \mathcal{H}_n^R \), which by the same argument as for \( \mathcal{H}_n^R \) above has no nontrivial intersection with the ideal generated by \( (X_0^2 \prod_{i=1}^n X_i - \mu_M) \) and can therefore be viewed as contained in \( \mathcal{H}_n^{\mu M} \). Certainly \( \mathcal{H}_n^R \cap \mathcal{H}_n^R X_0 = \{0\} \), so we have an \( \mathcal{H}_n^R \)-submodule of \( \mathcal{H}_n^{\mu M} \) which is isomorphic to \( \mathcal{H}_n^R \oplus \mathcal{H}_n^R X_0 \). But \( X_0 \mathcal{H}_n^R X_0 \subseteq \mathcal{H}_n^R \), as we see by considering that \( X_0^2 = \mu_M (\prod_{i=1}^n X_i^{-1}) \in \mathcal{H}_n^{\mu M} \) and \( X_0 T_0 X_0 = T_0^{-1} X_0^2 X_1 \). Thus \( \mathcal{H}_n^{\mu M} \cong \mathcal{H}_n^R \oplus \mathcal{H}_n^R X_0 \) as a left \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathcal{H}_n^R \)-module.

Set

\[
\psi : \mathcal{H}_n^P \to \mathcal{H}_n^P : \ h \mapsto X_0^{-1} h X_0
\]

and

\[
\sigma : \mathcal{H}_n^P \to \mathcal{H}_n^P : \quad T_i \mapsto T_i
X_0 \mapsto -X_0
X_i \mapsto X_i \quad \text{for } i \geq 1.
\]

Both automorphisms leave the ideal generated by \( (X_0^2 \prod_{i=1}^n X_i - \mu_M) \). For \( \psi \), this follows from the commutativity of \( \mathcal{P}_n \) and for \( \sigma \) from the fact that \( (-X_0)^2 = X_0^2 \). Thus \( \psi \) and \( \sigma \) define automorphisms on \( \mathcal{H}_n^{\mu M} \). Then \( \psi \) leaves \( \mathcal{H}_n^R \) invariant since

\[
X_0^{-1} T_0 X_0 = T_0 X_1^{-1} + (p - p^{-1})
\]

and \( \sigma \) fixes \( \mathcal{H}_n^R \) pointwise since it is the identity on the generators of \( \mathcal{H}_n^R \). Applying Lemma 1.1, the restriction to \( \mathcal{H}_n^R \) of an irreducible \( \mathcal{H}_n^{\mu M} \)-module \( M \) splits only if \( M^\sigma \cong M \).

**Lemma 1.3.** \( M^\sigma \cong M \) only if \(-1 \) occurs as an eigenvalue for some \( X_j, \ j = 1, \ldots, n \).
Proof. The element \( X_0 \prod_{1 \leq i \leq n} (1 + X_i) \) is central in \( \mathcal{H}_n^P \) as
\[
X_0(1 + X_1)T_0 = X_0T_0 + X_0X_1T_0
= T_0^{-1}X_0X_1 + T_0X_0 + (p - p^{-1})X_0X_1
= T_0(X_0 + X_0X_1)
\]
and
\[
(X_i + 1)(X_{i+1} + 1)T_i = X_iX_{i+1}T_i + X_iT_i + X_{i+1}T_i + T_i
= T_iX_iX_{i+1} + T_i^{-1}X_{i+1} + T_iX_i
+ (q - q^{-1})X_{i+1} + T_i
= T_i(X_i + 1)(X_{i+1} + 1)
\]
and all other factors commute with \( T_i \) anyway. If \(-1\) does not occur as an eigenvalue for any of the \( X_i, 1 \leq i \leq n \), \( X_0 \prod_{1 \leq i \leq n} (1 + X_i) \) acts by a nonzero scalar on \( M \) and by its negative on \( M^\sigma \), so the two are not isomorphic. \( \square \)

If \(-1\) occurs as an eigenvalue of some \( X_i \) on \( M \), it can indeed happen that \( M^\sigma \cong M \) but this will not always be the case.

Example 1.4. Consider the 2-dimensional module for \( \mathcal{H}_1^P \) on which the generators \( T_0, X_0, X_1 \) act by the matrices
\[
\begin{pmatrix}
p & 0 \\
0 & -p^{-1}
\end{pmatrix},
\begin{pmatrix}
o & 0 \\
0 & o
\end{pmatrix},
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]
respectively. This is obviously irreducible but splits upon restriction to \( \mathcal{H}_n^R \), the isomorphism between \( M \) and \( M^\sigma \) being given by conjugation with the matrix \( \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \). On the other hand, there is a two-dimensional representation for \( \mathcal{H}_2^P \) where \( T_0 \) and \( T_1 \) act as
\[
\begin{pmatrix}
(p - p^{-1})q^2 \\
q^2 + p^2
\end{pmatrix},
\begin{pmatrix}
-q^2 + p^2 \\
p(p^{-1} - q^2)
\end{pmatrix}
\]
and \( X_0, X_1 \) and \( X_2 \) act as
\[
\begin{pmatrix}
o & 0 \\
0 & -a_0q^2
\end{pmatrix},
\begin{pmatrix}
-q^2 & 0 \\
0 & -q^{-2}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]
respectively, which is also irreducible and which remains irreducible when restricted to $H_R^P$.

Since our main goal is to understand finite dimensional irreducible modules for affine Hecke algebras by looking at the action of the lattice, we will generally work with the algebra $H_n^P$ where a classification of irreducibles by the action of the lattice is at least theoretically possible. The affine Hecke algebra of type $\tilde{A}_n$ using the weight lattice of $GL_n$ is naturally embedded in $H_n$ as the subalgebra generated by $T_1, \ldots, T_{n-1}$ and $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ which will be denoted by $H_n^A$. A lot of work has been done on this algebra, upon which we will heavily rely in the following. We will review the theory in type $A$ when needed and point out similarities and differences along the way.

To simplify notation, we now introduce some abbreviations. A simultaneous eigenvector for a set of lattice operators $X_i, \ldots, X_r$ with respective eigenvalues $a_{i_1}, \ldots, a_{i_r}$ will be called an $(a_{i_1}, \ldots, a_{i_r})$-eigenvector for $X_i, \ldots, X_r$. Analogously, we will refer to the subspace of all $(a_{i_1}, \ldots, a_{i_r})$-eigenvectors for $X_i, \ldots, X_r$ as the $(a_{i_1}, \ldots, a_{i_r})$-eigenspace for $X_i, \ldots, X_r$. If the specification "for $X_i, \ldots, X_r$" is omitted, the tuple $(a_{i_1}, \ldots, a_{i_r})$ necessarily consists of as many entries as there are lattice operators and the lattice operators are taken in order, i.e. the $(a_0, a_1, \ldots, a_n)$-eigenspace in a module $M \in H_n$-mod denotes the $(a_0, a_1, \ldots, a_n)$-eigenspace for $X_0, X_1, \ldots, X_n$.

We will write $T_k T_{k+1} \cdots T_l$ for $k \leq l$ or $T_k T_{k-1} \cdots T_l$ for $k \geq l$ and $T_k \cdots T_0 \cdots T_l$ as $T_{k,l}$ and $T_{k,0,l}$ respectively, and adopt the analogous convention for $s_{k,l}$ and $s_{k,0,l}$.

**Lemma 1.5.** Let $N \in H_n$-mod$^l$.

(i) Let $u \in N$ be an $(a, b)$-eigenvector for $X_j, X_{j+1}$ where $a, b \in F$ satisfy $a^{-1}b \notin \{ q^{\pm 2} \}$. Then $(T_j - a^{-1}bT_j^{-1})u$ is a $(b, a)$-eigenvector for $X_j, X_{j+1}$.

(ii) Let $u \in N$ be an $(a_0, a)$-eigenvector for $X_0, X_1$, where $a \notin \{ q^2, 1 \}$. Then $(T_0 - aT_0^{-1})u$ is an $(a_0 a, a^{-1})$-eigenvector for $X_0, X_1$.

**Proof.**
(i) Since for $a^{-1}b \notin \{q^{\pm 2}\}$, $(T_j - a^{-1}bT_j^{-1})$ is an invertible element in $\mathcal{H}_m^{\text{fin}}$ (with inverse $(q^2 + q^{-2} - a^{-1}b - b^{-1}a)^{-1}(T_j - b^{-1}aT_j^{-1}))$, $(T_j - a^{-1}bT_j^{-1})u \neq 0$. We then compute

$$(X_j - b)(T_j - a^{-1}bT_j^{-1})u$$

$$= (X_j - b)((1 - a^{-1}b)T_j + (q - q^{-1})a^{-1}b)u$$

$$= ((1 - a^{-1}b)T_j^{-1}X_{j+1} - b(1 - a^{-1}b)T_j)u + (q - q^{-1})a^{-1}b(X_j - b)u$$

$$= (1 - a^{-1}b)T_j(X_{j+1} - b)u - (q - q^{-1})((1 - a^{-1}b)X_{j+1}u - a^{-1}b(X_j - b))u$$

$$= (q - q^{-1})(b - a^{-1}b^2)u + (q - q^{-1})a^{-1}b(a - b)u$$

$$= 0$$

and

$$(X_{j+1} - a)(T_j - a^{-1}bT_j^{-1})u$$

$$= (X_{j+1} - a)((1 - a^{-1}b)T_j + (q - q^{-1})a^{-1}b)u$$

$$= (1 - a^{-1}b)T_j(X_j - a)u + (q - q^{-1})X_{j+1}u + (q - q^{-1})a^{-1}b(X_{j+1} - a)u$$

$$= (q - q^{-1})(b - a^{-1}b^2 + a^{-1}b^2 - b)u$$

$$= 0.$$ 

(ii) Analogously as in (i), this follows from the fact that, whenever $a \notin \{p^2, 1\}$, $(T_0 - aT_0^{-1})$ is invertible in $\mathcal{H}_m^{\text{fin}}$ with inverse $(p^2 + p^{-2} - a - a^{-1})^{-1}(T_0 - a^{-1}T_0^{-1})$, so $(T_0 - aT_0^{-1})u \neq 0$ and computing
\[(X_0 - a_0a)(T_0 - aT_0^{-1})u\]
\[= (X_0 - a_0a)((1 - a)T_0 + (q - q^{-1})a)u\]
\[= (1 - a)(T_0X_0X_1 - (q - q^{-1})X_0X_1 - a_0aT_0)u\]
\[+ (q - q^{-1})a(X_0 - a_0a)u\]
\[= (1 - a)T_0(X_0X_1 - a_0a)u - (q - q^{-1})((1 - a)X_0X_1\]
\[a - a(X_0 - a_0a))u\]
\[= -(q - q^{-1})((1 - a)a_0a - a(a_0 - a_0a))u\]
\[= 0\]
and

\[(X_1 - a^{-1})(T_0 - aT_0^{-1})u\]
\[= (X_1 - a^{-1})((1 - a)T_0 + (q - q^{-1})a)u\]
\[= (1 - a)(T_0X_1^{-1} - a^{-1}) + (q - q^{-1})(X_1 + 1))u\]
\[+ (q - q^{-1})a(X_1 - a^{-1})u\]
\[= (q - q^{-1})((1 - a)(a + 1) + a(a - a^{-1}))u\]
\[= 0.\]

\[\square\]

**Lemma 1.6.** Let \(N \in \mathcal{H}_n\)-mod\(^\text{fd}\) and \(a \in F\). Let \(u \in N\) be an \((aq^2, a, a)\)-eigenvector for \(X_{j-1}, X_j, X_{j+1}\) which is annihilated by the element \((T_{j-1} + q^{-1})\). Then

\[v := ((T_{j-1} + q^{-1})T_j - 1)u\]

is an \((a, aq^2, a)\)-eigenvector for \(X_{j-1}, X_j, X_{j+1}\). Furthermore,

\[-\frac{1}{(q + q^{-1})}(T_{j-1} - q)v = u\]

and

\[(T_j + q^{-1})v = 0.\]

Also, if \(u \in N\) is an \((a, a, aq^2)\)-eigenvector for \(X_{j-1}, X_j, X_{j+1}\) which is annihilated by the element \((T_j - q)\), then

\[v := ((T_j - q)T_{j-1} - 1)u\]
is an \((a, aq^2, a)\)-eigenvector for \(X_{j-1}, X_j, X_{j+1}\), which is annihilated by 
\((T_{j-1} - q)\), and
\[
\frac{1}{(q + q^{-1})}(T_j + q^{-1})v = u.
\]
Analogous statements hold if we replace \(q\) with \(-q^{-1}\) in all expressions.

**Proof.** We have
\[
(X_{j-1} - a)((T_{j-1} + q^{-1})T_j - 1)u
= (T_{j-1}T_j^{-1}X_{j+1} - aT_{j-1}T_j
+ q^{-1}T_j(X_{j-1} - a) - (X_{j-1} - a))u
= (-a(q - q^{-1})(T_{j-1} + T_j) + a(q - q^{-1})^2
+ q^{-1}T_j(aq^2 - a) - (aq^2 - a))u
= (-a(q - q^{-1})(T_{j-1} + T_j) + a(q - q^{-1})^2
+ a(q - q^{-1})T_j - aq(q - q^{-1}))u
= -a(q - q^{-1})(T_{j-1} + q^{-1})u
= 0,
\]
\[
(X_j - aq^2)((T_{j-1} + q^{-1})T_j - 1)u
= (T_{j-1}T_j(X_{j-1} - aq^2) + (q - q^{-1})T_j^{-1}X_{j+1}
+ q^{-1}T_j^{-1}X_{j+1} - aqT_j - (X_j - aq^2))u
= (aqT_j^{-1} - aqT_j + aq(q - q^{-1}))u
= 0
\]
and
\[
(X_{j+1} - a)((T_{j-1} + q^{-1})T_j - 1)u
= ((T_{j-1} + q^{-1})(T_j(X_j - a) + (q - q^{-1})X_{j+1}
- (X_{j+1} - a))u
= a(q - q^{-1})(T_{j-1} + q^{-1})u
= 0.
\]
Furthermore,
\[
(T_{j-1} - q)((T_{j-1} + q^{-1})T_j - 1)u = (T_{j-1} - q)u
= (-q^{-1} - q)u
\]
and finally
\[(T_j + q^{-1})(T_{j-1} + q^{-1})T_j - 1)u\]
\[= (T_{j-1}(T_j + q^{-1})(T_{j-1} + q^{-1}) + q^{-1}T_j^2 + q^{-2}T_j - q^{-1}T_{j-1}^2 - q^{-2}T_{j-1} - (T_j + q^{-1}))u\]
\[= q^{-1}T_j^2 + q^{-2}T_j - q^{-3} + q^{-3} - (T_j + q^{-1})u\]
\[= (q^{-1}T_j - 1)(T_j + q^{-1})u\]
\[= q^{-1}(T_j - q)(T_j + q^{-1})u\]
\[= 0,

proving the claim.

The other statements as well as the cases where \( q \) is replaced by \(-q^{-1}\) follow analogously. \( \Box \)

We will now consider the behavior of elements in \( \mathcal{H}_n \) when we move lattice elements from one side to the other but we first need to define the Bruhat order on \( W_n^{\text{fin}} \). For \( x, y \in W_n^{\text{fin}}, x < y \) if and only if there exists a reduced expression \( y = u_1 \cdots u_k \) where \( u_j \in \{ s_i \mid i \in I \} \) for \( 1 \leq j \leq k \) and a subsequence \( 1 \leq m_1 < \cdots < m_l \leq k \) such that \( x = u_{m_1} \cdots u_{m_l} \). This defines a partial order on \( W_n^{\text{fin}} \) which is compatible with the length function.

**Lemma 1.7.** Let \( w \in W_n^{\text{fin}} \) and \( 0 \leq i \leq n \). Then
\[ T_wX_i \in X_{w^{-1}i}T_w + \sum_{\tilde{w} < w} \mathcal{P}_n T_{\tilde{w}} \]

where \( X_{w^{-1}i} \) is the element \( wX_i w^{-1} \) when we look at the action of the finite Weyl group on the lattice.

**Proof.** We prove this by induction on the length of \( w \), the case where \( w = 1 \) being trivial. For \( w \) of length one, we have \( T_w = T_j \) for some \( 0 \leq j \leq n - 1 \). If \( j \geq 1 \), we have
\[ T_jX_i = \begin{cases}  \ X_jT_j & \text{if } j \geq i - 1, i \\  \ X_{i+1}T_j - (q - q^{-1})X_{i+1} & \text{if } j = i \\  \ X_{i-1}T_j + (q - q^{-1})X_j & \text{if } j = i - 1 \end{cases} \].

If \( j = 0 \), we see that
\[ T_0 X_i = \begin{cases} 
X_i T_0 & \text{if } i \geq 2 \\
X_i^{-1} T_0 - (p^{-1} - p)X_1 + 1 & \text{if } i = 0 \\
X_0 X_1 T_0 - (p^{-1} - p)X_0 X_1 & \text{if } j = i - 1 
\end{cases} \]

so we see that in both cases the statements hold.

So assume that the statement holds for all elements of \( W_n \) with length less than \( n \) and assume that the length of \( w \) equals \( n \). Write \( T_w = T_{w'} T_j \) for some \( j \in I \), so the length of \( w' \) equals \( n - 1 \). Then if \( j \neq i, i - 1 \), \( s_j i = i \) and

\[
T_w X_i = T_{w'} T_j X_i \\
= T_{w'} X_i T_j \\
= X_{w'^{-1}} T_{w'} T_j + \text{terms in } \sum_{\tilde{w'} < w'} \mathcal{P}_n T_{\tilde{w'}} T_j \\
= X_{w'^{-1}} T_{w'} + \text{terms in } \sum_{\tilde{w'} < w'} \mathcal{P}_n T_{\tilde{w'}}
\]

as desired. If \( j = i \), \( s_i i = i + 1 \) and

\[
T_w X_i = T_{w'} T_i X_i \\
= T_{w'} X_{s_i i} T_i - (q - q^{-1}) T_{w'} X_{s_i i} \\
= X_{w'^{-1}} s_i T_{w'} T_i - (q - q^{-1}) X_{w'^{-1}} s_i T_{w'} \\
+ \text{terms in } \left( \sum_{\tilde{w'} < w'} \mathcal{P}_n T_{\tilde{w'}} T_i + \sum_{\tilde{w'} < w'} \mathcal{P}_n T_{\tilde{w'}} \right) \\
= X_{w'^{-1}} T_{w'} + \text{terms in } \sum_{\tilde{w} < w'} \mathcal{P}_n T_{\tilde{w}}.
\]
In the case where $j = i - 1$, $s_{i-1} = i - 1$ and

$$T_w X_i = T_{w'} T_{i-1} X_i$$

$$= T_{w'} X_{s_{i-1}} T_{i-1} + (q - q^{-1}) T_{w'} X_i$$

$$= X_{w'-1s_{i-1}} T_{w'} T_{i-1} + (q - q^{-1}) X_{w'-1i} T_{w'}$$

$$+ \text{ terms in } \left( \sum_{w' < w} \mathcal{P}_n T_{w'} T_i + \sum_{w' < w} \mathcal{P}_n T_{w'} \right)$$

$$= X_{w-1i} T_w + \text{ terms in } \sum_{\bar{w} < w} \mathcal{P}_n T_{\bar{w}}$$

so the assertion holds true in all cases. □
2. THE MAIN TOOLS – MACKEY THEOREM AND DUALITY

In a finite Coxeter group, a parabolic subgroup is generally defined as the subgroup generated by a subset of the Coxeter generators $s_i$. For $W_n^{\text{fin}}$ this means we take a subset

$$I = \{i_1, i_1 + 1, \ldots, i_1 + r_1 - 1, i_2, \ldots, i_2 + r_2 - 1, \ldots, i_l, \ldots, i_l + r_l - 1$$

$$| i_k + r_k < i_{k+1} \quad \forall 1 \leq k \leq l - 1 \}$$

$$\subseteq \{0, 1, \ldots n - 1 \}$$

and obtain $W_I^{\text{fin}} := \langle s_i \mid i \in I \rangle$ which is isomorphic to the product $W_i^{\text{fin}} \times S_{r_2} \times \cdots \times S_{r_l}$ if $i_1 = 0$ and to $S_{r_1} \times S_{r_2} \times \cdots \times S_{r_l}$ if $0 \notin I$.

Parabolic subgroups are extremely useful in representation theory as one can often assume knowledge about their representations in order to prove results about the actual groups by induction.

We generalize this concept to (extended) affine Weyl groups not by taking a subset of the Coxeter generators which would yield finite Coxeter groups of infinite index in the affine Weyl group — making induction difficult to handle — but by taking the semidirect product of a parabolic subgroup of the finite Weyl group with the full translation lattice. Thus we define $W_I := \langle s_i, X_j \mid i \in I, j \in \{0, \ldots, n\} \rangle$.

Analogously, the parabolic subalgebra $\mathcal{H}_I^{\text{fin}}$ of $\mathcal{H}_n^{\text{fin}}$ is defined as the subalgebra generated by $\{T_i \mid i \in I\}$ and the parabolic subalgebra $\mathcal{H}_I$ of $\mathcal{H}_n$ is the subalgebra generated by $\{T_i, X_j^{\pm 1} \mid i \in I, j \in \{0, \ldots, n\}\}$.

We will also write $(m_1, \ldots, m_l)$ where $m_1 + \cdots + m_l = n$ for

$$I = \{0, 1, \ldots, m_1 - 1, m_1 + 1, \ldots, m_1 + m_2 - 1, \ldots, \sum_{i=1}^{l-1} m_i + 1, \ldots, n - 1 \}$$

and write $\mathcal{H}_{m_1, \ldots, m_l}$ for $\mathcal{H}_I$ which is isomorphic to the tensor product $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2} \otimes \cdots \otimes \mathcal{H}_{m_l}$. Analogously, $\mathcal{H}_{m_1, \ldots, m_l}^A \cong \mathcal{H}_{m_1}^A \otimes \mathcal{H}_{m_2}^A \otimes \cdots \otimes \mathcal{H}_{m_l}^A$ denotes the parabolic subalgebra corresponding to

$$I = \{1, \ldots, m_1 - 1, m_1 + 1, \ldots, m_1 + m_2 - 1, \ldots, \sum_{i=1}^{l-1} m_i + 1, \ldots, n - 1 \}$$

$$=: (m_1^A, \ldots, m_l).$$
Those types of index sets are generally known as compositions of type $B_n$ and $A_n$ respectively, but we chose the above notation because we will use compositions of both types.

We will generally abbreviate induction and restriction functors between parabolic subalgebras as \( \text{ind}_J^I := \text{ind}_{H_J^I} \) and \( \text{res}_J^I := \text{res}_{H_J^I} \) for \( J \subseteq I \), and \( \text{ind}_{n_1, \ldots, n_k}^I := \text{ind}_{H_{n_1, \ldots, n_k}} \) and \( \text{res}_{n_1, \ldots, n_k}^I := \text{res}_{H_{n_1, \ldots, n_k}} \) for \((m_1, \ldots, m_l) \subseteq (n_1, \ldots, n_k)\). If we induce directly from a parabolic subalgebra of type $A$, we will always use the full expression.

For a parabolic subgroup $W_I^f$ of $W_n^f$, there are distinguished left and right coset representatives of minimal length, the sets of which will be denoted by $D_I$ and $D_I^{-1}$ respectively. For parabolic subgroups $W_I^f$ and $W_J^f$, $D_{I,J} := D_I^{-1} \cap D_J$ is then the set of distinguished minimal length $(W_I^f, W_J^f)$-double coset representatives. An account of this, including the following three properties of distinguished double coset representatives, can be found in [7], Chapter 2.1.

(i) For \( x \in D_{I,J} \), the subgroups \( W_I^f \cap xW_J^f x^{-1} =: W_{I \cap xJ}^f \) and \( x^{-1}W_I^f x \cap W_J^f =: W_{x^{-1}I \cap J}^f \) are parabolic subgroups of $W_n^f$.

(ii) For \( x \in D_{I,J} \), the map

\[
W_{I \cap xJ}^f \rightarrow W_{x^{-1}I \cap J}^f \\
w \mapsto x^{-1}wx
\]

defines a length preserving isomorphism.

(iii) For \( x \in D_{I,J} \), every \( w \in W_I^f x W_J^f \) can be written as \( w = u x v \) for unique elements \( u \in W_I^f \) and \( v \in W_J^f \cap D_{x^{-1}I \cap J}^{-1} \). Moreover, \( W_J^f \cap D_{x^{-1}I \cap J}^{-1} \) is the set of minimal length right coset representatives of $W_{x^{-1}I \cap J}^f$ in $W_J^f$.

**Lemma 2.1.** For \( x \in D_{I,J} \), the subspace $\mathcal{H}_I^f T_x \mathcal{H}_J^f$ has basis

\[
\{ T_w \mid w \in W_I^f x W_J^f \}
\]

**Proof.** Write \( w = u x v \) with \( u \in W_I^f \) and \( v \in W_J^f \cap D_{x^{-1}I \cap J}^{-1} \). Then, \( l(w) = l(u) + l(x) + l(v) \), so \( T_w = T_u T_x T_v \). An arbitrary element of $\mathcal{H}_I^f T_x \mathcal{H}_J^f$ looks like \( \sum_{a \in W_I^f, r \in W_J^f} a_{or} T_o T_x T_r \). So we need to show that any such \( T_o T_x T_r \) is contained in the span of the \( T_u T_x T_v \) where \( u \in W_I^f \) and \( v \in W_J^f \cap D_{x^{-1}I \cap J}^{-1} \). Now the element \( T_r \) can be written as $T_{r_1} T_{r_2}$.
where $T_{r_1} \in W_{x^{-1}I \cap J}^\text{fin}, T_{r_2} \in D_{x^{-1}I \cap J}$, so $T_{r_1} = T_{x^{-1}r_3}T_x$ with $r_3 \in I$. Therefore
\[ T_0 T_x T_r = T_0 T_x T_{x^{-1}r_3} T_x T_{r_2} = T_0 T_{x^{-1}r_3} T_x T_{r_2} \]
which is of the desired form.

Since $\{T_w \mid w \in W^\text{fin}\}$ forms a basis of $\mathcal{H}_n^\text{fin}$, the set
\[ \{T_w \mid w \in W_I^\text{fin} x W_J^\text{fin}\} \]
is certainly linearly independent which completes the proof. □

**Lemma 2.2.** For $x \in D_{I,J}$ the subspace $\mathcal{H}_I T_x \mathcal{H}_J^\text{fin}$ of $\mathcal{H}_n$ has basis
\[ B_{I,J}^x := \{X_0^{c_0} \cdots X_n^{c_n} T_w \mid (c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}, w \in W_I^\text{fin} x W_J^\text{fin}\}. \]
Moreover, as a vector space,
\[ \mathcal{H}_n = \bigoplus_{x \in D_{I,J}} \mathcal{H}_I T_x \mathcal{H}_J^\text{fin}. \]

**Proof.** Since $\mathcal{H}_I = P_n \mathcal{H}_I^\text{fin}$, it follows from the previous lemma that the elements of $B_{I,J}^x$ span $\mathcal{H}_I T_x \mathcal{H}_J^\text{fin}$. They are linearly independent by (9), proving the first statement. As $x$ runs through $D_{I,J}$, the indices of the $T_w$ occurring in $B_{I,J}^x$ run through all of $W^\text{fin}_n$, proving the second statement. □

We now fix some total order $<$ refining the Bruhat order $<$ on $D_{I,J}$. For $x \in D_{I,J}$, set
\begin{align}
\mathcal{B}_{\leq x} &= \bigoplus_{y \in D_{I,J}, y \leq x} \mathcal{H}_I T_y \mathcal{H}_J^\text{fin}, \\
\mathcal{B}_{< x} &= \bigoplus_{y \in D_{I,J}, y < x} \mathcal{H}_I T_y \mathcal{H}_J^\text{fin}, \\
\mathcal{B}_x &= \mathcal{B}_{\leq x} / \mathcal{B}_{< x}. 
\end{align}

This defines a filtration of $\mathcal{H}_n$ as an $(\mathcal{H}_I, \mathcal{H}_J)$-bimodule, since it follows from Lemma 1.7 that we can move lattice elements from the right to the left and only create terms that are smaller in the Bruhat order and therefore lower in the filtration. Property (ii) of double coset representatives above implies that for each $x \in D_{I,J}$, there exists an algebra
isomorphism

\[ \phi_{x^{-1}} : \mathcal{H}_{I \cap xJ} \rightarrow \mathcal{H}_{x^{-1}I \cap J} \]

\[ T_w \mapsto T_{x^{-1}wx} \]

\[ X_j \mapsto X_{x^{-1}j} \]

for \( 0 \leq i \leq n - 1 \) and \( w \in W^\text{fin}_{I \cap xJ} \). For \( N \in \mathcal{H}_{x^{-1}I \cap J} \)-mod, \( xN \) will denote the \( \mathcal{H}_{I \cap xJ} \)-module obtained by pulling back the action through \( \phi_{x^{-1}} \).

Now we can prove an affine version of the Mackey theorem, which differs from “classical” Mackey theorems in that it does not give a direct decomposition but only a filtration.

**Theorem 2.3. (“Mackey Theorem”)**

Let \( M \in \mathcal{H}_J \)-mod. Then \( \text{res}^I_{\cap xJ} \text{ind}^I_J M \) admits a filtration with subquotients isomorphic to \( \text{ind}^I_{I \cap xJ} (\text{res}^I_{x^{-1}I \cap J} M) \), one for each \( x \in D_{I,J} \). The subquotients can be taken in any order refining the Bruhat order on \( D_{I,J} \), in particular, since the double coset representative \( 1 \) is the smallest element in the ordering, \( \text{ind}^I_{I \cap xJ} \text{res}^I_{x^{-1}I \cap J} M \) appears as a submodule.

**Proof.** We already have a filtration of \( \mathcal{H}_n \) as \((\mathcal{H}_I, \mathcal{H}_J)\)-bimodule given in (13). Thus \( \text{res}^I_{\cap xJ} \text{ind}^I_J M = \mathcal{H}_n \otimes \mathcal{H}_J M \) inherits a filtration with subquotients isomorphic to \( B_x \otimes \mathcal{H}_J M \), the \( x \in D_{I,J} \) taken in any order refining the Bruhat order on \( D_{I,J} \). Now

\[
\text{ind}^I_{I \cap xJ} (\text{res}^I_{x^{-1}I \cap J} M) = \mathcal{H}_I \otimes \mathcal{H}_{I \cap xJ} \mathcal{H}_J \otimes \mathcal{H}_J M \\
\cong \mathcal{H}_I \otimes \mathcal{H}_{I \cap xJ} \mathcal{H}_J \otimes \mathcal{H}_J M,
\]

thus it suffices to show that \( \mathcal{H}_I \otimes \mathcal{H}_{I \cap xJ} \mathcal{H}_J \cong B_x \). In order to do this, we define a bilinear map

\[ \mathcal{H}_I \times \mathcal{H}_J \rightarrow B_x \]

\[ (h, h') \mapsto hT_x h' + B_{<x}. \]

Since, for \( w \in \mathcal{H}_{I \cap xJ}, T_w T_x = T_{wx} = T_{x^{-1}wx} = T_x T_{x^{-1}wx}, \) this map is \( \mathcal{H}_{I \cap xJ} \)-balanced and therefore induces a map \( \mathcal{H}_I \otimes \mathcal{H}_{I \cap xJ} \mathcal{H}_J \rightarrow B_x \).

By Lemma 2.1, a basis of \( \mathcal{H}_I \otimes \mathcal{H}_{I \cap xJ} \mathcal{H}_J \) is given by

\[ \{X_0^{c_0} \cdots X_n^{c_n} T_u \otimes T_v \mid (c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}, u \in W^\text{fin}_I, v \in W^\text{fin}_J \cap D_{x^{-1}I \cap J}\}. \]
the elements of which map to a basis of $B_x$ by Lemma 2.2, whence the map is actually an isomorphism. □

In general, for $N \in \mathcal{H}_n-\text{mod}^{fd}$ and $M \in \mathcal{H}_I-\text{mod}^{fd}$,

$$\text{Hom}_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_I} M, N) \cong \text{Hom}_{\mathcal{H}_I}(M, \text{Hom}_{\mathcal{H}_n}(\mathcal{H}_n, N))$$

and

$$\text{Hom}_{\mathcal{H}_I}(\mathcal{H}_I \otimes_{\mathcal{H}_n} N, M) \cong \text{Hom}_{\mathcal{H}_n}(N, \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M)).$$

We would like to express the coinduced module $\text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M)$ in terms of an induced module. Hence, for the rest of this chapter, we fix a subset $I$ of $\{0, 1, \ldots, n - 1\}$. Let $d$ be the longest element of $D_{I,I}$.

**Lemma 2.4.** Let $I = (m_1, \ldots, m_l)$. Then the longest double coset representative $d$ in $D_{I,I}$ is an involution and $dI \cap I = d^{-1}I \cap I = I$.

**Proof.** Let $w_0$ be the longest element of $W_n^{\text{fin}}$. This element has to map the short root $X_1$ in the basis of the root system to a short root in the inverse of this basis which is $X_1^{-1}, X_2^{-1}X_1, \ldots, X_n^{-1}X_{n-1}$. The only short root there is $X_1^{-1}$ and since $w_0$ acts as an isometry it follows that it sends each root to its negative, hence also each $X_i$ to $X_i^{-1}$. Set $k_i = \sum_{j=1}^{i} m_j$ and let $w_{0,I}$ be the longest element in

$$W_I^{\text{fin}} \cong W_{m_1} \times S_{m_2} \times \cdots \times S_{m_l},$$

which is the element sending $X_1, \ldots, X_{m_1}$ to their inverses as above and reversing the orders of $X_{k_i+1}, \ldots, X_{k_i+m_i+1}$ for $1 \leq i \leq l - 1$, as the element reversing the order of the numbers $1, \ldots, m_i$ is the longest element in the symmetric group $S_{m_i}$.

For the longest distinguished left coset representative $\tilde{d}$ in $D_I$, we have $w_0 = \tilde{d}w_{0,I}$ by the additivity of lengths for distinguished coset representatives. Since $w_{0,I}$ is equal to its inverse, we obtain $\tilde{d} = w_0w_{0,I}$. Computing the action of $\tilde{d}$ on the $X_i$, we see that it leaves $X_1, \ldots, X_{m_1}$ invariant and maps the ordered sets $(X_{k_i+1}, \ldots, X_{k_i+m_i+1})$ to the ordered sets $(X_{k_i+m_i+1}^{-1}, \ldots, X_{k_i+1})$ for $1 \leq i \leq l - 1$. From this presentation it is easy to see that $\tilde{d}^{-1}$ is equal to $\tilde{d}$, whence it is also the longest distinguished right coset representative and therefore the longest element $d$ in $D_{I,I}$. By direct computation it follows that $ds_i = s_i$ for
\[ 0 \leq i \leq m_1 - 1 \text{ and } ds_{k_{i+j}}d = s_{k_{i+1}}d \text{ for } 1 \leq j \leq m_{i+1} \text{ and } 1 \leq i \leq l-1, \]

which shows that \( dI \cap I = d^{-1}I \cap I = I \). \( \square \)

By property (ii) of distinguished double coset representatives, there is an isomorphism

(14) \[ \phi = \phi_{d^{-1}} : \mathcal{H}_I \rightarrow \mathcal{H}_I, \]

and for \( M \in \mathcal{H}_I\text{-mod} \), we denote by \( ^dM \) the \( \mathcal{H}_I \)-module obtained by twisting the action with \( \phi \).

We will need the homomorphism \( \theta : \mathcal{H}_n \rightarrow {^d\mathcal{H}}_I \) of \( \mathcal{H}_I, \mathcal{H}_I \)-bimodules that is given by first projecting \( \mathcal{H}_n \rightarrow B_d \) in (13) and then applying the isomorphism of \( B_d \rightarrow \mathcal{H}_I \otimes_{\mathcal{H}_I} {^d\mathcal{H}}_I \cong {^d\mathcal{H}}_I \) given in the proof of Theorem 2.3. Explicitly, this homomorphism is given by

\[ \theta(XT_w) = \begin{cases} 
\phi(X)T_{d^{-1}w} & \text{if } w \in dW_n^{\text{fin}}, \\
0 & \text{otherwise,} 
\end{cases} \]

for \( X \in \mathcal{P}_n, w \in W_n^{\text{fin}} \). Then the following holds.

**Lemma 2.5.** The map

\[ f : \mathcal{H}_n \rightarrow \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, {^d\mathcal{H}}_I) \]

\[ h \mapsto (h\theta : t \mapsto \theta(th)) \]

is an isomorphism of \( \mathcal{H}_n, \mathcal{H}_I \)-bimodules.

**Proof.** First, we need to show that \( f \) is an \( \mathcal{H}_n, \mathcal{H}_I \)-bimodule homomorphism. So, we check \( f(h)(t) = \theta(th) = h\theta(t) = hf(1)(t) \) for \( h \in \mathcal{H}_n \) and \( f(h')(t) = \theta(th') = \theta(t)h' = f(1)h'(t) \) for \( h' \in \mathcal{H}_I \).

Since, as left \( \mathcal{H}_I \)-module, \( {^d\mathcal{H}}_I \) is isomorphic to \( \mathcal{H}_I \) and \( \mathcal{H}_n \) is a free left \( \mathcal{H}_I \)-module on basis \( \{T_w \mid w \in D_I^{-1}\} \), the set \( K := \{\psi_w \mid w \in D_I^{-1}\} \), where \( \psi_w : \mathcal{H}_n \rightarrow {^d\mathcal{H}}_I : \psi_w(T_u) = \delta_{u,w} \) for \( u \in D_I^{-1} \), is a basis for \( \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, {^d\mathcal{H}}_I) \) as a free right \( \mathcal{H}_I \)-module.

So, we’ll be done if we can give a basis for \( \mathcal{H}_n \) as free right \( \mathcal{H}_I \)-module that is mapped to \( K \). But since \( \mathcal{H}_I = \mathcal{H}_I^{\text{fin}}\mathcal{P}_n \), a basis for \( \mathcal{H}_n^{\text{fin}} \) as free right \( \mathcal{H}_I^{\text{fin}} \)-module is automatically a basis for \( \mathcal{H}_n \) as free right \( \mathcal{H}_I \)-module. Therefore, we study the restrictions \( \theta' := \theta|_{\mathcal{H}_n^{\text{fin}}} \) and \( f' := f|_{\mathcal{H}_n^{\text{fin}}} \) to \( \mathcal{H}_n^{\text{fin}} \) of the above homomorphisms and want to construct a basis for \( \mathcal{H}_n^{\text{fin}} \) as free right \( \mathcal{H}_I^{\text{fin}} \)-module such that the basis
elements map to the $\psi' := \psi_w|_{H_n^\text{fin}}$. A basis like this can be found if we can show that $f'$ is an isomorphism of $(H_n^\text{fin}, H_I^\text{fin})$-bimodules. Suppose $f'$ is not an isomorphism, then there exists a nonzero $h$ in its kernel, i.e. $(f'(h))(t) = \theta'(th) = 0$ for all $t \in H_n^\text{fin}$. Now write $h = \sum_{y \in D_I, l(y) \leq l(x)} T_y h_y$ for some $x \in D_I, h_y \in H_I$. If $x = d$, we know that $f'(h)(1) = \theta(h) = h_d \neq 0$, so we can use downward induction on $l(x)$ to show $h = 0$. If $l(x) \leq l(d)$ we can find a transposition $s$ such that $sx \in D_I$ and $l(sx) \geq l(x)$, so $T_s h = \sum_{y \in D_I, l(y) \leq l(sx)} T_y h'_y$ and $h'_x = h_x \neq 0$, so by the inductive assumption we can deduce that $\theta'(H_n^\text{fin}h) = \theta'(H_n^\text{fin}T_s h) \neq 0$, whence $f'$ is indeed an isomorphism and we’re done. □

**Corollary 2.6.** For $M \in H_I$-mod, there is a natural isomorphism

$$\text{Hom}_{H_I}(H_n, M) \cong H_n \otimes_{H_I} dM$$

of $H_n$-modules.

**Proof.** Let $f : H_n \to \text{Hom}_{H_I}(H_n, dH_I)$ be the bimodule isomorphism constructed in Lemma 2.5 and recall that $d^2 = 1$. Then, there are natural isomorphisms

$$H_n \otimes_{H_I} dM \xrightarrow{f \otimes \text{id}} \text{Hom}_{H_I}(H_n, dH_I) \otimes_{H_I} dM$$

$$\cong \text{Hom}_{H_I}(H_n, dH_I \otimes_{H_I} dM)$$

$$\cong \text{Hom}_{H_I}(H_n, d^2 M)$$

$$\cong \text{Hom}_{H_I}(H_n, M),$$

the second isomorphism depending on the fact that $H_n$ is a free left $H_I$-module. □

On $H_n$, we can define an antiautomorphism $\tau$ defined on the generators as follows:

$$\tau : T_i \mapsto T_i,$$

$$X_j \mapsto X_j$$

for all $i = 0, \ldots, n-1, j = 0, \ldots, n$. Since the relations given in Chapter 1 are all invariant with respect to reversal of the order of generators, this does indeed define an antiautomorphism.
As any antiautomorphism, $\tau$ can be used to define an action of $\mathcal{H}_n$ on the $F$-dual $M^* = \text{Hom}_F(M,F)$ of a module $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ via $hf(m) = f(\tau(h)m)$. As $\tau$ leaves parabolic subalgebras of $\mathcal{H}_n$ invariant, it can also be used to define a duality on finite dimensional $\mathcal{H}_I$-modules for any subset $I$ of $\{0, \ldots, n-1\}$. If we think of representations in terms of matrices, the $\tau$-dual corresponds to taking the transposes of the representing matrices. Then we obtain another corollary of Lemma 2.5:

**Corollary 2.7.** For $M \in \mathcal{H}_I\text{-mod}^{\text{fd}}$, there is a natural isomorphism

$$\text{ind}_I^n(M)^\tau \cong \text{ind}_I^n(\text{d}(M^\tau)).$$

**Proof.** Since $\text{ind}_I^n$ is the unique left adjoint functor to $\text{res}_I^n$,

$$\text{Hom}_{\mathcal{H}_n}(\text{ind}_I^n M, N) \cong \text{Hom}_{\mathcal{H}_I}(M, \text{res}_I^n N)$$

$$\cong \text{Hom}_{\mathcal{H}_I}((\text{res}_I^n N)^\tau, M^\tau)$$

$$\cong \text{Hom}_{\mathcal{H}_I}(\text{res}_I^n N^\tau, M^\tau)$$

$$\cong \text{Hom}_{\mathcal{H}_n}(N^\tau, \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_n, M^\tau)))$$

$$\cong \text{Hom}_{\mathcal{H}_n}(N^\tau, \text{ind}_I^n(\text{d}(M^\tau)))$$

$$\cong \text{Hom}_{\mathcal{H}_n}((\text{ind}_I^n(\text{d}(M^\tau)))^\tau, N)$$

shows that $\text{ind}_I^n M \cong (\text{ind}_I^n(\text{d}(M^\tau)))^\tau$. Applying $\tau$ once again yields the claim. \qed
3. Formal Characters

In the following, we will make heavy use of the following lemma.

**Lemma 3.1.** For $F$-algebras $A$ and $B$, the irreducibles in $A \otimes_F B$-$\text{mod}^\text{fd}$ are exactly the outer tensor products $M \boxtimes N$ of irreducible $M \in A$-$\text{mod}^\text{fd}$, $N \in B$-$\text{mod}^\text{fd}$. Further, if $M \boxtimes N \cong M' \boxtimes N'$, then $M \cong M'$ and $N \cong N'$.

**Proof.** In [4], Theorem (10.38), this is stated for finite-dimensional algebras $A'$ and $B'$ such that at least one of $A'/\text{Rad}A'$ and $B'/\text{Rad}B'$ is split semisimple. A finite-dimensional representation

$$\rho : A \otimes B \to \text{End}_F(K)$$

certainly factors over the image of $\rho$, which is a finite-dimensional quotient $C$ of $A \otimes B$. Thus, since we’re working over an algebraically closed field, we’ll be done if we show that the representation factors over a finite-dimensional quotient of the form $A' \otimes B'$. Since $A$ is a subalgebra of $A \otimes B$, we get a map $A \to C$. Denote by $A'$ the image of this map, and construct $B'$ analogously. Since the maps are $F$-linear, this yields an $F$-balanced map $A' \times B' \to C$ inducing our desired factorization over $A' \otimes B'$. \qed

For $\underline{a} = (a_0, a_1, \ldots, a_n) \in F^{n+1}$, the one-dimensional $\mathcal{P}_n$-module on which $X_i$ acts as the scalar $a_i$ for $0 \leq i \leq n$ will also be denoted by $\underline{a}$. If an eigenvalue occurs several times this will be indicated by an exponent in parentheses. So $(a_0, a^{(n)})$ is the one-dimensional $\mathcal{P}_n$-module on which all $X_i$ for $i > 0$ act as $a$. Since $\mathcal{P}_n$ is commutative, the modules $(a_0, a_1, \ldots, a_n)$ exhaust all irreducibles in $\mathcal{P}_n$-$\text{mod}^\text{fd}$. This follows from the fact that commuting matrices can simultaneously brought to upper triangular form.

For $M \in \mathcal{P}_n$-$\text{mod}^\text{fd}$ and any $\underline{a} \in F^{n+1}$, let $M[\underline{a}]$ be the largest submodule of $M$ all of whose composition factors are isomorphic to $\underline{a}$, i.e. simultaneous generalized $(a_0, a_1, \ldots, a_n)$-eigenspace for $X_0, X_1, \ldots, X_n$.

**Lemma 3.2.** There are no nontrivial extensions between two non-isomorphic irreducible modules in $\mathcal{P}_n$-$\text{mod}^\text{fd}$. 


Proof. Suppose we have a nontrivial extension
\[ a \rightarrow M \rightarrow b, \]
where without loss of generality \( a_n \neq b_n \). If the restriction to every subalgebra generated by one \( X_i \) is diagonalizable, then all matrices representing the \( X_i \) are simultaneously diagonalizable and the extension is split. Thus, we can assume without loss of generality that \( X_1 \) has a Jordan block of size two. Thus the matrices representing \( X_1 \) and \( X_n \) can simultaneously be brought to the form
\[
\begin{pmatrix}
a_1 & c \\
0 & a_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
a_n & d \\
0 & b_n
\end{pmatrix}
\]
respectively. Now computing the product in both orders and comparing the upper right entry shows that necessarily \( a_n = b_n \). \( \square \)

**Corollary 3.3.** For any \( M \in \mathcal{P}_n \text{-mod}^{fd} \) we have \( M = \bigoplus_{a \in F^n} M[a] \) as a \( \mathcal{P}_n \)-module.

**Proof.** Since there are no nontrivial extensions between non-isomorphic irreducibles, every indecomposable module in \( \mathcal{P}_n \text{-mod}^{fd} \) is homogeneous, i.e. has composition factors only from one isomorphism class of irreducible modules, which proves the claim. \( \square \)

We define the *formal character* of \( M \in \mathcal{H}_n \text{-mod}^{fd} \) by:

(15) \[ \text{ch } M := [\text{res}^{\mathcal{H}_n}_{\mathcal{P}_n} M] \in K(\mathcal{P}_n \text{-mod}^{fd}). \]

Exactness of the functor \( \text{res}^{\mathcal{H}_n}_{\mathcal{P}_n} \) implies that \( \text{ch} \) induces a homomorphism

\[ \text{ch} : K(\mathcal{H}_n \text{-mod}^{fd}) \rightarrow K(\mathcal{P}_n \text{-mod}^{fd}) \]

between the corresponding Grothendieck groups. For \( M \in \mathcal{H}_t \text{-mod}^{fd} \), the definition is modified in the obvious way. Note that if we expand \( \text{ch } M = \sum_{a \in F^{n+1}} r_{\underline{a}}[(a_0, a_1, \ldots, a_n)] \) in terms of the basis for \( K(\mathcal{P}_n \text{-mod}^{fd}) \) given by the irreducibles, the coefficient \( r_{\underline{a}} \) is exactly the dimension of the generalized simultaneous \( \underline{a} \)-eigenspace \( M[\underline{a}] \) of \( X_0, \ldots, X_n \).

We can explicitly compute formal characters of induced modules.
Lemma 3.4. Let \( \underline{a} = (a_0, a_1, \ldots, a_n) \in F^{n+1} \). Then
\[
\text{ch ind}_{P_n}^{H_n} \underline{a} = \sum_{\underline{c} \in S_n} [(b_0(u, \underline{c}), a_{u-1}^{(1)}, \ldots, a_{u-1}^{(n)})]
\]
where \( b_0(u, \underline{c}) := a_0 \prod_{j \in \{1, \ldots, n\}} a_{u-1}(j) \).

Proof. This follows directly from the Mackey Theorem with \( I = J = \emptyset \).
\( \square \)

Lemma 3.5. ("Shuffle Lemma")

Let \( n = m + k \), and let \( M \in \mathcal{H}_m \)-mod\( ^{\text{fd}} \), \( K \in \mathcal{H}_k \)-mod\( ^{\text{fd}} \). Assume
\[
\text{ch} M = \sum_{\underline{a} \in F^{m+1}} r_{\underline{a}}[(a_0, \ldots, a_m)], \quad \text{ch} K = \sum_{\underline{b} \in F^k} s_{\underline{b}}[(b_1, \ldots, b_k)].
\]

Then
\[
\text{ch ind}_{m,k}^{n} M \boxtimes K = \sum_{\underline{a} \in F^m} \sum_{\underline{b} \in F^k} r_{\underline{a}} s_{\underline{b}} (\sum_{\underline{c}} (c_0, c_1, \ldots, c_n)),
\]
where the last sum is over all \( \underline{c} = (c_1, \ldots, c_n) \in F^{n+1} \) which are obtained by shuffling \( \underline{a} \) and \( \underline{b} \) := \( (b_1^{\epsilon_1}, \ldots, b_k^{\epsilon_k}) \) for \( \underline{c} = (\epsilon_1, \ldots, \epsilon_k) \in \{1, -1\}^k \), which means that there exist numbers \( 1 \leq u_1 < \cdots < u_m \leq n \) such that \( (c_{u_1}, \ldots, c_{u_m}) = (a_1, \ldots, a_m) \), \( (c_1, \ldots, c_{u_1}, \ldots, c_{u_m}, \ldots, c_n) = \underline{b} \) and that \( c_0 = a_0 \prod_{j \in \{1, \ldots, k\}} b_{(j)} \).

Proof. This also follows directly from the Mackey Theorem setting \( I = \emptyset \) and \( J = (m, k) \). \( \square \)

At this point, it is convenient to introduce the left coset representatives of the two maximal parabolic subgroups that will be most important in the following.

The set \( D_{(n-1,1)} \) of distinguished left coset representatives of \( W_{n-1}^{\text{fin}} \) in \( W_n^{\text{fin}} \) consists of 1, \( s_{j,n-1} \) for \( 0 \leq j \leq n-1 \) and \( s_{j,0,n-1} \) for \( 1 \leq j \leq n-1 \), see e.g. [6].

The other set that we will need is \( D_{(n^4)} \), which is the set of left coset representatives of \( S_n \) in \( W_n^{\text{fin}} \). We claim that \( D_{(n^4)} \) is the set consisting of all elements of the form
for subsets \( j = (j_1 > j_2 > \cdots > j_r) \subseteq \{0, \ldots, n - 1\} \). This set has \( 2^n \) elements since for each \( i \in \{0, \ldots, n - 1\} \), \( i \) can either be in \( j \) or not. Therefore the cardinality of the set is the correct one, so we are done if we can show that \( s_j \) is indeed a left coset representative, which is equivalent to saying that any element of a basis of the root system of type \( A_n \) corresponding to \( S_n \) is mapped to a positive root in the root system of type \( B_n \) (corresponding to \( W_n^{\text{fin}} \)) with respect to a basis of this root system that contains the chosen basis of the root system of \( S_n \). A basis for the root system of \( S_n \) is given by \( X_{i-1}X_{i+1} \), which can be extended to a basis of the root system of \( W_n^{\text{fin}} \) by adding the element \( X_1 \). So we have to check that the element \( s_j = s_{j_r,0} \cdots s_{j_1,0} \) maps any element of the form \( X_{i-1}X_{i+1} \) to a product of a power of \( X_1 \) and elements of the form \( X_{i-1}X_{i+1} \). But \( s_j \) inverts the first \( r \) entries in the ordered set \( (X_1, \ldots, X_n) \) and moves those inverses to the places \( j_1 + 1, j_2 + 1, \ldots, j_r + 1 \) respectively. So there are three cases to consider. The first is the case that \( i + 1 \leq r \). In this case, \( X_{i-1}X_{i+1} \) gets mapped to \( X_{j_i+1}X_{j_i+1}^{-1} \) which since \( j_i + 1 < j_i \) is the product \( X_{j_i+1}^{-1}X_{j_i+1+2}X_{j_i+1+3}^{-1}X_{j_i-1}X_{j_i}X_{j_i+1} \), hence a product of the desired form. The second case is the one where \( i = r \). In this case \( X_{i-1}X_{i+1} \) gets mapped to \( X_{j_r+1}X_1 \) which can be written as \( X_1^{j_r} \prod_{k=1}^{j_r} X_{k-1}X_{k+1} \) thus also corresponds to a positive root. In the final case, where \( i > r \), the root \( X_{i-1}X_{i+1} \) gets mapped to \( X_{i-r}X_{i-r+1} \) which is again a simple root. Therefore \( s_j \) is a left coset representative.

To illustrate the way of computing formal characters using the Shue Lemma, we will now give two examples of induced modules from of parabolic subalgebras of the types described above.

**Example 3.6.** First we take the irreducible module for \( H_{1,1} \) which is isomorphic to the outer tensor product of the one-dimensional \( H_{1,1} \)-module \( L(a_0, p^2) \) on which \( T_0 \) acts as \( p \), \( X_0 \) as \( a_0 \) and \( X_1 \) as \( p^2 \) and the one-dimensional \( H_{1,1}^{\mathrm{A}} \cong F[X_2] \) module \((c)\) on which \( X_2 \) acts by multiplication with \( c \). The set \( D_{(1,1),1} \), which is simply \( D_{(1,1)} \) can be ordered as \((1, s_1, s_0s_1, s_1s_0s_1)\) and the Shue Lemma yields
if and only if

\[ \text{ind}^{\mathcal{H}_2}_{\mathcal{H}_{n,1}} L(a_0, p^2) \boxtimes (c) = [(a_0, p^2, c)] + [(a_0, c, p^2)] + [(a_0c, c^{-1}, p^2)] + [(a_0c, p^2, c^{-1})]. \]

As a second example, we compute the character of the induced module

\[ \text{ind}^{\mathcal{H}_1}_{\mathcal{P}_0 \otimes \mathcal{H}_3^A} (a_0) \boxtimes L^A(-q^{-2}, -1, -q^2), \]

where \( L^A(-q^{-2}, -1, -q^2) \) denotes the one-dimensional \( \mathcal{H}_3^A \)-module, on which \( T_1 \) and \( T_2 \) act by multiplication with \( q \), and \( X_0, X_1, X_2, X_3 \) act as \( a_0, -q^{-2}, -1, -q^2 \) respectively.

Ordering the set \( D_{(3^A)} = D_{(3^A)} \) as

\[ (1, s_0, s_1s_0, s_2s_1s_0, s_0s_1s_0, s_0s_2s_1s_0, s_1s_0s_2s_1s_0, s_0s_1s_0s_2s_1s_0), \]

we obtain

\[
\begin{align*}
\text{ch ind}^{\mathcal{H}_3}_{\mathcal{P}_0 \otimes \mathcal{H}_3^A} (a_0) \boxtimes L^A(-q^{-2}, -1, -q^2) & = [(a_0, -q^{-2}, -1, -q^2)] + [(-a_0q^{-2}, -q^2, -1, -q^2)] \\
& \quad + [(-a_0q^{-2}, -1, -q^2, -q^2)] + [(-a_0q^{-2}, -1, -q^2, -q^2)] \\
& \quad + [(a_0q^2, -1, -q^2, -q^2)] + [a_0q^2, -1, -q^2, -q^2)] \\
& \quad + [(a_0q^2, -q^2, -1, -q^2)] + [(-a_0, -q^{-2}, -1, -q^2)].
\end{align*}
\]

For any affine Hecke algebra \( \mathcal{H} \), we know that its center \( Z(\mathcal{H}) \) is exactly the set of Laurent polynomials \( f \) in its lattice that are invariant under the action of the finite Weyl group on the lattice \([19], \S 2.9\). This is originally due to Lustzig. Given \( a \in F^{n+1} \), we associate the central character

\[ \chi_a : Z(\mathcal{H}_n) \to F, \quad f(X_0, \ldots, X_n) \mapsto f(a_0, \ldots, a_n), \]

so central characters are simply all algebra homomorphisms from \( Z(\mathcal{H}_n) \) to \( F \). Consider the left action of \( W_n^{\text{fin}} \) on \( F^{n+1} \) induced by the action on \( \mathcal{P}_n \), i.e. \( s_i(a_0, \ldots, a_n) = (a_0, \ldots, a_{i+1}, a_i, \ldots, a_n) \) for \( 1 \leq i \leq n - 1 \) and \( s_0(a_0, \ldots, a_n) = (a_0a_1, a_1^{-1}, a_2, \ldots, a_n) \). Writing \( a \sim b \) if \( a \) and \( b \) lie in the same orbit with respect to this action, we get:

**Lemma 3.7.** For \( a, b \in F^{n+1} \), \( \chi_a = \chi_b \) if and only if \( a \sim b \).

Thus the central characters of \( \mathcal{H}_n \) are actually labeled by the set \( F^{n+1}/\sim \) of \( W_n^{\text{fin}} \)-orbits on \( F_{n+1} \) and we set, for \( \gamma \in F^{n+1}/\sim \)

\[ \chi_{\gamma} := \chi_{a} \]
for any $a \in \gamma$. Accordingly, for $M \in \mathcal{H}_n \text{-mod}^{fd}$ we define

$$M[\gamma] = \{v \in M \mid (z - \chi_{\gamma}(z))^k v = 0 \text{ for all } z \in Z(\mathcal{H}_n) \text{ and } k \gg 0\}.$$ 

Observe this is an $\mathcal{H}_n$-submodule of $M$. Indeed, $(z - \chi_{\gamma}(z))^k v = 0$ implies $(z - \chi_{\gamma}(z))^k hv = h(z - \chi_{\gamma}(z))^k v = 0$ for all $h \in \mathcal{H}_n$. Now, for $a \in F^{n+1}$ with $a \in \gamma$, $Z(\mathcal{H}_n)$ acts on the $\mathcal{P}_n$-submodule $M[a]$ via the central character $\chi_{\gamma}$. Hence

$$M[\gamma] = \bigoplus_{a \in \gamma} M[a],$$

as a $\mathcal{P}_n$-module. So, recalling Corollary 3.3, we see:

**Lemma 3.8.** Any $M \in \mathcal{H}_n \text{-mod}^{fd}$ decomposes as

$$M = \bigoplus_{\gamma \in F^{n+1}/\sim} M[\gamma]$$

as an $\mathcal{H}_n$-module.

Thus the $\chi_{\gamma}, \gamma \in F^{n+1}/\sim$, exhaust the possible central characters that can arise in a finite dimensional $\mathcal{H}_n$-module, while Lemma 3.4 shows that every such central character does arise in some finite dimensional $\mathcal{H}_n$-module.

For $\gamma \in F^{n+1}/\sim$, we define $\mathcal{H}_n[\gamma] \text{-mod}^{fd}$ to be the full subcategory of $\mathcal{H}_n \text{-mod}^{fd}$ consisting of all modules $M$ with $M[\gamma] = M$. Lemma 3.8 in fact yields an equivalence of categories

$$(16) \quad \mathcal{H}_n \text{-mod}^{fd} \cong \bigoplus_{\gamma \in F^{n+1}/\sim} \mathcal{H}_n[\gamma] \text{-mod}^{fd}.$$ 

In the following, we will call $\mathcal{H}_n[\gamma] \text{-mod}^{fd}$ the block of $\mathcal{H}_n \text{-mod}^{fd}$ corresponding to $\gamma$.

Observe that none of this actually uses with which affine Hecke algebra we work, thus the analogous statements hold for any parabolic subalgebra of $\mathcal{H}_n$ as well as of $\mathcal{H}_n^R$, in particular we have the same notions of formal characters, central characters and blocks for $\mathcal{H}_n^A$.

We now need a type $A$ result for a special module that will be important in the following chapter. It is due to Kato [12], but for convenience, we include a proof. Denote by $L^A(a^{(n)}) := \text{ind}_{\mathcal{R}_n^A(a^{(n)})}^{\mathcal{H}_n^A}$ the principal series module in type $A$. 
Lemma 3.9. Let $a \in F$ and $I = (m_1^A, \ldots, m_r) \subseteq \{1, \ldots, n-1\}$.

(i) $L^A(a^{(n)})$ is irreducible, and it is the only irreducible module in its block.

(ii) All composition factors of $\text{res}_{H_{a_i}^{A_i}} H_{a_i}^{A_i} L^A(a^{(n)})$ are isomorphic to

$$L^A(a^{(m_1)}) \times \ldots \times L^A(a^{(m_r)}),$$

and $\text{soc res}_{H_{a_i}^{A_i}} H_{a_i}^{A_i} L^A(a^{(n)})$ is irreducible.

(iii) $\text{soc res}_{H_{a_i}^{A_i}} H_{a_i}^{A_i} L^A(a^{(n)}) \cong L^A(a^{(n-1)}).$

(iv) The size of any Jordan block of $X_n$ on $L^A(a^{(n)})$ is $n$.

Proof. We first show that the $a^{(n-1)}$-eigenspace for $X_1, \ldots, X_{n-1}$ is contained in $1 \otimes (a^{(n)})$, which is, of course, also contained in the $a$-eigenspace of $X_n$. This is certainly true for $n = 1$, since in this case $L^A(a) = (a)$ as $H_{a_i}^{A_i} = R_1$. So assume that the statement holds for $n-1$, i.e. the $a^{(n-2)}$-eigenspace for $X_1, \ldots, X_{n-2}$ in $L^A(a^{(n-1)})$ is contained in $1 \otimes (a^{(n-1)})$. We can write an $a^{(n-1)}$-eigenvector for $X_1, \ldots, X_{n-1}$ $w \in L^A(a^{(n)}) \cong \text{ind}_{H_{a_i}^{A_i}}^{H_{a_i}^{A_i}} L^A(a^{(n-1)}) \otimes (a)$ as

$$w = \sum_{j=1}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes w_0$$

with $w_j \in L^A(a^{(n-1)})$ for $0 \leq j \leq n-1$. Now suppose $l$ is minimal such that $w_l \neq 0$, then, if $l \leq n-1$

$$(X_{l+1} - a)(T_{i,n-1} \otimes w_i + T_{l+1,n-1} \otimes w_{l+1})$$

$$= T_{i,n-1}(X_l - a) \otimes w_l + (q - q^{-1}) T_{l+1,n-1} X_n \otimes w_l$$

$$+ T_{l+1,n-1}(X_n - a) \otimes w_{l+1}$$

$$+ \text{terms in } T_{j,n-1} \otimes L^A(a^{(n-1)}) \text{ for } j \geq l + 2 \text{ and } 1 \otimes L^A(a^{(n-1)}).$$

Since $X_n$ acts as $a$ on $L^A(a^{(n-1)})$, we see from the coefficient of $T_{l+1,n-1}$ that $w_l$ has to be zero, a contradiction. If $l = n - 1$, then for $l \leq n - 2$

$$(X_k - a)(T_{n-1} \otimes w_{n-1} + 1 \otimes w_0)$$

$$= T_{n-1}(X_k - a) \otimes w_{n-1} + 1 \otimes (X_k - a)w_0,$$
whence, \( w_{n-1}, w_0 \) have to be contained in \( 1 \otimes (a^{(n-1)}) \subseteq L^A(a^{(n-1)}) \) by the inductive hypothesis. But

\[
(X_{n-1} - a)(T_{n-1} \otimes w_{n-1} + 1 \otimes w_0)
= T_{n-1}(X_n - a) \otimes w_{n-1}
- (q - q^{-1})1 \otimes X_n w_{n-1} + 1 \otimes (X_{n-1} - a)w_0,
\]

requiring \((X_{n-1} - a)w_0 = (q - q^{-1})1 \otimes X_n w_{n-1}\), but, as we have just seen, \((X_{n-1} - a)w_0 = 0\). Hence, the \(a^{(n)}\)-eigenspace is contained in \(1 \otimes L^A(a^{(n-1)})\) and therefore, by induction, in \(1 \otimes (a^{(n)})\).

Now any nontrivial submodule of \(L^A(a^{(n)})\) has to contain a simultaneous eigenvector for \(X_1, \ldots, X_n\) which can only be an \(a^{(n)}\)-eigenvector as the formal character of \(L^A(a^{(n)})\) is \(n![a^{(n)}]\). Therefore any nontrivial submodule contains \(1 \otimes (a^{(n)})\), generating the whole of \(L^A(a^{(n)})\). This proves (i).

The fact that all composition factors of \(\text{res}_{H_n^A} L^A(a^{(n)})\) are isomorphic to \(L^A(a^{(m_1)}) \boxtimes \cdots \boxtimes L^A(a^{(m_r)})\) follows immediately from the Mackey Theorem and the irreducibility of the latter module by (i) and Lemma 3.1. Therefore, the socle of \(\text{res}_{H_n^A} L^A(a^{(n)})\) consists of a number of copies of \(L^A(a^{(m_1)}) \boxtimes \cdots \boxtimes L^A(a^{(m_r)})\). But any constituent of the socle contains a simultaneous eigenvector for \(R_n\), which we have seen to be 1-dimensional, hence the socle is irreducible, completing the proof of (ii).

By (ii) the socle of \(\text{res}_{H_{n-1}^A} L^A(a^{(n)})\) is isomorphic to \(L^A(a^{(n-1)}) \boxtimes (a)\), hence \(\text{soc res}_{H_{n-1}^A} L^A(a^{(n)})\) certainly contains a copy of \(L^A(a^{(n-1)})\). But again, every constituent of \(\text{soc res}_{H_{n-1}^A} L^A(a^{(n)})\) contains a simultaneous eigenvector of \(R_{n-1}\), which also is contained in \(1 \otimes (a^{(n)})\), hence \(\text{soc res}_{H_{n-1}^A} L^A(a^{(n)}) \cong L^A(a^{(n-1)})\).

For the proof of (iv), assume inductively that the size of any Jordan block of \(X_{n-1}\) on \(L^A(a^{(n-1)})\) is \(n - 1\) which we can do since the statement certainly holds for the \(H_n^A\)-module \((a)\). Hence there are \((n-2)!\) elements in \(L^A(a^{(n-1)})\), each generating a Jordan block of size \(n - 1\). Now for such an element \(v\) and any \(1 \leq l \leq n - 1\)

\[
(X_n - a)T_{l,n-1} \otimes v = T_{l,n-1}(X_{n-1} - a) \otimes v + (q - q^{-1})T_{l,n-2}(X_n) \otimes v
\]
whence

\[(X_n - a)^{n-1} T_{n-1} \otimes v = (q - q^{-1}) T_{n-2} \otimes (X_n)(X_{n-1} - a)^{n-2} v\]

which is nonzero but is annihilated by \((X_n - a)\). Hence we obtain \((n-1)(n-2)! = (n-1)!\) Jordan blocks of size \(n\) for \(X_n\), exhausting \(L^A(a^{(n)})\).

In type \(B\), we need certain restrictions on the eigenvalue \(a\) to prove a similar statement.

**Lemma 3.10.** Let \(a \in F \setminus \{\pm 1\}\). Then \(\text{ind}^{H_n}_{P_n}(a_0, a^{(n)})\) has an irreducible cosocle and this cosocle, denoted by \(L(a_0, a^{(n)})\), is the only irreducible module in \(H_n\)-mod containing \([(a_0, a^{(n)})]\) as a formal character.

**Proof.** The second statement follows from the first by Frobenius reciprocity, so it suffices to prove the first assertion. Note that

\[
\text{ind}^{H_n}_{P_n}(a_0, a^{(n)}) \cong \text{ind}^{H_n}_{P_0 \otimes H_n}(a_0, a^{(n)}) \cong \text{ind}^{H_n}_{P_0 \otimes H_n}(a_0) \otimes L^A(a^{(n)}).
\]

Now, applying the Shue Lemma, we see that in fact, the only characters of the form \([(a_0, a^{(n)})]\) in \(\text{ind}^{H_n}_{P_0 \otimes H_n}(a_0) \otimes L^A(a^{(n)})\) stem from the coset representative 1, thus

\[
\text{Hom}_{H_n}(\text{ind}^{H_n}_{P_n}(a_0, a^{(n)}), \text{cosoc ind}^{H_n}_{P_n}(a_0, a^{(n)})) \cong \text{Hom}_{P_0 \otimes H_n}((a_0) \otimes L^A(a^{(n)}), \text{res}_{P_0 \otimes H_n}^{H_n} \text{cosoc ind}^{H_n}_{P_n}(a_0, a^{(n)})) = F,
\]

proving the claim. \(\square\)

**Remark 3.11.** We will later see that \(\text{ind}^{H_n}_{P_n}(a_0, a^{(n)})\) is in fact irreducible if and only if \(a \notin \{p^{\pm 2}, \pm q^{\pm 1}\}\) or \(a \in \{\pm q^{\pm 1}\}\) and \(n = 1\). The proof of the "if" will be completed in Lemmas 6.1 and 7.5. The "only if" can be seen right away. Indeed, if \(a = p^2\), \(\text{ind}^{H_n}_{P_1}(a_0, p^2)\) has a one-dimensional submodule \(L(a_0p^2, p^{-2}) := (T_0 - p) \otimes (a_0, p^2)\) on which \(T_0, X_0\) and \(X_1\) act as \(p^{-1}, a_0p^2\) and \(p^{-2}\) respectively and an one-dimensional quotient which we will denote by \(L(a_0, p^2)\), on which \(T_0, X_0\) and \(X_1\) act as \(p, a_0\) and \(p^2\) respectively. The module \(\text{ind}^{H_n}_{P_1}(a_0p^2, p^{-2})\) has the same composition factors in reversed order.
For $a=q$, $\text{ind}_{\mathcal{P}_1}^k(a_0, q)$ has a matrix presentation where $T_0, X_0$ and $X_1$ act as

$$\begin{pmatrix}
\frac{p-p^{-1}}{1-q} & \frac{-(qp-p^{-1})(p-qp^{-1})}{p(q-1)p^{-1}} \\
1 & \frac{p-p^{-1}}{1-q}
\end{pmatrix}, \begin{pmatrix} a_0 & 0 \\ 0 & a_0q \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

respectively. This is easily seen to be irreducible. But for $n \geq 1$, $(T_1-q)(T_0-qT_0^{-1}) \otimes (a_0, q^{(n)})$ is contained in the $(q^{-1}, q^{(n-1)})$-eigenspace and generates a nontrivial submodule. Analogously, we obtain the statement for $a=q^{-1}$. 
4. Crystal operators

In this chapter we will, in analogy to the type $A$ situation, define maps between the sets of isomorphism classes of irreducibles in $\mathcal{H}_n$-mod$^\text{fd}$ and $\mathcal{H}_{n-1}$-mod$^\text{fd}$.

Let $M \in \mathcal{H}_n$-mod$^\text{fd}$ and $a \in F$. Define $\Delta_a M$ to be the generalized $a$-eigenspace of $X_n$ in $M$, i.e.

$$\Delta_a M = \bigoplus_{a \in F^{n+1}, a_n = a} M[a].$$

As $X_n$ is central in the parabolic subalgebra $\mathcal{H}_{n-1,1} \cong \mathcal{H}_{n-1} \otimes F[X_n^{\pm 1}]$ of $\mathcal{H}_n$, $\Delta_a M$ is an $\mathcal{H}_{n-1,1}$-submodule of $M$. Thus $\Delta_a$ defines a functor

$$\Delta_a : \mathcal{H}_n\text{-mod}^\text{fd} \to \mathcal{H}_{n-1,1}\text{-mod}^\text{fd},$$

which, on morphisms, is simply restriction. This functor is exact as the composite of restriction to a subalgebra and then taking a direct summand. Analogously, for $m \geq 0$, $\Delta_{a(m)}$ denotes the functor $\mathcal{H}_n\text{-mod}^\text{fd} \to \mathcal{H}_{n-m,m}\text{-mod}^\text{fd}$ that maps $M$ to simultaneous generalized $a$-eigenspace of $X_n,m+1, \ldots, X_n$.

By Lemmas 3.1 and 3.9, $\Delta_{a(m)} M$ is the largest submodule of $\text{res}^n_{n-m,m} M$ all of whose composition factors are of the form $N \boxtimes L^A(a^{(m)})$ for some irreducible $N \in \mathcal{H}_{n-m}$-mod$^\text{fd}$ and is indeed a direct summand of $\text{res}^n_{n-m,m} M$.

**Lemma 4.1.** For $N \in \mathcal{H}_{n-m}$-mod$^\text{fd}$, $M \in \mathcal{H}_n$-mod$^\text{fd}$, there is a functorial isomorphism

$$\text{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L^A(a^{(m)}), \Delta_{a(m)} M) \cong \text{Hom}_{\mathcal{H}_n}(\text{ind}^n_{n-m,m} N \boxtimes L^A(a^{(m)}), M).$$

**Proof.** By Lemma 3.1, all composition factors of $\text{res}^n_{n-m,m} M$ are of the form $K \boxtimes L$ for irreducible $K \in \mathcal{H}_{n-m}$-mod$^\text{fd}$ and $L \in \mathcal{H}_m^A$-mod$^\text{fd}$. An injection of the irreducible $N \boxtimes L^A(a^{(m)})$ into $\text{res}^n_{n-m,m} M$ can only map onto a submodule that is isomorphic to $N \boxtimes L^A(a^{(m)})$ and all composition factors with this property are contained in $\Delta_{a(m)} M$. Since $\Delta_{a(m)} M$ is a direct summand of $\text{res}^n_{n-m,m} M$, the assertion follows. \qed

The following is immediate from the definition:
Lemma 4.2. Let $M \in \mathcal{H}_n \text{-mod}^{fd}$ with

$$\text{ch } M = \sum_{a \in F^{n+1}} r_a[(a_0, \ldots, a_n)].$$

Then we have

$$\text{ch } \Delta_{a(m)} M = \sum_{b} r_b[(b_0, \ldots, b_n)],$$

summing over all $b \in F^{n+1}$ with $b_{n-m+1} = \cdots = b_n = a$.

Now for $a \in F$ and $M \in \mathcal{H}_n \text{-mod}^{fd}$, define

$$\epsilon_a(M) = \max\{m \geq 0 \mid \Delta_{a(m)} M \neq 0\}.$$ 

By Lemma 4.2, $\epsilon_a(M)$ is simply the length of the ‘longest $a$-tail’ in $\text{ch } M$.

Lemma 4.3. Let $M \in \mathcal{H}_n \text{-mod}^{fd}$ be irreducible, $a \in F$, $\epsilon = \epsilon_a(M)$. If $N \boxtimes L^A(a^{(m)})$ is an irreducible submodule of $\Delta_{a(m)} M$ for some $0 < m \leq \epsilon$, then $\epsilon_a(N) = \epsilon - m$.

Proof. The definition implies immediately that $\epsilon_a(N) \leq \epsilon - m$. For the reverse inequality, Lemma 4.1 and the irreducibility of $M$ imply that $M$ is a quotient of $\text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)})$. So applying the exact functor $\Delta_{a^{(s)}}$, we see that the non-zero module $\Delta_{a^{(s)}} M$ is a quotient of $\Delta_{a^{(s)}}(\text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)}))$. In particular, this shows that $\Delta_{a^{(s)}}(\text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)})) \neq 0$. Using the Shuffle Lemma and Lemma 4.2 it follows that $\epsilon_a(N) \geq \epsilon - m$. ☐

Lemma 4.4. Let $m \geq 0$, $a \in F \setminus \{\pm 1\}$ and $N \in \mathcal{H}_{n-m} \text{-mod}^{fd}$ be irreducible with $\epsilon_a(N) = 0$. Set $M = \text{ind}_{n-m,m}^n N \boxtimes L^A(a^{(m)})$. Then

(i) $\Delta_{a(m)} M \cong N \boxtimes L^A(a^{(m)})$;
(ii) $\text{cosoc } M$ is irreducible with $\epsilon_a(\text{cosoc } M) = m$;
(iii) All other composition factors $L$ of $M$ have $\epsilon_a(L) < m$.

Proof. (i) As, by Lemma 4.1,

$$\text{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L^A(a^{(m)}), \Delta_{a(m)} M) \cong \text{Hom}_{\mathcal{H}_n}(M, M) \neq 0$$

and $N \boxtimes L^A(a^{(m)})$ is irreducible, $\Delta_{a(m)} M$ contains a submodule isomorphic to $N \boxtimes L^A(a^{(m)})$. But, by the Shuffle Lemma and Lemma
4.2, \( \dim(\Delta_{a(m)} M) = \dim(N \boxtimes L^A(a^{(m)})) \), since no coset representatives other than 1 yield character tuples ending in \( m \) copies of \( a \), hence \( \Delta_{a(m)} M \cong N \boxtimes L^A(a^{(m)}) \).

(ii) Again by Lemma 4.1,
\[
\text{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L^A(a^{(m)}), \Delta_{a(m)} Q) \cong \text{Hom}_{\mathcal{H}_n}(M, Q) \neq 0
\]
for any non-zero quotient \( Q \) of \( M \), in particular for any constituent \( Q \) of the cosocle. But by the same argument as in (i), only one copy of \( N \boxtimes L^A(a^{(m)}) \) can occur, thus the cosocle is irreducible.

(iii) (i) and (ii) imply that \( \Delta_{a(m)} M \cong \Delta_{a(m)}(\text{soc} \ M) \). Therefore, \( \Delta_{a(m)} L = 0 \) for any other composition factor of \( M \) by exactness of \( \Delta_{a(m)} \).

\[\square\]

Lemma 4.5. Let \( M \in \mathcal{H}_n\text{-mod}^{\text{fd}} \) be irreducible, let \( a \in F \setminus \{\pm 1\} \), \( \epsilon = \epsilon_a(M) \) and \( 0 \leq m \leq \epsilon \). Then

(i) \( \Delta_{a(\epsilon)} M \cong N \boxtimes L^A(a^{(\epsilon)}) \) for some irreducible \( \mathcal{H}_{n-\epsilon}\text{-module} \) \( N \)
with \( \epsilon_a(N) = 0 \),

(ii) \( \text{soc} \Delta_{a(m)} M \cong L \boxtimes L^A(a^{(m)}) \) for some irreducible \( L \in \mathcal{H}_{n-m}\text{-mod}^{\text{fd}} \)
with \( \epsilon_a(L) = \epsilon_a(M) - m \).

In particular, if \( \pm 1 \) do not occur as eigenvalues of \( X_n \) on \( M \), the socle of \( \text{res}_{n-1,1}^n M \) is multiplicity-free.

Proof. (i) Pick an irreducible submodule \( N \boxtimes L^A(a^{(\epsilon)}) \) of \( \Delta_{a(\epsilon)} M \), cf. remark before Lemma 4.1. Then \( \epsilon_a(N) = 0 \) by Lemma 4.3. By Lemma 4.1
\[
\text{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L^A(a^{(m)}), \Delta_{a(m)} M) \cong \text{Hom}_{\mathcal{H}_n}(M, M) \neq 0,
\]
thus \( M \), being irreducible, is a quotient of \( \text{ind}_{n-\epsilon,\epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \). Hence, \( \Delta_{a(\epsilon)} M \) is a quotient of \( \Delta_{a(\epsilon)} \text{ind}_{n-\epsilon,\epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \). But, by Lemma 4.4(i), this is irreducible and isomorphic to \( N \boxtimes L^A(a^{(\epsilon)}) \), proving (i).

(ii) For every constituent \( L \boxtimes L^A(a^{(m)}) \) of \( \text{soc} \Delta_{a(m)} M \), Lemma 4.3 tells us that \( \epsilon_a(L) = \epsilon - m \), hence \( \Delta_{a(\epsilon-m)} L \boxtimes L^A(a^{(m)}) \) is a non-trivial submodule of \( \text{res}_{n-\epsilon,\epsilon-m,\epsilon}^n \Delta_{a(\epsilon)} M \). By (i), \( \Delta_{a(\epsilon)} M \) is irreducible of the form \( N \boxtimes L^A(a^{(\epsilon)}) \), so Lemma 3.9(ii) implies that
\[
\text{soc} \text{res}_{n-\epsilon,\epsilon-m,\epsilon}^n \Delta_{a(\epsilon)} M \cong N \boxtimes L^A(a^{(\epsilon-m)}) \boxtimes L^A(a^{(m)}).
\]
Therefore there can only be one such constituent in soc $\Delta_{a(m)}M$ and we’re done. \qed

Defining

$$e_a := \text{res}^{n+1}_{n-1} \circ \Delta_a : \mathcal{H}_n\text{-mod}^{\text{fd}} \to \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$$

and analogously

$$f_a = \text{ind}^{n}_{n-1,1} - \boxtimes(a) : \mathcal{H}_{n-1}\text{-mod}^{\text{fd}} \to \mathcal{H}_n\text{-mod}^{\text{fd}}$$

we obtain the following corollary.

**Corollary 4.6.** For $a \in F \setminus \{\pm 1\}$ and an irreducible $M \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ with $\epsilon_a(M) > 0$, soc $\epsilon_aM$ is irreducible, and $\epsilon_a(\text{soc} \epsilon_aM) = \epsilon_a(M) - 1$.

**Proof.** Choose a constituent $L$ of soc $\epsilon_aM$ which by Lemma 4.5 (ii) satisfies $\epsilon_a(L) = \epsilon_a(M) - 1$. The central element $Z := X_\emptyset X_1 \ldots X_n$ of $\mathcal{H}_n$ acts as a scalar on the whole of $M$ by Schur’s Lemma, and similarly the central element $Z' := X_\emptyset X_1 \ldots X_{n-1}$ of $\mathcal{H}_{n-1}$ acts as a scalar on $L$. Hence $X_n = Z^{-1}Z$ acts on $L$ as a scalar, too. The scalar must be $a$, so $L$ contributes a constituent $L \boxtimes (a)$ to soc $\Delta_aM$. By Lemma 4.5 (ii) this is irreducible, so soc $\epsilon_aM$ must also be irreducible and isomorphic to $L$. \qed

The following lemma provides a recipe for an inductive construction of irreducible modules in $\mathcal{H}_n\text{-mod}^{\text{fd}}$ from irreducibles in $\mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$.

**Lemma 4.7.** Let $m \geq 0$, $a \in F \setminus \{\pm 1\}$, let $N \in \mathcal{H}_n\text{-mod}^{\text{fd}}$ be irreducible and set $M = \text{ind}^{n+m}_{n,m}(N \boxtimes L^A(a^{(m)}))$. Then, cosoc $M$ is irreducible with $\epsilon_a(\text{cosoc} M) = \epsilon_a(N) + m$, and all other composition factors $L$ of $M$ have $\epsilon_a(L) < \epsilon_a(N) + m$.

**Proof.** Let $\epsilon = \epsilon_a(N)$. By Lemma 4.5, $\Delta_{a(\epsilon)}N \cong K \boxtimes L^A(a^{(\epsilon)})$ for an irreducible $K \in \mathcal{H}_{n-\epsilon}\text{-mod}^{\text{fd}}$ with $\epsilon_a(K) = 0$. By Lemma 4.1 and the irreducibility of $N$, $N$ is a quotient of $\text{ind}^{n}_{n-\epsilon,\epsilon}K \boxtimes L^A(a^{(\epsilon)})$. By transitivity of induction, $\text{ind}^{n+m}_{n,m}N \boxtimes L^A(a^{(m)})$ is then a quotient of $\text{ind}^{n+m}_{n-\epsilon,\epsilon+m}K \boxtimes L^A(a^{(\epsilon+m)})$. Now all assertions follow directly from Lemma 4.4. \qed
We can now define the desired crystal operators. Let $M$ be an irreducible module in $\mathcal{H}_n$-$\text{mod}^{fd}$. Define
\begin{equation}
\tilde{e}_a M := \text{soc } e_a M, \quad \tilde{f}_a M := \text{cosoc } \text{ind}_{n,1}^{n+1} M \boxtimes (a),
\end{equation}
For $a \neq \pm 1$, $\tilde{f}_a M$ is irreducible by Lemma 4.7 and $\tilde{e}_a M$ is irreducible or 0 by Corollary 4.6, hence the functors induce maps between the set of isomorphism classes of irreducibles in $\mathcal{H}_n$-$\text{mod}^{fd}$ and $\mathcal{H}_{n-1}$-$\text{mod}^{fd}$. Observe that Corollary 4.6 implies
\begin{equation}
\epsilon_a (M) = \max \{ m \geq 0 \mid \tilde{e}_a^m M \neq 0 \}
\end{equation}
and, by Lemma 4.7, we have
\begin{equation}
\epsilon_a (\tilde{f}_a M) = \epsilon_a (M) + 1.
\end{equation}

**Lemma 4.8.** Let $M \in \mathcal{H}_n$-$\text{mod}^{fd}$ be irreducible, $a \in F \setminus \{ \pm 1 \}$ and $m \geq 0$.

(i) $\text{soc } \Delta_{a(m)} M \cong (\tilde{e}_a^m M) \boxtimes L^A (a(m))$.
(ii) $\text{cosoc } \text{ind}_{n,m}^{n+m} M \boxtimes L^A (a(m)) \cong \tilde{f}_a^m M$.

**Proof.** (i) If $m > \epsilon_a (M)$, then both parts in the equality above are zero. Let $m \leq \epsilon_a (M)$. By Corollary 4.6 and Lemma 4.5, $(\tilde{e}_a M) \boxtimes (a)$ is a submodule of $\Delta_a M$. By applying this fact $m$ times, we deduce that $(\tilde{e}_a^m M) \boxtimes (a)^{\otimes m}$ is a submodule of $\text{res}_{n-m,1,...,1}^{n-m,m} \Delta_{a(m)} M$, whence we see that $(\tilde{e}_a^m M) \boxtimes L^A (a(m))$ is a submodule of $\Delta_{a(m)} M$ by Frobenius reciprocity and the fact that, by Lemma 3.9, in $\mathcal{H}_m^A$-$\text{mod}^{fd}$ the module $L^A (a(m))$ is the only irreducible containing the formal character $(a,...,a)$ . But Lemma 4.5 says that $\text{soc } \Delta_{a(m)} M$ is irreducible and therefore isomorphic to $(\tilde{e}_a^m M) \boxtimes L^A (a(m))$.

(ii) As $\tilde{f}_a^m M$ is certainly a quotient of $\text{ind}_{n,m}^{n+m} M \boxtimes (a) \boxtimes \cdots \boxtimes (a)$ by exactness of induction, it is, by transitivity of induction, also a quotient of $\text{ind}_{n,m}^{n+m} M \boxtimes L^A (a(m))$. By Lemma 4.7, this has an irreducible cosocle, which therefore has to be isomorphic to $\tilde{f}_a^m M$.

**Lemma 4.9.** Let $M \in \mathcal{H}_n$-$\text{mod}^{fd}$ and $N \in \mathcal{H}_{n-1}$-$\text{mod}^{fd}$ be irreducible modules and $a \in F \setminus \{ \pm 1 \}$. Then, $\tilde{e}_a M \cong N$ if and only if $\tilde{f}_a N \cong M$.

**Proof.** By Lemma 4.7, $\tilde{f}_a N \cong M$ is equivalent to
\[
\text{Hom}_{\mathcal{H}_n} (\text{ind}_{n-1,1}^n N \boxtimes (a), M) \cong \text{Hom}_{\mathcal{H}_{n-1}} (N \boxtimes (a), \Delta_a M)
\]
being non-zero, which means that \( N \boxtimes (a) \) appears in the socle of \( \Delta_a M \), which, by the irreducibility of the latter, is equivalent to \( N \cong \delta_a M \).

\[ \square \]

We immediately get the following corollary:

**Corollary 4.10.** Let \( M, N \in \mathcal{H}_n \)-mod\(^{fd} \) be irreducible and \( a \in F \setminus \{ \pm 1 \} \). Then

1. \( \delta_a \tilde{f}_a M \cong M \) and, if \( \epsilon_a(M) > 0 \), \( \tilde{f}_a \delta_a M \cong M \);  
2. \( \tilde{f}_a M \cong \tilde{f}_a N \) if and only if \( M \cong N \) and, if \( \epsilon_a(M), \epsilon_a(N) > 0 \), \( \delta_a M \cong \delta_a N \) if and only if \( M \cong N \).

We define the crystal graph \( \Gamma \) as the graph whose vertices correspond to isomorphism classes of irreducible modules in \( \bigoplus_{n \geq 0} \mathcal{H}_n \)-mod\(^{fd} \), where there is an edge \([N] \xrightarrow{a} [M]\) if and only if \( M \cong \tilde{f}_a N \). If an irreducible \( M \in \mathcal{H}_n \)-mod\(^{fd} \) is isomorphic to \( \tilde{f}_{a_n} \cdots \tilde{f}_{a_1}(a_0) \), we write \( M \cong L(a_0, a_1, \ldots, a_n) \). Note that these definitions make sense even in the case where \( a \in \{ \pm 1 \} \). Even though in this case \( \tilde{f}_a \) does not necessarily produce irreducible modules, as we will see in Chapter 7, we just don’t draw any edges from \( N \) if cosoc ind\(^n\) \( N \boxtimes (a) \) is not irreducible. This has the drawback that not every vertex in \( \Gamma \) is connected to a module for \( \mathcal{H}_0 \), but it enables us to use the labeling \( L(a) \) in the cases where 1 or \(-1\) occur in \( a \), but the corresponding cosocle of the induced modules are irreducible.

We will denote the full subcategory of \( \mathcal{H}_n \)-mod\(^{fd} \) consisting of those modules on which \((X_i \pm 1)\) acts invertibly for all \( 1 \leq i \leq n \) by \( \text{Rep}\mathcal{H}_n \). Then we obtain the following:

**Theorem 4.11.** The map \( \text{ch} : K(\text{Rep}\mathcal{H}_n) \to K(\mathcal{P}_n \text{-mod}^{fd}) \) is injective.

**Proof.** We need to show that the characters of the irreducible modules in \( \text{Rep}\mathcal{H}_n \) are linearly independent in \( K(\mathcal{P}_n \text{-mod}^{fd}) \). Proceed by induction on \( n \), the case \( n = 0 \) being trivial. Suppose \( n > 0 \) and there is a non-trivial \( \mathbb{Z}\)-linear dependence

\[
(21) \quad \sum c_L \text{ch} L = 0
\]
for some irreducible modules $L \in \text{Rep} \mathcal{H}_n$. As every irreducible $L$ has $\epsilon_a(L) > 0$ for at least one $a \in F$, it suffices to show by downward induction on $k = n, \ldots, 1$ that, for any fixed $a \in F \setminus \{±1\}$, $c_L = 0$ for all $L$ with $\epsilon_a(L) = k$.

In the case $k = n$, $\Delta_{a(n)} L = 0$ except if $L \cong L(a^{(n)})$, by Theorem 3.10. Applying $\Delta_{a(n)}$ to the equation (21) and using Lemma 4.2, we deduce that the coefficient of $\text{ch} L(a^{(n)})$ is zero.

Now suppose $1 \leq k < n$ and that we have shown $c_L = 0$ for all $L$ with $\epsilon_a(L) > k$. Apply $\Delta_{a(k)}$ to the equation to deduce that

$$\sum_{L \text{ with } \epsilon_a(L) = k} c_L \text{ch} \Delta_{a(k)} L = 0.$$ 

Now each such $\Delta_{a(k)} L$ is isomorphic to $(\tilde{e}_a^k L) \boxtimes L(a^{(k)})$, according to Lemmas 4.5 and 4.8(i). Moreover, $L \not\cong L'$ implies $\tilde{e}_a^k L \not\cong \tilde{e}_a^k L'$ by Corollary 4.10. Now the induction hypothesis on $n$ guarantees that all such coefficients $c_L$ are zero, as required. 

**Corollary 4.12.** If $L$ is an irreducible module in $\text{Rep} \mathcal{H}_n$, then $L \cong L^\tau$.

**Proof.** Since $\tau(X_i) = X_i$, $\tau$ leaves characters invariant. Hence it leaves irreducibles invariant since they are determined up to isomorphism by their character according to the theorem.

We now give three interpretations of the functions $\epsilon_a$.

**Theorem 4.13.** Let $a \in F \setminus \{±1\}$ and $M$ be an irreducible module in $\mathcal{H}_n \text{-mod}^{\text{fd}}$. Then

(i) $[e_a M] = e_a(M)[\tilde{e}_a M] + \sum c_r[N_r]$ where the $N_r$ are irreducible modules with $\epsilon_a(N_r) < \epsilon_a(\tilde{e}_a M)$;

(ii) $\epsilon_a(M)$ is the maximal size of a Jordan block of $X_n$ on $M$ with eigenvalue $a$;

(iii) The algebra $\text{End}_{\mathcal{H}_{n-1}}(e_a M)$ is isomorphic to the algebra of truncated polynomials $F[x]/(x^{\epsilon_a(M)})$.

**Proof.** Let $\epsilon = \epsilon_a(M)$ and $N = \tilde{e}_a^\epsilon M$.

(i) By Lemma 4.5 and Frobenius reciprocity, there is a short exact sequence

$$0 \longrightarrow R \longrightarrow \text{ind}_{n-\epsilon, \epsilon}^n N \boxtimes L^A(a^{(\epsilon)}) \longrightarrow M \longrightarrow 0,$$
where all composition factors $L$ of $R$ have $\epsilon_a(L) < \epsilon$ by Lemma 4.4(iii). Applying the exact functor $\Delta_a$, we obtain the exact sequence

$$0 \longrightarrow \Delta_a R \longrightarrow \Delta_a \text{ind}_{n-\epsilon,\epsilon} N \otimes L^A(a^{(\epsilon)}) \longrightarrow \Delta_a M \longrightarrow 0.$$  

By the Mackey Theorem

$$[\text{res}_{n-1,1}^{n} \text{ind}_{n,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})] = \left[\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes \text{res}_{H,1}^{H} L^A(a^{(\epsilon)})\right]$$

$$+ \left[\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} s_{n-1,n-\epsilon}(\text{res}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)}))\right]$$

$$+ \left[\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} s_{n-1,0,n-\epsilon}(N \otimes \text{res}_{H,1}^{H} L^A(a^{(\epsilon)}))\right]$$

The third subquotient does not contribute to $\Delta_a \text{ind}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})$ as its formal character ends on $a^{-1}$. Similarly, the direct summands of $\text{res}_{H,1}^{H} L^A(a^{(\epsilon)})$ other than $\Delta_a L^A(a^{(\epsilon)})$ and the direct summands of $\text{res}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})$ other than $\Delta_a N \otimes L^A(a^{(\epsilon)})$ cannot contribute to $\Delta_a \text{ind}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})$. As $\Delta_a N = 0$, we obtain

$$\Delta_a \text{ind}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)}) \cong \text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes \Delta_a L^A(a^{(\epsilon)}).$$

By considering characters, we see that

$$[\Delta_a L^A(a^{(\epsilon)})] = \epsilon[L^A(a^{(\epsilon-1)}) \otimes (a)],$$

hence

$$[\Delta_a \text{ind}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})] = \epsilon[\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes L^A(a^{(\epsilon-1)}) \otimes (a)].$$

By Lemma 4.8(ii), the cosocle of $\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes L^A(a^{(\epsilon-1)}) \otimes (a)$ is $(\tilde{f}_a^{(\epsilon-1)} N) \otimes (a)$ which is the same as $(\tilde{e}_a M) \otimes (a)$, and all other composition factors of this module are of the form $L \otimes (a)$ with $\epsilon_a(L) < \epsilon - 1$ by Lemma 4.4. Moreover, all composition factors of $\Delta_a R$ are of the form $L \otimes (a)$ with $\epsilon_a(L) < \epsilon - 1$. So we have now seen that

$$[\Delta_a M] = \epsilon[\tilde{e}_a M \otimes (a)] + \sum c_r [N_r \otimes (a)]$$

for irreducibles $N_r$ with $\epsilon_a(N_r) < \epsilon_a(\tilde{e}_a M)$, which implies (i).

(ii) By Lemma 4.4 (i) $\Delta_a M \cong N \otimes L^A(a^{(\epsilon)})$, so, by Lemma 3.9 (iv), we deduce that the maximal size of a Jordan block of $X_n$ on $\Delta_a M$ is $\epsilon$. Hence the maximal size of a Jordan block of $X_n$ on $\Delta_a M$ is at least $\epsilon$. On the other hand, $\Delta_a \text{ind}_{n-\epsilon,\epsilon}^{n} N \otimes L^A(a^{(\epsilon)})$ has a filtration with $\epsilon$ factors, each of which is isomorphic to $\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes L^A(a^{(\epsilon-1)}) \otimes (a)$. Since $(X_n - a)$ annihilates $\text{ind}_{n-\epsilon,\epsilon}^{n-1,1} N \otimes L^A(a^{(\epsilon-1)}) \otimes (a)$, it follows
that \((X_n - a)^\epsilon\) annihilates \(\Delta_n \text{ind}_{n, \epsilon}^n N \otimes L^A(a^{(\epsilon)})\). Therefore \((X_n - a)^\epsilon\) annihilates its quotient \(\Delta_n M\) and the maximal size of a Jordan block of \(X_n\) on \(\Delta_n M\) is at most \(\epsilon\).

(iii) As left multiplication by the element \((X_n - a)\) centralizes the subalgebra \(H_{n-1}\) of \(H_n\), we have an \(H_{n-1}\)-endomorphism

\[
\theta : e_a M \rightarrow e_a M : m \mapsto (X_n - a)m
\]

By (ii), \(\theta^{\epsilon-1} \neq 0\) and \(\theta^\epsilon = 0\), so \(1, \theta, \ldots, \theta^{\epsilon-1}\) span a subalgebra of \(\text{End}_{H_{n-1}}(e_a M)\) isomorphic to the algebra of truncated polynomials \(F[x]/(x^\epsilon)\). On the other hand, \(e_a M\) has irreducible socle \(\tilde{e}_a M\), and this appears in \(e_a M\) with multiplicity \(\epsilon\) by (i), restricting the dimension of \(\text{End}_{H_{n-1}}(e_a M)\) to be at most \(\epsilon\).

An interesting consequence is the following.

**Corollary 4.14.** Let \(M, N \in \text{Rep} H_n\) be non-isomorphic irreducible modules. Then, for \(a \in F \setminus \{\pm 1\}\), we have \(\text{Hom}_{H_{n-1}}(e_a M, e_a N) = 0\).

**Proof.** Suppose there is a non-zero homomorphism \(\theta : e_a M \rightarrow e_a N\). Then \(e_a N\) contains cosoc \(e_a M\) as a composition factor, which is isomorphic to \(\tilde{e}_a M\) since \(M\) is selfdual and \(\tau\) commutes with \(e_a\), hence by Theorem 4.13(i), \(e_a(\tilde{e}_a N) \geq e_a(\tilde{e}_a M)\). On the other hand, \(e_a N\) has simple socle \(\tilde{e}_a N\), so \(e_a M\) has soc \(e_a N \cong \tilde{e}_a N\) as a composition factor, which gives the inequality the other way round. Thus \(e_a(\tilde{e}_a N) = e_a(\tilde{e}_a M)\). But then, \(\tilde{e}_a M\) is a composition factor of \(e_a N\) with \(e_a(\tilde{e}_a M) = e_a(\tilde{e}_a N)\), hence by Theorem 4.13(i) again, \(\tilde{e}_a M \cong \tilde{e}_a N\). But this contradicts Corollary 4.10.
5. Facts About Representations of the Affine Hecke Algebra of Type A

At this point it is convenient to recall the most important facts from the representation theory of affine Hecke algebras of type A. The results in this section are compiled from [2], [17], [18], [9] and [8].

For $\lambda \in F$, define the full subcategory $\text{Rep}_\lambda \mathcal{H}_n^A$ of $\mathcal{H}_n^A$-$\text{mod}^{fd}$ to consist of those modules where all eigenvalues of the generators $P_{\lambda n}$ are from $I^+_\lambda := \{\lambda q^2 i \mid i \in \mathbb{Z}\}$. The significance of the "plus" will become clear later on when we move to type B.

The $\text{Rep}_\lambda \mathcal{H}_n^A$ are equivalent for all $\lambda \in F$, there are no nontrivial extensions between modules in $\text{Rep}_\lambda \mathcal{H}_n^A$ and $\text{Rep}_\lambda \mathcal{H}_n^A$ for $\lambda \neq \lambda'$, and $\text{ind}_{\mathcal{H}_n^{A_1} \otimes \mathcal{H}_n^{A_2}}^{\mathcal{H}_n^{A_{1+n_2}}} M \otimes N$, for irreducible $M$ and $N$ in $\text{Rep}_\lambda \mathcal{H}_n^{A_1}$ and $\text{Rep}_\lambda \mathcal{H}_n^{A_2}$ respectively, is always irreducible. The irreducible modules in $\text{Rep}_\lambda \mathcal{H}_n^A$ are well-understood and have a nice combinatorial description.

Let $\Gamma_{(i,i+k)} = L^A(aq^{2i}, aq^{2(i+1)}, \ldots, aq^{2(i+k)})$ denote the one-dimensional representation of $\mathcal{H}_{k+1}^A$ on which all $T_j$, for $2 \leq j \leq n$ act as $q$ and $X_j$ acts as $aq^{2(i+j-1)}$, and call this a segment of length $k+1$. Define a multisegment to be a concatenation of several segments denoting the one-dimensional representation for the tensor product of the corresponding algebras. The length of a multisegment is the sum of the lengths of the contained segments. We have two different orderings on multisegments, the so-called right and left order.

In the right order, $\Gamma_{(i,i+k)} > \Gamma_{(j,j+l)}$ if $i > j$ or if $i = j$ and $l > k$.

In the left order, $\Gamma_{(i,i+k)} > \Gamma_{(j,j+l)}$ if $i + k > j + l$ or $i + k = j + l$ and $j > i$.

Bernstein and Zelevinski [2] showed that there is a one-to-one correspondence between ordered multisegments of length $n$ and irreducible modules for the affine Hecke algebra of type A. This correspondence is given by inducing a multisegment $\Gamma$ up to $\mathcal{H}_n^A$ and taking the – always irreducible – cosocle of this induced module which is independent of whether we have chosen $\Gamma$ in right or left order. Denote this cosocle by $M_{\Gamma}$.
There is also a combinatorial description of some form of branching rules. Since the index of $\mathcal{H}^A_{n-1}$ in $\mathcal{H}^A_n$ is infinite, we substitute normal induction by functors $\text{ind}_a^A := \text{ind}_{\mathcal{H}^A_{n-1} / \mathcal{H}^A_n} : \text{Rep}_\lambda \mathcal{H}^A_{n-1} \to \text{Rep}_\lambda \mathcal{H}^A_n$ for every $a \in I_\lambda^+$. For irreducible $N$ in $\text{Rep}_\lambda \mathcal{H}^A_{n-1}$, the cosocle of $\text{ind}_a^A N$ is irreducible and we denote this by $\bar{f}_a^A N$. Dually, we define $\bar{\epsilon}_a^A N := \text{res}_{\mathcal{H}^A_{n-1}} \text{soc} \Delta_a N$ where $\Delta_a N$ is the generalized $a$-eigenspace of $X_n$ in $N$ as in type B. For irreducible $N$, $\bar{\epsilon}_a^A N$ is again an irreducible module. As in type B, we define $\epsilon_a(N)$ to be the largest number $r$ such that $\Delta_a^{(r)} N \neq 0$. Since the definitions of $\Delta_a$ and $\epsilon_a$ involve the action of the lattice, we keep the notation from type B and do not mark the symbols with an $A$.

Analogously, define $\text{ind}_a^{*A} := \text{ind}_{\mathcal{H}^A_{1,n-1} / \mathcal{H}^A_n} (a) \boxminus : \text{Rep}_\lambda \mathcal{H}^A_{n-1} \to \text{Rep}_\lambda \mathcal{H}^A_n$, $\bar{f}_a^{*A} N := \text{cosoc} \text{ind}_a^{*A} N$ and $\bar{\epsilon}_a^{*A} N := \text{res}_{\mathcal{H}^A_{n-1}} \text{soc} \Delta_a^{*} N$, where $\Delta_a^{*} N$ is the generalized $a$-eigenspace of $X_1$ on $N$ which is, analogously to $\Delta_a N$ an $\mathcal{H}^A_{1,n-1}$-submodule of $N$ since $X_1$ commutes with $T_2, \ldots, T_n$. Both $\bar{f}_a^{*A} N$ and $\bar{\epsilon}_a^{*A} N$ are irreducible if $N$ is. Lastly, $\epsilon_a(N)$ is defined as the maximal $r$ such that $\Delta_a^{(r)} N \neq 0$.

For the irreducible module $M_\Gamma$ in $\text{Rep}_\lambda \mathcal{H}^A_n$ there are combinatorial algorithms to compute $f_a^A M_\Gamma, \bar{f}_a^{*A} M_\Gamma, \bar{\epsilon}_a^{*A} M_\Gamma, \epsilon_a(M_\Gamma)$ and $\epsilon_a(M_\Gamma)$. To compute $\epsilon_a(M_\Gamma)$, write down the multisegment $\Gamma$ in right order. Then, write a $+$ for every segment ending on $aq^{-2}$ and a $-$ for every multisegment ending on $a$. In the resulting sequence of plus and minus signs successively cancel out all subsequences of the form $--$ until the leftover sequence is of the form $++\ldots+-\ldots-$. The number of un-canceled $-$ signs is $\epsilon_a(M_\Gamma)$. If we replace the segment $(aq^{-2i}, \ldots, aq^{-2}, a)$ which contributed the leftmost un-canceled $-$, by $(aq^{-2i}, \ldots, aq^{-2})$ we get the multisegment corresponding to $\bar{\epsilon}_a^{*A} M_\Gamma$. Denote this multisegment by $\bar{\epsilon}_a^{*A} \Gamma$. In case there is no $-$ sign left after cancellation $\bar{\epsilon}_a^{*A} M_\Gamma = 0$. If we replace the segment $(aq^{-2k}, \ldots, aq^{-2})$ which contributed the rightmost un-canceled $+$, by $(aq^{-2k}, \ldots, aq^{-2}, a)$ we get the multisegment $\bar{f}_a^{*A} \Gamma$ corresponding to $\bar{f}_a^{*A} M_\Gamma$. If there is no $+$ left after cancellation, we add a new segment $(a)$ to $\Gamma$ to obtain $\bar{f}_a^{*A} M_\Gamma$. 
To compute $\epsilon^*_a(M_\Gamma)$, write down the multisegment $\Gamma$ in left order. Then, write $a +$ for every segment starting on $aq^2$ and $a -$ for every multisegment starting on $a$. In the resulting sequence of plus and minus signs successively cancel out all subsequences of the form $+\cdots-$ until the leftover sequence is of the form $-\cdots+\cdots+$. The number of uncanceled $-$ signs is $\epsilon^*_a(M_\Gamma)$. If we replace the segment $(a, aq^2, \ldots, aq^{2i})$ which contributed the leftmost uncanceled $-$, by $(aq^2, \ldots, aq^{2i})$ we get the multisegment corresponding to $\tilde{e}^*_a M_\Gamma$. Denote this multisegment by $\tilde{e}^*_a \Gamma$. If there is no $-$ sign left after cancellation $\tilde{e}^*_a M_\Gamma = 0$. If we replace the segment $(aq^2, \ldots, aq^{2k})$ which contributed the rightmost uncanceled $+$, by $(a, aq^2, \ldots, aq^{2k})$ we get the multisegment $\tilde{f}^*_a \Gamma$ corresponding to $\tilde{f}^*_a M_\Gamma$. If there is no $+$ left after cancellation, we add a new segment $(a)$ to $\Gamma$ to obtain $\tilde{f}^*_a M_\Gamma$.

Note also that $\tilde{f}^*_a M_\Gamma \cong \text{soc ind}_a N$ and $\tilde{f}^*_a M_\Gamma \cong \text{soc ind}_a N$.

Ending this section, we would like to give a very general result on the characters of an $\mathcal{H}_n$-module obtained by inducing from $\mathcal{H}^A_n$.

**Lemma 5.1.** Let $N \in \mathcal{H}^A_n$-mod be irreducible. Set

$$M := \text{ind}_{\mathcal{H}_n}^{\mathcal{H}_n}(a_0) \boxtimes N.$$  

Then $\epsilon_a(M) \leq \epsilon_a(N) + \epsilon^*_a(N)$.

**Proof.** By the Shuffle Lemma, we obtain the formal character of $M$ by taking all characters of $N$ and then successively inverting the first entries of the tuple and moving them to the rear, multiplying $a_0$ by all inverted entries, see Chapter 3 for an example. From this, one easily sees that the maximal number of $a$'s at the end of a tuple from $\text{ch} M$ is less or equal to the maximal number of $a$'s at the end of a tuple from $\text{ch} N$ plus the maximal number of $a^{-1}$'s at the beginning of a tuple from $\text{ch} N$ that get inverted and moved to the rear. \qed
6. Classification For Subcategories

We define $\text{Rep}_\lambda \mathcal{H}_n$ for fixed $\lambda \in F$ to be the full subcategory of $\mathcal{H}_n^R \text{-mod}^{fd}$ where all eigenvalues of $\mathcal{R}_n$ are from the set

$$I_\lambda := \{\lambda q^{2i}, \lambda^{-1}q^{2i}| i \in \mathbb{Z}\}.$$

In this section, we consider the cases where $p^2, \pm q, \pm 1 \notin I_\lambda$. In these cases $I_\lambda^+ := \{\lambda q^{2i}| i \in \mathbb{Z}\}$ and $I_\lambda^- := \{\lambda^{-1}q^{2i}| i \in \mathbb{Z}\}$ are disjoint. Since we exclude the eigenvalue $-1$, Lemmas 1.1 and 1.3 guarantee that we can work with the algebra $\mathcal{H}_n^R$ instead of $\mathcal{H}_n$ without losing any information. This is more convenient since we want to exploit the subalgebra $\mathcal{H}_n^A$, which has finite index in $\mathcal{H}_n^R$ but not in $\mathcal{H}_n$.

Note that on $\mathcal{H}_n^A$ there is an algebra antiautomorphism $\kappa$ given by $T_i \mapsto T_{n+2-i}$ and $X_i \mapsto X_{n+1-i}$ inducing a duality on $\mathcal{H}_n^A \text{-mod}^{fd}$. It is easy to check on the generators that $\kappa$ is the composite of first taking the $\tau$-dual and then twisting with the longest coset representative $d = s_0s_{1,0} \cdots s_{j,0} \cdots s_{n-1,0}$. Now we compute the $\kappa$-dual of an irreducible $M_\Gamma \in \text{Rep}_\lambda \mathcal{H}_n^A$, where $\Gamma$ consists of segments $\Gamma_1, \ldots, \Gamma_r$ of length $n_1, \ldots, n_r$ respectively:

$$M_\Gamma^\kappa \cong \overset{d}{(M_\Gamma)}$$

$$\cong \overset{d}{(M_\Gamma)}$$

$$= \text{cosoc} \overset{d}{(\mathcal{H}_n^A \otimes \mathcal{H}_n^A \otimes \cdots \otimes \mathcal{H}_n^A \Gamma)}$$

$$\cong \text{cosoc} \mathcal{H}_n^A \otimes (\mathcal{H}_n^A \otimes \cdots \otimes \mathcal{H}_n^A)^d \Gamma$$

$$\cong \text{cosoc} \mathcal{H}_n^A \otimes \mathcal{H}_n^A \otimes \cdots \otimes \mathcal{H}_n^A \Gamma$$

$$\cong M_\Gamma,$$

where $\Gamma$ is the multisegment $\Gamma_r, \ldots, \Gamma_1$, and the segment $\Gamma_j$, for a segment $\Gamma_j = (a, \ldots, aq^{2k})$, is defined as $\overline{\Gamma}_j = (a^{-1}q^{-2k}, \ldots, a^{-1})$.

**Lemma 6.1.** Let $M_\Gamma \in \text{Rep}_\lambda \mathcal{H}_n^A$. Then $L_\Gamma := \text{ind}_{\mathcal{H}_n^A}^{\mathcal{H}_n^R} M_\Gamma \in \text{Rep}_\lambda \mathcal{H}_n$ is irreducible.

**Proof.** Since all characters of $M_\Gamma$ have entries in $I^-_\lambda$ and $I^-_\lambda \cap I^+_\lambda = \emptyset$, the Shuffle Lemma implies that the only summands in the formal character of $L_\Gamma$ exclusively containing entries from $I^-_\lambda$ are those afforded by
the coset representative 1. Thus

$$\text{Hom}_{\mathcal{H}_\lambda}(L_\Gamma, \text{cosoc } L_\Gamma) \cong \text{Hom}_{\mathcal{H}_\lambda}(M_\Gamma, \text{res}_{\mathcal{H}_\lambda} \text{cosoc } L_\Gamma) \cong F,$$

whence the cosocle of $L_\Gamma$ is irreducible.

Since

$$\text{Hom}_{\mathcal{H}_\lambda}(\text{soc } L_\Gamma, L_\Gamma) \cong \text{Hom}_{\mathcal{H}_\lambda}(L_\Gamma^*, \text{cosoc } L_\Gamma^*)$$

$$\cong \text{Hom}_{\mathcal{H}_\lambda}(\text{ind}_{\mathcal{H}_\lambda}^A(M_\Gamma^*), \text{cosoc } \text{ind}_{\mathcal{H}_\lambda}^A M_\Gamma^*)$$

$$\cong \text{Hom}_{\mathcal{H}_\lambda}(M_\Gamma, \text{res}_{\mathcal{H}_\lambda} \text{cosoc } \text{ind}_{\mathcal{H}_\lambda}^A M_\Gamma)$$

$$\cong F$$

by the same argument as above, the socle of $L_\Gamma$ is also simple and contains all generalized simultaneous eigenspaces of the lattice where all eigenvalues are from $I_\lambda^\pm$.

Now take a $(b_1, \ldots, b_n)$-eigenvector $u \in L_\Gamma$ such that $b_1, \ldots, b_n$ are all in $I_\lambda^\pm$. Then

$$v := \left( T_0 - b_n T_0^{-1} \right) \left( T_1 - b_{n-1} b_n T_1^{-1} \right) \left( T_0 - b_1 T_0^{-1} \right)$$

$$\cdots \left( T_{n-j} - b_j b_{n-j} T_{n-j}^{-1} \right) \cdots \left( T_0 - b_j T_0^{-1} \right)$$

$$\cdots \left( T_{n-1} - b_1 b_{n-1} T_{n-1}^{-1} \right) \cdots \left( T_1 - b_1 b_2 T_1^{-1} \right) (T_0 - b_1 T_0^{-1}) u$$

is a $(b_1^{-1}, \ldots, b_n^{-1})$-eigenvector by Lemma 1.5 since $b_j \notin \{p^{\pm 2}, 1\}$ and $b_j b_k \notin \{q^{\pm 2}\}$. Now all $b_i^{-1}$ belong to $I_\lambda^-$, therefore $v \in 1 \otimes M_\Gamma$ and generates $L_\Gamma$. Therefore any element in the socle of $L_\Gamma$ generates the whole module, so $L_\Gamma$ must be irreducible. \(\Box\)

**Lemma 6.2.** $\epsilon_a(\text{ind}_{\mathcal{H}_\lambda}^A M_\Gamma) = \begin{cases} 
\epsilon_a(M_\Gamma) & \text{if } a \in I_\lambda^- \\
\epsilon_a^{-1}(M_\Gamma) & \text{if } a \in I_\lambda^+
\end{cases}$

**Proof.** This follows directly from the proof of Lemma 5.1 and the fact that $I_\lambda^- \cap I_\lambda^+ = \emptyset$. \(\Box\)

**Lemma 6.3.** Let $L_\Gamma \in \text{Rep}_\lambda \mathcal{H}_\lambda$ be defined as in Lemma 6.1 and $a \in I_\lambda$.

Then

(i) $\tilde{f}_a L_\Gamma = \begin{cases} 
L_{\tilde{f}_a^\lambda \Gamma} & \text{if } a \in I_\lambda^- \\
L_{\tilde{f}_a^{-1} \lambda \Gamma} & \text{if } a \in I_\lambda^+
\end{cases}$
(ii) $\tilde{e}_a L_{\Gamma} = \begin{cases} 
 L_{\tilde{\varepsilon}_{a}^\lambda} & \text{if } a \in I^-_\lambda \\
 L_{\tilde{e}_{a}^{\lambda,-1}} & \text{if } a \in I^+_\lambda 
 \end{cases}$

**Proof.** (i) Without loss of generality we assume $a \in I^-_\lambda$ and compute $\tilde{f}_a L_{\Gamma}$ and $\tilde{f}_{a^{-1}} L_{\Gamma} = \text{soc ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} L_{\Gamma} \boxtimes (a)$. We know that

$$\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} L_{\Gamma} \boxtimes (a) \cong \text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} (\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} M_{\Gamma}) \boxtimes (a) \cong \text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} \text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} M_{\Gamma} \boxtimes (a),$$

so, as every composition factor in $\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} M_{\Gamma} \boxtimes (a)$ is in $\text{Rep}_{\lambda-1} \mathcal{H}_n^A$ and therefore yields an irreducible subquotient of $\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} L_{\Gamma} \boxtimes (a)$ upon induction to type B, $\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} L_{\Gamma} \boxtimes (a)$ has the same number of composition factors as $\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} M_{\Gamma} \boxtimes (a)$, labeled by the same multisegments. Since socle and cosocle of $\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} L_{\Gamma} \boxtimes (a)$ are irreducible, they have to coincide with

$$\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} \text{soc ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} M_{\Gamma} \boxtimes (a)$$

and

$$\text{ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} \text{cosoc ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} M_{\Gamma} \boxtimes (a)$$

respectively. The fact that $M_{\tilde{f}_{a^{-1}}^\lambda \Gamma} = \text{soc ind}^{\mathcal{H}_n}_{\mathcal{H}_{n-1,1}} M_{\Gamma} \boxtimes (a)$ completes the proof of (i).

(ii) follows directly from Lemma 4.9. □
7. Eigenvalues 1 and −1

In this chapter we investigate the cases when some of the $X_i$ have eigenvalues 1 and $-1$. First we want to consider a class of modules where the results are as in Chapter 4, even though the arguments using formal characters do no longer work.

**Lemma 7.1.** For all $N \in \mathcal{H}_{n-1} \text{-mod}$ and all $a \in F$,

$$(\text{ind}^n_{n-1,1} N \boxtimes (a))^\sigma \cong \text{ind}^n_{n-1,1} N^\sigma \boxtimes (a).$$

*Proof.* Since $1 \otimes N \boxtimes (a)$ generates both modules $\text{ind}^n_{n-1,1} N \boxtimes (a)$ and $(\text{ind}^n_{n-1,1} N \boxtimes (a))^\sigma$ as $\mathcal{H}_n$-modules it suffices to check that the twisted action on $1 \otimes N \boxtimes (a)$ is the same as the action on $1 \otimes N^\sigma \boxtimes (a)$. So let $h \in H_n$ and write $h = \sum_{w \in D_{n-1,1}} T_w h_w$ for $h_w \in \mathcal{H}_{n-1,1}$. Then, as $\sigma$ fixes all of $\mathcal{H}_n^{\text{fin}}$,

$$\sigma(h) \otimes n = \sigma \left( \sum_{w \in D_{n-1,1}} T_w h_w \right) \otimes n$$

$$= \sum_{w \in D_{n-1,1}} T_w \sigma(h_w) \otimes n$$

$$= \sum_{w \in D_{n-1,1}} T_w \otimes \sigma(h_w)n,$$

which proves the claim. □

**Theorem 7.2.** Let $M \in \text{Rep}_{-1} \mathcal{H}_n$ be irreducible with $\epsilon_{-1}(M) = r$ and $\text{ch} M \neq \text{ch} M^\sigma$. Then

(i) $\Delta_{-1(r)} M \cong K \boxtimes L^A(-1(r))$ for some irreducible $K \in \text{Rep}_{-1} \mathcal{H}_{n-r}$ with $\text{ch} K \neq \text{ch} K^\sigma$;

(ii) $M$ is the irreducible cosocle of $\text{ind}^n_{n-r,r} K \boxtimes L^A(-1(r))$;

(iii) $\text{soc} \Delta_{-1} M \cong N \boxtimes (-1)$ for some irreducible $N \in \text{Rep}_{-1} \mathcal{H}_{n-1}$ with $\text{ch} N \neq \text{ch} N^\sigma$;

(iv) $M$ is the irreducible cosocle of the non-irreducible module $f_{-1} N$;

(v) $M$ is uniquely determined by its formal character, in particular, $M \cong M^r$.

*Proof.* Since, by Lemma 7.1,

$$(f_a L)^\sigma \cong f_a L^\sigma$$
for all \( m \) and for all \( L \in \text{Rep}_{-1} \mathcal{H}_{m-1} \), whence \( L \cong L^\sigma \) immediately implies \( \tilde{f}_a L \cong (\tilde{f}_a L)^\sigma \), we can inductively assume that all statements hold for all irreducible modules \( L \not\cong L^\sigma \) in \( \text{Rep}_{-1} \mathcal{H}_m \) for \( m \leq n - 1 \), since a path to one of those modules in the crystal graph cannot pass over a module \( L \cong L^\sigma \). Thus assume that, if \( N \cong (-1) \) is an irreducible submodule of \( \Delta_{-1} M \), the statements hold for \( N \), i.e. there exists an irreducible \( K \in \text{Rep}_{-1} \mathcal{H}_{n-r} \) with \( \text{ch} K \neq \text{ch} K^\sigma \) such that

\[
\Delta_{-1(r-1)} N \cong K \boxtimes L^A(-1(r-1)),
\]

\[
N \cong \text{cosoc} \text{ind}^{n-1}_{n-r,r-1} K \boxtimes L^A(-1(r-1)),
\]

\[
N \cong N^\tau \text{ is uniquely defined by ch } N \text{ and }
\]

\[
K \cong K^\tau \text{ is uniquely defined by ch } K.
\]

Now we apply the Shue Lemma to compute \( \Delta_{-1(r)} \text{ind}^n_{n-1,1} N \boxtimes (-1) \).

\[
D_{(n-r,r),(n-1,1)} = \{1, s_{n-r,n-1}, s_{n-r,0,n-1}\},
\]

but considering characters we see that the subquotient of \( \text{res}^n_{n-r,r} \text{ind}^n_{n-1,1} N \boxtimes (-1) \) corresponding to \( s_{n-r,n-1} \) is isomorphic to \( \text{ind}^n_{n-r,r} s_{n-r,n-1} (\text{res}^n_{n-r-1,r} N \boxtimes (-1)) \) and does not have character values with an \( (-1) \)-tail of length \( r \). Thus it does not contribute to \( \Delta_{-1(r)} \text{ind}^n_{n-1,1} N \boxtimes (-1) \) and the latter only has subquotients isomorphic to

\[
\text{ind}^n_{n-r,r} r_{-1,1} (\Delta_{-1(r-1)} N) \boxtimes (-1)
\]

and

\[
\text{ind}^n_{n-r,r-1,r-1} s_{n-r,0,n-1} (\Delta_{-1(r-1)} N) \boxtimes (-1).
\]

The use of \( \Delta_{-1(r-1)} \) instead of \( \text{res}^n_{n-r,r-1,1} \) in the Mackey formula is validated by the fact that no other summand of \( \text{res}^n_{n-r,r-1,1} N \) can contribute to \( \Delta_{-1(r)} \text{ind}^n_{n-1,1} N \boxtimes (-1) \) for lack of \( (-1) \)'s at the end. Now, by induction, the first of those subquotients is isomorphic to \( K \boxtimes L^A(-1(r')) \) while the second has the same character as \( K^\sigma \boxtimes L^A(-1(r')) \) and is, by the induction hypothesis, isomorphic to the latter. By the induction hypothesis, we can compute

\[
(\text{ind}^n_{n-1,1} N \boxtimes (-1))^T \cong \text{ind}^n_{n-1,1} s_{n-1,0,n-1} (N^\tau \boxtimes (-1))^T
\]

\[
\cong \text{ind}^n_{n-1,1} s_{n-1,0,n-1} (N \boxtimes (-1))
\]

\[
\cong \text{ind}^n_{n-1,1} N^\sigma \boxtimes (-1).
\]
Now, if \( \text{ind}_{n-1,1}^n N \boxtimes (-1) \) were irreducible and therefore isomorphic to \( M \), \( M' \) would also have to be isomorphic to \( \text{ind}_{n-1,1}^n N^{\sigma} \boxtimes (-1) \) and therefore \( M' \cong M^{\sigma} \), a contradiction since
\[
\text{ch} \, M' = \text{ch} \, M \neq \text{ch} \, M^{\sigma}.
\]
Therefore, \( \text{ind}_{n-1,1}^n N \boxtimes (-1) \) is not irreducible. But since, for any quotient \( Q \) of \( \text{ind}_{n-1,1}^n N \boxtimes (-1) \) and in particular for any constituent of the cosocle of \( \text{ind}_{n-1,1}^n N \boxtimes (-1) \),
\[
\text{Hom}_{H_n}(\text{ind}_{n-r,r}^n K \otimes L^A(-1^{(r)}), Q)
\]
\[
\cong \text{Hom}_{H_n-r,r}(K \otimes L^A(-1^{(r)}), \Delta_{-1^{(r)}} Q)
\]
is nonzero, the same argument as in Lemma 4.4 shows that the cosocle of \( \text{ind}_{n-1,1}^n N \boxtimes (-1) \) is irreducible and therefore isomorphic to \( M \), which proves (iv).

The same argument applied to \( \text{ind}_{n-1,1}^n N^{\sigma} \boxtimes (-1) \) shows that
\[
\tilde{M} = \text{soc} \, \text{ind}_{n-1,1}^n N \otimes (-1)
\]
is also irreducible and has \( \epsilon_{-1}(\tilde{M}) = r \). Therefore \( \tilde{M} \) contributes the composition factor \( K^{\sigma} \otimes L^A(-1^{(r)}) \) to \( \Delta_{-1^{(r)}} \text{ind}_{n-1,1}^n N \boxtimes (-1) \) and
\[
\Delta_{-1^{(r)}} M \cong K \otimes L^A(-1^{(r)}),
\]
so (i) holds.

To see that \( N \boxtimes (-1) \) is actually the whole of \( \text{soc} \, \Delta_{-1} M \) it suffices to consider the fact that, if \( N' \boxtimes (-1) \) is an irreducible submodule of \( \Delta_{-1} M \), it has to satisfy \( \epsilon_{-1}(N') = r - 1 \) as in this case
\[
\text{Hom}_{H_n}(\text{ind}_{n-1,1}^n N' \boxtimes (-1), M) \neq 0.
\]
But then \( 0 \neq \Delta_{-1^{(r-1)}} N' \cong K' \otimes L^A(-1^{(r-1)}) \) whence, by transitivity of induction, \( \text{ind}_{n-r,r}^n K' \otimes L^A(-1^{(r)}) \) projects onto \( M \) or equivalently
\[
K' \otimes L^A(-1^{(r)}) \hookrightarrow \Delta_{-1^{(r)}} M,
\]
a contradiction, whence (iii) follows.

Now, suppose some \( M' \) has the same character as \( M \). Then \( \Delta_{-1^{(r)}} M' \) has the same character as \( \Delta_{-1^{(r)}} M \). By the induction hypothesis, we obtain
\[
\Delta_{-1^{(r)}} M' \cong K \otimes L^A(-1^{(r)}),
\]
so (v) will follow if we can show (ii), namely the irreducibility of \( \text{cosoc ind}_{n-r,r}^n K \otimes L^A(-1^{(r)}) \). This does certainly not contain several copies of \( M \) since

\[
\text{Hom}_{\mathcal{H}_{n-r,r}}(K \otimes L^A(-1^{(r)}), \Delta_{-1}^{(r)} M)
\]

\[
\cong \text{Hom}_{\mathcal{H}_n}(\text{ind}_{n-r,r}^n K \otimes L^A(-1^{(r)}), M)
\]

is one-dimensional, so assume it contains an irreducible submodule \( M' \) not isomorphic to \( M \). But again, choosing \( N' \otimes (-1) \) in the socle of \( \Delta_{-1} M' \) shows that necessarily

\[
\Delta_{-(r-1)} N' \cong K \otimes L^A(-1^{(r-1)}),
\]

whence \( N' \cong N \) and we’re done. \( \square \)

**Corollary 7.3.** In the situation of Theorem 7.2,

\[
\text{soc ind}_{n-1,1}^n N \otimes (-1) \cong \text{cosoc ind}_{n-1,1}^n N^\sigma \otimes (-1) \cong M^\sigma.
\]

We have seen that for \( a = -1 \), as long as \( M \not\cong M^\sigma \), everything works as in the case \( a \neq \pm 1 \), and \( M \not\cong M^\sigma \) is satisfied until, for the first time, the induced module \( \text{ind}_{n-1,1}^n N \otimes (-1) \) is irreducible.

**Lemma 7.4.** Let \( N \in \mathcal{H}_{n-1} - \text{mod}^{fd} \) be irreducible, \( a = 1 \) or \( a = -1 \) and \( N \cong N^\sigma \). Let \( \epsilon_a(N) = r - 1 \). Set \( M := \text{ind}_{n-1,1}^n N \otimes (a) \). Let \( (a_0, b, a^{(r-1)}) \) be a tuple of eigenvectors on \( N \). Then any \( (a_0, b, a^{(r)}) \)-eigenvector in \( M \) is either of the form

\[
1 \otimes v
\]

or

\[
T_{n-r,0,n-1} \otimes v + \sum_{j=0}^{n-r-1} T_{j,0,n-1} \otimes u_j + \sum_{j=1}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes u_0
\]

where \( v = \tilde{v} \otimes c \) for an \( (a_0, b, a^{(r-1)}) \)- or \( (a_0 a, b, a^{(r-1)}) \)-eigenvector \( \tilde{v} \) in \( N \), a generator \( c \) of the one-dimensional module \( (a) \) and some \( u_j, w_j, u_0 \) in \( N \otimes (a) \).

In other words, it has leading term 1 or \( T_{n-r,0,n-1} \).

In particular, the \( (a_0, b, a^{(r)}) \)-eigenspace in \( M \) has at most twice the dimension of that in \( N \otimes (a) \).
Proof. Directly from the action of the Weyl group on the lattice, we see that the leading term has to be of the form $T_{l,0,n-1}$ for $l \geq n-r$ or $T_{l,n-1}$ for $l \geq n-r+1$ for there to be an $a$-tail of length $r$.

For leading term $T_{l,0,n-1}$ with $l > n-r$

$$(X_l + 1)(T_{l,0,n-1} \otimes u_l + T_{l-1,0,n-1} \otimes u_{l-1})$$

$$= T_{l,0,n-1}(X_l + 1) \otimes u_l - (q - q^{-1})T_{l-1,0,n-1}X_l \otimes u_l$$

$$+ T_{l-1,0,n-1}(X_{n-1} + 1) \otimes u_{l-1} + \text{lower terms}$$

Since $X_{n-1} + 1$ acts as 0 on the whole of $N$ and since none of the lower terms can contribute to the coefficient of $T_{l-1,0,n-1}$, this shows that $u_l = 0$ which contradicts the assumption that $T_{l,0,n-1}$ is the leading term. Thus the leading term can only be $T_{n-r,0,n-1}$ in this case.

Equally, if the leading term is $T_{l,n-1}$ for $l \geq n-r+1$

$$(X_{l+1} - a)(T_{l,n-1} \otimes w_l + T_{l+1,n-1} \otimes w_{l+1})$$

$$= T_{l,n-1}(X_l - a) \otimes w_l + (q - q^{-1})T_{l+1,n-1}X_n \otimes w_l$$

$$+ T_{l+1,n-1}(X_n - a) \otimes w_{l+1} + \text{lower terms}$$

By the same argument as above, $w_l = 0$ and the leading term has to be 1. □

Lemma 7.5. Let $a \in \{\pm 1\}$. The simultaneous eigenspace of $X_0, \ldots, X_n$ in $\text{ind}_{P_n}^H(a_0, a^{(n)})$ is

(i) $1 \otimes (a_0, a^{(n)})$ if $a = 1$

(ii) contained in $(1 \oplus T_0) \otimes (a_0, a^{(n)})$ if $a = -1$.

As a consequence, $\text{ind}_{P_n}^H(a_0, a^{(n)})$ is irreducible.

Proof. We set

$$M_k := \mathcal{H}_{k,1,\ldots,1} \otimes_{P_n} (a_0, a^{(n)}) \cong \mathcal{H}_k \otimes_{P_k} (a_0, a^{(k)}) \boxtimes (a^{(n-k)})$$

and let $v$ be a common eigenvector for $X_0, \ldots, X_n$ in $M_1$. It is easily verified that if we let $\{w\}$ be a basis of the one-dimensional module $(a_0, a^{(n)})$, $v$ is a scalar multiple of $1 \otimes w$ if $a = 1$. If $a = -1$, $v$ is either a scalar multiple of $1 \otimes w$ and an $(a_0, -1^{(n)})$-eigenvector or a scalar multiple of $(T_0 - \frac{p-p^2}{2}) \otimes w$ and a $(-a_0, -1^{(n)})$-eigenvector.

(i) In case $a = 1$, inductively assume the statement holds for $M_{n-1}$. Then Lemma 7.4 yields that if we expand an $(a_0, 1^{(n)})$-eigenvector $u$ as
\[ \sum_{j=1}^{n-1} T_{j,0,n-1} \otimes u_j + \sum_{j=0}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes u_0 \] with \( u_j, w_j, u_0 \) in \( M_{n-1} \) the leading term must be \( T_{0,n-1} \) or 1. In case it’s 1, we’re done by induction. So we assume it’s \( T_{0,n-1} \). Then
\[
(X_1 - 1)(T_{0,n-1} \otimes u_0 + T_{1,n-1} \otimes w_1)
= T_{0,n-1} (X_n^{-1} - 1) \otimes u_0 + (q - q^{-1}) T_{1,n-1} (X_n + 1) \otimes u_0
+ T_{1,n-1} (X_n - 1) \otimes w_1
+ \text{terms in } T_{k,n-1} \otimes M_{n-1} \text{ for } k \geq 2 \text{ and in } 1 \otimes M_{n-1}.
\]

Since \( X_n \) acts as 1 on \( M_{n-1} \), this cannot be zero, so the leading term must not be \( T_{0,n-1} \).

(ii) As in (i), assume the statement holds for \( M_{n-1} \) and expand a simultaneous eigenvector \( u \) for \( X_0, \ldots, X_n \) as
\[
u = \sum_{j=1}^{n-1} T_{j,0,n-1} \otimes u_j + \sum_{j=0}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes u_0.
\]

Again it follows that the leading term is either \( T_{0,n-1} \) or 1. So, assume \( u \) is of the form \( \sum_{j=0}^{n-1} T_{j,n-1} \otimes w_j + 1 \otimes u_0 \) with \( w_j, u_0 \) in \( M_{n-1} \). In the following we omit the tensor signs for the sake of notational simplicity.

As the coefficient of \( T_{0,n-1} \) in \( (X_k + 1)u \) is \( (X_k - 1)u_0 \) for \( k \geq 2 \), \( (X_1 + 1)u_0 \) for \( k = 1 \) and \( (X_0 X_n + 1)u_0 \) for \( k = 0 \), we see that \( w_0 \) has to be a simultaneous eigenvector for \( X_0, X_1, \ldots, X_n \) and therefore, by induction, a scalar multiple of \( v \) (which is either \( 1 \otimes w \) or \( (T_0 - \frac{p-p^{-1}}{2}) \otimes w \)).

As, for \( k \geq 2 \), the coefficient of \( T_{k,n-1} \) in \( (X_k + 1)u \) is
\[
(X_n + 1)w_k + (q - q^{-1}) \sum_{j=0}^{k-1} T_{j,k-2} X_j w_j
\]
and \( X_n \) acts as \(-1\) on \( M_{n-1} \), we see that \( w_{k-1} = - \sum_{j=0}^{k-2} T_{j,k-2} w_j \).

From this we want to conclude that if we set \( z_1 := T_0 \) and recursively define \( z_k := z_{k-1} T_{k-1} - T_{k-1} z_{k-1} \), then
\[
\sum_{j=0}^{n-1} T_{j,n-1} w_j = z_n v.
\]

Indeed, we can show by induction on \( k \) that \( \sum_{j=0}^{k} T_{j,k} w_j = z_{k+1} v \). This formula certainly holds for \( k = 0 \). Now suppose it holds for all \( k' \leq k \)
then

\[
\sum_{j=0}^{k} T_{j,k}w_j = \sum_{j=0}^{k-1} T_{j,k}w_j + T_kw_k
\]

\[
= z_kT_kv + T_k(-\sum_{j=0}^{k-1} T_{j,k-1}w_j)
\]

\[
= z_kT_kv - T_kz_kv
\]

\[
= z_{k+1}v.
\]

Now \((X_n + 1)z_nv = 0\) and \((X_k + 1)z_nv = -(q - q^{-1})T_k^{-1}\cdots T_{n-2}^{-1}z_kv\) for \(1 \leq k \leq n - 1\), which is also verified by induction on \(n - k\). To do this, we first observe that

\[
z_nv = z_{n-1}T_{n-1}v - T_{n-1}z_{n-1}v
\]

\[
= z_{n-2}T_{n-2}T_{n-1}v - T_{n-2}z_{n-1}v - T_{n-2}T_{n-1}z_{n-2}v
\]

\[
= \cdots = T_0\cdots T_{n-1}v - \sum_{j=1}^{n-1} T_j\cdots T_{n-1}z_jv
\]
and apply \((X_k + 1)\) to this, obtaining

\[
(X_k + 1)z_n v = (q - q^{-1}) T_0 \cdots T_{k-2} T_k^{-1} \cdots T_{n-1} X_n v - (T_k^{-1} \cdots T_{n-1} X_n - T_k \cdots T_{n-1}) z_k v + (q - q^{-1}) \sum_{j=k+1}^{n-1} T_j \cdots T_{n-1} T_k^{-1} \cdots T_{j-2}^{-1} z_j v = -(q - q^{-1}) T_k^{-1} \cdots T_{n-1}^{-1} \cdot (T_0 \cdots T_{k-2} - \sum_{j=1}^{k-2} T_j \cdots T_{k-2} z_j v - z_{k-1} v)
\]

\[
= -(q - q^{-1}) T_k^{-1} \cdots T_{n-2}^{-1} z_k v - (q - q^{-1}) \sum_{j=k+1}^{n} T_k^{-1} \cdots T_{l-2}^{-1} T_l \cdots T_{n-1} z_k v + (q - q^{-1}) \sum_{j=k+1}^{n-1} T_j \cdots T_{n-1} T_k^{-1} \cdots T_{j-2}^{-1} z_k v
\]

where we have used the fact that \((X_k + 1)z_j v = 0\) for \(j \leq k - 1\) and the inductive assumption \((X_k + 1)z_j v = 0\) for \(j \geq k + 1\) in the first equality and the above presentation of \(z_{k-1} v\) in the third equality. So we need to find \(u_0 \in M_{n-1}\) such that

\[
(X_k + 1)u_0 = (q - q^{-1}) T_k^{-1} \cdots T_{n-2}^{-1} z_k v \text{ for } 1 \leq k \leq n - 1.
\]

Now we understand \(M_{n-1}\) as \(\mathcal{H}_{n-1,1} \otimes_{\mathcal{H}_{n-2,1,1}} M_{n-2} \otimes \langle a^{(2)} \rangle\) and write elements of \(M_{n-1}\) as \(\sum_{j=1}^{n-2} T_{j,0,n-2} x_j + \sum_{j=0}^{n-2} T_{j,n-2} y_j + x_0\) accordingly. Since the coefficients of \(T_{k-1,0,n-2}\) in \(T_k^{-1} \cdots T_{n-2}^{-1} z_k v\) are zero for all \(1 \leq k \leq n - 1\), we see as in Lemma 7.4 that the leading term in \(u_0\) must be \(T_{0,n-2}\) and \(u_0 = \sum_{j=0}^{n-2} T_{j,n-2} w_{0,j} + u_{0,0}\). But in \((X_1 + 1)u\)
terms of the form $T_{1,n-2}$ occur from

$$(X_1 + 1)(T_{0,n-2}w_0 + T_{1,n-2}w_0,1)
= T_{0,n-2}(X_{n-1}^1 + 1)w_0,0
+ (p - p^{-1})(T_{1}^{-1} \cdots T_{n-2}^{-1}X_{n-1} + T_{1,n-2})w_0,0
+ (T_{1}^{-1} \cdots T_{n-2}^{-1}X_{n-1} + T_{1,n-2})w_0,1
+ \text{terms in } T_{k,n-2}M_{n-2} \text{ for } k \geq 2$$

whence we see that the coefficient of $T_{1,n-2}$ in $(X_1 + 1)u$ is zero. But the coefficient of $T_{1,n-2}$ in $(X_1 + 1)z_n v = -(q - q^{-1})T_{1}^{-1} \cdots T_{n-2}^{-1}z_1$ is not equal to zero, so we have the desired contradiction and no such $u_0$ exists, whence no simultaneous eigenvector with leading term different from 1 can occur.  

\[ \square \]

**Lemma 7.6.** Let $N \in \mathcal{H}_{n-1}$-mod $\ell^d$ be irreducible and set

$$M := f_aN.$$  

Then $\dim \text{End}_{\mathcal{H}_n}(M) \leq 2$.

**Proof.** By Lemma 4.1, $\text{End}_{\mathcal{H}_n}(M) \cong \text{Hom}_{\mathcal{H}_{n-1,1}}(N \boxtimes (a), \Delta_a M)$. Let $r - 1 = \epsilon_a(N)$. By Lemma 7.4, the $(a, b, a^{(r)})$-eigenspace in $M$ has at most twice the dimension of that in $N \boxtimes (a)$, therefore the socle of $\Delta_a M$ can contain at most 2 copies of $N \boxtimes (a)$.  

\[ \square \]

**Corollary 7.7.** If $N$ in the above Lemma satisfies $N \cong N^\sigma$, $M = f_aN$ either has an irreducible cosocle or splits into a direct sum of two non-isomorphic irreducibles.

**Proof.** There are two cases to consider. The first case is that $N \not\cong N^\sigma$ and $a = -1$. In this case, $M$ has irreducible cosocle by Theorem 7.2. If either $a = 1$ or $a = -1$ and $N \cong N^\sigma$, then $N \cong N^\tau$ implies $M \cong M^\tau$ since

$$M^\tau \cong (\text{ind}_{n-1,1}^n N \boxtimes (a))^\tau
\cong \text{ind}_{n-1,1}^n s_{n-1,0,n-1} (N^\tau \boxtimes (a))^\tau
\cong \text{ind}_{n-1,1}^n s_{n-1,0,n-1} (N \boxtimes (a))
\cong \text{ind}_{n-1,1}^n N \boxtimes (a),$$

and the statement follows.  

\[ \square \]
Lemma 7.8. Let $N \cong N^r \in \mathcal{H}_{n-1} \text{-mod}^{\text{id}}$ be irreducible, $a \in \{\pm 1\}$ and $\epsilon_a(N) = r - 1$. Set $M := f_a N$. The generalized $(a^{(r)})$-eigenspace of $M$ is contained in the socle and cosocle of $M$, thus all other composition factors $K$ have $\epsilon_a(K) \leq r - 1$.

Proof. By the Mackey Theorem 2.3,

$$\Delta_{a^{(r)}}(M) = \left[ \text{ind}_{n-r, r-1, 1}^{n-r, r} \Delta_{a^{(r-1)}} N \boxtimes (a) \right]$$

and as in Lemma 7.4 we know that the $(a^{(r)})$-eigenspace of the generators $(X_{n-r+1}, \ldots, X_n)$ in the submodule $\text{ind}_{n-r, r-1, 1}^{n-r, r} \Delta_{a^{(r-1)}} N \boxtimes (a)$, which is by Theorem 2.3 contained in the socle of $\Delta_{a^{(r)}} M$, is contained in $1 \otimes \Delta_{a^{(r-1)}} N \boxtimes (a)$. Under the restriction of the projection of $M$ onto its cosocle, this certainly doesn’t map to zero, whence we have an injection of $\text{ind}_{n-r, r-1, 1}^{n-r, r} \Delta_{a^{(r-1)}} N \boxtimes (a)$ into $\Delta_{a^{(r-1)}} \text{cosoc } M$. Since $M$ is self-dual, the rest of the generalized $(a^{(r)})$-eigenspace must belong to the socle.

Lemma 7.9. For irreducible $N \cong N^r \in \mathcal{H}_{n-1} \text{-mod}^{\text{id}}$ and $b_1^{-1} b_2 \neq q^{\pm 2}$,

$$\text{cosoc ind}_{n, 1}^{n+1} \text{cosoc ind}_{n-1, 1}^{n} N \boxtimes (b_1) \boxtimes (b_2)$$

and

$$\text{cosoc ind}_{n, 1}^{n+1} \text{cosoc ind}_{n-1, 1}^{n} N \boxtimes (b_2) \boxtimes (b_1)$$

have the same irreducible constituents.

Proof. We consider

$$\text{cosoc ind}_{n-1, 1}^{n+1} N \boxtimes (b_1) \boxtimes (b_2) \cong \text{cosoc ind}_{n-1, 2}^{n+1} N \boxtimes L^A(b_1, b_2).$$

Any constituent $Q$ of the above has to have $\epsilon_{b_i}(Q) = \epsilon_{b_i}(N) + 1$ for $i = 1, 2$. The only composition factors $R$ of $f_{b_1} N$ that have $\epsilon_{b_1}(R) = \epsilon_{b_1}(N) + 1$ are contained in cosoc $f_{b_1} N$ or soc $f_{b_1} N$. The latter only happens if $b_1 \in \{\pm 1\}$ but then both cosocle and socle are isomorphic. Thus, all constituents of $\text{cosoc ind}_{n-1, 2}^{n+1} N \boxtimes L^A(b_1, b_2)$ are isomorphic to constituents of $\text{cosoc ind}_{n, 1}^{n+1} \text{cosoc ind}_{n-1, 1}^{n} N \boxtimes (b_1) \boxtimes (b_2)$. Since the argument is symmetric in $b_1$ and $b_2$ and

$$\text{cosoc ind}_{n, 1}^{n+1} \text{cosoc ind}_{n-1, 1}^{n} N \boxtimes (b_1) \boxtimes (b_2) \cong \text{cosoc ind}_{n-1, 1, 1}^{n+1} N \boxtimes (b_1) \boxtimes (b_2),$$

and
this yields the claim. □

We will now give an example where indeed the functor \( f_1 \) does not produce an irreducible module.

**Example 7.10.** We apply the functor \( f_1 \) to the two-dimensional irreducible module \( L(a_0, q^2) \cong \text{ind}_{P^1}^H(a_0, q^2) \in \mathcal{H}_1\text{-mod}^{fd} \) with basis \( \{w_1, w_2\} \) on which \( T_0, X_0, X_1 \) act by matrices

\[
\begin{pmatrix}
0 & 1 \\
1 & (p-p^{-1})
\end{pmatrix}, \quad
\begin{pmatrix}
a_0 & -(p-p^{-1})a_0q^2 \\
0 & a_0q^2
\end{pmatrix}, \quad
\begin{pmatrix}
q^2 & (p-p^{-1})(q^2+1) \\
0 & q^{-2}
\end{pmatrix}
\]

respectively. Since \( w_1 \in L(a_0, q^2) \) is an \((a_0, q^2)\)-eigenvector, Lemma 1.5 implies that, in the induced module \( M := f_1L(a_0, q^2) = \mathcal{H}_2 \otimes_{\mathcal{H}_{1,1}} L(a_0, q^2) \otimes (1), \) we find an \((a_0, 1, q^2)\)-eigenvector \((T_1 + q^{-1})w_1\). Again by Lemma 1.5, the vector \((T_0 - q^2T_0^{-1})w_1 \in L(a_0, q^2)\) is an \((a_0q^2, q^{-2})\)-eigenvector for \((X_0, X_1)\), and \((T_1 - q)(T_0 - q^2T_0^{-1})w_1 \in L(a_0, q^2)\) is an \((a_0, 1, q^{-2})\)-eigenvector in \( M \). Hence, by Frobenius reciprocity, we obtain homomorphisms

\[
\text{ind}_{P^2}^H(a_0, 1, q^2) \longrightarrow M
\]

and

\[
\text{ind}_{P^2}^H(a_0q^2, 1, q^{-2}) \longrightarrow M.
\]

Now we will show that both \( \text{ind}_{P^2}^H(a_0, 1, q^2) \) and \( \text{ind}_{P^2}^H(a_0q^2, 1, q^{-2}) \) have four-dimensional cosocles which are not isomorphic.

Indeed, \( L(a_0, 1) \cong \text{ind}_{P^1}^H(a_0, 1) \) is irreducible by Lemma 7.5, hence the only irreducible module with formal character \( 2[(a_0, 1)] \). Representing matrices for \( T_0, X_0, X_1 \) are

\[
\begin{pmatrix}
0 & 1 \\
1 & (p-p^{-1})
\end{pmatrix}, \quad
\begin{pmatrix}
a_0 & -(p-p^{-1})a_0 \\
0 & a_0
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 2(p-p^{-1}) \\
0 & 1
\end{pmatrix}
\]

Therefore

\[
(\text{ind}_{P^2}^H(a_0, 1, q^2))^\tau \cong \text{ind}_{1,1}^2 s_1s_0s_1(L(a_0, 1)^\tau \otimes (q^2)^\tau)
\]

\[
\cong \text{ind}_{1,1}^2 s_1s_0s_1(L(a_0, 1) \otimes (q^2))
\]

\[
\cong \text{ind}_{1,1}^2 L(a_0, 1) \otimes (q^{-2})
\]

\[
\cong \text{ind}_{P^2}^H(a_0q^2, 1, q^{-2}),
\]

so the two modules are dual to each other. To know both cosocles, we may therefore just as well compute both socles. The socle
of \( \text{ind}_{P_2}^{H_2}(a_0q^2,1,q^{-2}) \) has, by the above considerations, to contain an \((a_0,1,q^2)\)-eigenvector. As we have seen above \( X_1 \) acts on \( L(a_0,1) \) by a Jordan block of size two and by the same leading term argument as in Lemma 7.4, we see that there can only be one \((a_0,1,q^2)\)-eigenvector, which then, by Lemma 1.5 is

\[
w_1 := (T_1 + q^{-1})(T_0 - q^{-2}T_0^{-1})(T_1 + q^{-1})u,
\]

where \( u \) is the first basis vector in the above presentation of \( L(a_0,1) \).

On this vector, \( T_1 \) acts as the scalar \( q \), but

\[
w_2 := T_0w_1,
\]

\[
w_3 := T_1T_0w_1
\]

\[= qw_2 + 2\frac{p - p^{-1}}{q - q^{-1}}w_1
\]

\[= 2\frac{p - p^{-1}}{q - q^{-1}}q^2 + 1\frac{(T_0 - q^{-2}T_0^{-1})(T_1 + q^{-1})u}{q - q^{-1}}
\]

and

\[
w_4 := T_0T_1T_0w_1
\]

\[= qw_1 + \frac{p - p^{-1}}{q - q^{-1}}(q^2 + 1)w_2
\]

\[= 2\frac{(p - p^{-1})^2}{(q - q^{-1})^2}(q^2 + 1)(T_0 - q^{-2}T_0^{-1})(T_1 + q^{-1})u
\]

\[= c(T_1 + q^{-1})u,
\]

where \( c = 2\frac{(p - p^{-1})(1+q^2)(q^2p^2-1)(q^2-p^2)}{q^2(q+1)^2(q-1)^2p^2} \), yield, including \( w_1 \), four linearly independent vectors, whence the socle is four-dimensional. The calculations showing that the socle of \( \text{ind}_{P_2}^{H_2}(a_0,1,q^2) \) is four-dimensional are analogous.

Now we can conclude that the eight-dimensional module \( M \) has a composition series containing two four-dimensional non-isomorphic subquotients. Since \( L(a_0,q^2) \) is self-dual by Corollary 4.12 and uniquely
defined by its character, we know that

\[ M^\tau \cong \text{ind}_{1,1}^{\mathcal{H}_k} \left( \sigma_{s_0}^{s_1} (L(a_0, q^2)^\tau \boxtimes (1)^\tau) \right) \]

\[ \cong \text{ind}_{1,1}^{\mathcal{H}_k} \left( L(a_0, q^2) \boxtimes (1) \right) \]

\[ \cong \text{ind}_{1,1}^{\mathcal{H}_k} L(a_0, q^2) \boxtimes (1) \]

\[ \cong M, \]

so if \( M \) consists of two non-isomorphic constituents, it must be the direct sum of both, therefore

\[ \text{ind}_{1,1}^{\mathcal{H}_k} L(a_0, q^2) \boxtimes (1) \cong L(a_0, 1, q^2) \oplus L(a_0, 1, q^{-2}). \]

In particular, we can conclude that there exist \( h_1, h_2 \) with

\[ h_1(T_1 + q^{-1})w_1 + h_2(T_1 - q)(T_0 - q^2T_0^{-1})w_1 = w_1 \]

and, since we are doing computations in \( \mathcal{H}_2 \) without having factored out any annihilators,

\[ h_1(T_1 + q^{-1}) + h_2(T_1 - q)(T_0 - q^2T_0^{-1}) = 1 \quad \in \mathcal{H}_2. \]

Of course, we could have simply computed the submodules generated by the two given eigenvectors in \( M \) and shown that they are complements of each other, but the argument exploiting a self-dual module with at most two-dimensional endomorphism ring and two non-isomorphic composition factors carries us a little further.

We can generalize this example to prove a statement about a series of similar modules.

**Lemma 7.11.** Let

\[ L(a_0, q^2, \ldots, q^{2k}) = \tilde{f}_{q^{2k}} \cdots \tilde{f}_{q^2}(a_0) \]

\[ \cong \text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_k^*}^{\mathcal{H}_k} (a_0) \boxtimes M(q^2, \ldots, q^{2k}) \]

\[ \cong \text{ind}_{\mathcal{P}_0 \otimes \mathcal{H}_k^*}^{\mathcal{H}_k} (a_0q^{k(k+1)}) \boxtimes M(q^{-2k}, \ldots, q^{-2}) \]

\[ \cong \tilde{f}_{q^{-2}} \cdots \tilde{f}_{q^{-2k}}(a_0q^{k(k+1)}) \]

\[ = L(a_0q^{k(k+1)}, q^{-2k}, \ldots, q^{-2}) \]
and analogously
\[
L(a_0, q^{2k}, \ldots, q^2) = f q^2 \cdots f q^{2k}(a_0)
\]
\[
\cong \text{ind}_{P_0 \otimes H_k^R}^{H_k} (a_0) \boxtimes M(q^{2k}, \ldots, q^2)
\]
\[
\cong \text{ind}_{P_0 \otimes H_k^R}^{H_k} (a_0 q^{k+1}) \boxtimes M(q^{-2k}, \ldots, q^2)
\]
\[
\cong \tilde{f} q^{-2k} \cdots \tilde{f} q^{-2}(a_0 q^{k+1})
\]
\[
= L(a_0 q^{k+1}, q^{-2k}, \ldots, q^{-2k}).
\]

Then
\[
f_1 L(a_0, q^2, \ldots, q^{2k})
\]
\[
\cong L(a_0, 1, q^2, \ldots, q^{2k}) \oplus L(a_0 q^{k+1}, 1, q^{-2k}, \ldots, q^{-2})
\]
and
\[
f_1 L(a_0, q^{2k}, \ldots, q^2)
\]
\[
\cong L(a_0, 1, q^{2k}, \ldots, q^2) \oplus L(a_0 q^{k+1}, 1, q^{-2}, \ldots, q^{-2k}).
\]

Furthermore,
\[
\text{res}_{H_{k+1}^R} H_{k+1}^L (a_0, 1, q^2, \ldots, q^{2k}) \cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M(1, q^2, \ldots, q^{2k})
\]
\[
\cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M(q^{-2k}, \ldots, q^{-2}, 1),
\]
\[
\text{res}_{H_{k+1}^R} H_{k+1}^L (a_0 q^{k+1}, 1, q^{-2k}, \ldots, q^{-2}) \cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((1), (q^{-2k}, \ldots, q^{-2})
\]
\[
\cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((q^2, \ldots, q^{2k}), (1)),
\]
\[
\text{res}_{H_{k+1}^R} H_{k+1}^L (a_0, 1, q^{2k}, \ldots, q^2) \cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((q^{2k}, \ldots, (q^4), (1)^2)
\]
\[
\cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((q^{-2}, 1), (q^{-4}, \ldots, (q^{-2k})
\]
\[
\text{res}_{H_{k+1}^R} H_{k+1}^L (a_0 q^{k+1}, 1, q^{-2}, \ldots, q^{-2k}) \cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((1), (q^{-2}, \ldots, (q^{-2k})
\]
\[
\cong \text{ind}_{H_{k+1}^R}^{H_{k+1}^R} M((q^{2k}, \ldots, (q^2), (1)).
\]
Proof. The isomorphisms given in the beginning can be derived as in the proof of Lemma 6.1, where we only need that the set \( J \) of eigenvalues of the lattice on the module and the set \( J^- \) of inverses of those eigenvalues are disjoint, \( J \) does not contain \( p^2 \) or \( p^{-2} \) and no two elements \( a \in J \) and \( b \in J^- \) satisfy \( a^{-1}b \in \{q^\pm 2\} \).

In order to prove the remaining assertions, we proceed by induction on \( k \). In Example 7.10, we have seen that

\[
f_1(a_0, q^2) \cong L(a_0, 1, q^2) \oplus L(a_0q^2, 1, q^{-2}).
\]

As \( L(a_0, 1, q^2) \) and \( L(a_0q^2, 1, q^{-2}) \) are uniquely defined by their formal characters and \( \text{res}_{\mathcal{H}_2^R} L(a_0, 1, q^2) \) as well as \( \text{res}_{\mathcal{H}_2^R} L(a_0q^2, 1, q^{-2}) \) are irreducible by Lemma 7.7, it follows that

\[
\text{res}_{\mathcal{H}_2^R} L(a_0, 1, q^2) \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{(1,q^2)} \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{(q^{-2},1)}
\]

and

\[
\text{res}_{\mathcal{H}_2^R} L(a_0q^2, 1, q^{-2}) \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{(q^2,1)} \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{((1),(q^{-2}))}.
\]

By Lemma 7.9, \( \tilde{f}_a \) and \( \tilde{f}_1 \) commute if \( a \notin \{q^\pm 2\} \) so

\[
\tilde{f}_1(a_0, q^2, \ldots, q^{2k}) \cong \tilde{f}_{a_0} \tilde{f}_1(a_0, q^2, \ldots, q^{2(k-1)})
\]

\[
\cong \tilde{f}_{a_0} \tilde{f}_{a_0} L(a_0, 1, q^2, \ldots, q^{2(k-1)}) \oplus \tilde{f}_{a_0} \tilde{f}_{a_0} L(a_0q^{2(k-1)}, 1, q^{-2(k-1)}, \ldots, q^{-2})
\]

where

\[
\text{res}_{\mathcal{H}_2^R} L(a_0, 1, q^2, \ldots, q^{2(k-1)}) \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{(1,q^2,\ldots,q^{2(k-1)})} \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{(q^{-2(k-1)},\ldots,q^{-2},1)}
\]

and

\[
\text{res}_{\mathcal{H}_2^R} L(a_0q^{2(k-1)}, 1, q^{-2(k-1)}, \ldots, q^{-2}) \cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{((1),(q^{-2(k-1)},\ldots,q^{-2}))} \ cong \text{ind}_{\mathcal{H}_2^R} \text{M}_{((1),(q^{-2(k-1)},\ldots,q^{-2}))}.
\]

As \( L(a_0, q^2, \ldots, q^{2k}) \) is selfdual, Lemma 7.7 implies that

\[
\tilde{f}_1 a_0, q^2, \ldots, q^{2k}) \cong f_1 L(a_0, q^2, \ldots, q^{2k}).
\]
Since cosoc ind$^H_{k+1} \otimes \text{ind}^H_k M_{(q^2, \ldots, q^{2(k-1)})} \otimes (q^{2k})$ is irreducible, it is the same as
\[
\text{cosoc ind}^H_{k+1} \otimes \text{cosoc ind}^H_k M_{(q^2, \ldots, q^{2(k-1)})} \otimes (q^{2k}) \cong \text{cosoc ind}^H_{k+1} M_{(q^2, \ldots, q^{2k})}.
\]

But
\[
\text{ind}^H_{k+1} \otimes \text{ind}^H_k M(q^2, \ldots, q^{2k}) \otimes (1) \cong \text{ind}^H_{k+1} \otimes \text{ind}^H_k M(q^2, \ldots, q^{2k}) \otimes (1)
\]
splits into two non-isomorphic irreducibles, and $\text{ind}^H_{k+1} M(q^2, \ldots, q^{2k}) \otimes (1)$ has two composition factors $M_{(1,q^2,\ldots,q^{2k})}$ and $M_{(q^2,\ldots,q^{2k}), (1)}$, therefore $\text{ind}^H_{k+1} M_{(1,q^2,\ldots,q^{2k})}$ and $\text{ind}^H_{k+1} M_{(q^2,\ldots,q^{2k}), (1)}$ have to be irreducible and the former is isomorphic to
\[
\text{res}_{H_{k+1}}^H \tilde{f}_{q^{2k}} L(a_0, 1, q^2, \ldots, q^{2(k-1)}) = \text{res}_{H_{k+1}}^H L(a_0, 1, q^2, \ldots, q^{2k}).
\]

Analogously,
\[
\text{res}_{H_{k+1}}^H \tilde{f}_{q^{2k}} L(a_0, 1, q^2, \ldots, q^{2(k-1)}) \cong \text{soc ind}^H_{k+1} \otimes \text{ind}^H_k M_{(q^{-2(k-1)}, \ldots, q^{-2}, 1)} \otimes (q^{-2k})
\]
which is isomorphic to $\text{ind}^H_{k+1} M_{(q^{-2k}, \ldots, q^{-2}, 1)}$ by the same argument as above. It follows that the second direct summand is isomorphic to
\[
\tilde{f}_{q^{2k}} L(a_0 q^{k(k-1)}, 1, q^{-2(k-1)}, \ldots, q^{-2})
\cong \text{ind}_{P_0 \otimes H_{k+1}}^H (a_0 q^{k(k-1)}) \otimes M_{(1), (q^{-2k}, \ldots, q^{-2})}
\cong \text{ind}_{P_0 \otimes H_{k+1}}^H (a_0) \otimes M_{(q^2, \ldots, q^{2k}), (1)}
\]
and it remains to show that this is actually isomorphic to the module $L(a_0 q^{k(k-1)}, 1, q^{-2k}, \ldots, q^{-2})$.

But
\[
\text{ind}_{P_0 \otimes H_{k+1}}^H (a_0 q^{k(k-1)}) \otimes M_{(1), (q^{-2k}, \ldots, q^{-2})}
\cong \text{ind}_{P_0 \otimes H_{k+1}}^H (a_0 q^{k(k-1)}) \otimes \text{cosoc ind}^H_{k+1} M_{(1)} \otimes M_{(q^{-2k}, \ldots, q^{-2})}
and 
\[ L(a_0 q^{k(k+1)}, q^{-2k}, \ldots, q^{-2}) = \text{cosoc ind}_{\mathcal{H}_{1,k}} L(a_0 q^{k(k+1)}, 1, M(q^{-2k}, \ldots, q^{-2})) \]
since all cosocles taken are irreducible by the usual formal character argument. As the former module is irreducible, it is equal to its cosocle but
\[
\text{cosoc ind}_{\mathcal{H}_{1,k}} L(a_0 q^{k(k+1)}, 1, M(1) \boxtimes M(q^{-2k}, \ldots, q^{-2}))
\]
is certainly contained in
\[
\text{cosoc ind}_{\mathcal{H}_{1,k}} (\text{ind}_{\mathcal{H}_1} (a_0 q^{k(k+1)}) \boxtimes M(1)) \boxtimes M(q^{-2k}, \ldots, q^{-2})
\]
\[ \cong \text{cosoc ind}_{\mathcal{H}_{1,k}} L(a_0 q^{k(k+1)}, 1, M(q^{-2k}, \ldots, q^{-2})), \]
But this is also irreducible, hence the two cosocles are isomorphic. The statements about \( \text{ind}_{\mathcal{H}_{k+1}} L(a_0, q^{2k}, \ldots, q^2) \boxtimes (1) \) follow analogously.

The lemma suggests that maybe one could reach any module via a path in the crystal graph if only the right order is chosen on the labeling set of eigenvalues. This, however, is not true as the following example shows, the computations for which have been carried out using Magma V2.11-2.

**Example 7.12.** We apply the functor \( f_1 \) to the 16-dimensional irreducible \( \mathcal{H}_3 \)-module \( L(a_0, q^2, q^{-2}) \), which is selfdual, since \( L(a_0, 1) \) is and applying \( \tilde{f}_q^2 \) and \( \tilde{f}_q^{-2} \) again yields self-dual modules. The induced module
\[ M := f_1 L(a_0, 1, q^2, q^{-2}) \]
then is also selfdual. The element \((T_3 - q)u\) where \( u \) is the (unique) \((a_0, 1, q^2, q^{-2})\)-eigenvector in \( L(a_0, 1, q^2, q^{-2}) \) generates a 96-dimensional submodule \( M_1 \). Since 96 is more than half the dimension of \( M \), we either have a split or a socle of dimension less than or equal to 32, which is contained in \( M_1 \). Now this socle would have to contain an \((a_0, 1, q^2, q^{-2}, 1)\)-eigenvector but this eigenvector generates the whole of \( M_1 \). Hence \( M_1 \) is irreducible and we have a 32-dimensional complement \( M_2 \). Computing the formal character of \( M_2 \) shows that \( M_2 \) has \( \epsilon_a(M_1) = 0 \) for all \( a \neq 1 \), whence it can only be reached by applying
\( f_1 \) as the last functor to a submodule of \( \Delta_1 M_2 \). But, also by character calculations, \( \Delta_1 M_1 \) only has two composition factors both of which are isomorphic to \( L(a_0, 1, q^2, q^{-2}) \), thus \( M_1 \) cannot be reached by a path in the crystal graph as described in Chapter 4.

Lemma 7.9 motivates us to investigate more closely modules where the \( X_i \) have eigenvalues \( \pm 1 \) and \( \pm q^2 \).

We have already shown that for

\[
L(a_0, q^2) \cong L(a_0q^2, q^{-2}) \cong \text{ind}^H_2(a_0, q^2)
\]

\( f_1 L(a_0, q^2) \) splits into the direct sum of \( L(a_0, 1, q^2) \) generated by the \((a_0, 1, q^2)\)-eigenvector \((T_1 + q^{-1})v\) and \( L(a_0, 1, q^{-2}) \) generated by the \((a_0, 1, q^{-2})\)-eigenvector \((T_1 - q)(T_0 - q^2T_0^{-1})v\) for a generator \( v \) of the one-dimensional module \((a_0, q^2)\).

Next we consider \( f_1 L(a_0, (q^2)^{(2)}) \) and again choose a generator \( v \) of the one-dimensional \( P_2 \)-module \((a_0, (q^2)^{(2)})\). Now the \((a_0, q^2, 1, q^2)\)-eigenvector \((T_2 + q^{-1})v\) generates a submodule abstractly isomorphic to \( \mathcal{H}_3^{\text{fin}}(T_2 + q^{-1}) \) as an \( \mathcal{H}_3^{\text{fin}} \)-module, therefore it is a proper submodule. This again contains a submodule generated by the \((a_0, 1, (q^2)^{(2)})\)-eigenvector \((T_1 + q^{-1})(T_2 + q^{-1})v\) which is, as an \( \mathcal{H}_3^{\text{fin}} \)-module, isomorphic to the left ideal \( \mathcal{H}_3^{\text{fin}}(T_1 + q^{-1})(T_2 + q^{-1}) \) and therefore again a proper submodule. By the usual argument, self-duality of \( L(a_0, (q^2)^{(2)}) \) implies self-duality of \( f_1 L(a_0, (q^2)^{(2)}) \), whence using Corollary 7.7, a chain of proper nonzero submodules longer than one implies that the cosocle is irreducible.

Now we again apply \( f_1 \) to the now well-defined irreducible module \( L(a_0, (q^2)^{(2)}, 1) \). Here, by Lemma 1.6, we find an \((a_0, q^2, 1, q^2, 1)\)-eigenvector \(((T_2 + q^{-1})T_3 - 1)v\), where \( v \) denotes the outer tensor \( \tilde{v} \boxtimes c \) for \( \tilde{v} \) the image of the above \( v \) under the canonical projection

\[
f_1 L(a_0, (q^2)^{(2)}) \rightarrow L(a_0, (q^2)^{(2)}, 1)
\]

and \( c \) a generator of the one-dimensional \( F[X_{4}^{\pm 1}] \)-module \((1)\). Also by Lemma 1.6, this element generates the whole module. Then,

\[
(T_1 + q^{-1})(T_2 + q^{-1})T_3 - 1)v
\]
is an $(a_0, 1, (q^2)^{(2)}, 1)$-eigenvector and by the same process, we obtain an $(a_0, 1, q^{-2}, q^2, 1)$-eigenvector

$$(T_1 - q)(T_2 + q^{-1})T_3 - 1)(T_0 - q^2T_0^{-1})v.$$  

Using Magma V2.11-2, these have both been checked to generate proper submodules in $f_1L(a_0, (q^2)^{(2)}, 1)$, and we obtain $v$ as a linear combination of both by equation (22), so in this case the module splits into the direct sum of two irreducibles. In fact $L(a_0, (q^2)^{(2)}, 1)$ is isomorphic to $L(a_0, 1, q^{-2}, q^2)$ which we have already considered above. Now a natural generalization of this would be to conjecture that $f_1L(a_0, (q^2)^{(k)})$ is irreducible as long as $l \leq k$ and $f_1L(a_0, (q^2)^{(k)}, 1^{(k-1)})$ splits. Indeed, if we assume

$$M_{l-1} := \tilde{f}_1^{l-1}L(a_0, (q^2)^{(k)})$$

is irreducible and contains an $(a_0, (q^2)^{(k-l+1)}, (q^2, 1)^{(l-1)})$-eigenvector $v_{l-1}$, we obtain after again identifying $v_{l-1}$ with $v_{l-1} \otimes c \in M_{l-1} \otimes (1)$, an $(a_0, (q^2)^{(k-l)}, (q^2, 1)^{(l)})$-eigenvector

$$v_l := [(T_{k+2-l} + q^{-1})T_{k+3-l} - 1][T_{k+4-l} + q^{-1})T_{k+5-l} - 1] \cdots$$

$$\cdots [(T_{k+l-2} + q^{-1})T_{k+l-1} - 1] \otimes v_{l-1}$$

in $\text{ind}_{k+1-l,1}^{k+1}M_{l-1} \otimes (1)$, which generates the whole module. Now hope would be that then $(T_{k+1-l} + q^{-1})v_l$ and $(T_{k-l} + q^{-1})(T_{k+1-l} + q^{-1})v_l$ generate a chain of proper submodules (they are nonzero since they contain a nonzero leading term in the vector space decomposition

$$\text{ind}_{k+1-l,1}^{k+1}M_{l-1} \otimes (1) \cong \bigoplus_{j=1}^{k+l-1} T_{j,0,k+l-1} \otimes \mathcal{H}_{k+l-1} M_{l-1} \otimes (1)$$

$$\bigoplus_{j=0}^{k+l-1} T_{j,k+l-1} \otimes \mathcal{H}_{k+l-1} M_{l-1} \otimes (1)$$

$$\bigoplus 1 \otimes \mathcal{H}_{k+l-1} M_{l-1} \otimes (1),$$

which would then yield the first claim. Then analogously, as above, for $k = l$, we would get an $(a_0, (q^2, 1)^{(k)})$-eigenvector $v_k$ generating all of $\text{ind}_{2k-1,1}^{2k}M_{k-1} \otimes (1)$ and a submodule generated by $(T_1 + q^{-1})v_k$, but by the analogous argument as above, we would also have a submodule generated by the $(a_0, 1, q^{-2}, (q^2, 1)^{(k-1)})$-eigenvector $(T_1 - q)(T_0 - q^2T_0^{-1})v_k$, generating a complement. But it is generally not easy to see that the
generated submodules are indeed proper, i.e. that \((T_{k+1-l} + q^{-1})\) is not invertible modulo the annihilator of \(v_I\).

In the case where \(a = -1\), things work somewhat differently. We again start with the module \(L(a_0, -q^2) \cong L(-a_0q^2, -q^{-2})\) and apply the functor \(f_{-1}\). If we again set \(v\) to be the \((a_0, -q^2, -1)\)-eigenvector in \(L(a_0, -q^2) \otimes (-1)\) which is unique up to a scalar, we obtain submodules in \(f_{-1}L(a_0, -q^2)\) generated by the \((a_0, -1, -q^2)\)-eigenvector \(T_1 + q^{-1})v\) and the \((-a_0q^2, -1, -q^{-2})\)-eigenvector \((T_1 - q)(T_0 + q^2T_0^{-1})v\). The sum of those two submodules has codimension 2, so it is necessarily maximal. Applying \(f_1\) to \(L(a_0, -q^2, -1)\), we see that the new generalized \((-a_0, -q^2, -1, -1)\)-eigenspace is, by Lemma 1.6, spanned by

\[
w := (T_1 - q)(T_0 + T_0^{-1})((T_1 + q^{-1})T_2 - 1)v \text{ and } T_2w
\]

which is by Theorem 7.2 contained in the socle. But one can check that

\[
(T_1 - q)(T_0 + T_0^{-1})((T_1 + q^{-1})T_2 - 1)w
\]

is again a scalar multiple of \(v\), whence \(f_1L(a_0, -q^2, -1)\) is irreducible.

Similarly, we see that \(f_{-1}L(a_0, (q^2)^{(2)})\) is not irreducible as (letting \(v\) be the unique \((-q^2)^{(2)}, 1)\)-eigenvector in \(L(a_0, (-q^2)^{(2)} \otimes (-1))\) the \((a_0, -q^2, -1, -q^2)\)-eigenvector \((T_2 + q^{-1})v\) generates a proper nonzero submodule, and therefore \(f_{-1}L(a_0, (-q^2)^{(2)})\) has an irreducible cosocle \(L(a_0, (-q^2)^{(2)}, -1))\) by Theorem 7.2. Now in \(f_{-1}L(a_0, (-q^2)^{(2)}, -1),\) we again have an \((a_0, -q^2, -1, -q^2, -1)\)-eigenvector \((T_2 + q^{-1})T_3 - 1)v\) generating the whole module by Lemma 1.6 and it has been checked using Magma V2.11-2 that \((T_1 + q^{-1})((T_2 + q^{-1})T_3 - 1)v\) indeed generates a proper submodule – again being nonzero as in the case \(a = 1\) – so \(f_{-1}L(a_0, (-q^2)^{(2)}, -1)\) as well is not irreducible and, by Theorem 7.2 has an irreducible cosocle. Then, by the same argument as for \(f_1L(a_0, -q^2, -1)\), we see that the generalized \((-a_0, -q^2, -1, -q^2, -1, -1)\)-eigenspace is generated by

\[
(T_3 - q)(T_1 - q)(T_0 + T_0^{-1})((T_1 + q^{-1})T_2 - 1)((T_3 + q^{-1})T_4 - 1)v
\]

and

\[
T_4(T_3 - q)(T_1 - q)(T_0 + T_0^{-1})((T_1 + q^{-1})T_2 - 1)((T_3 + q^{-1})T_4 - 1)v,
\]
and that this generates the whole module which is therefore irreducible. Again we would hope that analogously, for any $k$, $f_1 \tilde{f}^l_1 L(a_0, (-q^2)^{(k)})$ be non-irreducible as long as $l \leq k$, then seeing that the module $f_1 \tilde{f}^k_1 L(a_0, (-q^2)^{(k)})$ would have to be irreducible.
APPENDIX A. GERMAN ABSTRACT


Kapitel 1. In Kapitel 1 werden die behandelten Algebren $\mathcal{H}_n$ und $\mathcal{H}_n^R \subset \mathcal{H}_n$ definiert und gezeigt, dass die irreduziblen Moduln sich nur dann unterscheiden, wenn Erzeuger des Gewichtsgitters mit Eigenwert $-1$ operieren. Ansonsten bleiben irreduzible Moduln für $\mathcal{H}_n$ unter Einschränkung auf $\mathcal{H}_n^R$ irreduzibel und analog lässt sich die Operation auf den Irreduziblen für $\mathcal{H}_n^R$ auf $\mathcal{H}_n$ ausdehnen.

Weitere wichtige Unteralgebren sind die Gewichtsgitter $\mathcal{P}_n$ und $\mathcal{R}_n$, die isomorph zu Laurent-Polynomringen in $n$ beziehungsweise $n+1$ Variablen sind. Ferner werden wichtige Begriffe und Notationen wie $A$-mod für die Kategorie der endlich dimensionalen Moduln einer Algebra $A$, die Grothendieckgruppe $K_0(A$-mod$)$, sowie technische Lemmata mit Rechnungen in $\mathcal{H}_n$ gesammelt.

Kapitel 2. In Kapitel 2 werden die wichtigsten Werkzeuge – eine Mackeyfiltrierung und eine Dualität – bereitgestellt. Die Mackeyfiltrierung beschreibt das Verhalten von Moduln für parabolische Unteralgebren $\mathcal{H}_I$ unter Induktion und anschließender Restriktion, liefert jedoch, im Gegensatz zum endlich-dimensionalen Fall keine Zerlegung in direkte Summanden.

Satz A.1 (Theorem 2.3). Seien $\mathcal{H}_I, \mathcal{H}_J \subseteq \mathcal{H}_n$ parabolische Unteralgebren, $D_{I,J}$ die Menge der ausgezeichneten Doppelnebenklassenvertreter, und $M \in \mathcal{H}_J$-mod$^d$. Dann hat $\text{res}^{\mathcal{H}_I}_{\mathcal{H}_J} \text{ind}^{\mathcal{H}_n}_{\mathcal{H}_I} M$ eine Filtration mit Subquotienten isomorph zu $\text{ind}^{\mathcal{H}_I}_{\mathcal{H}_{I\cap x,J}} x(\text{res}^{\mathcal{H}_J}_{\mathcal{H}_{x-1,J}} M)$ für jedes $x \in D_{I,J}$. Die Subquotienten können in jeder Ordnung genommen werden, die die
Bruhat-Ordnung auf $D_{I,J}$ verfeinert. Insbesondere ist $\text{ind}_{\mathcal{H}_{I,J}}^{\mathcal{H}_{I}^{\mathcal{H}_{J}}} M$ ein Untermodul, da der Doppelnebenklassenvertreter 1 in jeder solchen Ordnung das kleinste Element ist.

Im zweiten Teil dieses Kapitels wird eine Möglichkeit angegeben, einen koinduzierten Modul mithilfe eines induzierten Moduls auszudrücken. Hierzu wird der Anti-automorphismus $\tau$, der auf den Erzeugern die Identität ist, sowie der längste ausgezeichnete Nebenklassenvertreter $d$ in $D_{I,J}$ für eine parabolische Unteralgebra $\mathcal{H}_{I}$ benötigt. Man erhält so auch eine explizite Beschreibung der von $\tau$ induziertenDualität.

**Satz A.2** (Korollar 2.6). Für $M \in \mathcal{H}_{I}^{\text{-mod}^{\text{fd}}}$ existiert ein natürlicher Isomorphismus

$$\text{Hom}_{\mathcal{H}_{I}}(\mathcal{H}_{n}, M) \cong \mathcal{H}_{n} \otimes_{\mathcal{H}_{I}} d M$$

von $\mathcal{H}_{n}$-Moduln.

**Satz A.3** (Korollar 2.7). Für $M \in \mathcal{H}_{I}^{\text{-mod}^{\text{fd}}}$ existiert ein natürlicher Isomorphismus

$$\text{ind}_{\mathcal{H}_{I}}^{\mathcal{H}_{n}}(M)^{\tau} \cong \text{ind}_{\mathcal{H}_{I}}^{\mathcal{H}_{n}}(d(M^{\tau})).$$

**Kapitel 3.** In diesem Kapitel wird der formale Charakter eines Moduls $M \in \mathcal{H}_{n}^{\text{-mod}^{\text{fd}}}$ durch

$$(23) \quad \text{ch } M := [\text{res}_{\mathcal{P}_{n}}^{\mathcal{H}_{n}} M] \in K(\mathcal{P}_{n}^{\text{-mod}^{\text{fd}}}).$$

definiert, der eine wesentliche Rolle in der Beschreibung der irreduziblen Moduln spielt. Man kann formale Charaktere von induzierten Moduln explizit berechnen:

**Lemma A.4** (Lemma 3.4). Sei $a = (a_{0}, a_{1}, \ldots, a_{n}) \in F^{n+1}$. Dann ist

$$\text{ch } \text{ind}_{\mathcal{F}_{n}}^{\mathcal{H}_{n}} a = \sum_{\substack{u \in S_{n} \\varepsilon \in \{1, -1\}^{n}}} [(b_{0}(u, \varepsilon, a_{u-1}(1), \ldots, a_{u-1}(n))]$$

wobei $b_{0}(u, \varepsilon) := a_{0} \prod_{j \in \{1, \ldots, n\}} a_{u-1}(j)$.

**Lemma A.5.** ("Shuffle Lemma") 3.5

Sei $n = m + k$ und $M \in \mathcal{H}_{m}^{\text{-mod}^{\text{fd}}}$, $K \in \mathcal{H}_{k}^{\text{A-}^{\text{mod}^{\text{fd}}}}$. Sei ferner

$$\text{ch } M = \sum_{a \in F^{m+1}} r_{2}[(a_{0}, \ldots, a_{m})], \quad \text{ch } K = \sum_{b \in F^{k}} s_{2}[(b_{1}, \ldots, b_{k})].$$
Danngilt

\[
\text{ch ind}_{m,k}^{n} M \boxtimes K = \sum_{a \in F^m} \sum_{b \in F^k} \tau_a \delta_b (\sum_{\xi} (c_0, c_1, \ldots, c_n)),
\]

wobei die letzte Summe über alle \( \xi = (c_1, \ldots, c_n) \in F^{n+1} \) läuft, die man durch "mischen" von \( a \) und \( b \) für \( \xi = (\epsilon_1, \ldots, \epsilon_k) \in \{1, -1\}^k \) erhält, d.h. es existieren \( 1 \leq u_1 < \cdots < u_m \leq n \) so, dass

\[
(c_{u_1}, \ldots, c_{u_m}) = (a_1, \ldots, a_m), (c_1, \ldots, \widehat{c}_{u_1}, \ldots, \widehat{c}_{u_m}, \ldots, c_n) = b \xi \text{ und } c_0 = a_0 \prod_{j \in \{1, \ldots, k\}} b_{(j)}.
\]

Dann wird gezeigt, dass die zentralen Charaktere durch Bahnen \( \gamma \) unter der Operation der endlichen Weylgruppe auf \( F^{n+1} \) gekennzeichnet werden sowie ein Zerfallen eines Moduls in sogenannte Blöcke induzieren.

**Lemma A.6** (Lemma 3.8). Jedes \( M \in \mathcal{H}_n \text{-mod}^{fd} \) zerfällt als

\[
M = \bigoplus_{\gamma \in F^{n+1}/\sim} M[\gamma]
\]

als \( \mathcal{H}_n \)-Modul.

Am Ende des Kapitels wird noch der irreduzible induzierte Modul \( L^A(a^{(n)}) := \text{ind}_{\mathcal{R}_n}^{\mathcal{H}_n}(a^{(n)}) \) im Typ \( A \) eingeführt, der der einzige Irreduzible in seinem Block ist. Dieser wird mit dem folgenden Lemma auf den Typ \( B \) in den Fällen verallgemeinert, in denen das mit den bisher gesammelten Methoden möglich ist. Die übrigen Fälle werden im 7. Kapitel behandelt.

**Lemma A.7** (Lemma 3.10). Sei \( a \in F\setminus\{\pm 1\} \). Dann hat \( \text{ind}_{\mathcal{R}_n}^{\mathcal{H}_n}(a_0, a^{(n)}) \) einen irreduziblen Kosockel und dieser Kosockel, genannt \( L(a_0, a^{(n)}) \), ist der einzige irreduzible Modul in \( \mathcal{H}_n \text{-mod}^{fd} \), der \( (a_0, a^{(n)}) \) als formalen Charakter enthält.

**Kapitel 4.** Im vierten Kapitel werden die sogenannten Kristalloperatoren \( \tilde{e}_a = \text{soc} \circ e_a \) und \( \tilde{f}_a = \text{cosoc} \circ f_a \) für \( a \in F \) definiert. Hierbei ist \( f_a = \text{ind}_{\mathcal{H}_{n-1,1}}^{\mathcal{H}_n} - \otimes(a) \) mit dem eindimensionalen \( F[X_n^\pm 1] \)-Modul \( (a) \), auf dem \( X_n \) als Multiplikation mit \( a \) operiert und \( e_a \) der dazu adjungierte Funktor. Ein weiterer wichtiger Funktor ist

\[
\Delta_{a^{(n)}} : \mathcal{H}_n \text{-mod}^{fd} \rightarrow \mathcal{H}_{n-m,m} \text{-mod}^{fd},
\]
der die Komposition der Einschränkung auf $H_{n-m,m}$ mit der Projektion auf den direkten Summanden ist, dessen Kompositionsfaktoren alle von der Form $K \boxtimes L^{A}(a^{(m)})$ für irreduzible Moduln $K \in H_{n-m}^{\text{mod}}$ sind. Desweiteren spielt die Zahl $\epsilon_{a}(M) = \max\{m \geq 0 \mid \Delta_{a^{(m)}} M \neq 0\}$ eine wichtige Rolle.

Die Funktoren $\tilde{e}_{a}$ und $\tilde{f}_{a}$ induzieren für $a \neq \pm 1$ Abbildungen zwischen den Isomorphieklassen von $H_{n-1}$- und $H_{n}$-Moduln, da die Funktoren angewendet auf irreduzible Moduln wieder irreduzible Moduln ergeben.

**Lemma A.8** (Lemma 4.8). *Sei $M \in H_{n}^{\text{mod}}$ irreduzibel, $m \geq 0$ und $a \in F \setminus \{\pm 1\}$. Dann gilt:

(i) $\text{soc} \Delta_{a^{(m)}} M \cong (\tilde{e}_{a}^{m} M) \boxtimes L^{A}(a^{(m)})$.

(ii) $\text{cosoc ind}_{n,m}^{n+m} M \boxtimes L^{A}(a^{(m)}) \cong \tilde{f}_{a}^{m} M$.

Die Funktoren $\tilde{e}_{a}$ und $\tilde{f}_{a}$ sind invers zueinander:

**Lemma A.9** (Korollar 4.10). *Sei $M, N \in H_{n}^{\text{mod}}$ irreduzibel und $a \in F \setminus \{\pm 1\}$. Dann gilt

(i) $\tilde{f}_{a} M \cong \tilde{f}_{a} N$ genau dann, wenn $M \cong N$ gilt. Falls weiterhin $\epsilon_{a}(M), \epsilon_{a}(N) > 0$, gilt $\tilde{e}_{a} M \cong \tilde{e}_{a} N$ genau dann, wenn gilt $M \cong N$;

(ii) $\tilde{e}_{a} \tilde{f}_{a} M \cong M$ und, falls $\epsilon_{a}(M) > 0$, $\tilde{f}_{a} \tilde{e}_{a} M \cong M$.

Eine wichtige Konsequenz ist der folgende Satz über die Kategorie $\text{Rep} H_{n}$, in der die Erzeuger des Gewichtsgitters nicht mit Eigenwerten $\pm 1$ operieren.

**Satz A.10** (Theorem 4.11). *Die Abbildung

\[ \text{ch} : K(\text{Rep} H_{n}) \to K(\mathcal{P}_{n}^{\text{mod}}) \]

ist injektiv.*

**Kapitel 5.** In Kapitel 5 wird ein Überblick über die kombinatorische Klassifikation der irreduziblen Modulen und die Operation der Kristalloperatoren $\tilde{f}_{a}^{A}, \tilde{f}_{a}^{A^*}, \tilde{e}_{a}^{A}, \tilde{e}_{a}^{A^*}$ sowie der Funktionen $\epsilon_{a}$ und $\epsilon_{a}^{*}$ im Typ $A$ gegeben. Insbesondere die Ergebnisse von [17] werden gekürzt dargestellt und die Notation der Multisegmente erklärt. So ist jeder irreduzible Modul in $\text{Rep}_{\lambda-1} H_{n}^{A}$ durch ein Multisegment $\Gamma$ gekennzeichnet.
und wird mit $M_{\Gamma}$ bezeichnet, wobei $\text{Rep}_{\lambda^{-1}} \mathcal{H}_n^A$ die volle Unterkategorie von $\mathcal{H}_n^A$-mod ist, auf deren Moduln alle Eigenwerte der Erzeuger des Gewichtsgitters aus der Menge $I_\lambda^{-} := \{\lambda^{-1}q^2i \mid i \in \mathbb{Z}\}$ kommen.

**Kapitel 6.** In Kapitel 6 wird zunächst $\text{Rep}_\lambda \mathcal{H}_n$ für festes $\lambda \in F$ als die volle Unterkategorie von $H_n$-mod, in der alle Eigenwerte von $\mathcal{R}_n$ aus der Menge $I_\lambda := \{\lambda q^2, \lambda^{-1}q^2i \mid i \in \mathbb{Z}\}$ sind. Über diese Unterkategorie wird nun im Fall $p^2, \pm q, \pm 1 \notin I_\lambda$ gezeigt, dass alle irreduziblen Moduln für $\mathcal{H}_n^R$ gerade von der Form $\text{ind}_{\mathcal{H}_d^R}^H L$ für irreduzible Moduln $L$ aus $\text{Rep}_{\lambda^{-1}} \mathcal{H}_n^A$ sind und man somit die gleiche Parametrisierung durch Multisegmente wie im Typ A erhält.

**Lemma A.11** (Lemma 6.1). Sei $M_{\Gamma} \in \text{Rep}_\lambda \mathcal{H}_n^A$. Dann ist der induziert Modul $L_{\Gamma} := \text{ind}_{\mathcal{H}_d^R}^H M_{\Gamma}$ irreduzibel.

Dadurch erhält man auch eine genaue Beschreibung der Kristalloperatoren auf dieser Unterkategorie.

**Lemma A.12** (Lemma 6.2). $\epsilon_a(\text{ind}_{\mathcal{H}_d^R}^H M_{\Gamma}) = \begin{cases} \epsilon_a(M_{\Gamma}) & \text{if } a \in I_\lambda^{-} \\ \epsilon_{a^{-1}}(M_{\Gamma}) & \text{if } a \in I_\lambda^{+} \end{cases}$

**Lemma A.13** (Lemma 6.3). Sei $L_{\Gamma} \in \text{Rep}_\lambda \mathcal{H}_n$. Dann gilt

(i) $\tilde{f}_a L_{\Gamma} = \begin{cases} L_{\tilde{f}_a \lambda} & \text{if } a \in I_\lambda^{-} \\ L_{\tilde{f}_a^{-1} \lambda} & \text{if } a \in I_\lambda^{+} \end{cases}$

(ii) $\tilde{e}_a L_{\Gamma} = \begin{cases} L_{\tilde{e}_a \lambda} & \text{if } a \in I_\lambda^{-} \\ L_{\tilde{e}_a^{-1} \lambda} & \text{if } a \in I_\lambda^{+} \end{cases}$

**Kapitel 7.** Im siebten Kapitel werden die Fälle $a = \pm 1$ genauer untersucht, und Teilergebnisse und Gegenbeispiele gegeben. Das erste wichtige Ergebnis besagt, dass dieselben Aussagen wie im Fall $a \neq \pm 1$ gelten, wenn der Ausgangsmodul $N$ nicht isomorph zum dem Modul $N^\sigma$ ist. Hierbei ist $N^\sigma$ der Modul, der aus $N$ durch twisten mit dem Automorphismus $\sigma$ gewonnen wird, der die Identität auf $\mathcal{H}_n^R$ ist und $X_0$ auf ein Negatives abbildet.

**Satz A.14** (Theorem 7.2). Sei $M \in \text{Rep}_{-1} \mathcal{H}_n$ irreduzibel, $\epsilon_{-1}(M) = r$ und $\text{ch} M \neq \text{ch} M^\sigma$. Dann gilt:

(i) $\Delta_{-1(r)}M \cong K \otimes L^A(-1(r))$ für ein irreduzibles $K \in \text{Rep}_{-1} \mathcal{H}_{n-r}$ mit $\text{ch} K \neq \text{ch} K^\sigma$;
(ii) $M$ ist der irreduzible Kossockel von $\text{ind}_{n-r,r}^n K \otimes L^A(-1(r))$;
(iii) $\text{soc } \Delta_{-1}M \cong N \boxtimes (-1)$ für ein irreduzibles $N \in \text{Rep}_{-1} \mathcal{H}_{n-1}$ mit $\text{ch } N \neq \text{ch } N^\sigma$;
(iv) $M$ ist der irreduzible Kossockel von dem nicht irreduziblen Modul $f_{-1}N$;
(v) $M$ ist eindeutig bestimmt durch seinen formalen Charakter, insbesondere gilt $M \cong M^r$. 

Als nächstes wird gezeigt, dass der Endomorphismenring eines irreduziblen Moduls unter Anwendung von $f_a$ nicht beliebig groß werden kann.

**Lemma A.15** (Lemma 7.6). Sei $N \in \mathcal{H}_{n-1}\text{-mod}^{\text{fd}}$ irreduzibel. Setze $M := f_aN$. Dann gilt $\dim \text{End}(M) \leq 2$.

Im restlichen Kapitel werden Beispiele gerechnet, bei denen der induzierte Modul zergießt, es wird gezeigt, dass Serien von Moduln das gleiche Verhalten aufweisen, und schließlich werden Vermutungen über das Verhalten von weiteren Serien aufgestellt.


