Nonlinearity Assessment and Linear Control of Nonlinear Systems

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Vorwort

Die vorliegende Arbeit entstand während meiner Tätigkeit als wissenschaftlicher Mitarbeiter am Institut für Systemtheorie und Regelungstechnik (IST) der Universität Stuttgart.

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Stuttgart, im Dezember 2006 Tobias Schweickhardt

Contents

Abstract

Linear control may be favorable over nonlinear control because linear design techniques greatly facilitate the controller design process and because linear controllers impose lower requirements on the implementation and operation as compared to nonlinear controllers. It is therefore a tempting idea to use linear models and linear controller design methods also for nonlinear systems. It is for instance common practice in control engineering to use models obtained from Jacobi-linearization (i.e. first order Taylor series approximation of the state space equations) instead of complete nonlinear models. However, in order to guarantee the suitability of a linear model or the proper functioning of a linear controller in presence of the model/plant mismatch due to linearization, a rigorous justification is required. This thesis presents a general framework to design linear controllers for nonlinear systems based on linear models, that guarantees stability also for the nonlinear closed loop.

How to deal with a model/plant mismatch in a purely linear setup is very well known from linear robust control theory. Robust stability can be guaranteed if the gain of the modeling error satisfies a certain bound. This small gain condition remains valid in the case of a nonlinear plant that is modeled by a linear model. However, the small gain condition requires a global bound on the gains of the modeling error. Moreover, this bound must be small enough in order to prove stability. These requirements are often too strong to be satisfiable for general nonlinear systems. To avoid the conservatism connected to global analysis, the presented approach makes therefore use of gains over sets in order to be able to do the analysis and synthesis in a specified region of operation. Corresponding to the introduction of gains over sets in theory, a saturation block is introduced in the plant or the controller in implementation. The controller design for nonlinear systems is thus reduced to the controller design for linear saturated systems. Examples show that the remaining controller design problem is feasible with standard techniques from the literature.

Prior to controller design, a nominal linear model and the error bound have to be derived. While the linearization is a common choice as a linear model for a nonlinear system, it does not need to be the best choice for a given region of operation. Indeed, there are situations in which other linear models turn out to be better suited for controller design than the linearization. It is therefore interesting to search for the best linear model for a nonlinear system, characterized by a minimal modeling error. This minimal modeling error measures at the same time the disparity between the nonlinear system and the set of linear systems, and thus provides a way to assess the extend of nonlinearity of the plant. Hence,

the modeling error allows to (i) *evaluate* the quality of a *given* linear model, (ii) *derive* a *best* linear model that minimizes the modeling error, and (iii) use the modeling error of the best system as a *measure of nonlinearity* of the system. As there are different ways to define a modeling error, like an additive error, or multiplicative errors at the input or output etc., different according nonlinearity measures are defined under a common system-theoretic framework.

This thesis has two main areas of contribution. The first area is the derivation and assessment of linear models for nonlinear systems, and the second area is the utilization of this information for controller design. The main contribution of the first part of this thesis is to introduce a novel unifying framework for nonlinearity assessment, to provide a rigorous mathematical foundation, and to provide characterizations of the nonlinearity measures and best linear models for special system classes. The computation of nonlinearity measures and linear models is also discussed. In the second part of this thesis, stability conditions and controller design procedures for linear control of nonlinear systems are presented. The exact form of the nominal closed loop and of the stability condition depends on the type of modeling error chosen and we present results for all previously defined setups. Examples show the advantages and disadvantages of the proposed design procedure. In particular, the possible advantages of regional design and of the use of best models are illustrated.

The results of this thesis build a bridge between nonlinearity assessment and robust control theory. The key feature of the proposed methods is thereby to bring together nonlinearity measures, the development and assessment of linear models for nonlinear systems and the design of linear controllers for nonlinear systems under a unifying framework.

Deutsche Kurzfassung

Die lineare Regelung ist gegenüber der nichtlinearen Regelung vorteilhaft, weil lineare Methoden den Entwurfsprozess vereinfachen und lineare Regler niedrigere Anforderungen an die technische Umsetzung stellen als nichtlineare Regler. Es ist deshalb erstrebenswert, lineare Modelle und lineare Reglerentwurfsverfahren auch für nichtlineare Regelstrecken zu verwenden. So ist es gängige Praxis, statt der vollständigen nichtlinearen Modelle einfacherere Modelle zu verwenden, die aus den vollständigen Modellen durch Linearisierung (d.h. durch Taylor-Approximation erster Ordnung) hervorgehen. Um jedoch die Stabilität des Regelkreises und die Funktionsfähigkeit des linearen Reglers angesichts des durch die Modellvereinfachung verursachten Modellfehlers zu garantieren, und damit einen erfolgreichen Reglerentwurf sicherzustellen, bedarf es einer methodischen Rechtfertigung. In der vorliegenden Arbeit wird eine allgemeingültige Methodik vorgestellt, mit deren Hilfe anhand eines linearen Modells ein Regler für ein nichtlineares System entworfen werden kann, und die die Stabilität des geschlossenen Kreises garantiert.

Wie ein Modellfehler beim Regelungsentwurf berücksichtigt werden kann ist aus der Theorie linearer robuster Regelungen bekannt. Tatsächlich behält die Bedingung, die üblicherweise für robuste Stabilität linearer Systeme verwendet wird (das sog. “small-gain”-Kriterium), auch hinsichtlich nichtlinearer Systeme, welche durch lineare Modelle angenähert werden, Gültigkeit. Jedoch setzt die Anwendung der “small-gain”-Bedingung eine globale Schranke für die Signalverstärkung voraus. Das bedeutet anschaulich, dass das betreffende System “im Unendlichen linear beschränkt” sein muss. Darüber hinaus muss die Schranke, die sog. Verstärkung des Systems, klein genug sein, um die Stabilitätsbedingung zu erfüllen. Diese Anforderungen sind in der Praxis häufig zu hoch, um eine Anwendung der “small-gain”-Bedingung auf den Reglerentwurf für allgemeine nichtlineare Systeme zuzulassen. Um den mit der globalen Analyse und dem globalen Entwurf einhergehenden Konservatismus, d.h. die übermässig strengen Kriterien, zu vermeiden, findet in dieser Arbeit das Konzept der regionalen Verstärkung Anwendung. Dieses Konzept erlaubt die Analyse und Synthese von Regelungssystemen in einem vom Entwickler festgelegten Arbeitsbereich. Die Einführung des Konzepts der regionalen Verstärkung findet in der Praxis seine Entsprechung darin, dass entweder die Regelstrecke oder der Regler durch ein Sättigungsglied ergänzt wird. Der Reglerentwurf für nichtlineare Systeme wird auf diese Weise durch den Reglerentwurf für beschränkte lineare Systeme ersetzt. Wie Beispiele zeigen,

kann das verbleibende Regelungsproblem mit Standardmethoden aus der Literatur gelöst werden.

Bevor jedoch der Reglerentwurf durchgeführt werden kann, muss ein lineares Modell und der entsprechende Modellfehler hergeleitet werden. Während die (lokale) Linearisierung in der Praxis häufig als lineares Modell gewählt wird, muss sie für einen gegebenen Arbeitsbereich nicht das bestmögliche Modell darstellen. Vielmehr stellen sich in bestimmten Fällen andere Modelle als für den Reglerentwurf geeigneter heraus. Es ist deshalb von Interesse, für ein gegebenes nichtlineares System und einen gegebenen Arbeitsbereich das beste lineare Modell zu finden. Das beste linear Modell zeichnet sich dadurch aus, dass es den kleinsten Modellfehler auf dem gewählten Arbeitsbereich besitzt. Dieser kleinste Modellfehler ist gleichzeitig ein Maß für den Abstand des nichtlinearen Systems von der Klasse der linearen Modelle, und damit ein Maß für die "Nichtlinearität" des Systems. Der Modellfehler kann also dazu benutzt werden, erstens, die Qualität eines gegebenen linearen Modells zu bewerten, zweitens, ein bestes lineares Modell herzuleiten, und drittens, die Nichtlinearität eines nichtlinearen Systems zu messen. Es gibt verschiedene Arten, den Modellfehler festzulegen, wie auch in der linearen robusten Regelung z.B. additive Modellfehler und (invers-)multiplikative Modellfehler definiert werden können. In dieser Arbeit wird eine Methodik zur Modellbewertung und Modellherleitung eingeführt, die die Definition verschiedener Nichtlinearitätsmaße zu den entsprechenden Modellfehlern im Rahmen eines gemeinsamen systemtheoretischen Zugangs erlaubt.

Die Beiträge dieser Arbeit können also zwei Themenkomplexen zugeordnet werden. Der eine Komplex beschäftig sich mit der Herleitung und der Bewertung linearer Modelle für nichtlineare Systeme, und der zweite Komplex behandelt die Verwendung der dabei gewonnenen Information für den Reglerentwurf.

Bezüglich des ersten Komplexes besteht der Hauptbeitrag dieser Arbeit in der Einführung einer neuen, verallgemeinerten Herangehensweise zur Nichtlinearitätsbewertung, der Bereitstellung und Rechtfertigung der entsprechenden mathematischen Methoden sowie der Charakterisierung der Nichtlinearitätsmaße und der besten linearen Modelle für spezielle Systemklassen. Insbesondere wird gezeigt, unter welchen Bedingungen die oben beschriebenen Modellfehler definiert werden können, und es wird gezeigt, dass die mathematischen Eigenschaften dieser Definitionen den damit verbundenen Intentionen entsprechen. So wird zum Beispiel gezeigt, dass das beste lineare Modell im Sinne eines additiven Modellfehlers im Spezialfall eines gegen einen Punkt konvergierenden Arbeitsbereiches genau der Linearisierung an diesem Arbeitspunkt entspricht. Es wird gezeigt, wie aus dem stationären Verhalten eines Systems untere Schranken für die Nichtlinearitätsmaße berechnet werden können. Für den Fall statischer Nichtlinearitäten wird gezeigt, dass dynamische lineare Modelle nicht besser geeignet sind als statische lineare Modelle. Es werden Formeln für die Berechnung der Nichtlinearitätsmaße statischer Nichtlinearitäten angegeben. Der Begriff der Sektornichtlinearität führt dabei auf besonders einfach auszuwertende Formeln für Eingrößensysteme. Lineare Modelle und deren Modellfehler werden für die Klassen der Hammerstein-Systeme und der Wiener-Systeme angegeben. Es wird gezeigt, unter welchen Bedingungen ein exakt optimales lineares Modell, d.h. ein Modell dessen Modellfehler dem Nichtlinearitätmaß entspricht, existiert. Falls kein optimales

Modell existiert, kann immer ein lineares Modell gefunden werden, dessen Modellfehler das Nichtlinearitätsmaß beliebig genau annähert. Methoden zur computergestützen Berechnung von Nichtlinearitätsmaßen und optimalen bzw. suboptimalen linearen Modellen werden diskutiert.

Der Beitrag zum zweiten Themenkomplex besteht in der Herleitung von Stabilitätsbedingungen und Reglerentwurfsverfahren für die lineare Regelung nichtlinearer Systeme. Die genaue Form der Stabilitätsbedingung und des nominellen geschlossenen Kreises hängen dabei von der Art und Weise ab, wie der Modellfehler definiert wird. Es werden Stabilitätsbedingungen für alle im ersten Teil definierten Modellfehler angegeben. Anhand von Beispielen werden die Vor- und Nachteile der vorgeschlagenen Entwurfsmethodik gezeigt. Insbesondere wird die mögliche Verbesserung der Regelgüte durch die Verwendung eines optimalen linearen Modells illustriert. Zudem wird demonstriert, dass es notwendig sein kann, einen regionalen Entwurf statt eines globlen Entwurfs durchzuführen. Es werden Kriterien zur Auswahl der für ein gegebenes Regelproblem am besten geeigneten Definition des Modellfehlers diskutiert , und mögliche Erweiterungen der Methode für den Entwurf mit garantierter Regelgüte gezeigt.

Die Ergebnisse dieser Arbeit stellen eine Verbindung zwischen Methoden der Nichtlinearitätsbewertung und Methoden der robusten Regelung her. Der Vorteil der vorgestellten Methodik besteht dabei darin, Nichtlinearitätsmaße, die Herleitung und die Bewertung linearer Modelle für nichtlineare Systeme sowie den Entwurf von linearen Reglern für nichtlineare Systeme unter einem gemeinsamen systemtheoretischen Konzept zu vereinen.

CHAPTER 1

Introduction

Abstract models can describe and explain phenomenological observations in a structured and general way. Models play a fundamental role in science because they allow to predict natural events and design artificial (technical) mechanisms. A model is considered a good model, if it allows to describe experiments and phenomenological data sufficiently well.[1] In control engineering, the design and operation of technical systems is greatly facilitated by, and often necessitates, the use of models. Especially compliance with safety and performance specifications can be guaranteed by model-based analysis and design techniques. Thus, mathematical models play an important role in systems engineering, and a lot of effort is put into establishing and analyzing models of technical systems, as well as into putting up model-based design techniques.

An important property of mathematical models of dynamic systems is the linearity property. Due to the rather simple structure of linear systems, for many control-related engineering problems there are methods available that are theoretically sound as well as practically applicable. Due to the diverse and complex behaviour of nonlinear systems, tools for nonlinear systems analysis and controller design exhibit a much lower level of generality and efficiency. To cope with nonlinear analysis and control problems, there are two alternative approaches. For highly nonlinear systems, special methods have to be developed that possibly rely upon certain physical properties of the application or upon mathematical properties of a certain system class, like energy-shaping methods for mechanical control systems or feedback linearization. For mildly nonlinear problems, one can attempt to use linear models and linear controller design methods. However, this last approach requires a rigorous justification in order to guarantee the accuracy of a linear model or the proper function of a linear controller in presence of the nonlinear system behaviour.

In view of the preceding discussion, the following questions arise:

(1) Given (a model of) a dynamic system. Is the system linear? If not, is it “far from linear” or “close to linear”?
(2) Given (a model of) a dynamic system that is sufficiently close to linear. How can a good (the best) linear model be obtained?

[1] Martin Schetzen remarks with regard to the use of models in science [111]: “One only claims that experiments proceed in accordance with the model. In this sense, science is not directly concerned with reality. The question of reality is addressed in philosophy.”

(3) Given a control problem, i.e. a plant and a control objective. Is a linear controller adequate or is a nonlinear control algorithm needed?
(4) Given a control problem that can be solved with linear control. How can a linear controller actually be designed without the need for advanced nonlinear techniques?

The topic of this thesis is the assessment and the derivation of linear models for nonlinear systems, as well as the use of linear models for linear controller design for nonlinear systems. Therefore, this thesis touches upon all of the above four questions. The intention of this introduction is to give an overview over the research concerning these questions and to give an outline of the answers that are given in this thesis. To this end, the next section presents a literature review on the above four questions. In the subsequent Section 1.2, the objectives of this thesis, as well as its structure and key contributions are given.

1.1. Literature Review

The four questions above can be associated with the research areas of "Nonlinearity Assessment", "Linear Modeling for Nonlinear Systems", "Control-relevant Nonlinearity Assessment" and "Linear Control for Nonlinear Systems", and we will discuss previous work on these topics in that order. Overviews over the recent research in the area of nonlinearity assessment can also be found in [114, 118].

Nonlinearity Assessment. Linearity is a definite property that is characterized by the superposition and homogeneity principles. If these principles are satisfied by the input-output behaviour of a dynamic system, or more precisely of a model of a dynamic system, that system (model) is called linear. Otherwise it is called nonlinear[2]. If a mathematical model of a dynamic system is given, linearity can be checked with the model equations. For state space systems, linearity may be a matter of representation, and seemingly nonlinear systems can under certain conditions be brought to a linear form by coordinate transformations [80]. If no detailed mathematical model is available, measurement data can be used to evaluate whether the input-output behaviour of a process is linear or nonlinear, i.e. whether a linear process model can be used or not [49, 50].

Although the strict mathematical definition of linearity is a definite true/false property, it is sometimes interesting to ask whether a system is "close to linear" or "far from linear". Since the early days of control, it is known that feedback also reduces the effect of nonlinearity of the controlled process [9]. In order to be able to show that fact in an operator-theoretic setting, Desoer and Wang introduced the quantity

$$\delta_N^{\mathcal{U}} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \|Nu - Gu\| \tag{1.1}$$

as a nonlinearity measure [26]. The number given by $\delta_N^{\mathcal{U}}$ quantifies the deviation of the input-output behaviour of a system from linearity. In the definition (1.1), N is the nonlinear

[2]The term "nonlinear" is used in the literature for both, the class of *not necessarily linear* systems (i.e. including the class of linear systems), and the class of systems that are *not linear* (excluding linear systems). In this thesis, the term is also used in both meanings, but rendered more precise where necessary in order to avoid ambiguity.

system[3] under consideration, $\mathcal{U}$ denotes the set of considered input signals, and $\mathcal{G}$ is the set of causal stable linear systems. In [26], also a second measure can be found, where the above term is normalized by the norm of the input,

$$\tilde{\delta}_N^{\mathcal{U}} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{||Nu - Gu||}{||u||}. \tag{1.2}$$

This second measure represents the gain of the system that has as output the difference e between the output of the nonlinear system N and a linear model G, where G is chosen so as to minimize this gain. This second measure is used more frequently in the literature (e.g. [102, 94, 79, 116]).

A different context in which quantitative nonlinearity assessment of a system is of interest is approximate feedback linearization [2, 3]. In cases where an exact i/o-linearization by feedback is not possible, for example because of non-minimum phase behaviour of the plant, one may wish to design a feedback such that the closed loop is "as linear as possible". As it is known that the closed loop cannot be exactly linear, it is desirable to quantify the extent to which the approximate linearization has been successful. This desire was the reason for reintroducing (independently from [26]) the nonlinearity measure (1.2) and to define a new, relative-error-like nonlinearity measure

$$\phi_N^{\mathcal{U}} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{||Nu - Gu||}{||Nu||}, \tag{1.3}$$

in [2, 3]. Helbig *et al.* [57] point out that the definitions (1.1)-(1.3) are only meaningful for continuously operated (time-invariant) processes, and they give a possible generalization of the nonlinearity measure (1.3) for batch processes (time-varying systems). Closely related to definition (1.3) is the suggestion of the expression $||G - N||_\infty \,/\, ||N||_\infty$ as relative-error-like nonlinearity measure [94], where $||P||_\infty = \sup_{u\in L_\infty} \frac{||Pu||}{||u||}$ denotes the L_∞-induced operator norm. Sun and Hoo [130] define a nonlinearity measure for single-input-single-output (SISO) nonlinear systems by

$$\phi = \max\left\{\sup_{u\in\mathcal{U}} \left|\left|Nu - L_{up}u\right|\right|, \sup_{u\in\mathcal{U}} ||Nu - L_{lo}u||\right\} \tag{1.4}$$

where L_{up} and L_{lo} are linear systems such that the output y (in the time domain) of N lies at all times between the outputs y_{up} and y_{lo} of L_{up} and L_{lo}, $(y_{lo} - y)(y_{up} - y) \leq 0$.

Nonlinearity measures based on norms and gains, as those defined by eqns. (1.1)-(1.3) also play an important role in the derivation of linear models for nonlinear systems. Those aspects will be discussed below, and we first proceed with the discussion of general approaches to analyze the extend of nonlinearity of a system.

Before considering nonlinearity measures of dynamic operators, one can ask to first consider the simpler case of nonlinearity measures for (static) functions. A static nonlinearity

[3] More precisely, N is a mapping from one signal space to another signal space. No further assumptions or requirements on N need to be done at this point. If the physical setup requires to do so, the domain of N may be restricted to contain a certain class of signals, however. Continuous or bounded signals may be such a signal class, for instance.

measure allows for instance to assess the steady-state behaviour of a stable dynamical process. Such a nonlinearity measure for static mappings is proposed by Ogunnaike *et al.* [100]. The idea is to compare local linear models corresponding to different points of the operating range. The largest difference of the local models in the operating range, measured in terms of a matrix norm, is defined as a measure of nonlinearity. The idea of comparing different local models is frequently met in nonlinearity assessment [128, 132, 81].

A geometric viewpoint is taken in [46, 42], where the curvature of the steady state map is introduced as a measure of nonlinearity. The curvature measure can be extended to dynamic systems using Fréchet derivatives of operators [48, 42]. However, the approach requires the dynamic operator to be twice Fréchet-differentiable, which practically restricts its use to L_∞ signal spaces (see also discussion in Section 3.3). Moreover, the calculation of this dynamic curvature measure turns out to be impractical without significant simplifications.

Nikolaou and co-worker introduce an inner product for operators in order to quantify the nonlinearity of a dynamic system [98, 97]. The measure can be efficiently computed by Monte-Carlo-Simulations.

A different approach is presented by Hahn *et al.* [52, 53, 54], who introduce empirical controllability and observability Gramians in order to quantify the degrees of input-to-state and state-to-output nonlinearity respectively. Input-to-state nonlinearity refers to the mapping from input to state $u \to x$ in this context, while state-to-output nonlinearity refers to the mapping from the initial state to the output $x_0 \to y$. The chosen setup in [52, 53, 54] is discrete-time, and the empirical Gramians can be directly computed from process data. In order to get a measure for nonlinearity, the empirical Gramians from the nonlinear system are compared to Gramians from a linear model.

In this work, we will concentrate on deterministic signals and systems. It should nevertheless be mentioned that there are also linearity tests and measures for stochastic processes. These can also be used for nonlinearity assessment of dynamic systems, as is for instance done in [125, 127, 126, 128] using correlation analysis.

It is important to note that all nonlinearity measures reviewed so far are only defined for stable systems. This is due to different reasons. The dynamic operator-based nonlinearity measures discussed are (in general) only well-defined for stable systems as otherwise the used gain (or inner product) expressions are invalid. The steady-state measures assume that an asymptotically constant input implies an asymptotically constant output. Also, the empirical Gramians can only be defined for stable systems, and the same is true for the auto- and cross-correlations of input and output data used in statistical analysis. One way for nonlinearity analysis to overcome the restriction to stable systems is the framework of the gap metric [39]. Nonlinearity measures based on the gap metric and other approaches amenable to unstable systems are discussed in the section on control-relevant nonlinearity assessment.

As can be seen from the literature review, there are many different ideas how to define the degree of nonlinearity of a system. The approach presented in this work will take as a

starting point the nonlinearity measures (1.2) and (1.3). Although using these, and related, nonlinearity measures allows to apply the results of this thesis to finite-gain stable systems only, the nonlinearity measures exhibit two key feature which will be of central importance:

1. The input-output approach provides a very general setup to dynamic systems modeling and control. State-space techniques have been proven to be useful for many control-related problems, and it will be pointed out what the input-output results mean when specialized to state-space systems were adequate. However, the evolution of state-space systems will not be our primary concern, and, for instance, the influence of initial conditions on a systems behaviour will not be considered in this thesis. By the choice of an input-output approach, we will obtain results that are valid in a more general context, and we are able to draw the most general picture of the underlying ideas.

2. The analysis and synthesis of nonlinear systems is done in a non-global way. By their structural properties, there is no difference between local and global analysis and synthesis for linear systems, and there is no need to bother about we do the one or the other. But nonlinear systems may have very different properties in different regions in the state or signal space. Taking into account the behaviour in region B for doing controller design in region A, for instance, may lead to unnecessary restrictions. For example, if negative input flow rates are allowed in an otherwise stable CSTR, the resulting behaviour may be unstable [99], while from the physical context, a negative input flow will never happen. Furthermore, the validity of a given model may be restricted to the conditions under which it has been derived. Thus, we will use methods that always allows the engineer to specify a physical meaningful range for the input signals, the output signals, or the state variables.

A third important property of the nonlinearity measure (1.2) is its role in the derivation of linear models for nonlinear systems, as discussed next.

Linear Models for Nonlinear Systems. Question (2), “how can a good linear model be obtained”, is already more specific than question (1), “is a dynamic system far from linear or close to linear”: we do not merely ask to assess the degree of nonlinearity, but ask for a good (or the best possible) linear model for a nonlinear system. Of course there is a strong link between nonlinearity assessment and linear modeling for nonlinear systems. In fact, the gain-based nonlinearity measures discussed above just quantify the disparity, in a certain sense, between the nonlinear system and a best linear model.

But before discussing this relationship in detail, it is important to render the scope of our interest more precise. Almost all discussed nonlinearity measures require a detailed nonlinear reference model. Although some of them can in principle also be used solely based on process data, they are not especially tailored to problems that typically occur in identification practice like measurement noise, restricted number of experiments, for example due to large time constraints, or the necessity for “process-friendly” test inputs. Consequently, the question of linear modeling techniques for nonlinear systems can be divided into two categories. The first class of techniques solve the problem of *identification* of linear models, i.e. the problem of directly determining a linear model from process data and at the same time deriving a quality measure or confidence bounds for the model. The

second kind of techniques help to derive a simplified linear model from a given detailed nonlinear reference model, and can be described by the term model *complexity reduction*. Both topics have received considerable interest over the last decade. The identification of a best linear model for a nonlinear system has for instance been treated in [27, 112, 92]. The counter-intuitive examples given Enqvist and Ljung [33, 34] show that there is indeed the need to have techniques in system identification to deal with nonlinear plants. The focus of this thesis will however be on the model complexity reduction problem. That is, given a detailed nonlinear reference model, how can the quality of a given linear model be evaluated and how can a good, or the best, linear model be derived?

As pointed out above, the nonlinearity measures based on system gains are very closely related to this question. Indeed, the second nonlinearity measure from Desoer and Wang (1.1) corresponds to the distance between the nonlinear system N under consideration and a best linear approximation G^* defined by [26]

$$\sup_{u\in\mathcal{U}} \frac{||Nu - G^*u||}{||u||} = \tilde{\delta}_N^{\mathcal{U}} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{||Nu - Gu||}{||u||}. \tag{1.5}$$

Instead of considering the best linear model G^*, the expression

$$||N - G|| = \sup_{u\in\mathcal{U}} \frac{||Nu - Gu||}{||u||} \tag{1.6}$$

can be used to assess the quality of *any* model G. The expression $||N - G||$ corresponds to the gain (or induced norm) of the error system $E = N - G$. For instance, Kihas and Marquez assess the quality of a nonlinear system's local linearization with this measure [79]. But one can also derive the *best* linear approximation G^* of a given system in this setup. This direction of research will be further developed in this thesis for continuous-time systems. For discrete-time piecewise linear systems, an upper bound on the modeling error $||N - G||_\infty$ can be given [102]. In [94], the best linear model with respect to the l_∞-norm is given for discrete-time bi-gain systems, that are linear systems where the input matrix depends on the sign of the input. The existence of a best linear model for nonlinear finite impulse response filters is proven in the same paper. It is also of interest to study the achievable approximation quality for only one given pair of input and output signals, which is done in [90, 95, 91]. It can be shown that there are situations in which the best linear model defined this way is better suited for controller design than the model obtained by linearization around an equilibrium (or trajectory) [95]. A different example for this fact will be given later in this thesis in Section 4.2.

Mäkilä also introduces the notion of *near-linearity* [92]. A nonlinear system is called nearly linear if there exists a linear model G and a non-negative number ε such that

$$||Nu - Gu||_\infty \leq \varepsilon \ \forall u \in l_\infty \tag{1.7}$$

and in this case, the model G is called the *companion* of N. This definition corresponds to requiring that Desoer and Wang's nonlinearity measure (1.5) be finite for $\mathcal{U} = l_\infty$. The companion of a nonlinear model is unique [92]. It can be thought of as a complement (or even, opposite) of the local linearization of a nonlinear system: The derivative of an operator is the best local model in the sense that the output error, normalized by the

input, vanishes for small (in the norm) input signals. On the other hand, as the companion is defined by a bounded (not-normalized) output error for the entire signal space, the *normalized* output error must tend to zero for large signals.[4] Consequently, the Fréchet derivative and the companion are different in general [92]. A necessary condition for a strictly causal nonlinear finite impulse response (NFIR) system $y_{k+1} = f(u_k, u_{k-1}, \ldots, u_{k-N})$ to be nearly linear is that f may grow at most linearly in its arguments. If the companion exists, its coefficients can be calculated from f. Moreover, any nearly linear NFIR system is bounded-input-bounded-output stable and a controller stabilizes a nearly linear NFIR system if and only if it stabilizes the companion, in the BIBO sense with respect to l_∞. The concept of near-linearity is generalized to unstable systems in [93] using a factorization, and the issues of stabilization and identification are considered.

For the purpose of model complexity reduction, the inner product for operators defined in [97] turns out to be very useful. The key feature is that, equipped with a scalar product, a nonlinear operator can be projected on a linear subspace of linear systems, thus providing an easy characterization of the best linear model, which moreover is unique [98]. As mentioned in the context of nonlinearity measures, the weakness of this approach is that the corresponding norm derived from the inner product is not consistent with the signal norms (i.e., it is not an upper bound on any induced norm), and thus the measure of model quality turns out not to be helpful for stability analysis.

All methods to derive linear models for nonlinear systems are based on signal norms. The output error $e = Nu - Gu$ between the nonlinear system and its linear model is then either taken directly [26, 92] as a measure of the model quality, or it is used to define an inner product [97, 98], or a norm [26, 102, 94, 79] for operators. In this thesis, we will use operator norms to assess the model quality because, firstly, it is the most common way, and secondly, operator norms can (and will) be used to derive stability conditions.

So far, linearity was considered as a system property, and results on good models for nonlinear systems in terms of a small prediction error have been reviewed. Next we want to address the question whether or not the nonlinearity assessment of a plant gives a hint on the applicability and suitability of linear control strategies.

Control-relevant Nonlinearity Assessment. It has been recognized in the literature that, besides factors like the dynamic order and interaction, process nonlinearity is one of the main factors that determine the control problem difficulty [100]. In an industrial context, one tries to avoid nonlinear controller design because it requires more manpower and knowledge in the development process and it may require more complex (and therefore expensive) hardware and it may imply high demands on the operators at the stage of implementation and operation. Given a nonlinear plant, one is therefore interested to study in how far the controller design task is complicated by the nonlinear behaviour of the process.

[4] Interestingly, only for these two limiting cases the best linear model, i.e. the local linearization and the companion respectively, can be shown to be unique. Note that, if the nonlinear system N with $N0 = 0$ is continuous at the origin, any continuous model with $G0 = 0$ will imply a locally vanishing absolute error $\|Nu - Gu\|_\infty$. On the other hand, if N has finite gain, any finite-gain stable model G guarantees that the normalized error is bounded, i.e. the error system has finite gain. These relations once more show the complementarity of the derivative and the companion.

More specifically, one wants to find out whether linear controller design is suitable despite the nonlinear process behaviour.

In order to find an answer to this question, it might not be sufficient to quantify the nonlinearity of the input-output behaviour of the plant, due to the following two reasons. Firstly, it is not easy to judge from nonlinearity assessment of the plant alone whether the nonlinearity has a positive or negative effect on the closed loop. (One might, for instance, assign the systems $\dot{y} = -y - y^3 + u$ and $\dot{y} = -y + y^3 + u$ the same degree of nonlinearity, while their nonlinear terms have opposite effects on closed loop stability.) Secondly, the complexity of the control problem will in general depend on the performance objective, as has been demonstrated before [125, 127, 63, 59, 122]. For example, the disturbance attenuation problem may be more difficult for input disturbances than for output disturbances [59]. The performance criterion may render the control problem both, more or less nonlinear. Stack and Doyle [125] give an example of a nonlinear CSTR for which a linear controller achieves satisfactory performance. Fuller shows for a double integrator with a relay at the input that a linear controller performs almost as well as a nonlinear controller [37]. By virtue of variable transformations, it is easy to give examples that show that the optimal controller for a nonlinear plant can even be linear, given a nonlinear performance objective.[5] But contributions in the literature show that such a situation does not only occur in academic examples. Luenberger [87] provides for instance an economic control problem, where nonlinear dynamics together with a nonlinear objective lead to a linear optimal control law. On the other hand, even linear models may require a nonlinear controller, depending on the performance criterion. For example in diabetes control, low Glucose levels are more severe than higher Glucose levels, which results in an asymmetric performance criterion. As a result, nonlinear control will give a better performance than linear control even for a mildly nonlinear system [60]. Other examples for situations in which nonlinear controllers may be superior to linear controllers, even in presence of a linear plant, are L_1-optimal control [129], or special cases of stabilization of uncertain linear systems [103, 11].

The aim of control-relevant nonlinearity assessment is thus to assess whether linear control is adequate with respect to closed loop stability and performance. As closed-loop performance is very difficult to evaluate, many authors tackle the problem indirectly by assessing the nonlinearity of a suitable controller. As control-relevant nonlinearity assessment is done to avoid unnecessary nonlinear controller design, a primary goal is to come to conclusions *without* having to design a nonlinear controller.

The notion of control-law nonlinearity was introduced by Guay *et al.* [46, 42]. They acknowledge that the impact of plant nonlinearity on closed-loop stability and performance depends on the degree of nonlinearity of the process, the range and direction of operation and the choice of controller. As the inverse process steady-state map can be used to achieve

[5] Consider for example the nonlinear plant $\dot{y} = \frac{-y}{1+3y^2} + \frac{1}{1+3y^2}u^3$ and the non-quadratic integral performance objective $J = \int_0^\infty y^6 + 2y^4 + 2y^2 - 2yu^3 + u^6 \mathrm{d}\tau$. This problem is by the variable transformation $z = y + y^3$, $v = u^3 - y$ equivalent to the problem $\dot{z} = v$, $J = \int_0^\infty z^2 + v^2 \mathrm{d}\tau$. The solution to the last problem is known to be $v = -z$, from which follows $u^3 - y = -(y + y^3)$ and finally the linear optimal control law $u = -y$ for the original problem.

perfect set point tracking, it is analyzed by the curvature measure in order to obtain a measure of control law nonlinearity. In [47, 42], the approach of the control law nonlinearity measure is extended to a nonlinear interaction measure representing a generalization of the *relative gain array (RGA)* [12]. A drawback of both methods is that they capture only static effects and that they are defined only for stable systems.

Stack and Doyle [127] form the notion of control-relevant nonlinearity assessment, and give an approach that takes the plant dynamics and the performance objective into account. In order to leave most flexibility with respect to the performance criterion, they consider the classical optimal control problem with integral performance criterion. In order to circumvent the derivation of the exact solution for the optimal state feedback controller, the first order necessary conditions for optimal control [104, 82, 6, 15] are analyzed. It is noted that this system of equations, termed the "Optimal Control Structure", contains structural information about the optimal (state-feedback) control law. It is recognized that analysis of the OCS will result in an upper bound of the optimal control law nonlinearity [126]. While quantitative assessment of the OCS may therefore be misleading, the approach proves to be useful for analytic analysis of some system classes. For nonlinear processes with Hammerstein-structure, the OCS is of Wiener type and it is shown in [127] that the control-law becomes linear when the penalty on control action tends to infinity (corresponding to a cautious control action). In [126] the extension of the OCS approach to output feedback is tackled by combining the OCS and an extended K filter (EKF).

In order to more accurately characterize the optimal control law nonlinearity, the OCS approach is further developed in [122, 113, 114, 123]. When disturbances to the control system are present only in form of non-zero initial conditions, there is no difference between the optimal open-loop and the optimal closed-loop trajectories, and the trajectories can be used for nonlinearity assessment of the optimal feedback law. A further advantage of this technique is that the plant under consideration may be unstable. Using this technique, it is shown in [123] that the control-relevant nonlinearity of a nonlinear continuous stirred tank reactor (CSTR) corresponds to its open-loop nonlinearity.

As nonlinear internal model controllers (NIMC, [30]) can be derived easily, the NIMC framework represents also a way for control-relevant nonlinearity assessment. In [62, 61], the impact of performance requirements (in terms of the closed loop time constant) on controller nonlinearity is examined. While the nonlinearity measure of the IMC controller does not depend on the variation of the time constant for the considered Hammerstein example, the controller for the Wiener system shows a growing nonlinearity with increasing performance requirements. The trade-off between the use of linear control and performance is also demonstrated in [128] for an example system of Hammerstein type by assessing the achievable performance for multiple local models.

The IMC framework can also be used to directly compare the performance of linear and nonlinear controllers. In [31], two measures of control-relevant nonlinearity in terms of performance are proposed. The first measure gives the difference (in terms of weighted incremental norm over set) between a linear control system with linear controller and linear model and a nonlinear control system with linear controller and nonlinear plant. The

second definition gives the difference between a linearly controlled nonlinear plant and a nonlinearly controlled plant.

Hernjak [58] introduces a framework related to the problem of general model matching. In model matching, the problem is to design a controller such that the closed loop matches a prescribed behaviour (see e.g. [55]). Based on this idea, the general form

$$P_{\Xi}(G, H^*) = \inf_{C \in \mathcal{C}} \sup_{\xi \in \Xi} \frac{\|H(C)\xi - H^*\xi\|}{\|H^*\xi\|} \tag{1.8}$$

for a nonlinearity measure is introduced in [58], where H and H^* are the actual and desired closed loop behaviour and $\mathcal{C}$ is the set of linear stabilizing controllers. The difficulty in this framework is to choose a feasible H^*, or alternatively, to characterize the disparity between the actual and desired closed loop behaviour for the best *nonlinear* controller.

Instead of analyzing the nonlinearity numerically in the entire region of operation, the authors in [45, 44, 43] attempt to quantify the sensitivity of the closed loop performance to process nonlinearity locally at the operating point. The advantage of the approach is its computational efficiency, as the needed local linearizations and optimal controllers are easily computed. Furthermore, unstable plants can be considered. On the other hand, conclusions about a larger operating range are difficult to obtain, and the proposed interpretation of these measures as "performance sensitivity" seems misleading.

It has been said that gain-based nonlinearity measures have the disadvantage of only being defined for stable nonlinear systems. With the help of the gap metric, a distance can also be defined between two unstable systems [39]. This fact is used in [132, 133, 131] to define a nonlinearity measure for systems given in quasi-LPV (linear parameter varying) form. The nonlinearity measure is given by the maximal ν-gap between local linear models. This technique can also be used for deriving linear controllers (see below).

The approaches to control-relevant nonlinearity assessment are as diverse as the methods for nonlinearity assessment of a plant. As yet, there is no golden way to decide whether nonlinear controller design is necessary or beneficial for a given control problem, or whether a linear controller is adequate. But the discussion above makes clear that there may always be a trade-off between controller complexity and control system performance.

In this thesis, we are more interested in the way a linear controller can actually be derived for a nonlinear system than in the theoretical performance gain by using nonlinear control. The topic of control-relevant nonlinearity assessment as such will therefore only appear as one side-aspect.

Linear Control of Nonlinear Systems. If the control-relevant nonlinearity assessment indicates that linear control is adequate, the next step is to design a linear controller for the nonlinear system. In order to benefit from the preceding nonlinearity analysis, the controller design procedure should satisfy two criteria: Firstly, the controller design should not be much more involved than the controller design for a linear system. It should at least be much easier than a full nonlinear controller design, because a nonlinear controller design

is what we want to avoid. Secondly, the design procedure should guarantee stability, and ideally performance of the closed loop.

In order to guarantee closed loop stability, mainly three kinds of techniques are used in the literature: techniques based on small-gain arguments, techniques based on the passivity property, and techniques involving Lyapunov functions. The small-gain and passivity-based techniques allow to guarantee stability already by considering plant and controller separately. In order to verify (asymptotic) stability of a closed loop in state space form, Lyapunov theory is often used. Stability results that are derived using these techniques usually guarantee stability not only for one specific given nonlinear system, but for a family of nonlinear systems, characterized by a positivity, gain, or Lipschitz condition. Thus, the controller design methods using these techniques are in fact robust control methods that achieve robust stability for a whole class of nonlinear uncertainties. It must be noted that this approach includes a certain degree of *conservatism*. This still is the case if the robustness condition is *necessary and sufficient*, because robust stability is always achieved for a whole class of uncertainties. But we are only interested in stabilization of one specific member of this class, and for this member, the condition is always only *sufficient*. Of course, if all information about the nonlinearity is taken into consideration, results can be derived that are valid only for the given, known nonlinear system. The controller design can then be tailor-made to the given system, and will therefore be usually better than using robust control techniques. In fact, the conservatism in using robust control techniques is the price to pay in order to have a simple controller structure and design procedure. This remark is valid for all linear controller design methods for nonlinear systems that use robustness results.

A classic setup for linear control of nonlinear systems is the case of a linear dynamic system in feedback configuration with a memoryless nonlinear element, often referred to as the problem of absolute stability [1, 144, 145, 141, 25, 51]. Based on the classical stability results, also synthesis techniques can be found in the literature, e.g. [85, 4]. One particularly important type of static nonlinearity is input saturation, and many controller design methods in this area are related to the techniques of absolute stability [83, 84, 68].

In order to analyze robust stability in presence of nonlinear uncertainties in a more general setup than for linear dynamic systems interconnected to memoryless nonlinearities, one can consider nonlinear state space systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + \phi(x,u) \\ y &= Cx + \psi(x,u). \end{aligned} \tag{1.9}$$

By imposing different kinds of boundedness conditions, results can be obtained for practical global stabilization by state feedback [38] and output feedback [78]. Extensions of similar approaches that also guarantee L_2-performance of the closed loop are also available [5, 139]. Global approaches as those mentioned above have the disadvantage of requiring the nonlinearities to be linearly bounded, and practical systems often do not satisfy this assumption. It may therefore be necessary to perform a regional analysis and controller

design. Battilotti [7, 8] gives conditions for regional stabilizability of systems of the form (1.9) by state and output feedback in terms of Riccati equations.

Easier conditions can be based on the different variants of the small-gain and contraction mapping (incremental-small-gain) principles. For instance, using a nonlinear internal model control structure, finite gain of the closed loop can be guaranteed if $\|(N - L)Q\|_\Delta < 1$ with N the nonlinear process, L a linear model, and Q the IMC controller parameter that has been chosen based on the linear model [31]. This result is also valid for a regional controller design, the only disadvantage being that the set of reference signals can only be indirectly specified.

Doyle *et al.* [29] show for the application of a nonlinear CSTR how the linear methods based on the concept of the structured singular value can be generalized to performance synthesis for nonlinear systems.

Controllers with simple structure make up the far majority in control practice. Therefore, one is interested in applying such controllers also in presence of nonlinear processes. Morari [96] gives conditions for robust stability of linear control systems with integral control based on the steady-state gain. One conclusion is, that linear integral control cannot achieve set point tracking for nonlinear plants with input multiplicity (i.e. sign change of the steady state gain) [96]. However, if the nonlinear system is globally exponentially stable for any steady-state input and if it has no output multiplicities, the use of PI control is indeed possible [24]. For a given nonlinear system, proper function of an integral linear controller can also be shown using the bounding linear models from (1.4) [130]. The question of (locally) optimal tuning of parameters for a general controller with fixed structure is addressed by Kazantzis *et al.* [76]. Simple control structures may be theoretically well justified *and* prove useful in practice, as the application of PD/PID control with proof of stabilization for a nonlinear CSTR given in [88] demonstrates.

As for control-relevant nonlinearity assessment, the theory of optimal control also provides a possible approach to the design of linear controllers for nonlinear systems. Optimal trajectories for multiple initial conditions can be used to design suboptimal linear state feedback laws [109, 75, 115]. Although it has been demonstrated that the resulting linear controllers may perform almost as good as the optimal nonlinear controller [109, 75], no stability or performance guarantees can be obtained.

If one allows a nonlinear adaptation of a parameter of a fixed-structure linear controller, adaptive high-gain techniques can also be used for linear output feedback for nonlinear systems, as witnessed by the recent work in [105] and [16].

Rather heuristic approaches are the ones using multiple local models. Goh *et al.* [40] describe a general procedure how to design a linear controller if multiple linear models are given for the plant. The example of a fluidized catalytic cracking unit shows the practical applicability, although no rigorous justification is given for the approach. This is also the case for the methods given by Tan *et al.* [132, 133, 131]. For nonlinear systems in quasi linear parameter varying (LPV) form, the maximal ν-gap between local linear models is used in the loop-shaping design for the nominal linear model. The idea in using

quantitative feedback theory (QFT) for linear control of nonlinear plants is to capture the plant nonlinearity by linear uncertainties of the linear plant and by output disturbances [143]. However, as only a finite set of plants and disturbances are considered, again no stability and performance guarantees can be made.

As in the case of nonlinearity assessment, the primary concerns for the development of linear controller design techniques for nonlinear systems will be to use a setting that allows to treat general dynamic input/output-operators, and that allows to perform a synthesis for a specified region of operation. These two points will be the distinctive features of our approach as compared to the methods available in literature.

1.2. Structure and Contributions of the Thesis

The goal of this thesis is to establish methods

(1) to assess the suitability of a given linear model for a given nonlinear system and
(2) to derive optimal linear models for this system, as well as
(3) to design controllers for a given nonlinear system based on a given linear model.

Nonlinearity measures will be an important tool to tackle the above challenges, and they will prove to provide a rich and fruitful basis for further developments. In particular, two key features of the nonlinearity measures used here are (i) the (operator-theoretic) *input-output framework*, that constitutes a very general point of view, and (ii) the possibility of *regional*, instead of local or global analysis. The literature review has demonstrated the complexity of the problems associated with our goals, and it cannot be expected that the answers to all problems can be reduced to a single number. The importance will be to have a toolbox of compatible techniques, from which we can choose the most appropriate ones for a given problem.

Given these premises, the main contributions of this thesis are the following:

- We introduce a novel, unifying framework for nonlinearity assessment, for assessment and derivation of linear models for nonlinear systems, and for linear controller design for nonlinear systems. In particular, the derivation of an optimal linear model for a nonlinear system and the evaluation of its quality are done using the same technique. Moreover, the quality index can readily be used for the subsequent controller design based on the linear model.
- The framework is rigorously justified in a mathematical sense, giving the proof of the validity of the definition of nonlinearity measures and best linear models (Section 3.2) and the relation of optimal models to operator derivatives and state-space linearizations (Section 3.3).
- We give results to evaluate gains and nonlinearity measures (Sections 3.4 to 3.6) for special classes of systems, like systems with a given steady-state locus, memoryless systems, and systems composed of linear and nonlinear subsystems. Characterizations and the existence of the associated linear models are also given.

- Conditions for the stable control of the nonlinear system using design techniques for linear saturated systems are given that are similar to and as easy to use as techniques known from linear robust control theory (Sections 4.1 and 4.2).

This thesis is structured as follows:

Chapter 2 briefly describes the the mathematical background and notation for all of the subsequent studies.

Chapter 3 deals with the topics of nonlinearity assessment and derivation of linear models for nonlinear systems. The general framework and a family of nonlinearity measures is introduced in Section 3.1. Each nonlinearity measure corresponds to an associated setup to assess the model quality. The theoretical foundation as described above is discussed, and the results for special system classes are given. Some insight on system nonlinearity given by harmonic analysis is pointed out, followed by some remarks on the computation of nonlinearity measures and optimal linear models.

The controller design problem is then treated in Chapter 4. After stating the problem, a brief review of linear robust control principles is given. Then, an analogue development is given that leads to stability conditions and controller design procedures for all introduced model error setups. The design of linear controllers for nonlinear systems with guaranteed performance is beyond the scope of this work, but some remarks on possible approaches are given in Section 4.3. At the end of the chapter, the applicability of the method is shown on a practically relevant CSTR model.

The thesis closes with conclusions and an outlook to future research in Chapter 5.

Preliminary versions of some of the results presented here can be found in [116, 121, 120, 117, 119].

CHAPTER 2

Background and Notation

The purpose of this chapter is to introduce the basic mathematical notions and background information on signals and systems used throughout this thesis. These definitions and facts will be completed in later parts of the thesis where needed.

General Setup. In this thesis, the main focus will be on input-output properties of dynamic systems. Of course, in many practical applications, nonlinear systems are given in state-space form, and we will point out input-output properties of state-space systems where appropriate. But most of the time, we will more generally consider systems as mappings $P : \mathcal{U} \to \mathcal{Y}$ from one signal space $\mathcal{U}$ to another signal space $\mathcal{Y}$. We will assume throughout that $P0 = 0$. We want to emphasize that, if not otherwise stated, all developments given in this thesis are valid for vector valued signals and systems with multiple inputs and multiple outputs. The number of inputs is usually denoted by m, the number of outputs by n, and where needed the number of internal states by z. In that context we also want to note that we use the symbol $|\cdot|$ for the absolute value of a real number as well as for the vector norm in $\mathbb{R}^n$, which will be taken as the Euclidean norm for all examples. Also note that we will use the notation $P : u \in \mathcal{U} \mapsto y \in \mathcal{Y}$ for the mapping $P : \mathcal{U} \to \mathcal{Y}$ that maps u to y. The rest of this paragraph briefly recalls basic definitions as found in [140, 25, 136, 135], and can be skipped by the reader who is familiar with them.

In this work, we will consider as signal spaces the *Lebesgue-spaces* L_p^n that are defined to be the sets of measurable functions $u : \mathbb{R}^+ \to \mathbb{R}^n$ for which $\|u\|_p = \left(\int_0^\infty |u(t)|^p \, \mathrm{d}t\right)^{1/p} < \infty$ for $1 \leq p < \infty$ and $\|u\|_\infty = \operatorname{ess\,sup}_{t\geq 0} |u(t)| < \infty$ for $p = \infty$ respectively where $\mathbb{R}^+$ denotes the set of non-negative real numbers. Note that by equivalence of norms in $\mathbb{R}^n$ the Euclidean norm can be substituted by any other norm in the definition of the Lebesgue spaces without affecting them. For convenience we will drop the subscript in $\|u\|_p$ throughout, and we will only write it out to avoid ambiguity or if a result is given that only holds for a particular p.

The *truncation* u_T of a signal u is defined as

$$u_T(t) = \begin{cases} u(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases}$$

and the *extended Lebesgue-spaces* L_{pe}^n are defined as the sets of all measurable functions u for which $u_T \in L_p^n$ for all $T > 0$. The definition of the extended spaces allows to treat

persistent and even unbounded signals. While extended spaces are linear spaces[1], but they are no normed spaces.

As remarked in [140], the definition of extended signal spaces is in particular useful for causal systems. A mapping $P : L_{pe}^m \to L_{pe}^n$ is said to be *causal* if $(Pu)_T = (P(u_T))_T$, i.e. if two inputs are equal up to time T, then so are the corresponding outputs. In the input-output setting, a system is said to be (L_p-)*stable* if for every $u \in L_p^m$ the output satisfies $y = Pu \in L_p^n$ and it is said to be *L_p-stable with finite gain* (or short: to have finite gain) if there exists a finite constant γ such that $\|(Pu)_T\| \leq \gamma \|u_T\|$ for all $u \in L_{pe}^m$, $T > 0$, or, equivalently, if the induced operator norm (also called the *gain*) $\|P\| = \sup_{u \in L_{pe}^m, u_T \neq 0, T>0} \frac{\|(Pu)_T\|}{\|u_T\|}$ is finite. A system is said to be *finite incremental gain stable* if $\|P\|_\Delta = \sup_{u_1,u_2 \in L_{pe}^m, u_{1,T} \neq u_{2,T}, T>0} \frac{\|(Pu_1 - Pu_2)_T\|}{\|(u_1 - u_2)_T\|} < \infty$ and $\|P\|_\Delta$ is called the *incremental L_p-gain* of P. Note that if $P0 = 0$ then finite incremental gain stability implies finite gain stability ($\|P\|_\Delta \geq \|P\|$). We recall that an operator $P : L_{pe}^m \to L_{pe}^n$ has finite (incremental) gain on L_p if and only if it has finite (incremental) gain on L_{pe}, and the respective gains are equal [140].

Gains over Sets. It is often not useful to consider the system gains for nonlinear systems as defined above, as those definitions allow for signals that are not physically realizable [99, 20]. In practical applications, signals are always restricted in amplitude, energy or power. Therefore, for a subset $\mathcal{U} \subseteq L_{pe}^m$ we define the *L_p-gain of P over $\mathcal{U}$* as[2]

$$\|P\|^{\mathcal{U}} = \sup_{u \in \mathcal{U}, u_T \neq 0, T>0} \frac{\|(Pu)_T\|}{\|u_T\|}$$

and the *incremental L_p-gain of P over $\mathcal{U}$* as

$$\|P\|_{p,\Delta}^{\mathcal{U}} = \sup_{u_1,u_2 \in \mathcal{U}, u_1 \neq u_2, T>0} \frac{\|(Pu_1 - Pu_2)_T\|}{\|(u_1 - u_2)_T\|}.$$

We will again drop the subscript p when it is not ambiguous to do so. Whenever in the following we give results that hold for a certain value of p only, we will denote the corresponding (incremental) gain by $\|P\|_p$ and $\|P\|_{p,\Delta}$ respectively. Note that the gains and incremental gains over $\mathcal{U}$ satisfy the scalar multiplication property of norms and the triangle inequality, but they are in general no norms, as $\|P\|^{\mathcal{U}} = 0$ does not imply $P = 0$. We will call two operators P_1 and P_2 *identical on $\mathcal{U}$* if $\|P_1 - P_2\|^{\mathcal{U}} = 0$, and say P_2 is a *perfect model* for P_1 on $\mathcal{U}$ (and vice versa). Gains of systems will play a very important role in

[1] If the truncation of two signals has a bounded norm, also the truncation of a superposition or multiple of these signals is bounded.

[2] In the case that $u \in \mathcal{U}$ implies $u_T \in \mathcal{U}$ we can define the set $\mathcal{U}_T = \{u_T \in L_p : u \in \mathcal{U}, T \geq 0\}$ to get the identity

$$\|P\|_p^{\mathcal{U}} = \sup_{u \in \mathcal{U}, u_T \neq 0, T>0} \frac{\|(Pu)_T\|}{\|u_T\|} = \sup_{u \in \mathcal{U}_T, u \neq 0} \frac{\|Pu\|}{\|u\|}$$

and

$$\|P\|_{p,\Delta}^{\mathcal{U}} = \sup_{u_1,u_2 \in \mathcal{U}, u_1 \neq u_2, T>0} \frac{\|(Pu_1 - Pu_2)_T\|}{\|(u_1 - u_2)_T\|} = \sup_{u_1,u_2 \in \mathcal{U}_T, u_1 \neq u_2, T>0} \frac{\|Pu_1 - Pu_2\|}{\|u_1 - u_2\|}.$$

These relationships show that we may consider elements of the Banach spaces L_p instead of elements of the extended spaces L_{pe}. This fact will be used at some points in this work.

this work, as they allow not only to tell whether two systems have an identical behaviour, but they also allow to quantify how close their behaviour is if they are not identical.

In the sequel, we will frequently use the expression $\gamma = \sup_{x\in X} \frac{a(x)}{b(x)}$ with a slight abuse of notation as an equivalent to $\gamma^* = \inf \Gamma = \inf \{\gamma \in \mathbb{R} | a(x) \leq \gamma b(x), \forall x \in X\}$ and both expressions are defined to be $\gamma^* = \infty$ if the set Γ is empty. This definition thus allows us to have a well-defined expression also in the case $b(x) = 0$ for some $x \in X$. Note that, using this convention, $\|P\|^{\mathcal{U}} < \infty$ and $0 \in \mathcal{U}$ implies $P0 = 0$.

The composition (series connection) of two operators $P_1 : L^m_{pe} \to L^k_{pe}$ and $P_2 : L^k_{pe} \to L^n_{pe}$ will be denoted by $P_2 \circ P_1 = P_2 P_1 : u \in L^m_{pe} \mapsto P_2(P_1 u) \in L^n_{pe}$. With $P\mathcal{U}$ we will denote the set $P\mathcal{U} = \left\{y \in L^n_{pe} : y = Pu, u \in \mathcal{U}\right\}$ and for a given number $\rho \in \mathbb{R}^+$ we denote by $\rho\mathcal{U}$ the set $\rho\mathcal{U} = \{\rho u | u \in \mathcal{U}\}$.

At this point, a remark concerning the usage of the different signal and operator norms is in order. In a mathematical sense, the p-norms lead to the definition of different signal sets, and a signal in L_{p_1} is not necessarily contained in a different set L_{p_2}, $p_2 \neq p_1$. While the different structure of these sets may be of mathematical interest, in an engineering context, we are more often interested in the way how the system gains are quantified. This is due to the fact that different control objectives can be associated with different norms. For instance, the L_∞-gain of a system is a measure for the peak of the output signal, given the peak of the input, whereas the L_2-gain represents the amplification or attenuation of the energy or average power contained in a signal. Therefore, in what follows we will not pay so much attention on which signals are contained in which set. Consequently, in many situations we will assume that the input signals are elements of the signal set $L_\infty \cap L_p$, or $L_{\infty e}$, which is a subset of L_{pe} for any $p \geq 1$. In particular, we will often consider sets $\mathcal{U}(V) \subseteq L^m_{\infty e} \subseteq L^m_{pe}$ of input signals defined by $\mathcal{U}(V) = \{u \in L^m_{\infty e} : u(t) \in V \subseteq \mathbb{R}^m$ for all $t \geq 0\}$ (note that if V is bounded, then $\mathcal{U}(V) \subseteq L^m_\infty$).

Dissipation Theory and Gains of State-Space Systems. Dissipation theory provides a framework to relate input-output properties of nonlinear systems to their state-space representation (if there is one)

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{2.1}$$

with $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^z$ and $y(t) \in \mathbb{R}^n$. When considering those systems, we will throughout assume zero initial conditions and without loss of generality that $f(0,0) = 0$ and $h(0,0) = 0$. We will furthermore assume that for a given set $\mathcal{U}$, the solution of $x(t)$ and $y(t)$ is defined for all $u \in \mathcal{U}$ and $t \geq 0$. For a system given in state space form (2.1), a finite gain can be characterized via dissipation inequalities for $p < \infty$. The following definitions and facts closely follow the exposition in [135, 134]. But in contrast to the standard material, we give regional variants of the results in order to characterize the gains over sets. As a basic tool, we need the notion of dissipativity.

D 2.1. A state space system of the form (2.1) is said to be *dissipative* on $X \times V \subseteq \mathbb{R}^z \times \mathbb{R}^m$ with respect to a supply rate $s : (u, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto s \in \mathbb{R}$ if there exists a storage

function $S : \mathbb{R}^z \to \mathbb{R}^+$ such that for all u

$$S\left(x(T)\right) - S\left(x(0)\right) \leq \int_0^T s\left(u(t), y(t)\right) \mathrm{d}t \qquad \text{for all } T \geq 0. \tag{2.2}$$

Equation (2.2) is called an integral dissipation inequality. If the storage function S is continuously differentiable, $S \in C^1$, then the dissipation inequality can be written in its differential form

$$\dot{S} = \frac{\partial S}{\partial x} f\left(x(t), u(t)\right) \leq s\left(u(t), y(t)\right) \text{ for all } x(t) \in X, u(t) \in V. \tag{2.3}$$

This last form is much easier to verify, as the solution of the state space system does not need to be known. Under the condition of zero initial condition, the connection between dissipativity and finite gain on a set $\mathcal{U}(V) = \{u \in L_{pe}^m : u(t) \in V \, \forall T \geq 0\}$ can be established as follows [135].

P 2.2. *Let a nonlinear operator be given by the state-space system (2.1). Then* $\|P\|_p^{\mathcal{U}(V)} < \infty$ *if and only if there is a* $\gamma \geq 0$ *and a set* X*, invariant under the condition* $x(0) = 0$ *and* $u(t) \in V$*, such that the state-space system is dissipative on* $X \times V \subseteq \mathbb{R}^z \times \mathbb{R}^m$ *with respect to the supply rate*

$$s(u, y) = \gamma^p |u|^p - |y|^p$$

and admits a storage function with $S(0) = 0$*. Furthermore,* $\|P\|_p^{\mathcal{U}(V)}$ *is the largest lower bound of all such* γ*.*

The connection given above will turn out to be useful later to determine the gain over set of a system that is given in state-space form.

Although our main focus is on input-output properties of nonlinear systems, it is interesting to known that finite gain stability implies asymptotic stability of a state-space system. The equivalence of local exponential stability and input-output stability for small-amplitude and small-in-the-norm signals are given in [138, 19]. As we consider most of the time systems that are driven with inputs in a certain set, it is of interest to consider stability on the region of the state space that is reachable under these inputs. A connection between input-output stability and asymptotic stability is given by the following regional version of a result given in [135].

P 2.3. *Consider a zero-state detectable state-space system (2.1), i.e. a system for which* $x \to 0$ *whenever* $y \to 0$ *and* $u = 0$*. Let the system admit a positive definite* C^1 *storage function* $S(x)$ *to the supply rate* $s(u, y) = \gamma^p |u|^p - |y|^p$ *on the invariant set* X *under* $u(t) \in V$ *and* $x(0) = 0$*. If* X *is unbounded, we furthermore require that* S *be radially unbounded. Then* $x = 0$ *is an asymptotically stable equilibrium of* $\dot{x} = f(x, 0)$ *and the level sets* $\{S(x) < c\} \subseteq X$ *are contained in its region of attraction.*

The last relationship justifies the use of the input-output based approach even in cases where one might want to require asymptotic stability.

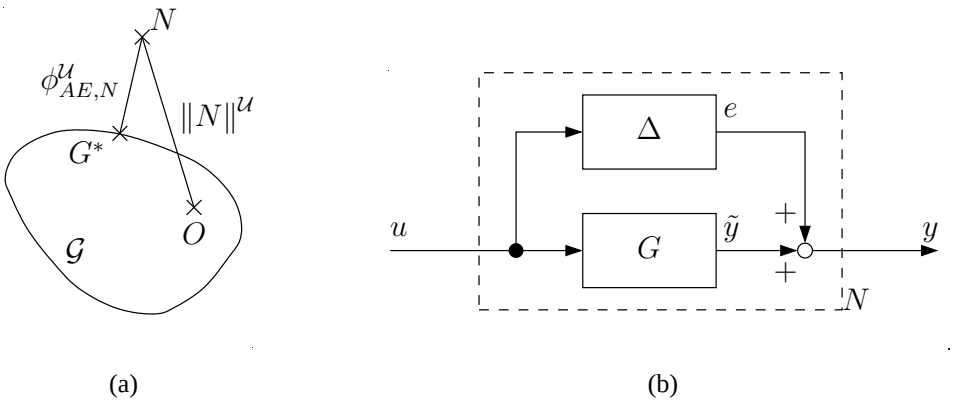
CHAPTER 3

Nonlinearity Measures and Linear Models for Nonlinear Systems

This chapter is devoted to the introduction and theoretical foundation of a unifying framework for nonlinearity measures, model assessment, and linear modeling of nonlinear systems. First, we motivate and define different nonlinearity measures. It is shown how these nonlinearity measures naturally characterize best linear models and how they give rise to general model quality indices. Some first properties of the nonlinearity measures are shown before we study the well-posedness of the proposed definitions in Section 3.2. It is of course of interest to ask how the nonlinearity measures and local linearizations of state-space systems and derivatives of the corresponding operators are related. These questions are answered in Section 3.3. After verifying that the chosen setup is well justified, we take advantage of the special structural properties of memoryless systems and systems exhibiting a certain steady-state behaviour to derive analytic results concerning the best linear models and nonlinearity measures of those systems in Sections 3.4 and 3.5. Linear models and upper bounds on the nonlinearity measures of composite systems such as systems of Hammerstein and Wiener structure are given in Section 3.6, before we extend the well-known [2] results on nonlinearity assessment by harmonic analysis. The chapter closes with the consideration of computational methods.

3.1. A Unified Framework for Nonlinearity Assessment

Nonlinearity measures have been introduced in the literature in order to quantify the accuracy of linear models for nonlinear systems. In most input-output approaches to nonlinearity assessment, the output error $e = Nu - Gu$, i.e. the difference between the nonlinear process output $y = Nu$ and the output $\tilde{y} = Gu$ of a linear model plays a central role, see also the discussion in the introduction, Section 1.1. A common approach is to quantify how small the gain of the corresponding error system, $\|N - G\|$, can be made by a suitable choice of the linear model G. As mentioned in the introduction, it is often not useful to perform a global analysis of nonlinear systems. In order to characterize an operating regime for the nonlinear system N, the error gain can be considered over a certain set $\mathcal{U}$ of admissible input signals only. Then the known definition of a nonlinearity measure [26, 2, 116]



(a) (b)

F 3.1. Different interpretations of the nonlinearity measure $\phi^{\mathcal{U}}_{AE,N} = \inf_{G\in\mathcal{G}_{m,n}} \|N - G\|^{\mathcal{U}}$: (a) as the distance of the nonlinear system N from the set of linear systems $\mathcal{G}_{m,n}$, and (b) as the minimal gain of the (nonlinear) error system Δ that corresponds to a linear model G, such that $N = G + \Delta$.

is obtained as

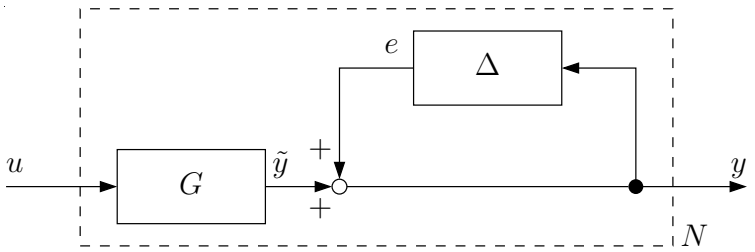
$$\phi^{\mathcal{U}}_{AE,N} \triangleq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} = \inf_{G\in\mathcal{G}_{m,n}} \|N - G\|^{\mathcal{U}} \tag{3.1}$$

where $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ is a causal, finite-gain stable (on $\mathcal{U}$) mapping. The symbol $\mathcal{G}_{m,n}$ denotes a set of linear causal transfer operators, and will be discussed in greater detail later in this section. A linear model G^* is called a best linear model if the minimum is achieved by G^*, i.e. $\|N - G^*\|^{\mathcal{U}} = \inf_{G\in\mathcal{G}_{m,n}} \|N - G\|^{\mathcal{U}}$ [26, 94]. Moreover, $\phi^{\mathcal{U}}_{AE,N}$ can be interpreted as the distance[1] of the element N from the set $\mathcal{G}_{m,n}$ of linear systems, see Fig. 3.1a).

The distance-from-set interpretation of the nonlinearity measure is very interesting and important in order to see the mathematical structure of this definition. However, we want to point out a different, system-theoretic interpretation, namely the similarity of this definition to the typical setups in linear robust control. The expression in (3.1) can also be interpreted as the gain of an error system $\Delta = N - G^*$, where the nonlinear process $N = G^* + \Delta$ is a parallel connection of its (best) nominal linear model G^* and an error term Δ, see Fig. 3.1b). As the definition of Δ corresponds to an additive error, we will from now on call the well known nonlinearity measure in (3.1) the *additive error nonlinearity measure* (AE-NLM) of N on $\mathcal{U}$.

A similar, relative-error like nonlinearity measure is obtained if the norm of the error signal is normalized by the output of the nonlinear system instead of its input [2, 57]. The

[1] In fact, the set of finite-gain stable (not necessarily linear) mappings together with the system gain as a norm is a Banach space [140]. The same is true when the gain on $\mathcal{U}$ is used as norm and two systems are considered identical when they are identical on $\mathcal{U}$, so the expression $d(N_1, N_2) = \|N_1 - N_2\|^{\mathcal{U}}$ defines a distance. The set $\mathcal{G}_{m,n}$ of linear operators is a subset of this Banach space, more precisely it is a linear subspace, and the distance of N from $\mathcal{G}_{m,n}$ is given by $d(\mathcal{G}_{m,n}, N) = \inf_{G\in\mathcal{G}_{m,n}} d(G, N) = \inf_{G\in\mathcal{G}_{m,n}} \|N - G\|^{\mathcal{U}}$. Note that, as no inner product is defined for general nonlinear operators, projection methods cannot be used to characterize (i.e. examine existence and uniqueness of) a best linear model G^*.



F 3.2. Representation of a general nonlinear system N as the interconnection of a nominal linear model G^* and a nonlinear inverse multiplicative output error Δ. The nonlinearity measure $\phi^{\mathcal{U}}_{IMOE,N}$ from eq. (3.2) can be interpreted as the minimal gain of Δ.

corresponding nonlinearity measure reads

$$\phi^{\mathcal{U}}_{IMOE,N} \triangleq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}. \tag{3.2}$$

This second nonlinearity measure can also be interpreted as the gain of an error system Δ, where the real plant N is represented by an interconnection of Δ and a nominal linear model G^*. From the definition in (3.2) it becomes apparent that the output of the system Δ must be $Nu - Gu$ and its input is Nu. This setup corresponds to the interconnection depicted in Fig. 3.2, which we will refer to as the inverse multiplicative output error setup. We therefore call the quantity defined by (3.2) the *inverse multiplicative output error nonlinearity measure* (IMOE-NLM) of N on $\mathcal{U}$.

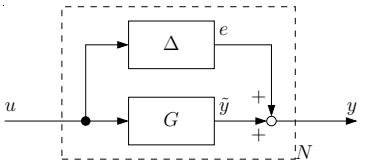
Similar definitions can be also posed for other interconnection structures of the nominal linear model G and the error term Δ. A collection of interconnections together with the corresponding definitions of the nonlinearity measures is given in Fig. 3.3. The common setup for all measures is the following. For a given linear model G, an error system Δ is defined such that the interconnection of G and Δ reproduces the behaviour of the nonlinear system N. An index for the quality of the model G is given by the gain of the error system $\|\Delta\|^{\mathcal{X}}$, where $\mathcal{X}$ is the set of signals that may occur as input to Δ, when u is restricted to the given set of admissible inputs $\mathcal{U}$. Both, Δ and $\mathcal{X}$ depend on the chosen type of interconnection. A linear model is called an optimal, or best, linear model, if it minimizes this error. This minimal error gain

$$\phi^{\mathcal{U}}_{N} = \inf_{G\in G_{m,n}} \|\Delta\|^{\mathcal{X}} \tag{3.3}$$

corresponds to the quality index of the best linear model, and is at the same time a measure for the nonlinearity of the process.

The expressions of the nonlinearity measures are defined for causal mappings N having finite gain on $\mathcal{U}$. They can also be well-defined in some cases for unstable N (consider for example the trivial case of an unstable linear system). In this work, however, we will, with few exceptions, restrict ourselves to stable systems with finite gain.

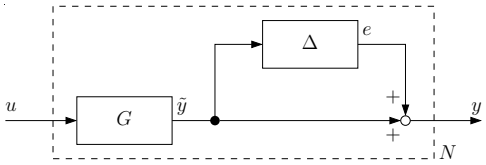
a)



Additive error
nonlinearity measure (AE-NLM)

$$\phi^{\mathcal{U}}_{AE,N} \triangleq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|u_T\|}$$
$$\Delta = N-G$$

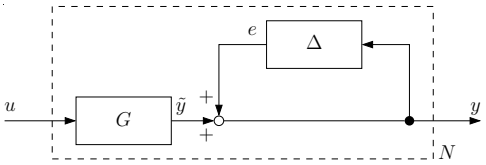
b)



Multiplicative output error
nonlinearity measure (MOE-NLM)

$$\phi^{\mathcal{U}}_{MOE,N} \triangleq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|}$$
$$\Delta = NG^{-1}-I$$

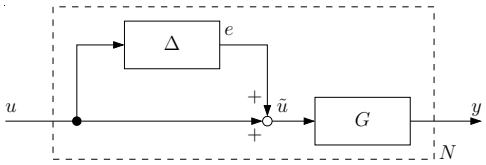
c)



Inverse multiplicative output error
nonlinearity measure (IMOE-NLM)

$$\phi^{\mathcal{U}}_{IMOE,N} \triangleq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$$
$$\Delta = I-GN^{-1}$$

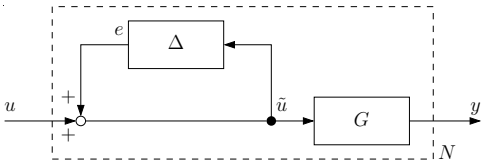
d)



Multiplicative input error
nonlinearity measure (MIE-NLM)

$$\phi^{\mathcal{U}}_{MIE,N} \triangleq \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|u_T\|}$$
$$\Delta = G^{-1}N-I$$

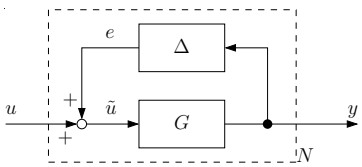
e)



Inverse multiplicative input error
nonlinearity measure (IMIE-NLM)

$$\phi^{\mathcal{U}}_{IMIE,N} \triangleq \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\left\|\left(G^{-1}Nu\right)_T\right\|}$$
$$\Delta = I-N^{-1}G$$

f)



Feedback error
nonlinearity measure (FE-NLM)

$$\phi^{\mathcal{U}}_{FE,N} \triangleq \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|(Nu)_T\|}$$
$$\Delta = G^{-1}-N^{-1}$$

F 3.3. Different possible representations of a general nonlinear system N as the interconnection of a nominal linear model G and an error system Δ, together with the definitions of the corresponding nonlinearity measures and the informal definitions of the error systems.

All nonlinearity measures are by definition bounded below by zero. Also, some trivial upper bounds can be given.

P 3.1. *Assume the system $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ is finite gain stable on $\mathcal{U}$. Then the following statements are true:*

(i) $\phi^{\mathcal{U}}_{AE,N} \le \phi^{\mathcal{U}}_{IMOE,N} \cdot \|N\|^{\mathcal{U}}$

(ii) $\phi^{\mathcal{U}}_{AE,N} \le \|N\|^{\mathcal{U}}$, $\phi^{\mathcal{U}}_{IMOE,N} \le 1$

(iii) $\phi^{\mathcal{U}}_{MIE,N} \le 1$, $\phi^{\mathcal{U}}_{MOE,N} \le 1$

(iv) *If $\phi^{\mathcal{U}}_{IMIE,N}$ is finite, then $\phi^{\mathcal{U}}_{IMIE,N} \le 1$.*

(v) $\phi^{\mathcal{U}}_{MIE,N} \le \phi^{\mathcal{U}}_{FE,N} \cdot \|N\|^{\mathcal{U}}$.

P . (i) It is easy to see that

$$\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|} = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|u_T\|}\frac{\|u_T\|}{\|(Nu)_T\|} \ge \frac{1}{\|N\|^{\mathcal{U}}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|u_T\|}$$

and the relationship $\phi^{\mathcal{U}}_{IMOE,N} \ge \phi^{\mathcal{U}}_{AE,N} / \|N\|^{\mathcal{U}}$ follows.

(ii) By considering a sequence of systems $G_i \in \mathcal{G}_{m,n}$ with $\|G_i\|^{\mathcal{U}} \to 0$ we have $\phi^{\mathcal{U}}_{AE,N} \le \|N\|^{\mathcal{U}}$, $\phi^{\mathcal{U}}_{IMOE,N} \le 1$.

(iii) The inequality

$$\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|u_T\|} \le 1 + \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu\right)_T\right\|}{\|u_T\|} \le 1 + \left\|G^{-1}\right\|^{N\mathcal{U}} \|N\|^{\mathcal{U}}$$

proves the bound $\phi^{\mathcal{U}}_{MIE,N} \le 1$ by letting $\left\|G^{-1}\right\|^{N\mathcal{U}} \to 0$. The bound $\phi^{\mathcal{U}}_{MOE,N} \le 1$ follows by

$$\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|} \le 1 + \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu)_T\|}{\|u_T\|}\frac{\|u_T\|}{\|(Gu)_T\|} \le 1 + \|N\|^{\mathcal{U}} \left(\inf_{u\in\mathcal{U},T>0} \frac{\|(Gu)_T\|}{\|u_T\|}\right)^{-1}$$

and considering linear models with arbitrarily large "minimal gain" $\inf_{u\in\mathcal{U},T>0} \frac{\|(Gu)_T\|}{\|u_T\|}$.

(iv) For the next result, suppose there is a $G^{-1} \in \mathcal{G}^{inv}_{m,n}(N\mathcal{U})$ with

$$\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} = M < \infty$$

(if there isn't, there is nothing to prove). Then

$$M = \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} \ge \sup_{u\in\mathcal{U},T>0} \frac{\|u_T\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} - 1$$

and by linearity of the space $\mathcal{G}_{m,n}$ there is a $\tilde{G}^{-1} = \rho G^{-1} \in \mathcal{G}^{inv}_{m,n}(N\mathcal{U})$ for any $\rho > 0$ and

$$\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(\tilde{G}^{-1}Nu-u\right)_T\right\|}{\left\|\left(\tilde{G}^{-1}Nu\right)_T\right\|} \le 1 + \sup_{u\in\mathcal{U},T>0} \frac{\|u_T\|}{\left\|\left(\rho G^{-1}Nu\right)_T\right\|} \le 1 + \frac{1+M}{\rho}.$$

As ρ can be arbitrarily large, the infimum over $\sup_{u\in\mathcal{U},T>0}\frac{\|(G^{-1}Nu-u)_T\|}{\|(G^{-1}Nu)_T\|}\leq M<\infty$ must be less or equal to one.

(v) The relation $\phi^{\mathcal{U}}_{FE,N}\geq\phi^{\mathcal{U}}_{MIE,N}/\|N\|^{\mathcal{U}}$ follows from

$$\begin{aligned}\sup_{u\in\mathcal{U},T>0}\frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|(Nu)_T\|} &= \sup_{u\in\mathcal{U},T>0}\frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|u_T\|}\frac{\|u_T\|}{\|(Nu)_T\|}\\ &\geq \frac{1}{\|N\|^{\mathcal{U}}}\sup_{u\in\mathcal{U},T>0}\frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|u_T\|}\end{aligned}$$

and the proof is complete. □

The given bounds can be interpreted in the way that all quality indices corresponding to a multiplicative interconnection structure give some kind of relative error, and that it is always possible to choose a linear model such that this relative error does not exceed 100%. In the additive error setup, one can form the relative error $\phi^{\mathcal{U}}_{AE,N}/\|N\|^{\mathcal{U}}$ from the AE-NLM and the gain of the system N, and this relative error will always be smaller than the IMOE-NLM. For the feedback error setup, no such interpretation can be given, and the measure is also not guaranteed to be bounded.

Classes of linear systems. At this point, we want to address the class of linear models we are considering in greater detail. It is important to keep in mind that the goal is to derive a linear model for a nonlinear system in order to reduce the complexity of the model. We will therefore most of the time deal with the set $\mathcal{G}_{m,n}$ of causal linear time-invariant finite-dimensional systems $G:L^m_{pe}\to L^n_{pe}$ given by

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)+Du(t)\end{aligned}\tag{3.4}$$

with $u(t)\in\mathbb{R}^m$, $x(t)\in\mathbb{R}^z$, $y(t)\in\mathbb{R}^n$, and according dimensions of the real matrices A, B, C and D. For three of the setups, inverses of linear systems appear in the definition of the nonlinearity measures. This set of inverses of operators in $\mathcal{G}_{m,n}$ is defined by

$$\mathcal{G}^{inv}_{m,n}(\tilde{\mathcal{Y}})=\left\{G^{-1}:\tilde{y}\in\tilde{\mathcal{Y}}\subseteq L^n_{pe}\to u\in L^m_{pe}\,\middle|\,\exists G:\tilde{\mathcal{Y}}\subseteq GL^m_{pe}\text{ and }\tilde{y}=Gu\text{ for all }u\in L^m_{pe}\right\}.$$

This last definition is done this way in order to have for each $G^{-1}\in\mathcal{G}^{inv}_{m,n}(\tilde{\mathcal{Y}})$ a $G\in\mathcal{G}_{m,n}$ such that $G^{-1}G$ is the identity on $\tilde{\mathcal{Y}}$.

But the basic definitions would also be possible for general causal linear infinite-dimensional or time-varying systems. To choose a class of time-varying linear systems might for instance be an advantage for linear modeling of transient (or batch) processes. We will however concentrate on the class of systems (3.4), giving some hints on generalizations where possible. Thus, if the nonlinearity measures are used for time-variant (linear or nonlinear) systems, not only nonlinearity is quantified by the measures, but also time-variance.

Moreover, the following proposition shows that in approximating stable nonlinear systems, unstable linear models are not superior to stable linear models.

P 3.2. *Assume the system $N : \mathcal{U} \subseteq L_{pe}^m \to L_{pe}^n$ is finite gain stable on $\mathcal{U}$. Then the following statements are true:*

(i) *The values of $\phi_{AE,N}^{\mathcal{U}}$ and $\phi_{IMOE,N}^{\mathcal{U}}$ do not change when the set $\mathcal{G}_{m,n}$ is replaced by the set $\left\{G \in \mathcal{G}_{m,n} : \|G\|^{\mathcal{U}} < 2\|N\|^{\mathcal{U}}\right\}$.*

(ii) *The value of $\phi_{IMIE,N}^{\mathcal{U}}$ does not change if the set $\mathcal{G}_{m,n}$ is replaced by the set $\{G \in \mathcal{G}_{m,n} : \|G\|^{G^{-1}N\mathcal{U}} < 2\|N\|^{\mathcal{U}}\}$.*

(iii) *The values of $\phi_{MIE,N}^{\mathcal{U}}$, $\phi_{MOE,N}^{\mathcal{U}}$ and $\phi_{FE,N}^{\mathcal{U}}$ do not change when the set $\mathcal{G}_{m,n}$ is replaced by the set $\left\{G \in \mathcal{G}_{m,n} : \|G\|^{\mathcal{U}} < \infty\right\}$.*

P . (i) For any G with $\|G\|^{\mathcal{U}} > 2\|N\|^{\mathcal{U}}$ we have that $\|N - G\|^{\mathcal{U}} > \|N\|^{\mathcal{U}}$. But from Proposition 3.1, we know that $\phi_{AE,N}^{\mathcal{U}} \leq \|N\|^{\mathcal{U}}$, and we need not to consider systems with $\|G\|^{\mathcal{U}} > 2\|N\|^{\mathcal{U}}$ for the AE-NLM. In a similar way, if $\|G\|^{\mathcal{U}} > 2\|N\|^{\mathcal{U}}$ we can see that

$$\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \geq \sup_{u\in\mathcal{U},T>0} \frac{\|(Gu)_T\|}{\|(Nu)_T\|} - 1 \geq \frac{\|G\|^{\mathcal{U}}}{\|N\|^{\mathcal{U}}} - 1 > 1$$

which shows that we need only to consider models with $\|G\|^{\mathcal{U}} \leq 2\|N\|^{\mathcal{U}}$ for the IMOE-NLM. Note that this last development used a somewhat sloppy notation (which we will keep up in the following for readability). A more precise way to show the inequality is the following. Remember that with $\gamma = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$ we mean that $\gamma = \inf\{\gamma \in \mathbb{R} : \|(Nu - Gu)_T\| \leq \gamma\|(Nu)_T\|, u \in \mathcal{U}, T > 0\}$. But $\|(Nu)_T\| - \|(Gu)_T\| \leq \|(Nu - Gu)_T\|$ and $\|(Nu)_T\| \leq \|N\|^{\mathcal{U}}\|u_T\|$ for all $u \in \mathcal{U}$ and $T > 0$ by definition of $\|N\|^{\mathcal{U}}$ show that $\|(Gu)_T\| \leq (1 + \gamma)\|N\|^{\mathcal{U}}\|u_T\|$. Hence, by definition of $\|G\|^{\mathcal{U}}$ we have $\|G\|^{\mathcal{U}} \leq (1 + \gamma)\|N\|^{\mathcal{U}}$, and finally $\gamma > 1$ if $\|G\|^{\mathcal{U}} > 2\|N\|^{\mathcal{U}}$.

(ii) If $\|G\|^{G^{-1}N\mathcal{U}} > 2\|N\|^{\mathcal{U}}$ we have

$$\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} \geq \sup_{u\in\mathcal{U},T>0} \frac{\|u_T\|}{\|(Nu)_T\|} \frac{\|(Nu)_T\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} - 1 \geq \frac{\|G\|^{G^{-1}N\mathcal{U}}}{\|N\|^{\mathcal{U}}} - 1 > 1$$

and if the infimum is finite, it is also achieved by a $G^{-1} \in \mathcal{G}_{m,n}^{inv}(N\mathcal{U})$ for which $\|G\|^{G^{-1}N\mathcal{U}} \leq 2\|N\|^{\mathcal{U}}$.

(iii) If G is not finite gain stable on $\mathcal{U}$, then for any $\gamma > 0$ there are $u \in \mathcal{U}, T > 0$ such that $\|(Gu)_T\| > \gamma\|u_T\|$ and thus

$$\frac{\|(Nu - Gu)_T\|}{\|(Gu)_T\|} \geq 1 - \frac{\|(Nu)_T\|}{\|(Gu)_T\|} \geq 1 - \frac{\|N\|^{\mathcal{U}}\|u_T\|}{\|(Gu)_T\|} \geq 1 - \frac{\|N\|^{\mathcal{U}}}{\gamma}$$

and the supremum over $u \in \mathcal{U}, T > 0$ can not be less than one (MOE setup). In the same way for any $\gamma > 0$ there are $y \in N\mathcal{U}, T > 0$ such that $\|y_T\| > \gamma \left\|\left(G^{-1}y\right)_T\right\|$ and thus

$$\frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|u_T\|} \geq 1 - \frac{\left\|\left(G^{-1}Nu\right)_T\right\|}{\|u_T\|} \geq 1 - \frac{\|N\|^{\mathcal{U}}}{\gamma}$$

and thus considering only stable models does not imply a loss of generality in the MIE setup. Finally, if G is not finite gain stable on $\mathcal{U}$, then for any $\gamma > 0$ there are $y \in N\mathcal{U}, T > 0$ such that $\|y_T\| > \gamma \left\|\left(G^{-1}y\right)_T\right\|$. Then, for any $\gamma > 0$

$$\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|(Nu)_T\|} \geq \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|} - \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu\right)_T\right\|}{\|(Nu)_T\|} \geq \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|} - \frac{1}{\gamma}$$

and thus $\sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\|(Nu)_T\|} \geq \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|}$. If the last expression is unbounded, there is nothing more to prove. If $\sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|} < \infty$, then

$$\begin{aligned}
\phi^{\mathcal{U}}_{FE,N} &= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|(Nu)_T\|} \\
&= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|u_T\|} \frac{\|u_T\|}{\|(Nu)_T\|} \\
&\leq \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|} \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|u_T\|} \\
&\leq \phi^{\mathcal{U}}_{MIE,N} \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|} \leq \sup_{u\in\mathcal{U},T>0} \frac{\|(u)_T\|}{\|(Nu)_T\|}
\end{aligned}$$

shows that unstable models are not better suited than stable ones in the FE setup and the proof is complete. □

As a consequence of the above proposition, we can in the following always restrict $\mathcal{G}_{m,n}$ to include only stable linear systems without loss of generality. In particular in the AE, IMOE and IMIE setups, we need only to consider systems whose gain is at most $2\|N\|^{\mathcal{U}}$, rendering the search for optimal linear models easier.

Summary and Discussion. This section introduced model quality indices and nonlinearity measures. These measures correspond to the gains of error systems Δ, if the given nonlinear system N is represented as the interconnection of a (best) nominal linear model G (or G^*) and Δ. As class of linear systems, we consider time-invariant linear systems that are equivalent to a state-space form, i.e. that are also causal and finite-dimensional. We have moreover shown that we do not need to consider unstable linear models, as the nonlinear system N is also assumed to be stable.

At a first glance, the introduction of different setups seems to be a disadvantage, but it will become clear in later parts of the thesis that the presence of different tools allows to pick the most useful one, depending on a priori known system structure, a possible control objective, or based on the computational difficulty. Moreover, four of the measures correspond to relative errors, allowing a particularly intuitive interpretation. Thus, the most suited setup can be chosen in face of a particular application.

3.2. Well-Posedness of Interconnections and Existence of Best Models

It is not guaranteed under all circumstances that there exists an optimal nominal linear model G^* and an error term Δ such that the given interconnection exactly returns the behaviour of the nonlinear plant N. Firstly, there is no guarantee that the infimum in the definition of the nonlinearity measures is actually attained by a member of class $\mathcal{G}$ or $\mathcal{G}^{inv}$ respectively. Moreover, the error model Δ or the interconnection might not be well-defined.

Well-posedness of interconnections. We will first consider the question: Given a $G \in \mathcal{G}_{m,n}$, under which conditions one can define the system Δ, such that the interconnection is well-posed[2] and that its input-output behaviour is equal to N?

For the additive error nonlinearity measure, the definition of $\Delta : u \in \mathcal{U} \subseteq L^m_{pe} \mapsto e = Nu - Gu \in L^n_{pe}$ is possible for any $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ and $G \in \mathcal{G}_{m,n}$, and the interconnection $y = Gu + \Delta u$ is well-posed and results in $y = Nu$. This implies also that a $G \in \mathcal{G}_{m,n}$ can be given that achieves the modeling error given by $\phi^{\mathcal{U}}_{AE,N}$ up to an arbitrarily small number.

In order to discuss the question of well-posedness for the other interconnections and nonlinearity measures, we will need to use inverses of systems. For mappings that are not one-to-one and onto, two different inverses can be defined. From a system-theoretic view, the left-inverse is equivalent to a filter, that reconstructs the input of a system based solely on the information of its output. This is possible if and only if the corresponding mapping is one-to-one (injective). The second type of inverse is the right-inverse. It corresponds to an open-loop controller that generates a (not necessarily unique) input signal, such that the plant output tracks a given reference signal. The tracking problem has a solution if and only if the signal to be tracked lies in the image of the mapping representing the plant. It can be solved for a whole class of desired outputs if and only if the mapping is surjective to the set of admissible reference signals. Both types of inverses play a role for the well-posedness of the interconnections.

E . In order to clarify ideas, consider the examples of the static linear mappings

$$\text{A)}\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{B)}\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \qquad \text{C)}\ y = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where all variables are scalar. It is easy to see that mapping A) is both, injective and surjective onto $\mathbb{R}^2$. Mapping B) is injective, i.e. from the knowledge of the output $(y_1\, y_2)^T$ the input is given by $u = y_1 = 0.5y_2$. However, it is not surjective onto $\mathbb{R}^2$, as only outputs

[2] Well-posedness, as it is used here, means that the operator equations of the interconnection have a unique solution.

with $y_1 = 0.5y_2$ can be generated, and the inverse is not defined everywhere in $\mathbb{R}^2$. Mapping C) finally is not injective, but surjective onto $\mathbb{R}$. The family of right-inverses is given by $u_2 = \lambda$, $u_1 = y - 2\lambda$, where λ may depend on y. Interestingly, there are not only multiple right-inverses, but all but one of them are nonlinear mappings. The analogous properties for dynamic operators can be seen by considering their range and kernel. ■

In the following we give conditions for well-posedness for all of the interconnection setups. We start out with the multiplicative output error nonlinearity measure, for which we require N not to have more inputs than outputs in order to make a general statement.

P 3.3 (Well-posedness of MOE setup). *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be a causal, finite-gain stable mapping with $m \leq n$ and let $G \in \mathcal{G}_{m,n}$ be* injective *over $\mathcal{U}$. Then there exists an error system Δ such that the interconnection in Fig. 3.3b) is well-defined and the corresponding input-output operator is equal to N. Moreover, the gain of Δ with respect to all signals occurring in the interconnection for $u \in \mathcal{U}$, i.e. the MOE model quality index, is $\|\Delta\|^{G\mathcal{U}} = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|}$.*

P . As $G \in \mathcal{G}_{m,n}$ is injective, we can define the error system as $\Delta = NG^{-1} - I$ where G^{-1} is the (possibly unstable) left-inverse of G. The corresponding interconnection is obviously well-posed and $y = (I + \Delta)Gu = Nu$. Furthermore, we can write $\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|} = \sup_{\tilde{y}\in G\mathcal{U},T>0} \frac{\|(NG^{-1}\tilde{y}-\tilde{y})_T\|}{\|\tilde{y}_T\|}$. □

We could also drop the requirement of $m \leq n$ (it was not used in the proof). However, in most meaningful cases, this condition is implied by injectivity of G on $\mathcal{U}$. So we mentioned it for the sake of clarity. This remark will be valid for the upcoming results in a similar way. Note also that G^{-1}, and hence Δ, are not necessarily stable.[3] Nevertheless, Δ (and the interconnection) have finite gain *on* $\mathcal{U}$. This makes sense, as we are not interested in some kind of internal stability, but only in well-posedness. Our goal is not to *realize* N as the interconnection of G and Δ, but only to *represent* it that way. We go on with this idea in mind and derive a similar result for the inverse multiplicative output error nonlinearity measure, where we first require the nonlinear operator to be left-invertible (injective).[4]

P 3.4 (Well-posedness of IMOE setup). *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be an* injective, *causal and finite-gain stable mapping with $m \leq n$ and let $G \in \mathcal{G}_{m,n}$ be* injective *over $\mathcal{U}$. Then there exists an error system Δ such that the interconnection in Fig. 3.3c) is well-defined and the corresponding input-output operator is equal to N. Moreover, the gain of Δ with respect to all signals occurring in the interconnection, i.e. the IMOE model quality index, is $\|\Delta\|^{N\mathcal{U}} = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$.*

[3] Although G^{-1} may be improper, G^{-1} and Δ are causal as G and N are assumed to contain no dead-time.

[4] For conditions for invertibility of nonlinear input-output operator given in state-space form, see [67, 66].

P . As N is injective on $\mathcal{U}$, it has a well-defined left-inverse N^{-1} on $N\mathcal{U}$. Define $\Delta = I - GN^{-1}$ and consider the equations of the interconnection

$$\begin{aligned} \tilde{y} + \Delta y &= y \\ \tilde{y} &= Gu \end{aligned}$$

or equivalently $G\left(u - N^{-1}y\right) = 0$. Clearly, as N is a single-valued map, $y = Nu$ is the unique solution of $u - N^{-1}y = 0$. By injectivity of G, it is also the unique solution of $G\left(u - N^{-1}y\right) = 0$. Furthermore, we have $\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|} = \sup_{\tilde{y}\in N\mathcal{U},T>0} \frac{\left\|(\tilde{y}-GN^{-1}\tilde{y})_T\right\|}{\|\tilde{y}_T\|}$. □

We can remove the assumption about the injectivity of N by allowing Δ to be a relation instead of a mapping.

In some parts of the thesis it will be necessary to consider relations on signal sets instead of mappings. A relation $R \subseteq L_{pe}^m \times L_{pe}^n$ represents the generalization of a nonlinear mapping in the sense that its output does not need to be unique [135]. In fact, a relation just puts constraints on its ports, and strictly spoken the ports cannot be separated into input and output any more. However, we will always have two ports, one corresponding to an input, and one corresponding to a non-unique output. The gain on $\mathcal{U}$ of a relation is then accordingly defined as

$$\|R\|^{\mathcal{U}} = \sup_{(u,y)\in R, u\in\mathcal{U},T>0} \frac{\|y_T\|}{\|u_T\|}$$

and its incremental gain on $\mathcal{U}$ as

$$\|R\|_{\Delta}^{\mathcal{U}} = \sup_{(u_1,y_1),(u_2,y_2)\in R,u_1,u_2\in\mathcal{U},T>0} \frac{\left\|(y_1 - y_2)_T\right\|}{\|(u_1 - u_2)_T\|}.$$

We then define the modeling error as

$$\Delta = \left\{(y, e) \in L_{pe}^n \times L_{pe}^n | \exists u \in \mathcal{U} : y = Nu = Gu + e\right\}$$

and the feedback part of the interconnection becomes

$$(I - \Delta)^{-1} = \left\{(\tilde{y}, y) \in L_{pe}^n \times L_{pe}^n | y = \tilde{y} + e, (y, e) \in \Delta\right\}.$$

Now, for any $\tilde{y} \in G\mathcal{U}$, there is a (unique, by injectivity of G) $u \in \mathcal{U}$ with $\tilde{y} = Gu$. Hence, there is a unique pair $(y, e) \in \Delta$ such that $y = Gu + e = Nu$ (as N and G are single-valued) and $(I - \Delta)^{-1} : \tilde{y} \mapsto y = NG^{-1}\tilde{y}$ is a single-valued mapping. Moreover, the gain of Δ satisfies $\|\Delta\|^{N\mathcal{U}} = \sup_{(y,e)\in\Delta,y\in N\mathcal{U},T>0} \frac{\|e_T\|}{\|y_T\|} = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$.

In a similar manner, well-posedness is shown next for the other setups. Readers only interested in a summary of the results may therefore directly go to the end of this subsection.

The case of the multiplicative input error nonlinearity measure can be treated next in a straightforward manner for systems with at least as many inputs as outputs.

P 3.5 (Well-posedness of MIE setup). *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be a causal, finite-gain stable mapping with $m \geq n$ and let the operator $G \in \mathcal{G}_{m,n}$ be* surjective *to $N\mathcal{U}$. Then there exists an error system Δ such that the interconnection in Fig. 3.3d) is well-defined and the corresponding input-output operator is equal to N. Moreover, the gain of Δ with respect to all signals occurring in the interconnection, i.e. the MIE model quality index, is $\|\Delta\|^{\mathcal{U}} = \sup_{u\in\mathcal{U},T>0} \frac{\|(G^{-1}Nu-u)_T\|}{\|u_T\|}$.*

P . As G is surjective to $N\mathcal{U}$, there is a right-inverse $G^{-1} \in \mathcal{G}^{inv}_{m,n}(N\mathcal{U})$ such that the error system $\Delta = G^{-1}N - I$ is well-defined on $\mathcal{U}$. The interconnection is obviously well-posed, and $y = G(I + \Delta)u = Nu$. The statement on the gain is true by definition of the gain on $\mathcal{U}$. □

Note that surjectivity of G on $N\mathcal{U}$ implies for instance that G may not have a higher relative degree than N. Moreover, the model quality index as given in Proposition 3.5 is not necessarily uniquely defined, because the right inverse does not need to be unique. This defect can however be remedied by taking the minimum over all right-inverses.

The case of the inverse multiplicative input error setup is again a little bit more sophisticated, and in addition to $m \geq n$ we require N to be left-invertible in a first step, in order to relax it again later.

P 3.6 (Well-posedness of IMIE setup). *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be an* injective, *causal and finite-gain stable mapping $m \geq n$ and let the operator $G \in \mathcal{G}_{m,n}$ be* injective and surjective *(one-to-one and onto) from $\mathcal{U}$ to $N\mathcal{U}$. Then there exists an error system Δ such that the interconnection in Fig. 3.3e) is well-defined and the corresponding input-output operator is equal to N. Moreover, the gain of Δ with respect to all signals occurring in the interconnection, i.e. the IMIE model quality index, is $\|\Delta\|^{G^{-1}N\mathcal{U}} = \sup_{u\in\mathcal{U},T>0} \frac{\|(G^{-1}Nu-u)_T\|}{\|(G^{-1}Nu)_T\|}$.*

P . The system N is injective on $\mathcal{U}$ and therefore it has a well-defined left-inverse N^{-1} on $N\mathcal{U}$. Define $\Delta = I - N^{-1}G$ and consider the equations of the interconnection

$$\begin{aligned} \tilde{u} &= u + \Delta\tilde{u} \\ y &= G\tilde{u}. \end{aligned}$$

The first equation yields $u = N^{-1}G\tilde{u}$, the unique solution of which is $\tilde{u} = G^{-1}Nu$ (N^{-1} is injective as N is single-valued, G^{-1} can be applied as it is defined on $N\mathcal{U}$ and it is unique as G is injective). The second equation gives $y = Nu$ and we can write $\sup_{u\in\mathcal{U},T>0} \frac{\|(G^{-1}Nu-u)_T\|}{\|(G^{-1}Nu)_T\|} = \sup_{\tilde{u}\in G^{-1}N\mathcal{U},T>0} \frac{\|(\tilde{u}-N^{-1}G\tilde{u})_T\|}{\|\tilde{u}_T\|}$. □

We can remove the assumption about the injectivity of N and G again by allowing Δ to be a relation. We choose a right inverse G^{-1} of G and define

$$\Delta = \left\{(\tilde{u}, e) \in L^m_{pe} \times L^m_{pe} | \tilde{u} = G^{-1}N(\tilde{u} - e)\right\},$$

so the feedback part is

$$(I-\Delta)^{-1}=\left\{(u,\tilde{u})\in L_{pe}^{m}\times L_{pe}^{m}|\tilde{u}=u+e,(\tilde{u},e)\in\Delta\right\}.$$

From $\tilde{u}-e=u,(\tilde{u},e)\in\Delta$ follows $\tilde{u}=G^{-1}Nu$, which corresponds to a single-valued mapping, and finally $y=G\tilde{u}=Nu$. Furthermore, $\|\Delta\|^{G^{-1}N\mathcal{U}}=\sup_{(\tilde{u},e)\in\Delta,\tilde{u}\in G^{-1}N\mathcal{U},T>0}\frac{\|e_T\|}{\|y_T\|}=\sup_{u\in\mathcal{U},T>0}\frac{\|(G^{-1}Nu-u)_T\|}{\|(G^{-1}Nu)_T\|}$.

Finally, we consider the case of a feedback error system.

P 3.7 (Well-posedness of FE setup). *Let $N:\mathcal{U}\subseteq L_{pe}^{m}\to L_{pe}^{n}$ be an* injective, *causal and finite-gain stable mapping $m\geq n$ and let the operator $G\in\mathcal{G}_{m,n}$ be* injective and surjective *(one-to-one and onto) from $\mathcal{U}$ to $N\mathcal{U}$. Then there exists an error system Δ such that the interconnection in Fig. 3.3f) is well-defined and the corresponding input-output operator is equal to N. Moreover, the gain of Δ with respect to all signals occurring in the interconnection, i.e. the FE model quality index, is $\|\Delta\|^{N\mathcal{U}}=\sup_{u\in\mathcal{U},T>0}\frac{\|(G^{-1}Nu-u)_T\|}{\|(Nu)_T\|}$.*

P . The system N is injective on $\mathcal{U}$ and therefore it has a well-defined inverse N^{-1} on $N\mathcal{U}$, and the same applies for G, as it is one-to-one and onto $N\mathcal{U}$. Define $\Delta=G^{-1}-N^{-1}$ and consider the equations of the interconnection

$$\begin{aligned}\tilde{u}&=\Delta y+u\\ y&=G\tilde{u}.\end{aligned}$$

Substituting the second equation in the first yields $\tilde{u}=\left(G^{-1}-N^{-1}\right)G\tilde{u}+u$ and thus $u=N^{-1}G\tilde{u}$, the unique solution of which is $\tilde{u}=G^{-1}Nu$ (N^{-1} is injective as N is single-valued), or $y=Nu$. Furthermore we can write $\sup_{u\in\mathcal{U},T>0}\frac{\|(G^{-1}Nu-u)_T\|}{\|(Nu)_T\|}=\sup_{y\in N\mathcal{U},T>0}\frac{\|(G^{-1}y-N^{-1}y)_T\|}{\|y_T\|}$. □

We can again remove the assumption about the injectivity of N and G by allowing Δ to be a relation. We choose a right inverse G^{-1} of G and define

$$\Delta=\left\{(y,e)\in L_{pe}^{n}\times L_{pe}^{m}|y=Nu,e=G^{-1}Nu-u,u\in\mathcal{U}\right\}.$$

The feedback interconnection is

$$(I-G\Delta)^{-1}G=\left\{(y,u)\in L_{pe}^{n}\times L_{pe}^{m}|y=G(u+e),(y,e)\in\Delta\right\}$$

and from $y=G(u+e),(y,e)\in\Delta$ follows $y=Nu$, which corresponds to a single-valued mapping. Furthermore, $\|\Delta\|^{N\mathcal{U}}=\sup_{(y,e)\in\Delta,y\in N\mathcal{U},T>0}\frac{\|e_T\|}{\|y_T\|}=\sup_{u\in\mathcal{U},T>0}\frac{\|(G^{-1}Nu-u)_T\|}{\|(Nu)_T\|}$.

The above studies show that the AE, MOE, and IMOE setups are always well-defined if G is injective (left-invertible) on $\mathcal{U}$, which usually requires $m\leq n$. On the other hand, the MIE, IMIE and FE setups require G to be surjective (right-invertible) on $N\mathcal{U}$, which will in general only be the case for $n\leq m$. Note that these results concern the validity of the model quality index $\|\Delta\|^{\mathcal{X}}$. It has yet to be verified that the definitions of the nonlinearity measures and best linear models are also justified.

Well-posedness of nonlinearity measures. In the last paragraph, conditions for the six different setups were derived in order to guarantee that the error system Δ and the model quality index $\|\Delta\|^{\mathcal{X}}$ can be properly defined. Assume the nonlinearity measure $\phi_N^{\mathcal{U}}$ of a nonlinear plant in one of the error model setups (interconnection structures) is known. The question we want to address now is: Given any arbitrarily small number $\varepsilon > 0$, is there a linear model G such that $\|\Delta\|^{\mathcal{X}} \leq \phi_N^{\mathcal{U}} + \varepsilon$? I.e. can the value of the nonlinearity measure be approximated arbitrarily well by the quality index of a linear model that guarantees well-posedness of the interconnection and the error system Δ?

The answer to this question is "yes" for all of the interconnection structures.

For the additive error (AE) setup, the answer is obvious, as the error model and interconnection are always well defined, i.e. for the choice of any $G \in \mathcal{G}_{m,n}$. For the MIE, IMIE and FE setups, the answer is also easy to see. Take as example the MIE setup and nonlinearity measure. By definition of the set $\mathcal{G}_{m,n}^{inv}(N\mathcal{U})$, the choice of a $G^{-1} \in \mathcal{G}_{m,n}^{inv}(N\mathcal{U})$ implies that there is a $G \in \mathcal{G}_{m,n}$ that is surjective to $N\mathcal{U}$. As surjectivity is (necessary and) sufficient for the interconnection and error model to be well-defined, there is nothing more to prove. The same reasoning holds for the IMIE and FE setups.

However, the situation is different for the MOE and IMOE setups. Here, we allow in the definition of the nonlinearity measures any linear model $G \in \mathcal{G}_{m,n}$. Well-posedness of the error system Δ and the interconnection is however only assured if G is injective. Therefore, it will be shown next that we can drop the assumption of injectivity of G in the definition of the nonlinearity measure without loss of generality. We have to keep, however, the requirement that the nonlinear process under consideration has at least as many outputs as inputs, $n \geq m$. This is practically no restriction, as for $n < m$, the nonlinearity measure will almost always return a value of 1, rendering the information useless (see also the discussion on the IMOE-NLM of memoryless operators in Section 3.4).

P 3.8 (Well-posedness of IMOE nonlinearity measure). *Let $N : \mathcal{U} \subseteq L_{pe}^m \to L_{pe}^n$ be a causal and finite-gain stable mapping with $m \leq n$. Let $\mathcal{G}_{m,n}^{inj}(\mathcal{U}) \subset \mathcal{G}_{m,n}$ be the set of injective operators $G \in \mathcal{G}_{m,n}$. Then*

$$\phi_{IMOE,N}^{\mathcal{U}} = \inf_{G \in \mathcal{G}_{m,n}} \sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} = \inf_{G \in \mathcal{G}_{m,n}^{inj}(\mathcal{U})} \sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|},$$

i.e. in the limit, the models of class $\mathcal{G}_{m,n}^{inj}(\mathcal{U})$ achieve the same IMOE model quality index as the models of class $\mathcal{G}_{m,n}$.

P . By

$$\frac{\left\|\left(Nu - \tilde{G}u\right)_T\right\|}{\|(Nu)_T\|} - \left\|\tilde{G} - G\right\| \frac{\|u_T\|}{\|(Nu)_T\|} \leq \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \leq \frac{\left\|\left(Nu - \tilde{G}u\right)_T\right\|}{\|(Nu)_T\|} + \left\|\tilde{G} - G\right\| \frac{\|u_T\|}{\|(Nu)_T\|}$$

it is easy to see that $\frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$ is continuous in G for all $u, T > 0$. Since the supremum of continuous functions is lower semi-continuous, the function

$$\|\Delta\|^{N\mathcal{U}} = \sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|}$$

is lower semi-continuous. The result follows from the fact that for $m \leq n$, the set $\mathcal{G}_{m,n}^{inj}(\mathcal{U})$ is dense in $\mathcal{G}_{m,n}$. □

It is straight-forward to give an analogue result for the MOE-NLM.

P 3.9 (Well-posedness of MOE nonlinearity measure). *Under the same assumptions as in Proposition 3.8,*

$$\phi_{MOE,N}^{\mathcal{U}} = \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|} = \inf_{G\in\mathcal{G}_{m,n}^{inj}(\mathcal{U})} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Gu)_T\|},$$

i.e. in the limit, the models of class $\mathcal{G}_{m,n}^{inj}(\mathcal{U})$ achieve the same MOE model quality index as the models of class $\mathcal{G}_{m,n}$.

P . With the inequalities

$$\frac{\|(Nu-Gu)_T\| - \left\|\left(Gu-\tilde{G}u\right)_T\right\|}{\|(Gu)_T\| + \left\|\left(Gu-\tilde{G}u\right)_T\right\|} \leq \frac{\left\|\left(Nu-\tilde{G}u\right)_T\right\|}{\left\|\left(\tilde{G}u\right)_T\right\|} \leq \frac{\|(Nu-Gu)_T\| + \left\|\left(Gu-\tilde{G}u\right)_T\right\|}{\|(Gu)_T\| - \left\|\left(Gu-\tilde{G}u\right)_T\right\|}$$

the result follows as in Proposition 3.8. □

Existence of optimal linear models. We have proven so far that there always is a linear model G that satisfies the well-posedness assumptions and that achieves a model quality index *arbitrarily close to* the corresponding nonlinearity measure. Next, we are concerned with the question of existence of best linear models in the sense that the value of the nonlinearity measure can be matched *exactly*. We start out with considering the AE-NLM, for which things turn out to be particularly simple, based on the following fact.[5]

L 3.10. *Let $N : \mathcal{U} \subseteq L_{pe}^m \to L_{pe}^n$ be a causal, finite-gain stable mapping. Then the function $\|N-G\|^{\mathcal{U}}$ is continuous in $G \in \mathcal{G}_{m,n}$.*

P . Let $\tilde{G} = G + G_e$. Sub-additivity of the supremum, i.e. $\sup_x (a(x)+b(x)) \leq \sup_x a(x) + \sup_x b(x)$, and the triangle inequality show that $\left|\|N-G\|^{\mathcal{U}} - \left\|N-\tilde{G}\right\|^{\mathcal{U}}\right| \leq \|G_e\|^{\mathcal{U}}$. Hence, $\|G_e\|^{\mathcal{U}} \to 0$ implies $\left|\|N-G\|^{\mathcal{U}} - \left\|N-\tilde{G}\right\|^{\mathcal{U}}\right| \to 0$. □

[5]From the lemma, we can also conclude that general, possibly infinite-dimensional, stable delay-free transfer operators $G : L_{pe}^m \to L_{pe}^n$ defined by

$$(Gu)(t) = G_{hf}u(t) + (g*u)(t) = G_{hf}u(t) + \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau$$

with the instantaneous gain $G_{hf} \in \mathbb{R}^{n\times m}$ and a measurable kernel $g : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ with $\|g\|_1 < \infty$ do not result in a smaller modeling error in the additive error setup than systems from $\mathcal{G}_{m,n}$. This can be seen by considering that the Laguerre functions are dense in L_1 [89], and that any finite superposition of Laguerre functions corresponds to a finite-dimensional system. It is, on the other hand, well-known [137] that distributed linear systems containing time-delay cannot be approximated by delay-free (finite-dimensional or not) linear systems.

It is known that if the additive error nonlinearity measure of a system N vanishes, $\phi^{\mathcal{U}}_{AE,N} = 0$, the system N admits a perfect linear model [26]. In the general case, it is not yet known in which cases there is a best linear approximation G^* with $\|N - G^*\|^{\mathcal{U}} = \inf_{G\in\mathcal{G}_{m,n}} \|N - G\|^{\mathcal{U}}$. However, if we consider only approximations from a finite-dimensional set

$$\mathcal{G}^N_{m,n} = \Big\{ G \in \mathcal{G}_{m,n} \Big| \quad G = \sum_{i=1}^{N} \alpha_i G_i,\ \alpha \in \mathbb{R}^N \\ G_i \in \mathcal{G}_{m,n} \text{ linear independent},\ \|G_i\|^{\mathcal{U}} \le \gamma_i < \infty \Big\}$$

then we can prove existence of a solution of the corresponding finite-dimensional minimization problem in $\mathcal{G}^N_{m,n}$.

T 3.11. *Let the finite-gain stable operator $N : u \in \mathcal{U} \subseteq L^m_{pe} \mapsto y \in L^n_{pe}$ be given and let $\mathcal{G}^N_{m,n}$ be as above. Then there exists a best linear approximation $G^* \in \mathcal{G}^N_{m,n}$ with $\|N - G^*\|^{\mathcal{U}} = \inf_{G\in\mathcal{G}^N_{m,n}} \|N - G\|^{\mathcal{U}}$.*

P . Following Proposition 3.1 we can restrict the minimum search without loss of generality to the set $\left\{ G \in \mathcal{G}^N_{m,n} \middle| \|G\|^{\mathcal{U}} \le 2\|N\|^{\mathcal{U}} \right\}$. Furthermore, the mapping $(\alpha_1, ..., \alpha_n) \mapsto \left\| \sum_{i=1}^N \alpha_i G_i \right\|^{\mathcal{U}}$ is a norm on $\mathbb{R}^N$, so by equivalence of norms in $\mathbb{R}^N$ the search over $\mathcal{G}^N_{m,n}$ is equivalent to the search over the set $\tilde{\mathcal{G}}^N_{m,n} = \left\{ G \in \mathcal{G}^N_{m,n} \middle| G = \sum_{i=1}^N \alpha_i G_i, \alpha_i \le M < \infty \right\}$ for some M. Since $\|N - G\|^{\mathcal{U}}$ is continuous in G, its minimum is achieved over the finite-dimensional, closed and bounded (and hence, compact) set $\tilde{\mathcal{G}}^N_{m,n}$ (Weierstraß' theorem, e.g. [86]). □

A similar result can be obtained for the IMOE-NLM.

T 3.12. *Let $N, \mathcal{G}^N_{m,n}$ be as above, and let N be injective at 0, i.e. $(Nu)_T = 0 \Rightarrow u_T = 0$. Then there exists a best linear approximation $G^* \in \mathcal{G}^N_{m,n}$ with*

$$\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu - G^*u)_T\|}{\|(Nu)_T\|} = \inf_{G\in\mathcal{G}^N_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|}.$$

P . We need again only to consider bounded $G \in \mathcal{G}_{m,n}, \|G\| \le 2\|N\|^{\mathcal{U}}$. By Proposition 3.8, the function $\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$ is lower semi-continuous in G, and existence of the minimum of $\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|}$ over the compact set $\tilde{\mathcal{G}}_N$ follows from Weierstraß' theorem. □

Note that the best model is not necessarily unique. Note also that Weierstraß' theorem needs a compact set, which is a much stronger requirement in infinite dimensions than closedness and boundedness. Therefore, the existence result does not carry over to the general case[6]. Moreover, the strong results above cannot be generalized to all nonlinearity

[6]Consider for instance the case of an infinite-dimensional linear system whose integral kernel is the infinite sum of (linearly independent) Laguerre polynomials. Then for any finite-dimensional linear model of order z, there is a finite-dimensional linear model of higher order than z that achieves a smaller modeling error.

measures, although lower semi-continuity in G of the expressions $\|\Delta\|^{\mathcal{X}}$ can be established for all of the nonlinearity measures. However, the bound that is needed on $\|G\|$ or $\left\|G^{-1}\right\|$ respectively in order to establish the existence results can only be given for the AE- and IMOE-setups. On the other hand, for the computation of the nonlinearity measures (see also Section 3.8), the existence of the exact optimal model is not very important. We are very well satisfied by the fact that the values of the nonlinearity measures can be approximated arbitrarily well by sub-optimal models.

Summary. In this section, we have verified that the definition of the model quality index is well-justified under some intuitive conditions on the linear model G. Under very weak conditions, the nonlinearity measures reflect what they are intended to reflect, namely the achievable quality of linear models for a nonlinear system. This means that it is always possible to find a linear model that satisfies the well-posedness conditions and that has a model quality index arbitrarily close to the nonlinearity measures. The last results give more restrictive conditions that guarantee that the value of the nonlinearity measure can be matched exactly. While being an interesting theoretical result, the existence of an exact best linear model will not be necessary for the usefulness of the setup in practice.

3.3. Operator Derivatives and Local Linear Models

In this section we study the relationship between the best linear models in the novel setup, and the commonly used class of linear models, the derivatives. More specifically, we will describe conditions under which the two are identical. On the way to these results, we point out some traps when using derivatives in an input-output framework for dynamical systems.

The definitions of nonlinearity measures allow to describe the nonlinear behaviour in a *specified* region of operation, i.e. in a *regional* way. However, it is also insightful to consider the *local* behaviour of nonlinear systems close to an operating point. We first want to underline the fact that two kinds of subsets of the signal spaces L_p are frequently encountered: balls of the type $\left\{u \in L_p^m \,|\, \|u\| \leq r\right\}$ characterized by the maximal norm r, and subsets of the form $\left\{u \in L_p^m \,|\, |u(t)| \leq A\ \forall t\right\}$ characterized by maximal signal amplitudes (or in a more general setting by their maximal range as in $\mathcal{U}(V) = \{u \in L_{\infty e}^m : u(t) \in V \subseteq \mathbb{R}^m$ for all $t \geq 0\}$). Both points of view are merged when considering the space L_∞, but for L_p, $p < \infty$ they may differ significantly. One can search for local linear models using either characterization and is thus led to two different kinds of local behaviour. We start with the norm-wise approach and recall accordingly the definition of Fréchet-derivative on Banach spaces (see e.g. [86]).

D 3.13. Consider an operator $P : \mathcal{U} \subseteq \mathcal{X} \to \mathcal{Y}$ with the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{U}$ open. P is said to be *Fréchet-differentiable* at $x_0 \in \mathcal{U}$ if there exists a continuous linear operator $DP_{x_0} : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{\|h\|_{\mathcal{X}} \to 0} \frac{\left\|P(x_0 + h) - Px_0 - DP_{x_0}h\right\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

DP_{x_0} is called the *Fréchet-derivative* of P at x_0.

The Fréchet-derivative DP is the (unique) perfect local linear approximation to P in the following sense.

P 3.14. *Consider an L_p-stable mapping $N : \mathcal{U} \subset L_p^m \to L_p^n$ defined on the bounded set $\mathcal{U}$ containing a neighborhood of the origin of L_{pe}^m. Under these conditions there exists a perfect local continuous linear model G_{loc}^* with*

$$\lim_{\rho \to 0} \sup_{u \in \rho \mathcal{U}} \frac{\left\| N(u) - G_{loc}^*(u) \right\|}{\|u\|} = 0 \tag{3.5}$$

if and only if N is Fréchet-differentiable at $u = 0$. Furthermore, G_{loc}^* is unique and we have $G_{loc}^* = DN_0$.

P . As $\mathcal{U}$ is bounded and as it contains a ball around the origin, the result follows directly from the definition of the Fréchet-derivative and from the fact that the Fréchet-derivative is always unique [86]. □

This fact has been observed before [26], but without stating the uniqueness of the best local model. The notion of Fréchet-derivative can be generalized to the extended spaces by defining for any $T \geq 0$ $(DP_{x_0} h)_T = (DP_{(x_0)_T} h_T)_T$ [140], and Proposition 3.14 is easily generalized accordingly.

Note that we did *not* state that the best linear model is from the set $\mathcal{G}_{m,n}$ (the Fréchet-derivative might for example be a time-variant system or contain time-delays). Moreover, the concept of Fréchet-derivative can lead to problems as for $p < \infty$ a small L_p-norm does not imply a small signal amplitude: it can be shown that for $p < \infty$ a static operator $P : x \in L_p^m \mapsto y \in L_p^n$, $y(t) = g(x(t))$ defined by the function g is Fréchet-differentiable if and only if P (i.e. g) is linear [110]. We therefore consider a second type of derivative frequently encountered in local analysis of dynamic systems.

D 3.15. Consider an operator $P : \mathcal{U} \subseteq \mathcal{X} \to \mathcal{Y}$ with the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{U}$ open. P is said to be *Gâteaux-differentiable* at $x_0 \in \mathcal{U}$ if for all $h \in \mathcal{X}$ there exists a continuous linear operator $\delta P_{x_0} : \mathcal{X} \to \mathcal{Y}$ such that

$$\delta P_{x_0} h = \lim_{t \to 0} \frac{P(x_0 + th) - P x_0}{t}.$$

δP_{x_0} is called the *Gâteaux-derivative* of P at x_0.

The Gâteaux-derivative generalizes the notion of directional derivative. Its definition is different form Fréchet-differentiability in the fact that the convergence does not have to be uniform in $\mathcal{U}$ (and may thus depend on the "direction" in which x_0 is approached). Therefore, the existence of a Fréchet-derivative implies the existence of a Gâteaux-derivative, and in this case both derivatives are equal [10]. This is for instance the case if the Gâteaux-derivative is continuous in x_0. The notion of Gâteaux-derivative can also be generalized to the extended spaces by defining for any $T \geq 0$ $(\delta P_{x_0} h)_T = (\delta P_{(x_0)_T} h_T)_T$ [35].

The linearization of a state-space system along trajectories corresponds to the Gâteaux-derivative with respect to the $L_{\infty e}$-space of the corresponding input-output operator N [36, 35]. Under the more restrictive assumption of a Lipschitz-continuous right-hand side, it coincides with the Gâteaux-derivative w.r.t. the L_{2e}-space as well. The following result states that the state-space linearization is also the perfect local linear model in the sense of the additive error nonlinearity measure. To get the desired statement, it is necessary to consider admissible inputs with $u(t) \in V$ for all $t \geq 0$ where $V \subseteq \mathbb{R}^m$ is *bounded*, e.g. $\{v \in \mathbb{R}^m | \, |v| \leq A \in \mathbb{R}^+ \, \forall t\}$.

T 3.16. *For any $p \in [1, \infty]$ consider an operator $N : \mathcal{U} \subseteq L_{pe}^m \cap L_\infty^m \to L_{pe}^n$ given by the state-space realization*

$$\begin{aligned} \dot{x} &= f(x,u), \;\; x(0) = 0 \\ y &= h(x,u). \end{aligned}$$

Assume that $u \in \mathcal{U} \Rightarrow u_T \in \mathcal{U} \, \forall T > 0$. Suppose that f and h are C^1 in both arguments, that for any given $u \in \mathcal{U}$, the state and output trajectories x and y respectively are uniquely defined and absolutely continuous, that $f(0,0) = 0$ and $h(0,0) = 0$, and that $\dot{x} = f(x,0)$ is locally exponentially stable. Under these conditions there exists a unique (on $\mathcal{U}$) perfect local linear model G_{loc}^* of class $\tilde{\mathcal{G}}_{m,n}$ with

$$\lim_{\rho \to 0} \left\| Nu - G_{loc}^* u \right\|^{\rho \mathcal{U}} = \lim_{\rho \to 0} \phi_{AE,N}^{\rho \mathcal{U}} = 0 \tag{3.6}$$

and $G_{loc}^* : L_{pe}^m \to L_{pe}^n$ is given by the linear time-invariant state-space system

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu, \;\; \tilde{x}(0) = 0 \\ \tilde{y} &= C\tilde{x} + Du \end{aligned}$$

with $A = \frac{\partial f}{\partial x}(0,0)$, $B = \frac{\partial f}{\partial u}(0,0)$, $C = \frac{\partial h}{\partial x}(0,0)$ and $D = \frac{\partial h}{\partial u}(0,0)$.

P . Instead of considering $\mathcal{U}$, we can consider the sets $\mathcal{U}_T = \{u_T : u \in \mathcal{U}\} \subseteq L_p \cap L_\infty$ for all $T > 0$ (see Section 2). We introduce the error variables $y_e = y - \tilde{y}$ and $x_e = x - \tilde{x}$ and consider the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{x}_e &= Ax_e + \tilde{f}(x,u) \\ y_e &= Cx_e + \tilde{h}(x,u) \end{aligned}$$

with $\tilde{f}(x,u) = f(x,u) - Ax - Bu$ and $\tilde{h}(x,u) = h(x,u) - Cx - Du$. We have to show that, given any $\varepsilon > 0$ there is a ρ such that $\|y_e\| < \varepsilon \|u\|$ for all $u \in \mathcal{U}_T$ with $\|u\|_\infty < \rho$.

With $g(t) = Ce^{At}$ we have $y_e = g * \tilde{f}(x,u) + \tilde{h}(x,u)$ and thus $\|y_e\| \leq \|g\|_1 \cdot \left\| \tilde{f}(x,u) \right\| + \left\| \tilde{h}(x,u) \right\|$. The definition of $\tilde{f}(x,u)$ implies that for any $\varepsilon_f > 0$ there is a ρ_f with $\left\| \tilde{f}(x,u) \right\| < \varepsilon_f (\|x\| + \|u\|)$ for $\|x\|_\infty, \|u\|_\infty < \rho_f/2$, and analogously for $\tilde{h}(x,u)$ ($\tilde{f}$ and $\tilde{h}$ are functions of higher order than one). Hence for all $\varepsilon_f, \varepsilon_h > 0$ there are constants ρ_f, ρ_h such that $\|y_e\| \leq \|g\|_1 \cdot \varepsilon_f (\|x\| + \|u\|) + \varepsilon_h (\|x\| + \|u\|)$ whenever $\|x\|_\infty, \|u\|_\infty < \min\{\rho_f, \rho_h\}/2$. That G_{loc}^* is a best local linear model follows by observing that, under the given assumptions, there are $\mu, c > 0$ such that $\|x\|_p < c \|u\|_p$ for all $\|u\|_\infty < \mu$ and any $p \in [0, \infty]$ [136,

proof of Thm. 6.3.15]. (Given any ε, take for instance: $\varepsilon_f = \frac{\varepsilon}{2(1+c)\|g\|_1}$, $\varepsilon_h = \frac{\varepsilon}{2(1+c)}$, so that $\varepsilon = \|g\|_1 \cdot \varepsilon_f (1+c) + \varepsilon_h (1+c)$. Then $\|u\|_\infty < \rho = \min\{\mu, \frac{\rho_f}{2}, \frac{\rho_h}{2}, \frac{\rho_f}{2c}, \frac{\rho_h}{2c}\}$ implies $\|u\|_\infty < \mu$, $\|u\|_\infty < \min\{\frac{\rho_f}{2}, \frac{\rho_h}{2}\}$, $\|x\|_\infty < c\|u\|_\infty < \min\{\frac{\rho_f}{2}, \frac{\rho_h}{2}\}$ and finally $\|y_e\| \leq \|g\|_1 \cdot \varepsilon_f (c+1)\|u\| + \varepsilon_h (c+1)\|u\| = \varepsilon\|u\|$.) Consider a second linear model candidate G_2 that is different from G^*_{loc} on $\mathcal{U}$, i.e. $\left\|G_2 - G^*_{loc}\right\|^{\mathcal{U}} > 0$. Then $\lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}} \frac{\|Nu - G_2 u\|}{\|u\|} \geq \left\|G_2 - G^*_{loc}\right\|^{\mathcal{U}} - \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}} \frac{\|Nu - G^*_{loc}u\|}{\|u\|} > 0$ shows that G^*_{loc} is unique. □

Note that the local results we obtained so far concern the additive error nonlinearity measure and state the existence of a unique *perfect* local linear model. That does not only imply that there is no equally good linear model, but also the corresponding value of the nonlinearity measure is zero, indicating a perfect match. Of course, only the input-output behaviour of the best linear model is unique, while different state space representation do exist.

The state-space linearization does not need to be a perfect local linear model with respect to the IMOE-NLM (see example below). This is due to the fact that for the IMOE-NLM, the existence of a perfect local linear model is not guaranteed in general. If it exists, however, it can be shown to be unique and to be identical to the state-space linearization.

C 3.17. *If for a nonlinear operator $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ with a non-vanishing local gain $\lim_{\rho\to 0} \|N\|^{\rho\mathcal{U}} > 0$ there is a perfect local model G with*

$$\lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} = 0$$

then this model is unique and

$$G = G^*_{loc} \text{ with } \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\left\|Nu - G^*_{loc}u\right\|}{\|u\|} = 0.$$

P . The equation

$$\begin{aligned} \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} &= \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} \frac{\|u_T\|}{\|(Nu)_T\|} \\ &\geq \lim_{\rho\to 0} \frac{1}{\|N\|^{\rho\mathcal{U}}} \cdot \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} \end{aligned}$$

shows that $\lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu-Gu)_T\|}{\|(Nu)_T\|} = 0 \Rightarrow \lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}, T>0} \frac{\|(Nu-Gu)_T\|}{\|u_T\|} = 0$ and uniqueness follows as in Theorem 3.16. □

The corollary shows that only the state-space linearization can be a perfect local linear model in the IMOE setup (but it does not need to be one), while the example below shows that the state-space linearization does not need to be a perfect local linear model.

E . Consider the following systems with their state-space linearizations

$$N_1 : \begin{cases} \dot{x} = -x + u \\ y = x + x^3 \end{cases} \qquad G^*_{1,loc} : \begin{cases} \dot{x} = -x + u \\ y = x \end{cases}$$

and

$$N_2 : \left\{ \begin{array}{rcl} \dot{x} & = & -x+u \\ y & = & x+x^3-u \end{array} \right. \qquad G^*_{2,loc} : \left\{ \begin{array}{rcl} \dot{x} & = & -x+u \\ y & = & x-u. \end{array} \right.$$

Clearly, all systems have finite gain for any bounded operating region $u \in \mathcal{U}(V)$,$V = [\underline{v}, \overline{v}]$. It is also true, by Theorem 3.16, that $G^*_{1,loc}$ ($G^*_{2,loc}$) is the unique perfect local linear model with respect to the additive error setup of N_1 (N_2). Note that for system N_1, $\|x_T\|_\infty \leq \|u_T\|_\infty$ and

$$\limsup_{\rho\to 0} \sup_{u\in\rho\mathcal{U}} \frac{\left\|\left(Nu - G^*_{loc}u\right)_T\right\|}{\|(Nu)_T\|} \leq \lim_{\rho\to 0} \sup_{|x|\leq\rho \max\{\underline{v},\overline{v}\}} \frac{\left|x^3\right|}{\left|x+x^3\right|} = 0.$$

Thus, $G^*_{1,loc}$ is a (unique, by Corollary 3.17) perfect local model for N_1 w.r.t. the IMOE setup. This is not necessarily the case, as shown by system N_2. Considering the steady-state, we get $\phi^{\mathcal{U}(V)}_{IMOE,N_2} = 1$ and $\lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}} \frac{\left\|\left(N_2u - G^*_{2,loc}u\right)_T\right\|}{\|(N_2u)_T\|} \geq \lim_{\rho\to 0} \sup_{v\in\rho V} \frac{|v^3|}{|v^3|} = 1$. At least for the case $p = 2$ it can be shown with SOS techniques (see Section 3.8) that $\lim_{\rho\to 0} \sup_{u\in\rho\mathcal{U}} \frac{\left\|\left(N_2u - G^*_{2,loc}u\right)_T\right\|}{\|(N_2u)_T\|} \leq 1$, and the best local models result in the highest possible value of the nonlinearity measure. Moreover, the zero operator achieves the same value of the error system gain, and the best model is not unique. ■

In this context it is also in order to note that the gain of a perfect local linear model G^*_{loc} with respect to the AE-NLM gives a lower bound on the gain of the nonlinear system N.

P 3.18. *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be a finite gain stable mapping with $\mathcal{U}$ satisfying $\rho\mathcal{U} \subseteq \mathcal{U}$ for $0 \leq \rho \leq 1$. The existence of a perfect local linear model G^*_{loc} for the set $\mathcal{U}$ such that $\lim_{\rho\to 0} \left\|N - G^*_{loc}\right\|^{\rho\mathcal{U}} = 0$ implies $\|N\|^{\mathcal{U}} \geq \left\|G^*_{loc}\right\|^{\mathcal{U}}$.*

P . By the definition of a perfect local linear model, for any $\varepsilon > 0$ there is a ρ such that

$$\left\|N - G^*_{loc}\right\|^{\rho\mathcal{U}} \leq \varepsilon.$$

The inequality $-\|(Nu)_T\| + \left\|\left(G^*_{loc}u\right)_T\right\| \leq \left\|\left(Nu - G^*_{loc}u\right)_T\right\|$ and linearity of G^*_{loc} show that for any $\varepsilon > 0$ there is a ρ such that $\|N\|^{\mathcal{U}} \geq \|N\|^{\rho\mathcal{U}} = \left\|G^*_{loc}\right\|^{\rho\mathcal{U}} - \left\|N - G^*_{loc}\right\|^{\rho\mathcal{U}} \geq \left\|G^*_{loc}\right\|^{\mathcal{U}} - \varepsilon$. □

The above proposition is true for the linearization in the sense of Fréchet as well as for the state-space linearization given above, depending on the chosen set $\mathcal{U}$ (note that $\mathcal{U} \subseteq L^m_\infty$ does *not* contain a neighborhood of the origin in L^m_p for $p < \infty$). We can now state that the achievable linear model error with respect to the additive error nonlinearity measure is at best half the model error corresponding to a perfect local linear model.

P 3.19. *Let $N : \mathcal{U} \subseteq L^m_{pe} \to L^n_{pe}$ be a finite gain stable mapping with $\mathcal{U}$ satisfying $\rho\mathcal{U} \subseteq \mathcal{U}$ for $0 \leq \rho \leq 1$. The existence of a perfect local linear model G^*_{loc} for the set $\mathcal{U}$ such that $\lim_{\rho\to 0} \left\|N - G^*_{loc}\right\|^{\rho\mathcal{U}} = 0$ implies $\phi^{\mathcal{U}}_{AE,N} \geq \frac{1}{2} \left\|N - G^*_{loc}\right\|^{\mathcal{U}}$.*

P . From the definition of the perfect local model it is obvious that for any linear operator $G \in \mathcal{G}_{m,n}$, $G^*_{loc} - G$ is a perfect linear model for $N - G$. Hence, by Proposition 3.18, $\left\|N - G^*_{loc}\right\|^{\mathcal{U}} - \|N - G\|^{\mathcal{U}} \leq \left\|G - G^*_{loc}\right\|^{\mathcal{U}} \leq \|N - G\|^{\mathcal{U}}$ for all $G \in \mathcal{G}$. □

Summary. In this section, the local behaviour of nonlinear systems has been considered. It has been shown that care must be taken when using operator derivatives, as the choice of the signal norm determines the meaning of the term "local" in the mathematical sense. In a control context, the most useful meaning of local behaviour is the behaviour for signals of small amplitude. As could be expected, with respect to this notion the first-order Taylor approximation of the state space equations deliver a perfect local model for the additive error setup. While it may also be a perfect local model for other setups, it has been shown that this is not necessarily the case.

3.4. Memoryless Systems and Steady-State Behaviour of Dynamical Systems

Next, we derive results on nonlinearity measures and best linear models based on special structural properties of systems. We first consider systems with a given steady-state response, before we turn our attention to memoryless operators. In this section, the exposition concerns the additive error setup only, while the next section gives corresponding results for all other setups. In this and the next section, it will be a standing assumption[7] on the set $\mathcal{U}$ that $\|G\|^{\mathcal{U}} < \infty$ implies $\|G\| < \infty$ for all $G \in \mathcal{G}_{m,n}$, i.e. if a system G has finite gain on $\mathcal{U}$, then it has finite gain on L^m_{pe}.

Recall that the additive error nonlinearity measure just corresponds to the gain of the system $N - G^*$. Therefore, we derive results for the gain of a general nonlinear system that can then be applied to the system $N - G^*$.

Steady-State Behaviour of Dynamical Systems. The first result we are going to present concerns the gain of stable nonlinear systems featuring a steady state response to a steady state input. The intuitive idea is that the system gain is bounded from below by the gain of the function that describes the steady state locus. This idea is formalized as follows.

T 3.20. *Consider a nonlinear system defined by the causal, finite gain stable mapping $P : u \in \mathcal{U} \subseteq L^m_{pe} \mapsto y \in L^n_{pe}$. Consider the function $f : V \subseteq \mathbb{R}^m \to \mathbb{R}^n$ satisfying $|f(v)| < \infty$ for all $v \in V$ and assume that P has a steady state locus on V with respect to $\mathcal{U}$ given by f in the sense that for any $v \in V$ there is a $u \in \mathcal{U} \subseteq L^m_{pe}$ with $u(t) \to v$ and $y(t) = (Pu)(t) \to f(v)$ for $t \to \infty$. If $p = \infty$ assume furthermore that $|u(t)| < |v|$ for all $t \geq 0$. Under these conditions*

$$\|P\|^{\mathcal{U}} \geq |f|^V \triangleq \sup_{v \in \mathcal{V}} \frac{|f(v)|}{|v|}. \tag{3.7}$$

[7] Note that a linear state space system is stable if and only if the matrix A is Hurwitz, and in such a case, if $\lim_{t\to\infty} u(t) = u_0$ and $u(t)$ is bounded for all times then $\lim_{t\to\infty} y(t) = y_0 = \left(CA^{-1}B + D\right)u_0 = Ku_0$.

P . We define the set $\mathcal{U}^* \subseteq \mathcal{U}$ of functions $u \in L^m_{pe}$ with $u(t) \to v \in V$ for $t \to \infty$. Consider the following (in)equalities

$$\begin{aligned} \|P\|^{\mathcal{U}} &= \sup_{u\in\mathcal{U},T>0} \frac{\|(Pu)_T\|}{\|u_T\|} \\ &\geq \sup_{u^*\in\mathcal{U}^*} \limsup_{T\to\infty} \frac{\|(Pu^*)_T\|}{\left\|u^*_T\right\|} \\ &= \sup_{v\in\mathcal{V}} \frac{|f(v)|}{|v|}. \end{aligned}$$

The first inequality is obvious. To prove the second equality, we first consider the case $p < \infty$. For any $u^* \in \mathcal{U}^*$ we have $(Pu^*)(t) \to f(v)$ as $t \to \infty$. Note that, if $x(t) \to a$ for $t \to \infty$ then for any $\varepsilon > 0$ there is a T_1 such that $|x(t) - a| < \varepsilon$ for all $t > T_1$. Hence, for any $\varepsilon > 0$

$$\left(\limsup_{T\to\infty} \frac{\|(x-a)_T\|}{T}\right)^p \leq \limsup_{T\to\infty} \frac{\left\|(x-a)_{T_1}\right\|^p + \varepsilon^p(T-T_1)}{T} = \varepsilon^p$$

and $\lim_{T\to\infty} \frac{\|(x-a)_T\|}{T^{1/p}} = 0$. From $\|(x-a)_T\| \geq \|\|x_T\| - \|a_T\|\| \geq \left|\|x_T\| - |a|\,T^{1/p}\right| \geq 0$ follows that $\lim_{T\to\infty} \frac{\|x_T\|}{T^{1/p}} = |a|$ and thus

$$\lim_{T\to\infty} \frac{\|(Pu^*)_T\|}{\left\|u^*_T\right\|} = \frac{|f(v)|}{|v|}$$

and the result is proven for $p < \infty$. If $p = \infty$ we additionally require for $u \in \mathcal{U}^*$ that $|u(t)| < |v|$ for all $t \geq 0$. Then $\left\|u^*_T\right\|_\infty = |v|$ and $\lim_{T\to\infty} \|(Pu^*)_T\|_\infty \geq |f(v)|$ and we directly get

$$\limsup_{T\to\infty} \frac{\|(Pu^*)_T\|_\infty}{\left\|u^*_T\right\|_\infty} \quad \geq \quad \frac{|f(v)|}{|v|}$$

and the proof is complete. □

A nonlinear operator can have multiple steady-states, so f is not necessarily unique. Theorem 3.20 then applies to any such function. Theorem 3.20 also allows to infer that a dynamic system with the described steady state behaviour has infinite gain if the corresponding steady state locus does.

The above result can now be directly applied to derive a lower bound for the AE-NLM.

C 3.21. *Consider the same conditions as in Theorem 3.20. Then*

$$\phi^{\mathcal{U}}_{AE,N} = \inf_{G\in\mathcal{G}_{m,n}} \|Nu - Gu\|^{\mathcal{U}} \geq \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in V} \frac{|f(v) - Kv|}{|v|}. \tag{3.8}$$

P . Follows immediately from Theorem 3.20. □

The formulae given by (3.7) and (3.8) allow to derive lower bounds on the gain and the additive error nonlinearity measure at very low computational cost, as only a static optimization problem has to be solved.

Memoryless Systems. The next theorem deals with the gain of memoryless operators, i.e. operators given by a static nonlinear function. In fact, it is shown that the lower bound given by Theorem 3.20 is equal to the gain of the nonlinear memoryless operator defined by the function $f(v)$. To make things clearer, we will use as an example the family of signal sets $\mathcal{U}(V)$ introduced above. Recall that $\mathcal{U}(V)$ contains all functions in $L^m_{\infty e} \subseteq L^m_{pe}$ that have values in $V \subseteq \mathbb{R}^m$ for all times.

T 3.22. *Let the operator $P_f : u \in L^m_{pe} \mapsto y \in L^n_{pe}$ be given by*

$$y(t) = (P_f u)(t) \triangleq f(u(t)) \ \forall t \tag{3.9}$$

where $f : V \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a function satisfying $|f(v)| < \infty$ for all $v \in V$. Let the set $\mathcal{U}$ be such that $u \in \mathcal{U} \Rightarrow u(t) \in V \, \forall t$. Assume that for any $v \in V$ there is a $u \in \mathcal{U} \subseteq L^m_{pe}$ with $u(t) \to v$ and $(P_f u)(t) \to f(v)$ for $t \to \infty$. If $p = \infty$ assume furthermore that $|u(t)| < |v|$ for all $t \geq 0$. Then the following equivalence holds

$$\left\|P_f\right\|^{\mathcal{U}} = \left\|P_f\right\|^{\mathcal{U}(V)} = |f|^V = \sup_{v \in V} \frac{|f(v)|}{|v|}. \tag{3.10}$$

P . First note that $\mathcal{U}(V)$ satisfies the assumptions on $\mathcal{U}$, so $\mathcal{U} = \mathcal{U}(V)$ is just a special case. Let $\gamma \triangleq \sup_{v \in V} |f(v)| / |v| \geq 0$ and note that the inequality $|f(v)| \leq \gamma |v|$ holds for any $v \in V$. For $p = \infty$

$$\frac{\left\|(P_f u)_T\right\|_\infty}{\|u_T\|_\infty} = \frac{\operatorname{ess\,sup}_{0 \leq t \leq T} |f(u(t))|}{\operatorname{ess\,sup}_{0 \leq t \leq T} |u(t)|} \leq \gamma$$

and similarly for $p < \infty$

$$\frac{\|(Pu)_T\|}{\|u_T\|} = \frac{\left(\int_0^T |f(u(t))|^p \, \mathrm{d}t\right)^{\frac{1}{p}}}{\left(\int_0^T |u(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}} \leq \gamma$$

for all $u \in \mathcal{U}$ and $T > 0$ and thus

$$\sup_{u \in \mathcal{U}, T > 0} \frac{\left\|(P_f u)_T\right\|}{\|u_T\|} \leq \gamma = \sup_{v \in V} \frac{|f(v)|}{|v|}.$$

The result then follows by application of Theorem 3.20, considering that the assumptions imply that f is a steady state of P_f on V with respect to $\mathcal{U}$ (and $\mathcal{U}(V)$). □

In direct application, Theorem 3.22 is most useful when considering a set $\mathcal{U}(V)$ of admissible inputs. However, we will also use it later as an intermediate result for sets with less structure. The minimum assumptions on $\mathcal{U}$ require that $u(t) \in V$ in order to get the upper bound. Also the steady-state behaviour is here rather a condition on $\mathcal{U}$ than on f. It is for instance fulfilled if for any $v \in V$ there is an input signal that tends to v (if f is continuous, otherwise we need to have a $u(t) \equiv v$).

Note that (3.10) holds regardless of the value of $p \in [1, \infty]$, and so all L_p-gains of P_f and R_r respectively are equal. This is due to the definition of the signal norm, where the vector

norm is always taken to be the Euclidean norm. As all norms in $\mathbb{R}^n$ are equivalent, the input-output stability (bounded-input-bounded-output or finite-gain stability) of a nonlinear mapping is independent of the chosen norm in $\mathbb{R}^n$, and, contrary to dynamic systems, finite-gain stability of memoryless operators is also independent of the chosen signal space L_{pe}. Especially the L_∞-norm is often defined to be $\|u\|_\infty = \operatorname{ess\,sup}_{t\geq 0} \max_{k\leq n} |u_k(t)|$. As a matter of fact, only the norm properties of the Euclidean norm $|\cdot|$ are used for the proofs in this paper, and the Euclidean norm $|\cdot|$ can be substituted throughout by any other norm in $\mathbb{R}^n$, e.g. $\|x\|_p = \left(\sum_{k=1}^n |x_k(t)|^p\right)^{1/p}$. Only the values of the respective expressions are affected by a change of the norm in $\mathbb{R}^n$, but not the relations amongst them.

E . The memoryless operator given by $f(v) = \sqrt[3]{v}$ is L_∞-stable, but has no finite L_∞-gain due to the infinite slope for small v. Considering the input $u = (1+t)^{-3/p}$ it can easily be seen that the same system is not L_p-stable (and thus has no finite L_p-gain) for any $p < \infty$. As the steady-state map gives a lower bound on the gain of a nonlinear dynamical system, the system $\dot{y} = -y^3 + u$ does not have finite gain even though it is globally asymptotically stable. Moreover, for any initial condition y_0, $y(t) = \left(1 + 2ty_0^2\right)^{-\frac{1}{2}} y_0$ and thus $\int_0^\infty |y(t)|^p \, \mathrm{d}t = \infty$ for $p \leq 2$. As any state is reachable in finite time with bounded input, the system $\dot{y} = -y^3 + u$ is also not L_p-stable for $p \leq 2$. ■

Equipped with the results for gains above, it can now be shown that for memoryless nonlinear operators dynamic linear approximations are not superior in terms of the nonlinearity measure to memoryless linear approximations, i.e. to linear models from the set

$$\mathcal{G}_{m,n}^0 = \{G_0 : u \mapsto G_0 u | (G_0 u)(t) = Ku(t),\ K \in \mathbb{R}^{n\times m}\}.$$

Moreover the infinite-dimensional minimax problem is reduced to a finite-dimensional one.

C 3.23. *Let the operator $N_f : u \in L_{pe}^m \mapsto y \in L_{pe}^n$ and the set $\mathcal{U}$ satisfy the same assumptions as given in Theorem 3.22. Then the following equivalence holds*

$$\phi_{AE,N_f}^{\mathcal{U}} = \phi_{AE,N_f}^{\mathcal{U}(V)} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|}. \tag{3.11}$$

P . As $\mathcal{G}_{m,n}^0 \subset \mathcal{G}_{m,n}$ and from Theorem 3.22 we have

$$\phi_{AE,N}^{\mathcal{U}} = \inf_{G\in\mathcal{G}_{m,n}} \left\|N_f - G\right\|^{\mathcal{U}} \leq \inf_{G\in\mathcal{G}_{m,n}^0} \left\|N_f - G_0\right\|^{\mathcal{U}} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|}.$$

On the other hand, Theorem 3.20 ascertains that

$$\inf_{G\in\mathcal{G}_{m,n}} \left\|N_f - G\right\|^{\mathcal{U}} \geq \inf_{G\in\mathcal{G}_{m,n}} \left\|N_f - G_0\right\|^{\mathcal{U}} = \inf_{G\in\mathcal{G}_{m,n}^0} \left\|N_f - G_0\right\|^{\mathcal{U}}$$

which completes the proof. □

We can now also conclude that a best linear model for a nonlinear memoryless operator indeed exists, and that the associated optimization problem is convex.

C 3.24. *Let the finite-gain stable operator $N : u \in \mathcal{U} \subseteq L^m_{pe} \mapsto y \in L^n_{pe}$ be given. Then there exists a best linear static approximation $K^* \in \mathbb{R}^{n\times m}$ with*

$$\sup_{u\in\mathcal{U}} \frac{\|Nu - K^*u\|}{\|u\|} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{u\in\mathcal{U}} \frac{\|Nu - Ku\|}{\|u\|}. \tag{3.12}$$

and the optimization problem in K is convex.

P . The result follows from Theorem 3.11 with the required form, up to ordering,

$$K = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}E_{ij}$$

where E_{ij} is the matrix with an entry of one in the (i, j)-th element and zero in all other entries. □

Note that this theorem is also valid for a *dynamic* nonlinear system that is approximated by a linear static system.

It is worth noting that the best linear approximation in Corollary 3.23 is adapted to the worst case of functions with values in V in the set of considered inputs $\mathcal{U}$. If different function sets are considered, the result may be different. In the L_2-case it is, for instance, straightforward to show that the optimal static linear approximation $K \in R^{n\times m}$ of *one* pair of input/output trajectories $(u, y) \in L^m_p \times L^n_p$ is given by

$$K^* = \left(\int_0^\infty u(t)u^T(t)\mathrm{d}t\right)^{-1}\left(\int_0^\infty y(t)u^T(t)\mathrm{d}t\right)$$

and does *not* correspond to the solution of the optimization problem (3.12). Moreover, the best dynamic approximation will in most cases be *superior* to a static approximation.

Scalar Memoryless Systems. In the case of a scalar function $f : \mathbb{R} \to \mathbb{R}$ the additive error nonlinearity measure can be determined more explicitly. This result is established by the following theorem. Moreover, it establishes the uniqueness of the best linear approximation.

T 3.25. *For a given function $f : V \subseteq \mathbb{R} \to \mathbb{R}$ define*

$$k^+ \triangleq \sup_{v\in V} \frac{f(v)}{v}$$

$$k^- \triangleq \inf_{v\in V} \frac{f(v)}{v}$$

and define k^* by

$$k^* \triangleq \begin{cases} \frac{1}{2}(k^+ + k^-) & \text{if } |k^+k^-| < \infty \\ 0 & \text{else} \end{cases}.$$

Then,

$$\phi^{\mathcal{U}(V)}_{AE,N_f} = \inf_{k\in\mathbb{R}} \sup_{v\in V} \frac{|f(v) - kv|}{|v|} = \begin{cases} \frac{1}{2}(k^+ - k^-) & \text{if } |k^+|, |k^-| < \infty \\ \infty & \text{else} \end{cases}$$

and if the infimum is finite it is uniquely achieved for $k = k^*$.

Note that we do not require continuity or differentiability of f, but for consistency we define $k^+ = \infty$ or $k^- = -\infty$ respectively if $f(0) \neq 0$. The nonlinearity measure is defined to be infinite in this case (it is, indeed, not well-defined, as N_f has no finite gain).

P . In the scalar case, the additive error nonlinearity measure of a static function can be reformulated as

$$\inf_{k\in\mathbb{R}} \sup_{v\in V} \frac{|f(v) - kv|}{|v|} = \inf_{k\in\mathbb{R}} \sup_{v\in V} \left| \frac{f(v)}{v} - k \right|.$$

The cases in which $|k^+| = \infty$ or $|k^-| = \infty$ are obvious. In all other cases we have for $k = k^*$

$$\sup_{v\in V} \left| \frac{f(v)}{v} - k^* \right| = \left| k^+ - k^* \right| = \left| k^- - k^* \right| = \frac{1}{2}\left(k^+ - k^-\right).$$

Note that we always have $k^- \leq k^* \leq k^+$. We can now distinguish the two cases

$$k^- \leq k^* < k: \quad \sup_{v\in V} \left|\tfrac{f(v)}{v} - k\right| \quad \geq k - k^- > k^* - k^- = \frac{1}{2}\left(k^+ - k^-\right)$$

$$k < k^* \leq k^+: \quad \sup_{v\in V} \left|\tfrac{f(v)}{v} - k\right| \quad \geq k^+ - k > k^+ - k^* = \frac{1}{2}\left(k^+ - k^-\right).$$

We have shown that no other k than k^* can achieve the same or a smaller value of the nonlinearity gain and the proof is complete. □

What we rediscovered here using a novel framework are of course the famous sector conditions used in the theory of absolute stability, e.g. to derive the circle criterion [145]. As

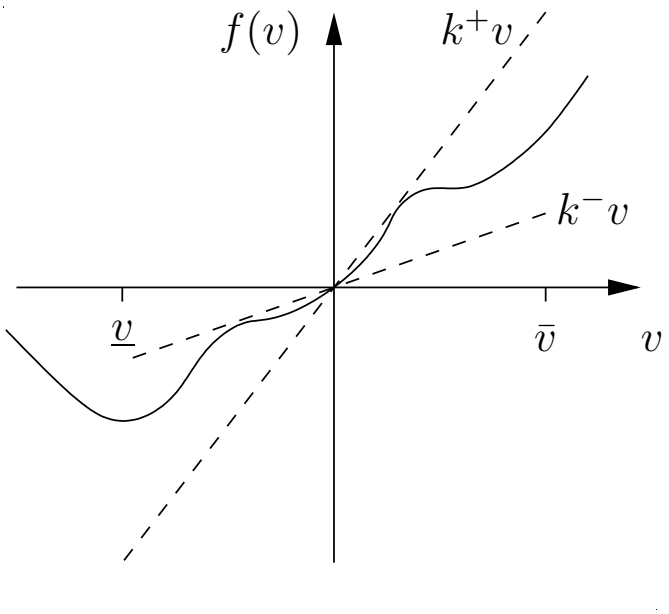
$$k^- v^2 \leq f(v)v \leq k^+ v^2 \; \forall v \in V$$

the sector in which f lies is given by the slopes k^+ and k^- and by definition, those bounds are the tightest bounds possible. Note that f is to lie in the sector $[k^-, k^+]$ only for $v \in \mathcal{V}$ and may lie outside for $v \notin V$, see Fig. 3.4.

E 3.26. We consider the example of a static nonlinearity $f(v) = v + v^3$ in the range of $\mathcal{V} = [0, 2]$. One sector bound is given by the derivative at the origin ($k^- = 1$), and the second bound by the point at the border of the considered interval ($k^+ = 5$). With the given formula, the best linear approximation is $k^* = 3$ and the nonlinearity measure becomes $\phi^{\mathcal{U}}_{AE,N_f} = 2$.

Now consider the nonlinearity measure for only one input signal $\mathcal{U} = \left\{u(t) = 2\,t\,e^{1-t}\right\}$. The input signal takes values in the interval $[0, 2]$ and lies in the range $\mathcal{V}$ that was considered before. For the given pair of trajectories, the best static approximation is given by $\tilde{k}^* = \frac{\int uy\mathrm{d}t}{\int u^2\mathrm{d}t} = 3.77$ which results in a nonlinearity measure of $\inf_{k\in R} \frac{\|N_f u - ku\|}{\|u\|} = \frac{\int u^2\mathrm{d}t \int y^2\mathrm{d}t - \left(\int uy\mathrm{d}t\right)^2}{\left(\int u^2\mathrm{d}t\right)^2} = 1.14$. The latter static approximation performs better for this specific input trajectory than the first approximation (which gives a measure of $\|y - k^*u\| / \|u\| = 1.37$), but it will of course be worse for other signals with values in $\mathcal{V}$, e.g. for small step functions of height 0.01 we have $\left\|y - \tilde{k}^*u\right\| / \|u\| = 2.76 > \|y - k^*u\| / \|u\| = 1.99$.

Still better for the single trajectory case is a dynamic linear approximation. The Laplace transform of both, input and output of the operator N_f can easily be calculated and an



F 3.4. A nonlinear function and its sector bounds for $V = [\underline{v}, \overline{v}]$.

LTI system that realizes the corresponding input-output-behaviour is given by the transfer function $G(s) = \frac{Y(s)}{U(s)} = 1 + \frac{24e^2(1+s)^2}{(3+s)^4}$ and this linear model achieves a perfect match: $N_f u = Gu$. ■

Notion of a Generalized Sector. Interestingly, an analogy shows that the notion of a sector can be generalized to dynamic systems in the case of L_2-signals. Note that the inequality $k^- v^2 \leq f(v)v \leq k^+ v^2 \; \forall v \in \mathcal{V}$ can also be written as $(k^- v - f(v))\,(k^+ v - f(v)) \leq 0 \; \forall v \in V$. In the same way, a sector for dynamic systems can be defined by $\langle Nu - G_1 u | Nu - G_2 u \rangle \leq 0 \; \forall u \in \mathcal{U}$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in L_2. The geometric interpretation is that "G_1 and G_2 lie on different sides of N", i.e. the vectors $Nu - G_1 u$ and $Nu - G_2 u$ point in opposite directions for all $u \in \mathcal{U}$. Let $\mathcal{S}$ be the set of all pairs $\{G_1, G_2\}$ defining a sector for N in the aforementioned sense. We then have the following alternative characterization of the additive error nonlinearity measure in L_2.

P 3.27. *Let $N : u \in \mathcal{U} \subseteq L^m_{pe} \mapsto y \in L^n_{pe}$ be a finite-gain stable operator and let the set $\mathcal{S}$ be defined as*

$$\begin{aligned} \mathcal{S}_N^{\mathcal{U}} \quad = \quad & \Big\{ \{G_1, G_2\} : G_1, G_2 \in \mathcal{G}_{m,n}, \\ & \langle Nu - G_1 u | Nu - G_2 u \rangle \leq 0 \; \forall u \in \mathcal{U} \Big\}. \end{aligned}$$

Under this condition

$$\phi^{\mathcal{U}}_{AE,N} = \inf_{G \in \mathcal{G}_{m,n}} \|N - G\|_2^{\mathcal{U}} = \inf_{\{G_1, G_2\} \in \mathcal{S}_N^{\mathcal{U}}} \|G_1 - G_2\|_2^{\mathcal{U}} .$$

P . The result follows from

$$\begin{aligned} & \exists G_1 \in \mathcal{G}_{m,n}, G_2 \in \mathcal{G}_{m,n} : && \left\langle Nu - G_1 u \middle| Nu - G_2 u \right\rangle \leq 0 \; \forall u \in \mathcal{U} \\ \iff \quad & \exists G \in \mathcal{G}_{m,n}, \Gamma \in \mathcal{G}_{m,n} : && \|Nu - Gu\|^2 - \|\Gamma u\|^2 \leq 0 \; \forall u \in \mathcal{U} \\ \iff \quad & \exists G \in \mathcal{G}_{m,n}, \gamma \in \mathbb{R} : && \|Nu - Gu\| \leq \gamma \|u\| \; \forall u \in \mathcal{U} \end{aligned}$$

where the first step is obvious by $G = \frac{1}{2}(G_1 + G_2)$ and $\Gamma = \frac{1}{2}(G_1 - G_2)$ and the second step is verified by considering $\Gamma = \gamma I$ and $\gamma = \sup_{u \in \mathcal{U}} \frac{\|\Gamma u\|}{\|u\|}$ respectively. □

The proof essentially reveals that, when searching for linear systems $\{G_1, G_2\}$ that define a generalized sector for the nonlinear system N, it is sufficient to consider systems differing only by a multiple of the identity mapping. The smallest factor occurring is at the same time a measure of the "aperture" of the sector and corresponds to the nonlinearity measure. Note that this definition of sector is different from the one frequently seen in literature [145, 140], where a generalized sector is defined as $\left\langle Nu - au \middle| Nu - bu \right\rangle \leq 0$.

Summary. In this section, it was shown for the gain and the additive error nonlinearity measure how lower bounds can be derived by the steady state behaviour of a system. Moreover, the gain and the nonlinearity measure of a memoryless system can be characterized by a "static" optimization problem that has a particularly simple solution in the case of scalar systems. The result for the lower bound allows to very easily derive a first estimate of the degree of nonlinearity of a general system. The results for memoryless systems show that in a practical context, dynamic linear models cannot approximate a memoryless nonlinear system better than a memoryless linear model. Moreover, these results will turn out to be useful for the characterization of systems composed of dynamic linear subsystems and memoryless nonlinear subsystems in Section 3.6.

3.5. More Results on Memoryless Systems and Steady-State Behaviour

This section is devoted to the derivation of similar results as above for the other setups for nonlinearity assessment and linear modeling. In some cases, it will however be necessary to slightly modify the statement due to the different mathematical properties of the setups. Most importantly, we will have to deal with relations instead of mappings, and give the respective generalizations of Theorems 3.20 and 3.22 first. After the results for all error setups are presented, a discussion follows that points out the main differences for the different setups. At the end of this section we give conditions under which the results can be extended to classes of infinite-dimensional linear models.

Gains of Relations. The result giving a lower bound on the gain of a nonlinear system by its steady state behaviour can be transferred to the case of a relation without technical difficulties.

C 3.28. *Consider a nonlinear system defined by the causal, finite gain stable relation $R \subseteq L^m_{pe} \times L^n_{pe}$ that is defined for inputs in $\mathcal{U}$, i.e. $u \in \mathcal{U} \Rightarrow \exists y \in L^n_{pe} : (u,y) \in R$. Consider the relation $r \subseteq \mathbb{R}^m \times \mathbb{R}^n$, defined on $V \subseteq \mathbb{R}^m$, i.e. $v \in V \Rightarrow \exists z \in \mathbb{R}^n : (v,z) \in r$. Assume that R has a steady state locus on V given by r in the sense that for any $(v,z) \in r, v \in V$ there is a pair $(u,y) \in R$ with $u \in \mathcal{U}$ such that $u(t) \to v$ and $y(t) \to z$ for $t \to \infty$. If $p = \infty$ assume further more that $|u(t)| < |v|$ for all $t \geq 0$. Under these conditions*

$$\|R\|^{\mathcal{U}} \geq |r|^V \triangleq \sup_{(v,z)\in r, v\in V} \frac{|z|}{|v|}.$$

P . Considering the relations

$$\begin{aligned}
\|R\|^{\mathcal{U}} &= \sup_{(u,y)\in R, u\in\mathcal{U}, T>0} \frac{\|y_T\|}{\|u_T\|} \geq \sup_{(u^*,y^*)\in R, u^*\in\mathcal{U}^*} \limsup_{T\to\infty} \frac{\left\|y^*_T\right\|}{\left\|u^*_T\right\|} = \sup_{(v,z)\in r, v\in V} \frac{|z|}{|v|} \\
&\geq \sup_{(u^*,y^*)\in R, u^*\in\mathcal{U}^*} \limsup_{T\to\infty} \frac{\left\|y^*_T\right\|}{\left\|u^*_T\right\|} \\
&= \sup_{(v,z)\in r, v\in V} \frac{|z|}{|v|}
\end{aligned}$$

the proof follows as in Theorem 3.20. □

The gain of a memoryless relation follows immediately from Theorem 3.22, and we state it without proof.

C 3.29. *Let the relation $R_r \subseteq L^m_{pe} \times L^n_{pe}$ be given by*

$$R_r \triangleq \left\{ (u,y) \in L^m_{pe} \times L^n_{pe} \middle| (u(t), y(t)) \in r \; \forall t \geq 0 \right\}$$

where $r \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a relation defined on V. Assume that for any $(v,z) \in r, v \in V$ there is a $(u,y) \in R, u \in \mathcal{U}$ with $u(t) \to v$ and $y \to z$ for $t \to \infty$. If $p = \infty$ assume further more that $|u(t)| < |v|$ for all $t \geq 0$. Then the following equivalence holds

$$\|R_r\|^{\mathcal{U}} = \|R_r\|^{\mathcal{U}(V)} = |r|^V \triangleq \sup_{(v,z)\in r, v\in V} \frac{|z|}{|v|}. \tag{3.13}$$

We are now equipped with the necessary tools to give the results corresponding to the ones from Section 3.4 for the other nonlinearity measures.

Results for the IMOE-NLM. Similar results as for the AE-NLM concerning the steady-state locus and the nonlinearity measure of memoryless operators can be given for the IMOE-NLM.

T 3.30. *Consider the same conditions as given in Theorems 3.22 and 3.20. Then* $\phi^{\mathcal{U}}_{IMOE,N} \geq \phi^{\mathcal{U}}_{IMOE,N_f} = \phi^{\mathcal{U}(V)}_{IMOE,N_f} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in V} \frac{|f(v)-Kv|}{|f(v)|}$.

P . Define the relation $R(G) = \left\{(y,e) \in L^n_{pe} \times L^n_{pe} \middle| y = Nu, e = Nu - Gu, u \in \mathcal{U}\right\}$. For any stable $G \in \mathcal{G}_{m,n}$, $r(K) = \left\{(z,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \middle| z = f(v), \eta = f(v) - Kv, v \in V\right\}$ is a steady-state of $R(G)$ on $f(V)$. Then, by Corollary 3.28,

$$\sup_{u\in\mathcal{U},T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} = \|R(G)\|^{N\mathcal{U}} \geq |r(K)|^{f(V)} = \sup_{v\in V} \frac{|f(v) - Kv|}{|f(v)|}$$

and the relation also holds if N is substituted by N_f. As for any $(z,\eta) \in r(K), z \in f(V)$, there is a $(y,e) \in R, y \in N_f\mathcal{U}$ such that $y(t) \to z$ and $e(t) \to \eta$ for $t \to \infty$, and for all $y \in N_f\mathcal{U}$, $y(t) \in f(V)$, Corollary 3.29 delivers

$$\begin{aligned} \phi^{\mathcal{U}}_{IMOE,N_f} &= \inf_{G\in\mathcal{G}_{m,n}} \|R(G)\|^{N_f\mathcal{U}} \\ &\leq \inf_{G_0\in\mathcal{G}^0_{m,n}} \|R(G_0)\|^{N_f\mathcal{U}} = \inf_{K\in\mathbb{R}^{n\times m}} |r(K)|^{f(V)} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in V} \frac{|f(v) - Kv|}{|f(v)|} \end{aligned}$$

and the fact that $\mathcal{U}(V)$ satisfies the same assumptions as $\mathcal{U}$ completes the proof. □

According to the definition in Chapter 2 the expression

$$\phi^{\mathcal{U}(V)}_{IMOE,N_f} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in V} \frac{|f(v) - Kv|}{|f(v)|}$$

is to be understood in the sense of

$$\phi^{\mathcal{U}(V)}_{IMOE,N_f} = \inf\left\{\varphi \in \mathbb{R} | \exists K \in \mathbb{R}^{n\times m} : |f(v) - Kv| \leq \varphi\,|f(v)|\ \forall v \in \mathcal{V}\right\}.$$

If the IMOE nonlinearity measure is finite, then from $f(v) = 0$ follows $f(v) = Kv = 0$. On the other hand, $Kv = 0$ implies that either $f(v) = 0$ or $\phi = 1$. As a consequence, either the set $\mathcal{V}_0$ of all points v for which $f(v) = 0$ (and all limit points $\bar{v}$ of sequences $\{v_i\}$ for which $f(v_i) \to 0$) is equal to $\ker K \cap \mathcal{V}$ or we have that $\phi^{\mathcal{U}(V)}_{IMOE,N_f} = 1$. For rectangular systems with more inputs than outputs ($m > n$), the relation $f(v) = 0 \leftrightarrow Kv = 0$ is very unlikely to hold for any K and thus the IMOE nonlinearity measure, being almost always equal to one, is of restricted use for such systems. This fact is consistent with the well-posedness conditions in Section 3.2, where we required the linear model to be injective. But injectivity also requires in general (except for pathological cases) that $m \leq n$. In the cases where $m \leq n$, the set $\mathcal{V}_0$ will usually have only few elements, and the nonlinearity measure is likely to deliver useful information.

In the case of scalar functions, we can again obtain an explicit solution.

T 3.31. *Let f, k^+ and k^- be as above and define k^* by*

$$\begin{cases} \frac{1}{k^*} \triangleq \frac{1}{2}\left(\frac{1}{k^+} + \frac{1}{k^-}\right) & \text{if } 0 < k^+k^-, |k^+|, |k^-| < \infty \\ k^* \triangleq 0 & \text{else} \end{cases}.$$

Then

$$\phi^{\mathcal{U}(V)}_{IMOE,N_f} = \inf_{k\in\mathbb{R}} \sup_{v\in V} \frac{|f(v) - kv|}{|f(v)|} = \begin{cases} \left|\frac{k^+ - k^-}{k^+ + k^-}\right| & \text{if } 0 < k^+k^-, |k^+|, |k^-| < \infty \\ 0 & \text{if } k^+ = k^- = 0 \\ 1 & \text{else} \end{cases}.$$

and the infimum is uniquely achieved for $k = k^*$ if k^+ and k^- are finite.

P . In the scalar case, the nonlinearity measure of a static function can be reformulated as

$$\inf_{k\in\mathbb{R}} \sup_{v\in V} \frac{|f(v) - kv|}{|f(v)|} = \inf_{k\in\mathbb{R}} \sup_{v\in V} \left|1 - \frac{k}{f(v)/v}\right|.$$

We first consider the case $k^+k^- \leq 0$. The case $k^+ = k^- = 0$ is obvious. If either $k^+ = 0$ or $k^- = 0$ then $k = 0$ is the only value for which $\phi^{\mathcal{U}(V)}_{IMOE,N_f}$ is finite and $\phi^{\mathcal{U}(V)}_{IMOE,N_f} = 1$, so it suffices to consider the case $k^- < 0 < k^+$. If $k = 0$ we have again $\phi^{\mathcal{U}(V)}_{IMOE,N_f} = 1$. If $k > 0$ then $\phi^{\mathcal{U}(V)}_{IMOE,N_f} \geq |1 - k/k^-| > 1$ and if $k < 0$ then $\varphi^V_f \geq |1 - k/k^+| > 1$. Hence $\phi^{\mathcal{U}(V)}_{IMOE,N_f} = 1$ and this value is uniquely achieved for $k = 0$.
Next, consider the case when either $|k^+| = \infty$ or $|k^-| = \infty$. Clearly, for any $k \neq 0$: $\sup_{v\in V} \left|1 - \frac{k}{f(v)/v}\right| \geq 1$ but $\sup_{v\in V} |f(v) - kv| / |f(v)| = 1$ for $k = 0$ (but this time there may be other k achieving the same value).
We have now to verify the case $0 < k^+k^-$. For $k = k^*$ we have

$$\sup_{v\in V} \left|1 - \frac{k^*}{f(v)/v}\right| = \left|1 - \frac{k^*}{k^+}\right| = \left|1 - \frac{k^*}{k^-}\right| = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|.$$

Note that we always have $k^- \leq k^* \leq k^+$. We can now distinguish the two cases

$$\begin{aligned}
k^- \leq k^* < k: \qquad & \sup_{v\in V} \left|1 - \frac{k}{f(v)/v}\right| \geq \left|1 - \frac{k}{k^-}\right| = \left|\frac{1}{k^-}\right| (k - k^-) \\
& > \left|\frac{1}{k^-}\right| (k^* - k^-) = \left|\frac{k^+ - k^-}{k^+ + k^-}\right| \\
k < k^* \leq k^+: \qquad & \sup_{v\in V} \left|1 - \frac{k}{f(v)/v}\right| \geq \left|1 - \frac{k}{k^+}\right| = \left|\frac{1}{k^+}\right| (k^+ - k) \\
& > \left|\frac{1}{k^+}\right| (k^+ - k^*) = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|.
\end{aligned}$$

We have shown that no other k than k^* can achieve the same or a smaller value of the nonlinearity measure and the proof is complete. □

If k^+ or k^- are infinite, the nonlinearity measure is actually not well-defined, as the corresponding operator N_f has no finite gain. Thus, the result above generalizes the nonlinearity measure and states that a memoryless operator without finite gain is consistently attributed a value of $\phi^{\mathcal{U}(V)}_{IMOE,N_f} = 1$.

Results for the MOE-NLM. The respective result for the MOE-NLM is the following.

T 3.32. *Consider the same conditions as given in Theorems 3.22 and 3.20. Then* $\phi^{\mathcal{U}}_{MOE,N} \geq \phi^{\mathcal{U}}_{MOE,N_f} = \phi^{\mathcal{U}(V)}_{MOE,N_f} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v)-Kv|}{|Kv|}$.

P . Define the relation $R(G) = \left\{(\tilde{y}, e) \in L^n_{pe} \times L^n_{pe} \middle| \tilde{y} = Gu, e = Nu - Gu, u \in \mathcal{U}\right\}$. For any stable $G \in \mathcal{G}_{m,n}$, note that $r(K) = \left\{(\tilde{z}, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \middle| \tilde{z} = Kv, \eta = f(v) - Kv, v \in V\right\}$ is a steady state of $R(G)$ on KV. Then, by Corollary 3.28,

$$\sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Gu)_T\|} = \|R(G)\|^{G\mathcal{U}} \geq |r(K)|^{KV} = \sup_{v \in V} \frac{|f(v) - Kv|}{|Kv|}$$

and the relation also holds if N is substituted by N_f. As for any $(\tilde{z}, \eta) \in r(K), \tilde{z} \in KV$, there is a $(\tilde{y}, e) \in R, \tilde{y} \in G_0\mathcal{U}$ such that $\tilde{y}(t) \to \tilde{z}$ and $e(t) \to \eta$ for $t \to \infty$, and for all $\tilde{y} \in G_0\mathcal{U}$, $\tilde{y}(t) \in KV$, Corollary 3.29 delivers

$$\begin{aligned} \phi^{\mathcal{U}}_{MOE,N_f} &= \inf_{G \in \mathcal{G}_{m,n}} \|R(G)\|^{N_f\mathcal{U}} \\ &\leq \inf_{G_0 \in \mathcal{G}^0_{m,n}} \|R(G_0)\|^{N_f\mathcal{U}} = \inf_{K \in \mathbb{R}^{n \times m}} |r(K)|^{f(V)} = \inf_{K \in \mathbb{R}^{n \times m}} \sup_{v \in V} \frac{|f(v) - Kv|}{|f(v)|} \end{aligned}$$

which completes the proof. □

In the case of scalar functions, we can again obtain an explicit solution.

T 3.33. *Let f, k^+ and k^- be as above and define k^* by*

$$k^* \triangleq \frac{1}{2}(k^+ + k^-).$$

Then

$$\phi^{\mathcal{U}(V)}_{MOE,N_f} = \inf_{k \in \mathbb{R}} \sup_{v \in V} \frac{|f(v) - kv|}{|kv|} = \begin{cases} \left|\frac{k^+ - k^-}{k^+ + k^-}\right| & \text{if } 0 < k^+k^- \\ 0 & \text{if } k^+ = k^- = 0 \\ 1 & \text{else} \end{cases}.$$

and the infimum is achieved for $k = k^*$ if $0 \leq k^+k^-$ and uniquely achieved if at least one of k^+, k^- is non-zero.

P . In the scalar case, the nonlinearity measure of a static function can be reformulated as

$$\inf_{k \in \mathbb{R}} \sup_{v \in V} \frac{|f(v) - kv|}{|kv|} = \inf_{k \in \mathbb{R}} \sup_{v \in V} \left|\frac{f(v)/v}{k} - 1\right|.$$

If $k^- < 0 < k^+$ then $\sup_{v \in V} \left|\frac{f(v)/v}{k} - 1\right| \geq \max\left\{\left|\frac{k^+}{k} - 1\right|, \left|\frac{k^-}{k} - 1\right|\right\} > 1$ for any k and $\sup_{v \in V} \left|\frac{f(v)/v}{k} - 1\right| \to 1$ for $|k| \to \infty$. If $k^+ = k^- = 0$ then $f(v) \equiv 0$ and the result is obvious. We have now to verify the case $0 \leq k^+k^-$, $|k^*| > 0$. For $k = k^*$ we have

$$\sup_{v \in V} \left|\frac{f(v)/v}{k^*} - 1\right| = \left|\frac{k^+}{k^*} - 1\right| = \left|\frac{k^-}{k^*} - 1\right| = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|.$$

Note that we always have $k^- \leq k^* \leq k^+$. If k has a different sign than k^+, k^- then clearly

$$\sup_{v \in \mathcal{V}} \left|\frac{f(v)/v}{k} - 1\right| > 1$$

and we only can have to consider the four cases

$$0 \le k^- \le k^* < k: \qquad \sup_{v\in V}\left|\frac{f(v)/v}{k}-1\right| \ge 1-\frac{k^-}{k} > 1-\frac{k^-}{k^*} = \frac{k^+-k^-}{k^++k^-}$$

$$k^- \le k^* < k \le 0: \qquad \sup_{v\in V}\left|\frac{f(v)/v}{k}-1\right| \ge \frac{k^-}{k}-1 > \frac{k^-}{k^*}-1 = -\frac{k^+-k^-}{k^++k^-}$$

$$0 \le k < k^* \le k^+: \qquad \sup_{v\in V}\left|\frac{f(v)/v}{k}-1\right| \ge \frac{k^+}{k}-1 > \frac{k^+}{k^*}-1 = \frac{k^+-k^-}{k^++k^-}$$

$$k < k^* \le k^+ \le 0: \qquad \sup_{v\in V}\left|\frac{f(v)/v}{k}-1\right| \ge 1-\frac{k^+}{k} > 1-\frac{k^+}{k^*} = -\frac{k^+-k^-}{k^++k^-}$$

We have shown that no other k than k^* can achieve the same or a smaller value of the nonlinearity measure and the proof is complete. □

Results for the MIE-NLM. The result establishing the steady-state behaviour as a lower bound cannot be derived for the MIE-NLM, IMIE-NLM and FE-NLM in a similar way. This is due to the fact that the system $G^{-1}N - I$ may not have the desired steady-state behaviour. Consider for instance the case of linear systems with zeros on the imaginary axis, where convergence of the output to a constant value does not imply convergence of the input. It is possible, however, to give an upper bound for the three measures for memoryless operators.

T 3.34. *Consider the same conditions as given in Theorem 3.22 and assume $m \ge n$. Then $\phi^{\mathcal{U}}_{MIE,N_f} \le \inf_{\tilde{K}\in\mathbb{R}^{m\times n}} \sup_{v\in V} \frac{|\tilde{K}f(v)-v|}{|v|}$.*

P . Define the set $\mathcal{K}^+ = \left\{\tilde{K}\in\mathbb{R}^{m\times n} : \tilde{K}^T\tilde{K} \text{ is regular}\right\}$, i.e. the set of all matrices in $\mathbb{R}^{m\times n}$ that have full column rank. The set of memoryless linear operators $\mathcal{G}^0_{m,n}$ is a subset of $\mathcal{G}_{m,n}$, and the matrix $K = \left(\tilde{K}^T\tilde{K}\right)^{-1}\tilde{K}^T \in \mathbb{R}^{n\times m}$ satisfies $z = Kv$ for all $v = \tilde{K}z$ (and K is onto $\mathbb{R}^n$ as K has full row rank), so the set $\mathcal{G}^{0,inv}_{m,n} = \left\{G_0^{-1} : y \mapsto G_0^{-1}y | (G_0^{-1}y)(t) = \tilde{K}y(t),\ \tilde{K}\in\mathcal{K}^+\right\}$ is a subset of $\mathcal{G}^{inv}_{m,n}$. Hence, from Theorem 3.22,

$$\begin{aligned}\phi^{\mathcal{U}}_{MIE,N_f} &= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}N_fu-u\right)_T\right\|}{\|u_T\|} \\ &\le \inf_{G_0^{-1}\in\mathcal{G}^{0,inv}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G_0^{-1}N_fu-u\right)_T\right\|}{\|u_T\|} = \inf_{\tilde{K}\in\mathcal{K}^+}\sup_{v\in\mathcal{V}} \frac{\left|\tilde{K}f(v)-v\right|}{|v|}.\end{aligned}$$

The result follows from density of $\mathcal{K}^+$ in $\mathbb{R}^{m\times n}$ (using standard regularization arguments) and the continuity of the last expression in $\tilde{K}$. □

Note that if $\inf_{v\in V}\frac{|f(v)|}{|v|} > 0$, a common bound on $\tilde{K}$ can be given and the existence of an optimal $\tilde{K}^*$ with

$$\sup_{v\in V}\frac{\left|\tilde{K}^* f(v)-v\right|}{|v|} = \inf_{\tilde{K}\in\mathbb{R}^{m\times n}}\sup_{v\in V}\frac{\left|\tilde{K} f(v)-v\right|}{|v|}$$

is guaranteed. The corresponding nominal model $K = \tilde{K}^{-1}$ only exists, if $\tilde{K}$ has a right inverse. Also, the model is not necessarily an optimal model, as equality in $\phi^{\mathcal{U}}_{MIE,N_f} \leq \inf_{\tilde{K}\in\mathbb{R}^{m\times n}}\sup_{v\in V}\frac{|\tilde{K}f(v)-v|}{|v|}$ cannot generally be guaranteed. In the case of scalar functions, an explicit solution of the upper bound for the MIE-NLM can be given.

T 3.35. *Let f, k^+ and k^- be as above and define k^* by*

$$k^* \triangleq \frac{1}{2}\left(k^+ + k^-\right).$$

Then

$$\phi^{\mathcal{U}(V)}_{MIE,N_f} \leq \phi^{\mathcal{U}(V)}_{MOE,N_f} = \inf_{\tilde{k}\in\mathbb{R}}\sup_{v\in V}\frac{\left|\tilde{k}f(v)-v\right|}{|v|} = \begin{cases} \left|\frac{k^+-k^-}{k^++k^-}\right| & \text{if } 0 < k^+k^- \\ 0 & \text{if } k^+ = k^- = 0 \\ 1 & \text{else} \end{cases}.$$

and if $0 < k^+k^-$ the infimum is uniquely achieved for $\tilde{k} = 1/k^*$.

P . In the scalar case, the nonlinearity measure of a static function can be reformulated as

$$\inf_{\tilde{k}\in\mathbb{R}}\sup_{v\in V}\frac{\left|\tilde{k}f(v)-v\right|}{|v|} = \inf_{\tilde{k}\in\mathbb{R}}\sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right|.$$

If $k^- < 0 < k^+$ then $\sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| \geq \max\left\{\left|k^+\tilde{k}-1\right|,\left|k^-\tilde{k}-1\right|\right\} > 1$ for any $\tilde{k}\neq 0$ and

$$\sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| = 1$$

for $\tilde{k} = 0$. If $k^+ = k^- = 0$ then $f(v)\equiv 0$ and the result is obvious.

We have now to verify the case $0 \leq k^+k^-$, $|k^*| > 0$. For $\tilde{k} = 1/k^*$ we have

$$\sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| = \left|\frac{k^+}{k^*}-1\right| = \left|\frac{k^-}{k^*}-1\right| = \left|\frac{k^+-k^-}{k^++k^-}\right|.$$

Note that we always have $k^- \leq k^* \leq k^+$. If $\tilde{k}$ has a different sign than k^+, k^- then clearly

$$\sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| > 1$$

and we only can have to consider the four cases

$$
\begin{aligned}
0 \le k^- \le k^* < \frac{1}{\tilde{k}}: \qquad & \sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| \ge 1-\tilde{k}k^- > 1-\frac{k^-}{k^*} = \frac{k^+-k^-}{k^++k^-}\\
k^- \le k^* < \frac{1}{\tilde{k}} \le 0: \qquad & \sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| \ge \tilde{k}k^- - 1 > \frac{k^-}{k^*} - 1 = -\frac{k^+-k^-}{k^++k^-}\\
0 \le \frac{1}{\tilde{k}} < k^* \le k^+: \qquad & \sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| \ge \tilde{k}k^+ - 1 > \frac{k^+}{k^*} - 1 = \frac{k^+-k^-}{k^++k^-}\\
\frac{1}{\tilde{k}} < k^* \le k^+ \le 0: \qquad & \sup_{v\in V}\left|\frac{f(v)}{v}\tilde{k}-1\right| \ge 1-\tilde{k}k^+ > 1-\frac{k^+}{k^*} = -\frac{k^+-k^-}{k^++k^-}
\end{aligned}
$$

We have shown that no other $\tilde{k}$ than k^* can achieve the same or a smaller value of the nonlinearity measure and the proof is complete. □

Results for the IMIE-NLM. The respective result for the IMIE-NLM is the following.

T 3.36. *Consider the same conditions as given in Theorem 3.22 and assume $m \ge n$. Then $\phi^{\mathcal{U}}_{IMIE,N_f} \le \inf_{\tilde{K}\in\mathbb{R}^{m\times n}} \sup_{v\in V} \frac{\left|\tilde{K}f(v)-v\right|}{\left|\tilde{K}f(v)\right|}$.*

P . Define the set $\mathcal{K}^+ = \left\{\tilde{K}\in\mathbb{R}^{m\times n} : \tilde{K}^T\tilde{K} \text{ is regular}\right\}$, i.e. the set of all matrices in $\mathbb{R}^{m\times n}$ that have full column rank. The set of memoryless linear operators $\mathcal{G}^0_{m,n}$ is a subset of $\mathcal{G}_{m,n}$, and the matrix $K = \left(\tilde{K}^T\tilde{K}\right)^{-1}\tilde{K}^T \in \mathbb{R}^{n\times m}$satisfies $z = Kv$ for all $v = \tilde{K}z$ (and K is onto $\mathbb{R}^n$ as K has full row rank), so the set $\mathcal{G}^{0,inv}_{m,n} = \left\{G_0^{-1} : y \mapsto G_0^{-1}y | (G_0^{-1}y)(t) = \tilde{K}y(t),\ \tilde{K}\in\mathcal{K}^+\right\}$ is a subset of $\mathcal{G}^{inv}_{m,n}(N\mathcal{U})$. Define the relation

$$
R(G^{-1}) = \left\{(\tilde{u},e)\in L^n_{pe}\times L^n_{pe} \middle| \tilde{u} = G^{-1}N_f u, e = G^{-1}N_f u - u, u\in\mathcal{U}\right\}
$$

and for any $G_0^{-1}\in\mathcal{G}^{0,inv}_{m,n}$ let the relation

$$
r(\tilde{K}) = \left\{(\tilde{v},\eta)\in\mathbb{R}^m\times\mathbb{R}^m \middle| \tilde{v} = \tilde{K}f(v), \eta = \tilde{K}f(v) - v, v\in V\right\}.
$$

As for any $(\tilde{v},\eta)\in r(\tilde{K}), \tilde{v}\in\tilde{K}f(V)$, there is a $(\tilde{u},e)\in R, \tilde{u}\in G_0^{-1}N_f\mathcal{U}$ such that $\tilde{u}(t)\to\tilde{v}$ and $e(t)\to\eta$ for $t\to\infty$, and for all $\tilde{u}\in G_0^{-1}N_f\mathcal{U}$, $\tilde{u}(t)\in\tilde{K}f(V)$, Corollary 3.29 delivers

$$
\begin{aligned}
\inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})}\sup_{u\in\mathcal{U},T>0}\frac{\left\|(G^{-1}N_fu-u)_T\right\|}{\left\|(G^{-1}N_fu)_T\right\|} &\le \inf_{G_0^{-1}\in\mathcal{G}^{0,inv}_{m,n}}\left\|R(G_0^{-1})\right\|^{G_0^{-1}N_f\mathcal{U}}\\
&= \inf_{\tilde{K}\in\mathcal{K}^+}\left|r(\tilde{K})\right|^{\tilde{K}f(V)} = \inf_{\tilde{K}\in\mathcal{K}^+}\sup_{v\in V}\frac{\left|\tilde{K}f(v)-v\right|}{\left|\tilde{K}f(v)\right|}.
\end{aligned}
$$

Take any matrix $\bar{K} \in \mathbb{R}^{n\times m}$ and assume $\frac{|\bar{K}f(v)|}{|v|} \geq \gamma > 0$ (otherwise the nonlinearity measure is unbounded). With $\bar{K} = \tilde{K} + \Delta K$, $\tilde{K} \in \mathcal{K}^+$, $\Delta K \in \mathbb{R}^{n\times m}$ we then have

$$\sup_{v\in V} \frac{\left|\frac{|\tilde{K}f(v)-v|}{|\tilde{K}f(v)|} - \frac{|\Delta K f(v)|}{|\tilde{K}f(v)|}\right|}{1+\frac{|\Delta K f(v)|}{|\tilde{K}f(v)|}} \leq \sup_{v\in V} \frac{\left|\bar{K}f(v)-v\right|}{\left|\bar{K}f(v)\right|} \leq \sup_{v\in V} \frac{\frac{|\tilde{K}f(v)-v|}{|\tilde{K}f(v)|} + \frac{|\Delta K f(v)|}{|\tilde{K}f(v)|}}{\left|1-\frac{|\Delta K f(v)|}{|\tilde{K}f(v)|}\right|}.$$

Clearly, $\sup_{v\in V} \frac{|\Delta K f(v)|}{|\tilde{K}f(v)|} = \gamma < \infty$ as $\tilde{K}$ has full column rank. Because $\mathcal{K}^+$is dense in $\mathbb{R}^{m\times n}$, we can make γ arbitrarily small by choosing $\tilde{K}$ and ΔK appropriately, which shows that

$$\inf_{\tilde{K}\in\mathcal{K}^+} \sup_{v\in V} \frac{\left|\tilde{K}f(v)-v\right|}{\left|\tilde{K}f(v)\right|} = \inf_{K\in\mathbb{R}^{m\times n}} \sup_{v\in V} \frac{\left|\bar{K}f(v)-v\right|}{\left|\bar{K}f(v)\right|}.$$

□

In the case of scalar functions, an explicit solution of the upper bound can again be given.

T 3.37. *Let f, k^+ and k^- be as above and define k^* by*

$$\frac{1}{k^*} \triangleq \frac{1}{2}\left(\frac{1}{k^+} + \frac{1}{k^-}\right)$$

If $\inf_{v\in V}\left|\frac{f(v)}{v}\right| > 0$ *then*

$$\phi^{\mathcal{U}(V)}_{IMIE,N_f} \leq \phi^{\mathcal{U}(V)}_{IMOE,N_f} = \inf_{\tilde{k}\in\mathbb{R}} \sup_{v\in V} \frac{\left|\tilde{k}f(v)-v\right|}{\left|\tilde{k}f(v)\right|} = \begin{cases} \left|\frac{k^+-k^-}{k^++k^-}\right| & \text{if } 0 < k^+k^-, |k^+|, |k^-| < \infty \\ 1 & \text{else} \end{cases}.$$

and the infimum is uniquely achieved for $\tilde{k} = 1/k^*$ if $0 < k^+k^- < \infty$. If $\inf_{v\in V}\left|\frac{f(v)}{v}\right| = 0$ *then* $\inf_{\tilde{k}\in\mathbb{R}} \sup_{v\in V} \frac{|\tilde{k}f(v)-v|}{|\tilde{k}f(v)|} = \infty$.

P . In the scalar case, the nonlinearity measure of a static function can be reformulated as

$$\inf_{\tilde{k}\in\mathbb{R}} \sup_{v\in V} \frac{\left|\tilde{k}f(v)-v\right|}{\left|\tilde{k}f(v)\right|} = \inf_{\tilde{k}\in\mathbb{R}} \sup_{v\in V} \left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right|.$$

If $\inf_{v\in V}\left|\frac{f(v)}{v}\right| = 0$ then $\sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| = \infty$ for any $\tilde{k}$. If $\inf_{v\in V}\left|\frac{f(v)}{v}\right| > 0$ and $k^- < 0 < k^+$, $\sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| \geq 1$ for any choice of $\tilde{k}$, and $\tilde{k} \to \infty$ (and/or $\left|\frac{f(v)}{v}\right| \to \infty$) results in

$$\sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| \to 1.$$

We have now to verify the case $0 < k^+k^-$. For $\tilde{k} = 1/k^*$ we have

$$\sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| = \left|1-\frac{k^*}{k^+}\right| = \left|1-\frac{k^*}{k^-}\right| = \left|\frac{k^+-k^-}{k^++k^-}\right|.$$

Note that we always have $k^- \leq k^* \leq k^+$. We can now distinguish the two cases

$$\begin{aligned} k^- \leq k^* < \frac{1}{\tilde{k}}: \qquad & \sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| \geq \left|1-\frac{1}{\tilde{k}}\frac{1}{k^-}\right| = \left|\frac{1}{k^-}\right|(\frac{1}{\tilde{k}}-k^-) \\ & > \left|\frac{1}{k^-}\right|(k^*-k^-) = \left|\frac{k^+-k^-}{k^++k^-}\right| \\ \frac{1}{\tilde{k}} < k^* \leq k^+: \qquad & \sup_{v\in V}\left|1-\frac{1}{\tilde{k}\frac{f(v)}{v}}\right| \geq \left|1-\frac{1}{\tilde{k}}\frac{1}{k^+}\right| = \left|\frac{1}{k^+}\right|(k^+-\frac{1}{\tilde{k}}) \\ & > \left|\frac{1}{k^+}\right|(k^+-k^*) = \left|\frac{k^+-k^-}{k^++k^-}\right|. \end{aligned}$$

We have shown that no other $\tilde{k}$ than $\tilde{k} = 1/k^*$ can achieve the same or a smaller value of the nonlinearity measure and the proof is complete. □

Results for the FE-NLM. The last results concern the feedback error setup, and are comparable to the results for the MIE and IMIE nonlinearity measures.

T 3.38. *Consider the same conditions as given in Theorem 3.22 and assume $m \geq n$. Then $\phi^{\mathcal{U}}_{FE,N_f} \leq \inf_{\tilde{K}\in\mathbb{R}^{m\times n}} \sup_{v\in V} \frac{|\tilde{K}f(v)-v|}{|f(v)|}$.*

P . Define the set $\mathcal{K}^+ = \left\{\tilde{K}\in\mathbb{R}^{m\times n} : \tilde{K}^T\tilde{K} \text{ is regular}\right\}$, i.e. the set of all matrices in $\mathbb{R}^{m\times n}$ that have full column rank. The set of memoryless linear operators $\mathcal{G}^0_{m,n}$ is a subset of $\mathcal{G}_{m,n}$, and the matrix $K = \left(\tilde{K}^T\tilde{K}\right)^{-1}\tilde{K}^T \in \mathbb{R}^{n\times m}$satisfies $z = Kv$ for all $v = \tilde{K}z$ (and K is onto $\mathbb{R}^n$ as K has full row rank), so the set $\mathcal{G}^{0,inv}_{m,n} = \left\{G_0^{-1} : y \mapsto G_0^{-1}y | (G_0^{-1}y)(t) = \tilde{K}y(t),\ \tilde{K}\in\mathcal{K}^+\right\}$ is a subset of $\mathcal{G}^{inv}_{m,n}(N\mathcal{U})$. Define the relation

$$R(G^{-1}) = \left\{(\tilde{u},y)\in L^n_{pe}\times L^n_{pe}\middle|\tilde{u} = G^{-1}y-u, y = N_f u, u\in\mathcal{U}\right\}$$

and for any $G_0^{-1}\in\mathcal{G}^{0,inv}_{m,n}$ let the relation

$$r(\tilde{K}) = \left\{(\tilde{v},z)\in\mathbb{R}^m\times\mathbb{R}^m\middle|\tilde{v} = \tilde{K}f(v)-v, z = f(v), v\in V\right\}.$$

As for any $(\tilde{v},z)\in r(\tilde{K}), \tilde{v}\in\tilde{K}f(V)$, there is a $(\tilde{u},y)\in R, \tilde{u}\in G_0^{-1}N_f\mathcal{U}$ such that $\tilde{u}(t)\to\tilde{v}$ and $y(t)\to z$ for $t\to\infty$, and for all $\tilde{u}\in G_0^{-1}N_f\mathcal{U}$, $\tilde{u}(t)\in\tilde{K}f(V)$, Corollary 3.29 delivers

$$\begin{aligned} \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(N\mathcal{U})}\sup_{u\in\mathcal{U},T>0}\frac{\left\|(G^{-1}N_fu-u)_T\right\|}{\left\|(N_fu)_T\right\|} &= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}}\left\|R(G^{-1})\right\|^{N_f\mathcal{U}} \\ &\leq \inf_{G_0^{-1}\in\mathcal{G}^{0,inv}_{m,n}}\left\|R(G_0^{-1})\right\|^{N_f\mathcal{U}} \\ &= \inf_{\tilde{K}\in\mathcal{K}^+}\left|r(\tilde{K})\right|^{f(V)} = \inf_{\tilde{K}\in\mathcal{K}^+}\sup_{v\in V}\frac{\left|\tilde{K}f(v)-v\right|}{|f(v)|}. \end{aligned}$$

The result follows from density of $\mathcal{K}^+$ in $\mathbb{R}^{m\times n}$ (using standard regularization arguments) and the continuity of the last expression in $\tilde{K}$. □

In the case of scalar functions, an explicit solution of the upper bound can be given.

T 3.39. *For a given function* $f : V \subseteq \mathbb{R} \to \mathbb{R}$ *let*

$$\begin{aligned} \tilde{k}^+ &\triangleq \sup_{v \in V} \frac{v}{f(v)} \\ \tilde{k}^- &\triangleq \inf_{v \in V} \frac{v}{f(v)} \end{aligned}$$

and define $\tilde{k}^*$ *by*

$$\tilde{k}^* \triangleq \frac{1}{2}\left(\tilde{k}^+ + \tilde{k}^-\right).$$

If $\inf_{v \in V} \left|\frac{f(v)}{v}\right| > 0$ *then*

$$\phi_{FE,N_f}^{\mathcal{U}(V)} \leq \inf_{\tilde{k} \in \mathbb{R}} \sup_{v \in V} \frac{\left|\tilde{k} f(v) - v\right|}{|f(v)|} = \begin{cases} \frac{1}{2}\left(\tilde{k}^+ - \tilde{k}^-\right) & \text{else} \\ \infty & \text{if } \tilde{k}^+ = \infty \text{ or } \tilde{k}^- = \infty \end{cases}$$

and if $0 < k^+ k^-$ *the infimum is uniquely achieved for* $\tilde{k} = \tilde{k}^*$.

P . In the scalar case, the feedback error nonlinearity measure of a static function can be reformulated as

$$\inf_{\tilde{k} \in \mathbb{R}} \sup_{v \in V} \frac{\left|\tilde{k} f(v) - v\right|}{|f(v)|} = \inf_{\tilde{k} \in \mathbb{R}} \sup_{v \in V} \left|\tilde{k} - \frac{1}{f(v)/v}\right|.$$

If $\tilde{k}^+ = \infty$ or $\tilde{k}^- = -\infty$ then $\sup_{v \in V} \left|\tilde{k} - \frac{1}{f(v)/v}\right| = \infty$ for any $\tilde{k}$. In all other cases, for $\tilde{k} = \tilde{k}^*$ we have

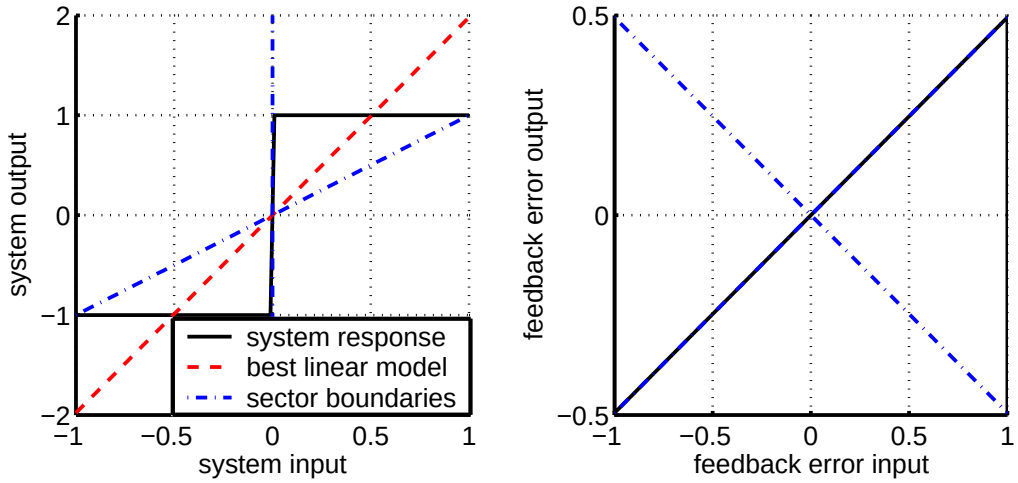
$$\sup_{v \in V} \left|\tilde{k} - \frac{1}{f(v)/v}\right| = \left|\tilde{k}^* - \tilde{k}^+\right| = \left|\tilde{k}^* - \tilde{k}^-\right| = \left(\tilde{k}^+ - \tilde{k}^-\right).$$

Note that we always have $\tilde{k}^- \leq \tilde{k}^* \leq \tilde{k}^+$. By considering the two cases

$$\begin{aligned} \tilde{k}^- \leq \tilde{k}^* < \tilde{k} : &\qquad \sup_{v \in V} \left|\tilde{k} - \frac{1}{f(v)/v}\right| \geq \tilde{k} - \tilde{k}^- > \tilde{k}^* - \tilde{k}^- = \frac{1}{2}\left(\tilde{k}^+ - \tilde{k}^-\right) \\ \tilde{k} < \tilde{k}^* \leq \tilde{k}^+ : &\qquad \sup_{v \in V} \left|\tilde{k} - \frac{1}{f(v)/v}\right| \geq \tilde{k}^+ - \tilde{k} > \tilde{k}^+ - \tilde{k}^* = \frac{1}{2}\left(\tilde{k}^+ - \tilde{k}^-\right). \end{aligned}$$

We have shown that no other $\tilde{k}$ than $\tilde{k} = \tilde{k}^*$ can achieve the same or a smaller value of the nonlinearity measure and the proof is complete. □

R . The results on scalar memoryless systems for the feedback error setup allow an interesting interpretation. The formula given is a complete analogue to the case of the additive error setup, if the *inverse* nonlinear function and the *inverse* of the linear model is considered with prescribed *output* range. As a consequence, the scalar static feedback error nonlinearity measure is the only given nonlinearity measure that can be generalized to unstable systems (i.e. functions that are not linearly bounded), while still delivering non-trivial values.



F 3.5. Input-output relationship for the sign-function and best linear model (left) and error system Δ (right) for the feedback error setup.

E . The sign function

$$y = f(u) = \begin{cases} -1 & : u < 0 \\ 1 & : u \geq 0 \end{cases}$$

clearly has no finite gain, $k^+ = \infty$, so all other nonlinearity measures are unbounded or return their maximal possible value for any given operating range. The feedback error nonlinearity measure, however, can be defined and one obtains $\tilde{k}^+ = 1$, $\tilde{k}^- = 0$ and $\tilde{k}^* = \frac{1}{2}\left(\tilde{k}^+ + \tilde{k}^-\right) = 0.5$ for an operating range of $|u| \leq 1$, to which belongs the best linear gain $k^* = 1/\tilde{k}^* = 2$. The system response together with the best linear model and the sector boundaries are depicted in Fig. 3.5 (left graph). The nonlinearity measure is obtained as $\phi^{\mathcal{U}(V)}_{FE,N_f} = \frac{1}{2}\left(\tilde{k}^+ - \tilde{k}^-\right) = 0.5$, which is the gain of the uncertainty shown in Fig. 3.5 (right graph). Note that the error system is a relation in this case (the black upright line at the right side of the graph belongs also to Δ). Therefore, the sector in which the error system lies is given by the two bisectors, the first one coinciding with a part of Δ. This examples shows that the feedback error setup might open the possibility to define a nonlinearity measure also for unstable *dynamic* systems. We will however not pursue this idea here any further. ■

Discussion. In Section 3.4 it was shown for the additive error setup that a lower bound on the nonlinearity measure can be derived by the steady state behaviour of a system, and that the nonlinearity measure of a memoryless system can be characterized by a "static" optimization problem that has a particularly simple solution in the case of scalar systems. The extension of this result in this section was only complete for the MOE and IMOE setups. For those two setups, similar (but different) results were given. For the remaining three nonlinearity measures, only results for memoryless systems could be given. It is believed that the given characterization is tight, although it was only shown that the given formulae are upper bounds on the nonlinearity measures for memoryless systems. This is

T 1. Nonlinearity measures and best linear models of the memoryless nonlinear operator $y = u + 2u^3$.

	Best linear model	Nonlinearity measure
AE	$y = k^* u = \frac{1}{2}(k^+ + k^-)\,u = 2u$	$\phi^{\mathcal{U}}_{AE,N_f} = \frac{1}{2}(k^+ - k^-) = 1$
MOE	"	$\phi^{\mathcal{U}}_{MOE,N_f} = \left\lvert\frac{k^+ - k^-}{k^+ + k^-}\right\rvert = 0.5$
MIE	"	"
IMOE	$y = k^* u = \frac{2k^+k^-}{k^+ + k^-}u = 1.5u$	"
IMIE	"	"
FE	"	$\phi^{\mathcal{U}}_{FE,N_f} = \frac{1}{2}\left(\frac{1}{k^-} - \frac{1}{k^+}\right) = \frac{1}{3}$

due to the appearance of inverses of linear operators, that in general do not possess the required steady-state behaviour.

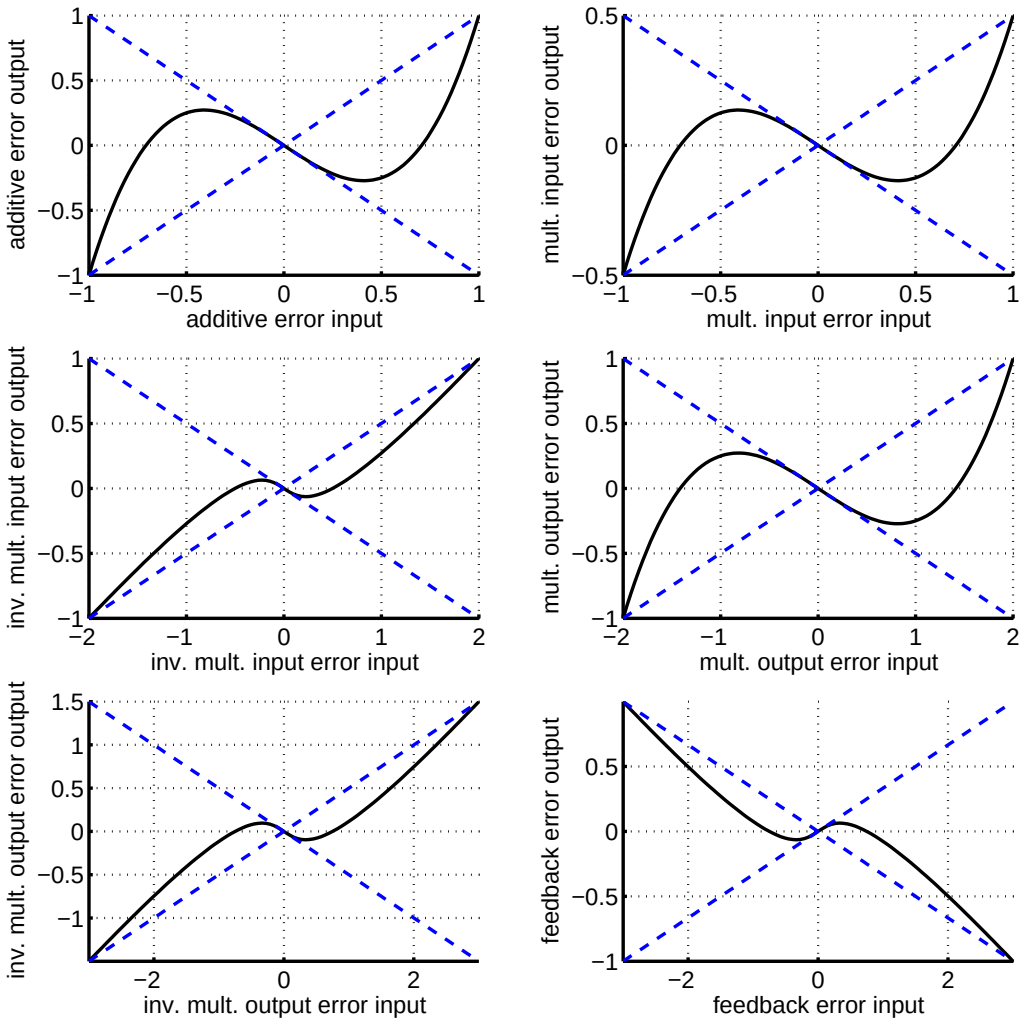
In order to illustrate the similarities and differences between the results on the static scalar nonlinearity measures for the various setups, we give an example.

E 3.40. Consider the memoryless nonlinear function

$$y = f(u) = u + 2u^3$$

and an operating region of $|u| \leq 1$. It can be readily seen that $k^+ = 3$ and $k^- = 1$. For the feedback error case, we have $\tilde{k}^+ = 1/k^+$ and $\tilde{k}^- = 1/k^-$ because the nonlinearity strictly lies inside the first and third quadrant. With these values, the best linear models and nonlinearity measures of the different setups can be determined as collected in Table 1. It is also instructive to have a look at the corresponding error systems Δ. It can be seen from Fig. 3.6 that the error systems for the different setups are very similar. But note that the scales on the inputs and the outputs are different for the different setups. The reason is that the input signal to the nonlinear system N_f is also the input signal to the error systems only for the AE and MIE setups. The inputs to the error systems in IMOE and FE setups correspond to the output of N_f, while the inputs to the error systems in the IMIE and MOE setups are internal signals. For a similar reason, also the outputs have different ranges. Taking the ranges into account, the nonlinearity measures given in Table 1 can be verified. The values of the nonlinearity measures just correspond to the gain of the depicted error systems. From the figure we can also see a typical characteristic of error systems Δ that correspond to *best* linear models G^*: the sector in which the nonlinear error lies (dashed) is symmetric to the input-axis. This behaviour implies that the gain of the error system cannot be reduced by adding or subtracting a linear term, and corresponds to the associated linear models being the best possible models. ■

Generalization to Infinite-Dimensional Systems. At the end of this section, we address the question whether the given results can be extended when a more general class of linear



F 3.6. Error systems Δ for the memoryless nonlinear operator $y = u + 2u^3$ and the best linear models in different setups.

models is considered. This is the class of causal linear time-invariant (LTI) delay-free transfer operators $G : L_{pe}^m \to L_{pe}^n$ defined by

$$(Gu)(t) = G_{hf}u(t) + (g * u)(t) = G_{hf}u(t) + \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau \tag{3.14}$$

with the instantaneous gain $G_{hf} \in \mathbb{R}^{n\times m}$ and a measurable kernel $g : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$, that has already been addressed in Section 3.1.

The results given in this and the last section concerning the steady-state behaviour of nonlinear systems and the values of the nonlinearity measures in the additive and the (inverse) multiplicative output error setups made use of the fact that if the input of a finite gain stable linear system converges to a certain value, then the output asymptotically tends to a value that is given by a linear function of the asymptotic input value, i.e. from $u \to u_0$ follows $y \to Ku_0$, $K \in \mathbb{R}^{n\times m}$. This behaviour is not so easy to verify for general convolution

operators. However, if such a linear convolution operators is such that $\|g\|_1 < \infty$[8], this assumption is also satisfied.

P 3.41. *Consider a convolution operator given by (3.14) that satisfies* $\|g\|_1 < \infty$. *If* $u \in L_\infty$ *and* $\lim_{t\to\infty} u(t) = u_0$ *then* $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(Gu)(t) = Ku_0$ *with* $K = G_{hf} + \int_0^\infty g(\tau)d\tau$.

P . Let $\lim_{t\to\infty} u(t) = u_0$, $\tilde{u}(t) = u(t) - u_0$ and rewrite the convolution as $(g * u)(t) = \int_0^t g(\tau)u_0 \mathrm{d}\tau + (g * \tilde{u})(t)$. If $u(t)$ is bounded, i.e. $u \in L_\infty^m$ then $\|g * (\tilde{u}_T)\|_1 \leq \|g\|_1 \|\tilde{u}_T\|_1 < \infty$ for all $T > 0$ and $(g * \tilde{u})$ is uniformly continuous [64], so Barbalat's lemma proves that $(g * (\tilde{u}_T))(t) \to 0$ for $t \to \infty$ [77]. From $u(t) \to u_0$, for any $\varepsilon > 0$ there is a $T > 0$ such that $\|g * (\tilde{u} - \tilde{u}_T)\|_\infty \leq \|g\|_1 \|\tilde{u} - \tilde{u}_T\|_\infty < \varepsilon$, which proves that $(g * \tilde{u})(t) \to 0$ for $t \to \infty$ and thus $\lim_{t\to\infty} Gu(t) = G_{hf}u_0 + \int_0^\infty g(\tau)\mathrm{d}\tau\, u_0 = Ku_0$. □

The proposition shows that the results of Sections 3.4 and 3.5 remain true if instead of stable finite-dimensional linear models, stable convolution models are considered and a L_1 or L_∞ setup is chosen.

3.6. Composite Systems

In this section, some results on nonlinearity measures of interconnections of two subsystems are established. The goal is to derive upper bounds on such interconnections based on the measures of the subsystems alone. Moreover, if linear models are known for the subsystems, as is the case for interconnections of dynamic linear systems and memoryless nonlinear systems, these results immediately deliver linear model candidates that achieve the modeling error given by these upper bounds.

Parallel connections. We first consider the case of a parallel connection $N_1 + N_2$ of two finite gain stable mappings N_1 and N_2. The proposition below shows that the AE-NLM of the parallel connection is bounded by the sum of the individual nonlinearity measures. For this relation, no equivalent property can be shown for the other nonlinearity measures.

P 3.42 (AE-NLM of parallel connections). *For two finite-gain stable mappings* $N_1, N_2 : \mathcal{U} \subseteq L_{pe}^m \to L_{pe}^n$ *the "triangle inequality"* $\phi^{\mathcal{U}}_{AE,N_1+N_2} \leq \phi^{\mathcal{U}}_{AE,N_1} + \phi^{\mathcal{U}}_{AE,N_2}$ *holds.*

P . We have

$$\begin{aligned}\phi^{\mathcal{U}}_{AE,N_1+N_2} &= \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U}} \frac{\|(N_1+N_2)u - Gu\|}{\|u\|} \\ &= \inf_{G_1,G_2\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U}} \frac{\|(N_1+N_2)u - (G_1+G_2)u\|}{\|u\|}\end{aligned}$$

[8]Note that the condition $\|g\|_1 = \int_0^t \|g(\tau)\|\,\mathrm{d}\tau < \infty$ is sufficient for a system (3.14) to be stable, but it is also necessary only for the cases of $p = 1, \infty$ [25].

$$
\begin{aligned}
&\leq \inf_{G_1,G_2\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U}} \frac{\|N_1u - G_1u\| + \|N_2u - G_2u\|}{\|u\|} \\
&\leq \inf_{G_1,G_2\in\mathcal{G}_{m,n}} \sup_{u,w\in\mathcal{U}} \left(\frac{\|N_1u - G_1u\|}{\|u\|} + \frac{\|N_2w - G_2w\|}{\|w\|} \right) \\
&= \phi^{\mathcal{U}}_{AE,N_1} + \phi^{\mathcal{U}}_{AE,N_2}.
\end{aligned}
$$

□

As has been observed before [26], if one system, say $N_1 = L \in \mathcal{G}_{m,n}$, is linear and one is nonlinear $N_2 = N$, we trivially get $\phi^{\mathcal{U}}_{AE,N+L} = \phi^{\mathcal{U}}_{AE,N}$. If the nonlinear system in parallel to the linear system is memoryless, Theorems 3.23 and 3.24 give the best finite-dimensional approximation G_0 (a constant gain matrix) and the associated linear model is $G = L + G_0$.

Series connections, Hammerstein and Wiener systems. Next, we consider series connections of systems. In particular, we are interested in series connections of a linear and a nonlinear system. The first case is a series connection $L \circ N$ of a nonlinear system N followed by a linear system L. If the nonlinear subsystem is memoryless, this is the classical Hammerstein system structure. We therefore call a series connection $L \circ N$ a Hammerstein-like system. For this class of composite systems, the following proposition establishes results for the AE-NLM, the MIE-NLM and the IMIE-NLM.

P 3.43 (Nonlinearity measures of Hammerstein-like series connections). *For a series connection $L \circ N$ of a nonlinear system $N : \mathcal{U} \subseteq L^m_{pe} \to L^k_{pe}$ and a linear system $L \in \mathcal{G}_{k,n}$, finite-gain stable on $\mathcal{U}$ and $N\mathcal{U}$ respectively,*

$$
\phi^{\mathcal{U}}_{AE,LN} \leq \|L\|\, \phi^{\mathcal{U}}_{AE,N}.
$$

Moreover, if L is injective on $N\mathcal{U}$, then

$$
\begin{aligned}
\phi^{\mathcal{U}}_{MIE,LN} &\leq \phi^{\mathcal{U}}_{MIE,N}, \text{ and} \\
\phi^{\mathcal{U}}_{IMIE,LN} &\leq \phi^{\mathcal{U}}_{IMIE,N}.
\end{aligned}
$$

P . As $\left\{ L\tilde{G} \middle| \tilde{G} \in \mathcal{G}_{m,k} \right\} \subseteq \mathcal{G}_{m,n}$ for any given $L \in \mathcal{G}_{k,n}$, the first result follows from

$$
\begin{aligned}
\phi^{\mathcal{U}}_{AE,LN} &= \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(L(Nu) - Gu)_T\|}{\|u_T\|} \\
&\leq \inf_{\tilde{G}\in\mathcal{G}_{m,k}} \sup_{u\in\mathcal{U},T>0} \frac{\left\| \left(L(Nu) - L(\tilde{G}u) \right)_T \right\|}{\|u_T\|} \leq \|L\|\, \phi^{\mathcal{U}}_{AE,N}.
\end{aligned}
$$

For the other results, observe that there is a left-inverse L^{-1} to L and $\left\{\tilde{G}^{-1}L^{-1}|\tilde{G}^{-1} \in \mathcal{G}^{inv}_{m,k}\right\} \subseteq \mathcal{G}^{inv}_{m,n}$ for any such $L \in \mathcal{G}_{k,n}$, and so

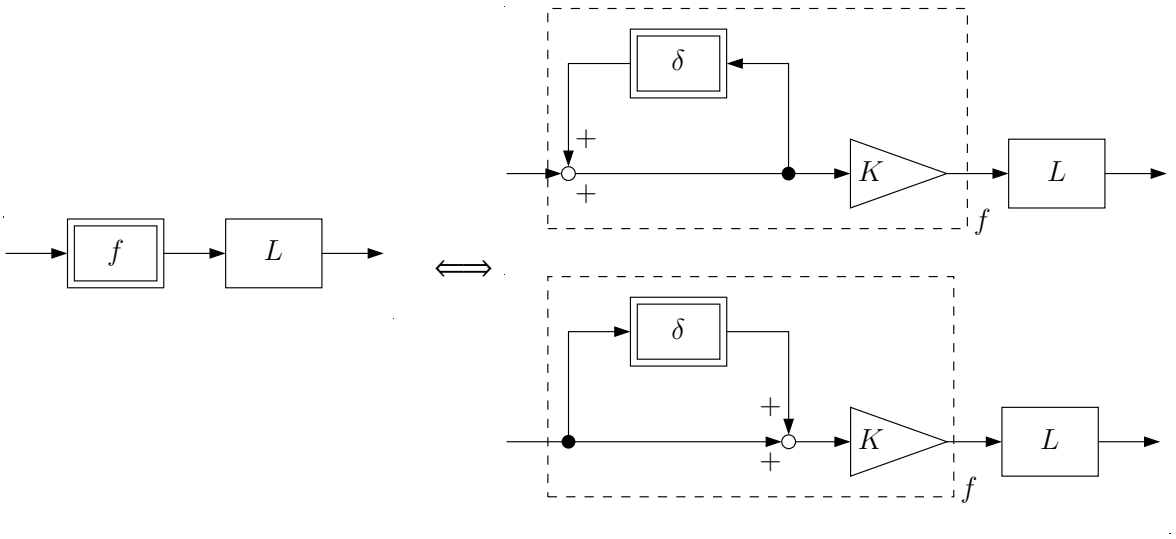
$$\begin{aligned}\phi^{\mathcal{U}}_{MIE,LN} &= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}LNu-u\right)_T\right\|}{\|u_T\|} \\ &\leq \inf_{\tilde{G}^{-1}\in\mathcal{G}^{inv}_{m,k}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(\tilde{G}^{-1}L^{-1}LNu-u\right)_T\right\|}{\|u_T\|} = \phi^{\mathcal{U}}_{MIE,N}\end{aligned}$$

and analogously $\phi^{\mathcal{U}}_{IMIE,LN} \leq \phi^{\mathcal{U}}_{IMIE,N}$. □

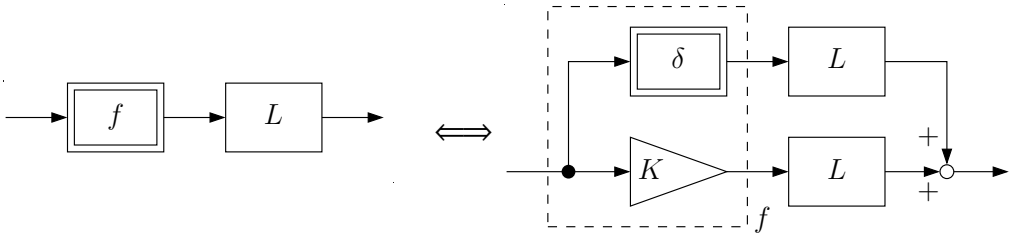
Note that injectivity of L requires $k \leq n$. The results for the IMIE- and MIE-NLM can be generalized for unstable L in a straightforward manner. If the nonlinear subsystem is static, the structure of the composed system is of Hammerstein-type and the upper bound can be calculated easily with the help of the previously given formulae for static systems. From Fig. 3.7 it can be seen that the corresponding linear model is given by $G = L \circ G_0$, where the memoryless linear system G_0 is a constant gain matrix K. Figure 3.8 shows the derivation of the model for the additive error setup. Note that linearity of L allows to move the summation point from before L to behind L, so the model candidate is also $G = L \circ G_0$, as above. But contrary to the MIE and IMIE setups, the error system Δ is given by $L \circ N_\delta$, where $\delta(v) = f(v) - Kv$, leading to the appearance of the additional factor $\|L\|$ in the estimate of the upper bound of the error system gain.

E . Consider the Hammerstein system composed of the input nonlinearity N_f given by $f(x) = 0.05x + x^3$ and the linear dynamics $L(s) = \frac{1}{(s+1)^3}$ for an operating range of $|u| \leq 1$. In this case, Proposition 3.43 delivers the upper bound $\phi^{\mathcal{U}}_{MIE,LN_f} \leq \phi^{\mathcal{U}}_{MIE,N_f}$. As the nonlinearity lies in the sector $[k^-, k^+] = [0.05, 1.05]$, Theorem 3.35 provides the value of $\phi^{\mathcal{U}}_{MIE,N_f} \leq \left|\frac{k^+-k^-}{k^++k^-}\right| = 0.91$ for the multiplicative input nonlinearity measure for N_f, together with the optimal linear gain of $k^* = \frac{1}{2}(k^+ + k^-) = 0.55$. From these data we obtain the linear model $G(s) = \frac{0.55}{(s+1)^3}$ (for an illustration see Fig. 3.7), and the associated modeling error is bounded by $\|\Delta\|^{\mathcal{U}} \leq 0.91$. An important point is that this linear model is much better than the model obtained by linearizing the nonlinearity. The linearization admits the model $G_{loc}(s) = \frac{0.05}{(s+1)^3}$, which delivers the modeling bound $\|\Delta\|^{\mathcal{U}} \leq 20$. Although for technical reasons, the bounds on the nonlinearity measure and model quality index respectively cannot be shown to be sharp, it is believed that they in fact do reflect the exact values in these cases. The significant difference between the best linear model and the linearization can be seen from Fig. 3.9. The steady state gain of the local linearization is by far too small. On the other hand, the gain of the best linear model is larger than the gain of the nonlinear system for small input amplitudes, while it is smaller for large inputs. Thus, the best linear model achieves a compromise between optimal approximation for small signal behaviour and for large signal behaviour. ■

Upper bounds can also be derived in the case of a series connection $N \circ L$ of a linear system $L \in \mathcal{G}_{m,k}$ followed by a nonlinear system $N : \mathcal{U} \subseteq L^k_{pe} \to L^n_{pe}$. If the nonlinear subsystem



F 3.7. Illustration of the derivation of a linear model candidate for Hammerstein systems in the IMIE and MIE setup.



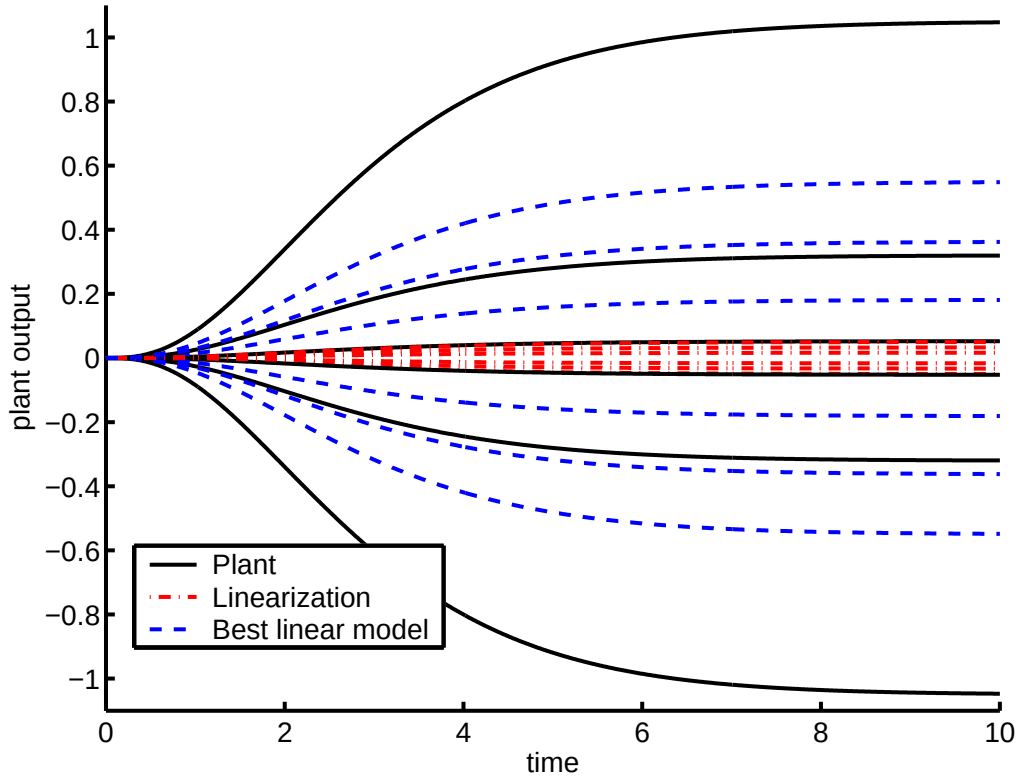
F 3.8. Illustration of the derivation of a linear model candidate for Hammerstein systems in the additive error setup.

is memoryless, this is the classical Wiener system structure. We therefore call a series connection $N \circ L$ a Wiener-like system. The given bounds concern the AE setup and the MOE and IMOE setups, i.e. the setups that have the error term after the linear model.

P 3.44 (Nonlinearity measures of Wiener-like series connections). *Let $L \in \mathcal{G}_{m,k}$ and the nonlinear system $N : \mathcal{U} \subseteq L^k_{pe} \to L^n_{pe}$ be finite gain stable systems. Then $\phi^{\mathcal{U}}_{MOE,NL} \leq \phi^{L\mathcal{U}}_{MOE,N}$ and $\phi^{\mathcal{U}}_{IMOE,NL} \leq \phi^{L\mathcal{U}}_{IMOE,N}$, and if L is injective on $\mathcal{U}$, then $\phi^{\mathcal{U}}_{AE,NL} \leq \|L\| \, \phi^{L\mathcal{U}}_{AE,N}$.*

P . The first result follows from

$$\begin{aligned} \phi^{\mathcal{U}}_{MOE,NL} &= \inf_{G \in \mathcal{G}_{m,n}} \sup_{u \in \mathcal{U}, T>0} \frac{\|(NLu - Gu)_T\|}{\|(Gu)_T\|} \\ &\leq \inf_{\tilde{G} \in \mathcal{G}_{k,n}} \sup_{u \in \mathcal{U}, T>0} \frac{\left\|\left(NLu - \tilde{G}Lu\right)_T\right\|}{\left\|\left(\tilde{G}Lu\right)_T\right\|} \end{aligned}$$



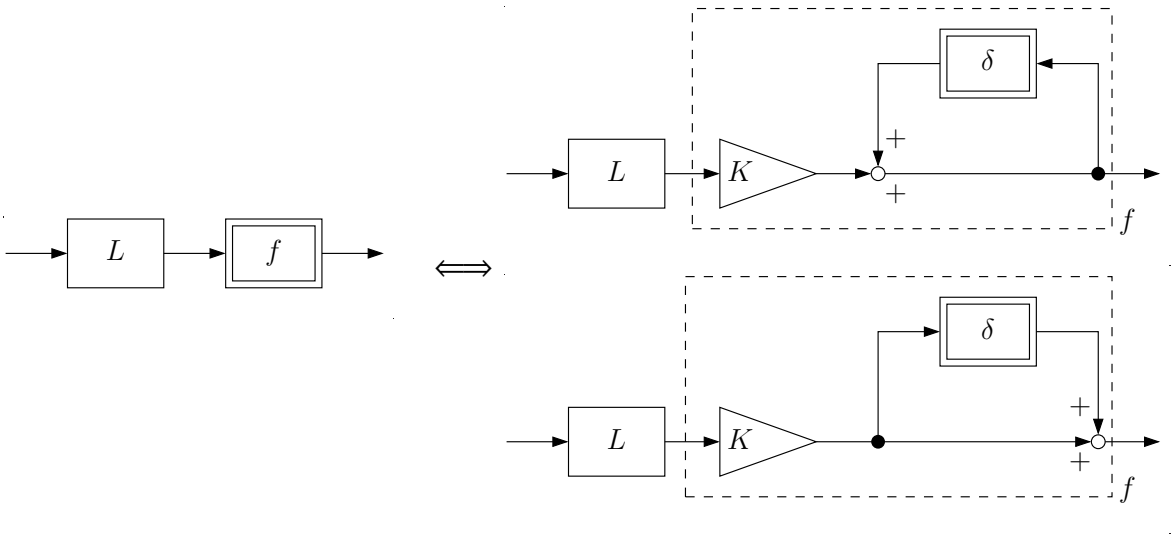
F 3.9. Step responses of the Hammerstein system, its best local model on the considered operating range, and the local linearization.

$$= \inf_{\tilde{G}\in\mathcal{G}_{k,n}} \sup_{\tilde{u}\in L\mathcal{U},T>0} \frac{\left\|\left(N\tilde{u}-\tilde{G}\tilde{u}\right)_T\right\|}{\left\|\left(\tilde{G}\tilde{u}\right)_T\right\|} = \phi^{L\mathcal{U}}_{MOE,N}$$

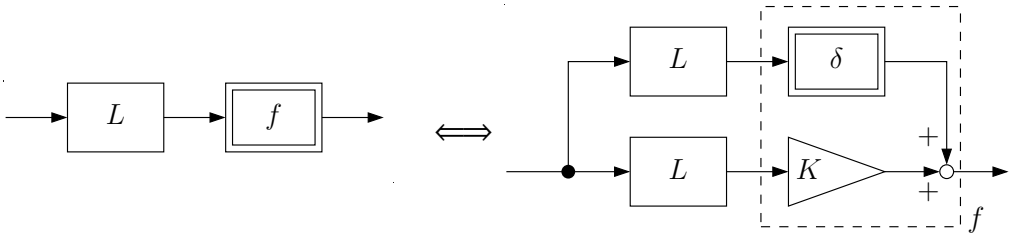
and the second result follows in an analogous way. The last result is obtained from

$$\begin{aligned}
\phi^{\mathcal{U}}_{AE,NL} &= \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(N(Lu)-Gu)_T\|}{\|u_T\|} \\
&\leq \inf_{\tilde{G}\in\mathcal{G}_{k,n}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(NLu-\tilde{G}Lu\right)_T\right\|}{\|u_T\|} \\
&\overset{\text{injectivity of } L}{=} \inf_{\tilde{G}\in\mathcal{G}_{k,n}} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(NLu-\tilde{G}Lu\right)_T\right\|}{\|u_T\|} \frac{\|(Lu)_T\|}{\|(Lu)_T\|} \\
&\leq \inf_{\tilde{G}\in\mathcal{G}_{k,n}} \sup_{u\in\mathcal{U},w\in\mathcal{U},T>0} \frac{\left\|\left(NLu-\tilde{G}Lu\right)_T\right\|}{\|(Lu)_T\|} \frac{\|(Lw)_T\|}{\|w_T\|} \\
&\leq \|L\|\,\phi^{L\mathcal{U}}_{AE,N}.
\end{aligned}$$

□



F 3.10. Illustration of the derivation of a linear model candidate for Wiener systems in the IMOE and MOE setup.



F 3.11. Illustration of the derivation of a linear model candidate for Wiener systems in the additive error setup.

For the set $L\mathcal{U}$, an easy estimate can be given in the form

$$L\mathcal{U} \subseteq \left\{ \tilde{y} \in L_{pe}^{n} \middle| |\tilde{y}(t)| \leq \|L\|_{\infty} \cdot \sup \{ \|u\|_{\infty}, u \in \mathcal{U} \} \right\}$$

if all signals in the set $\mathcal{U}$ have bounded amplitude. This estimate can be used to calculate an upper bound on the nonlinearity measures for Wiener systems based on the static nonlinearity and the gain of the linear dynamic system part.

Figure 3.10 shows how the corresponding linear model $G = G_0 \circ L$ can be derived, where the memoryless linear system G_0 is a constant gain matrix K. Figure 3.8 shows the derivation of the model for the additive error setup. Note that linearity of L allows to move the branching point from behind L to before L, so the model candidate is also $G = L \circ G_0$, as above. But contrary to the MOE and IMOE setups, the error system Δ is given by $N_\delta \circ L$, where $\delta(v) = f(v) - Kv$, leading to the appearance of the set $L\mathcal{U}$ instead of $\mathcal{U}$ for the operating region of the memoryless error system N_δ.

The results above show one more possible use of the nonlinearity measures. Suppose different nonlinearity measures are given for one nonlinear system. Then it is possible

to tell whether the system structure is closer to a Hammerstein-like system or closer to a Wiener-like system (or to none of them) by comparing, e.g., the values of the MOE and the MIE nonlinearity measure.

Feedback interconnection. As the third common interconnection, we study the feedback interconnection of a linear system L in the forward path and a nonlinear system N in the feedback path. The transfer operator of this interconnection can be written as $L(I-NL)^{-1}$. If the nonlinear system is memoryless, this is the classical Lurie system structure. As expected, we can derive an upper bound on the feedback error nonlinearity measure for this type of system.

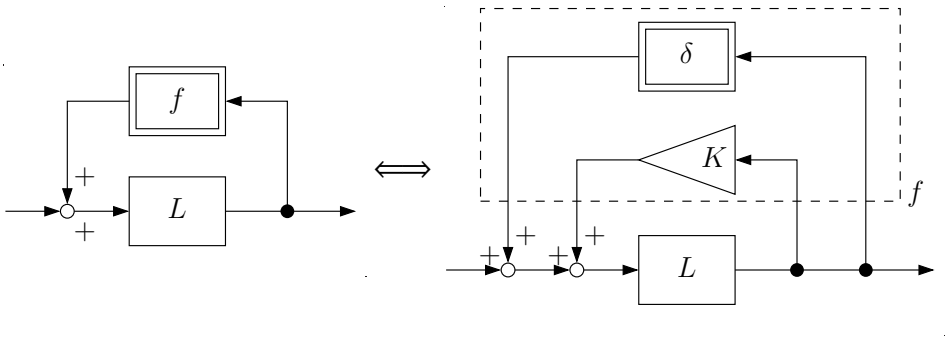
P 3.45 (Feedback interconnection of a linear and a nonlinear system). *Let $L \in \mathcal{G}_{m,n}$ be strictly proper and injective, let the nonlinear system $N : \mathcal{U} \subseteq L^n_{pe} \to L^m_{pe}$ be finite gain stable and the feedback interconnection $L(I-NL)^{-1}$ be well-posed, injective and finite-gain stable. Then $\phi^{\mathcal{U}}_{FE,L(I-NL)^{-1}} \leq \phi^{L(I-NL)^{-1}\mathcal{U}}_{AE,N}$.*

P . Insert the expression $L(I-NL)^{-1}$ for the feedback system in the definition of the FE-NLM and use the notation $\mathcal{Y} = L(I-NL)^{-1}\mathcal{U}$ to get

$$\begin{aligned}
\phi^{\mathcal{U}}_{FE,L(I-NL)^{-1}} &= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(\mathcal{Y})} \sup_{u\in\mathcal{U},T>0} \frac{\left\|\left(G^{-1}L(I-NL)^{-1}u-u\right)_T\right\|}{\left\|(L(I-NL)^{-1}u)_T\right\|} \\
&= \inf_{G^{-1}\in\mathcal{G}^{inv}_{m,n}(\mathcal{Y})} \sup_{y\in\mathcal{Y},T>0} \frac{\left\|\left(G^{-1}y-(I-NL)L^{-1}y\right)_T\right\|}{\|y_T\|} \\
&\leq \inf_{G^{-1}=(I-\tilde{G}L)L^{-1}\in\mathcal{G}^{inv}_{m,n}(\mathcal{Y})} \sup_{y\in\mathcal{Y},T>0} \frac{\left\|\left((I-\tilde{G}L)L^{-1}y-(I-NL)L^{-1}y\right)_T\right\|}{\|y_T\|} \\
&= \inf_{\tilde{G}\in\mathcal{G}_{m,n}} \sup_{y\in\mathcal{Y},T>0} \frac{\left\|\left(Ny-\tilde{G}y\right)_T\right\|}{\|y_T\|} = \phi^{\mathcal{Y}}_{AE,N}.
\end{aligned}$$

The supremum over $\mathcal{U}$ can be replaced be the supremum over $\mathcal{Y}$, because $L(I-NL)^{-1} : \mathcal{U} \to \mathcal{Y}$ is one-to-one and onto. In the second step, we made use of the special form of a linear model $G^{-1} = (I-\tilde{G}L)L^{-1}$. In order for this step to be valid, we have to verify that the system $G = L(I-\tilde{G}L)^{-1}$ is well-posed (i.e. it is a unique mapping) and surjective onto $\mathcal{Y}$. Well-posedness follows from strict properness of L and properness of $\tilde{G}$ [146]. Moreover, G is surjective onto $\mathcal{Y}$ if $(I-\tilde{G}L)^{-1}$ is surjective onto $(I-NL)^{-1}\mathcal{U}$, or equivalently, if $(I-\tilde{G}L)$ is defined for all $\tilde{u} \in (I-NL)^{-1}\mathcal{U}$, which clearly is the case. From Proposition 3.1 we know that we do not have to care whether to consider only stable models or not, so $\tilde{G}$ does not need to stabilize L for the argument to be valid and the proof is complete. □

If the nonlinear subsystem N is memoryless, the formula for the AE-NLM of memoryless systems can be used to derive an upper bound of the FE-NLM of the interconnection. The provenance of the associated model is illustrated in Fig. 3.12. It can be seen that the linear



F 3.12. Illustration of the derivation of a linear model candidate in the FE-NLM setup for feedback systems composed of a linear dynamic subsystem L in the forward path and a nonlinear memoryless system N_f in the feedback path.

model G is the feedback interconnection of L and the optimal memoryless model G_0 of N_f, represented by a constant gain matrix K, $G = L(I - G_0 L)^{-1}$.

Summary. In this section, we derived linear models for systems that are composed of a linear and a nonlinear subsystem. For such systems, linear models and bounds on the corresponding nonlinearity measures and model quality indices can be given by considering the linear and the nonlinear part separately. In case the nonlinear subsystem is memoryless, these results can be used with the help of the formulae given in Section 3.4. As compared to previous results [61], these results show the advantage of the general framework for nonlinearity assessment introduced in this thesis. In particular, the selection of a compatible error setup for a system with known structure is an advantage. Moreover, for Hammerstein and Wiener systems this approach also allows to handle systems with unstable linear dynamics.

3.7. Nonlinearity Measures and Harmonic Analysis

In this section we discuss a numerical approach on how to compute a lower bound for single-input single-output systems in the L_2-setting by means of harmonic analysis. The given method was first proposed in [2, 3] in a slightly less general setting.

We consider the special class of sinusoidal inputs $u(t) = A\sin(\omega t)$ where the sets of admissible frequencies and amplitudes are specified according to the desired operating regime. In particular we suggest the following lower bound on the nonlinearity measure.

D 3.46. For a finite-gain stable mapping $N : \mathcal{U} \subseteq L_{pe} \to L_{pe}$ and a set

$$\mathcal{U}_s = \{u(t) = A\sin(\omega t),\ A \in \mathcal{A} \subseteq \mathbb{R}^+, \omega \in \Omega \subseteq \mathbb{R}^+, \omega > 0\} \subseteq \mathcal{U}.$$

we define

$$\chi_{AE,N}^{\mathcal{U}_s} \triangleq \sup_{u\in\mathcal{U}_s} \inf_{G\in\mathcal{G}_{1,1}} \limsup_{T\to\infty} \frac{\| (Nu - Gu)_T \|_2}{\|u_T\|_2}. \tag{3.15}$$

Note that maximization and minimization are interchanged in (3.15) as compared to (3.1), and the supremum over time is replaced by a limit, and hence $\phi^{\mathcal{U}}_{AE,N} \geq \chi^{\mathcal{U}_s}_{AE,N}$.

It can often be assumed that the response of a stable nonlinear system N to a sinusoidal input with fixed amplitude $A \in \mathcal{A}$ and fixed frequency $\omega \in \Omega$ is the sum of (i) the continuous periodic steady-state response y_{SS}, and (ii) the transient response $y_{tr} \to 0$ that decays to zero, $y_N = Nu = y_{tr} + y_{SS}$. The periodic signal y_{SS} can then be expanded in a Fourier series

$$y_{SS}(t) = A_0(\omega, A) + \sum_{k=1}^{\infty} A_k(\omega, A) \cdot \sin(k\omega t + \varphi_k) \tag{3.16}$$

where the amplitude coefficients A_k are functions of the amplitude and the frequency of the input signal.[9] Given the Fourier coefficients of the responses of N to inputs in $\mathcal{U}_S$, the lower bound on the additive error nonlinearity measure can be calculated easily.

T 3.47. *Let $N : \mathcal{U} \subset L_{pe} \to L_{pe}$ be a finite-gain stable mapping exhibiting asymptotically periodic outputs to sinusoidal inputs. If $\mathcal{U}_s$ and $\chi^{\mathcal{U}_s}_{AE,N}$ are as in (3.15), then*

$$\chi^{\mathcal{U}_s}_{AE,N} = \sup_{\substack{A \in \mathcal{A} \\ \omega \in \Omega}} \frac{1}{A} \sqrt{2A_0^2(\omega, A) + \sum_{k=2}^{\infty} A_k^2(\omega, A)}, \tag{3.17}$$

where $A_k(\omega, A)$ are the amplitude coefficients of the k^{th} harmonics of the steady-state response y_{SS} to the input $u(t) = A\sin(\omega t)$.

P . Consider an input signal with fixed amplitude A and fixed frequency ω. First of all, from $y_{tr} \to 0$ we have that (as in the proof of Theorem 3.20) $\lim_{T\to\infty} \sqrt{\frac{1}{T}} \| (y_{tr})_T \|_2 = 0$ and thus

$$\lim_{T\to\infty} \sqrt{\frac{1}{T}} \| (Gu - Nu)_T \|_2 = \lim_{T\to\infty} \sqrt{\frac{1}{T}} \| (Gu - y_{SS})_T \|_2.$$

We rewrite the periodic steady-state response as $y_{SS}(t) = a_0 + \sum_{k=1}^{\infty} \{a_1 \sin(k\omega t) + b_1 \cos(k\omega t)\}$. The periodic steady-state response of a stable linear system is $(Gu)(t) = y_L(t) = \alpha \sin(\omega t) + \beta \cos(\omega t)$. By orthogonality of the sine and cosine functions involved, we can write

$$\lim_{T\to\infty} \sqrt{\frac{1}{T}} \left\| (y_{SS} - y_L)_T \right\|_2 = \sqrt{a_0^2 + \frac{1}{2}(a_1 - \alpha)^2 + \frac{1}{2}(b_1 - \beta)^2 + \frac{1}{2}\sum_{k=2}^{\infty} \left(a_k^2 + b_k^2\right)}$$

where we used the fact that $\lim_{T\to\infty} \frac{1}{T} \left\| (\sin(k\omega t + \varphi_k))_T \right\|_2^2 = \frac{1}{2}$. Moreover, $\lim_{T\to\infty} \frac{1}{T} \|u_T\|_2^2 = \frac{A^2}{2}$, and we have

$$\limsup_{T\to\infty} \frac{\| (Nu - Gu)_T \|_2}{\|u_T\|_2} = \frac{\sqrt{a_0^2 + \frac{1}{2}(a_1 - \alpha)^2 + \frac{1}{2}(b_1 - \beta)^2 + \frac{1}{2}\sum_{k=2}^{\infty} \left(a_k^2 + b_k^2\right)}}{\sqrt{\frac{1}{2}A}}.$$

[9] Classes of nonlinear systems where this decomposition holds are for instance given in [110, 35]. For systems given in state-space form, this assumption can be verified by the attractivity of the center manifold corresponding to an exosystem that models the sinusoidal excitation signal [72].

Straightforward calculation shows that the last expression is minimized by any linear system that satisfies the principle of harmonic balance [136], i.e. $y_L(t) = -a_1 \sin(\omega t) - b_1 \cos(\omega t)$. With $A_k^2(\omega, A) = a_k^2 + b_k^2$ the result follows. □

When choosing finite sets of amplitudes and frequencies, the quantity $\chi_{AE,N}^{\mathcal{U}_s}$ constitutes a computationally attractive opportunity to calculate a lower bound on the AE-NLM for dynamic systems. The numerical computation of $\chi_N^{\mathcal{U}_s}$ only involves the Fourier analysis of the simulated output and therefore the computation can also be done for systems described by differential-algebraic equations, systems with non-smooth nonlinearities, or systems governed by partial differential equations. From (3.15) it follows immediately that if N is a linear I/O-operator, the value of the lower bound is also equal to zero. From $\chi_N^{\mathcal{U}_s} = 0$ it can however only be concluded that for each input signal $u \in \mathcal{U}_s$ there exists a perfect linear system G_u^* (depending on the respective u). Linearity on $\mathcal{U}_s$ follows only under the additional condition that $\bigcap_{u\in\mathcal{U}_s}\{G_u^*\} \neq \emptyset$.

Similar results can be derived for the IMOE-NLM and the MOE-NLM.

C 3.48. *Assume the same conditions as in Theorem 3.47, and define*

$$\chi_{MOE,N}^{\mathcal{U}_s} \triangleq \sup_{u\in\mathcal{U}_s} \inf_{G\in\mathcal{G}_{1,1}} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|(Gu)_T\|} \tag{3.18}$$

and

$$\chi_{IMOE,N}^{\mathcal{U}_s} \triangleq \sup_{u\in\mathcal{U}_s} \inf_{G\in\mathcal{G}_{1,1}} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|}. \tag{3.19}$$

Then $\phi_{MOE,N}^{\mathcal{U}} \geq \chi_{MOE,N}^{\mathcal{U}_s}$ *and* $\phi_{IMOE,N}^{\mathcal{U}} \geq \chi_{IMOE,N}^{\mathcal{U}_s}$ *and*

$$\chi_{MOE,N}^{\mathcal{U}_s} = \chi_{IMOE,N}^{\mathcal{U}_s} = \sup_{\substack{A\in\mathcal{A}\\ \omega\in\Omega}} \sqrt{1 - \frac{A_1^2(\omega, A)}{2A_0^2(\omega, A) + \sum_{k=1}^{\infty} A_k^2(\omega, A)}}.$$

P . Redo the proof of Theorem 3.47 and calculate the optimal α, β according to the definitions of the MOE and IMOE-NLM respectively. □

For nonlinear systems that do not produce a steady-state offset A_0, the lower bounds of the MOE- and IMOE-NLM are in fact a variant of the total harmonic distortion (THD), that gives the sum of powers of the harmonic signals divided by the sum of powers of all frequencies (including the fundamental frequency).

The following proposition shows that similar definitions for the MIE- and IMIE-setup turn out not to be useful, because they only deliver the trivial lower bound of zero.

P 3.49. *Given the same assumptions as above, we have*

$$\chi_{MIE,N}^{\mathcal{U}_s} \triangleq \sup_{u\in\mathcal{U}_s} \inf_{G^{-1}\in\mathcal{G}_{1,1}^{inv}} \limsup_{T\to\infty} \frac{\left\|\left(G^{-1}Nu - u\right)_T\right\|}{\|u_T\|} = 0 \tag{3.20}$$

and

$$\chi_{IMIE,N}^{\mathcal{U}_s} \triangleq \sup_{u\in\mathcal{U}_s} \inf_{G^{-1}\in\mathcal{G}_{1,1}^{inv}} \limsup_{T\to\infty} \frac{\left\|\left(G^{-1}Nu-u\right)_T\right\|}{\left\|\left(G^{-1}Nu\right)_T\right\|} = 0. \tag{3.21}$$

P . For any sinusoidal input $u(t) = A\sin(\omega t)$, and thus for any output $y_{SS}(t) = A_0(\omega, A) + \sum_{k=1}^{\infty} A_k(\omega, A)\cdot\sin(k\omega t + \varphi_k)$ of the nonlinear system, we can construct a proper, stable, minimum-phase system $G^{-1} \in \mathcal{G}_{1,1}^{inv}$ such that $G^{-1}\left(A_1(\omega, A)\cdot\sin(\omega t + \varphi_1)\right) = A\sin(\omega t) = u(t)$ and all other terms are attenuated to any desired level. As there is a uniform bound on all A_k, the result follows. □

The lower bounds for the MOE and IMOE nonlinearity measures lead to one more interesting fact concerning the suitability of those measures for Hammerstein systems, that has been observed for the IMOE nonlinearity measure in [61, 62]. Consider a nonlinear memoryless system that is not odd (i.e. $f(-v) \neq -f(v)$ for some $v \in \mathcal{V}$) followed by a strictly proper linear system with a non-vanishing steady-state gain. For sinusoidal inputs, the memoryless nonlinearity generates a periodic signal of the same frequency superposed with a constant bias. For very high input frequencies, the linear system attenuates the periodic signal to an arbitrarily small level, and lets pass only the constant bias. In this case, the measures $\chi_{MOE,N}^{\mathcal{U}_s}$ and $\chi_{IMOE,N}^{\mathcal{U}_s}$ are equal to one as the only non-vanishing term is A_0, and thus the MOE and IMOE nonlinearity measures also take their maximal value. This fact shows again that the value of nonlinearity measures can deliver insight in the system structure. Vice versa, given a system with known structure the result helps to select the appropriate nonlinearity measure for systems analysis.

If the aim is not the derivation of a linear model, and one is contented with a rough estimate in form of a lower bound on the nonlinearity measure, harmonic analysis is a computationally very efficient way for nonlinearity assessment. As only simulation data is needed for the computation, it is also attractive for plants that are only given by simulation code.

3.8. Computational Aspects

The results in Sections 3.4 to 3.6 allow to compute the nonlinearity measures and best linear models of memoryless systems and Wiener and Hammerstein systems. If only the nonlinearity measure is of interest and no linear model is desired, the methods based on harmonic analysis presented before (Section 3.7) often provide good estimates for general nonlinear systems, and these estimates are very easy to obtain. Also, the computation of a lower bound of general nonlinear systems by considering the steady state behaviour (Sections 3.4 and 3.5) promises a first insight at a very low computational cost. If a model is already given as a series of linear and higher order terms, this information can be used in some cases to directly compute the nonlinearity measures (or, more precisely, model quality indices), as done in [56] using the technique of functional expansion. But we are of course also interested in having a method to obtain the nonlinearity measure and a best linear model for general nonlinear systems. In this section we consider the problem of computing nonlinearity measures for general nonlinear systems.

Problem structure. The exposition so far has made clear that nonlinearity measures are equal to the gain of the error system corresponding to the best linear model. Consequently, there are strong links between the computation of nonlinearity measures and the computation of system gains over sets. Consider the structure of a general nonlinearity measure as pointed out in Section 3.1, Eq. (3.3),

$$\phi_N^{\mathcal{U}} = \inf_{G \in G_{m,n}} \|\Delta\|^{\chi} .$$

The problem of computing the quantity $\phi_N^{\mathcal{U}}$ can be decomposed in two parts, an outer minimization problem to get the best linear model, and an inner maximization problem in order to obtain, for each linear model candidate, the corresponding error gain.

It has been observed before that the outer minimization problem is convex in G for the quantities that we call the additive error nonlinearity measure and the inverse multiplicative output error nonlinearity measure for a finite set $\mathcal{U}$ [2, 3, 57]. This result can be generalized to arbitrary sets $\mathcal{U}$, and in a similar manner the minimization problems for the multiplicative input nonlinearity measure and the feedback error nonlinearity measure are convex in G^{-1}.

P 3.50. *Let $N : \mathcal{U} \subseteq L_{pe}^m \to L_{pe}^n$ be a causal, finite-gain stable mapping. Then the additive error model quality index*

$$\|\Delta\|^{\mathcal{U}} \triangleq \sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|u_T\|}$$

and the inverse multiplicative output error model quality index

$$\|\Delta\|^{N\mathcal{U}} \triangleq \sup_{u \in \mathcal{U}, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|}$$

are convex in $G \in \mathcal{G}_{m,n}$. Furthermore, the functions corresponding to the error system gain in the MIE setup

$$\delta_{MIE,N}^{\mathcal{U}}(G^{-1}) \triangleq \sup_{u \in \mathcal{U}, T>0} \frac{\left\| \left(G^{-1}Nu - u \right)_T \right\|}{\|u_T\|}$$

and in the FE setup

$$\delta_{FE,N}^{\mathcal{U}}(G^{-1}) \triangleq \sup_{u \in \mathcal{U}, T>0} \frac{\left\| \left(G^{-1}Nu - u \right)_T \right\|}{\|(Nu)_T\|}$$

are convex in $G^{-1} \in \mathcal{G}_{m,n}^{inv}(N\mathcal{U})$.

P . Convexity for the AE setup follows from

$$\begin{aligned} \|N - (\lambda G_1 + (1-\lambda)G_2)\|^{\mathcal{U}} &= \sup_{u \in \mathcal{U}, T>0} \frac{\|((\lambda + (1-\lambda))\, Nu - (\lambda G_1 + (1-\lambda)G_2)\, u)_T\|}{\|u_T\|} \\ &\leq \sup_{u \in \mathcal{U}, T>0} \left(\frac{\lambda \|(Nu - G_1 u)_T\| + (1-\lambda)\|(Nu - G_2 u)_T\|}{\|u_T\|} \right) \\ &\leq \lambda \|N - G_1\|^{\mathcal{U}} + (1-\lambda)\|N - G_2\|^{\mathcal{U}} . \end{aligned}$$

for all $\lambda \in [0, 1]$. The proof for the other setups is done in a similar way and is omitted. □

The two last expressions, $\delta^{\mathcal{U}}_{MIE,N}(G^{-1})$ and $\delta^{\mathcal{U}}_{FE,N}(G^{-1})$ are strictly spoken not model quality indices, as they are functions of G^{-1} and not of G. It was shown in Section 3.2 that all problems are lower semi-continuous in G or G^{-1} respectively, so that by virtue of a finite-dimensional parameterization, the minimization problems can be treated in a numerically efficient way.

A Direct Method. In order to avoid the computation of the norm for each model candidate in the process of minimization, Allgöwer [2, 3] proposed to select a representative finite subset $\mathcal{U}_N \subset \mathcal{U}$ from the infinite-dimensional set $\mathcal{U}$ that characterizes the region of operation. The minimax problem of computing the nonlinearity measure can then be reformulated into a convex constrained minimization problem that can be efficiently solved with numeric standard tools, like sequential quadratic programming [3]. Although this method proved to be efficient and to deliver good results in many applications [2, 3, 57, 60, 62, 59, 122, 113, 114, 123], no bounds on the approximation error can be given due to the finite approximation of $\mathcal{U}$.

An Iterative Method. Once an efficient computational scheme to derive the gain $\|\Delta\|^{\mathcal{X}}$ along with a (suboptimal) maximizing input ($u \in \mathcal{U}, T \in \mathbb{R}^+$) as a function of the linear model G (or its inverse G^{-1}, respectively) is available, an appealing scheme to compute the nonlinearity measure is the one given by Sourlas and Manousiouthakis [124]. In order to fix the ideas, we give the algorithm for the additive error nonlinearity measure, but the application of the procedure to other definitions is straightforward. The overall iterative algorithm is as follows:

(1) Start with an arbitrary model candidate G_0, e.g. $G_0 = I$ of $G_0 = G^*_{loc}$, and an empty set of input signals $U_0 = \{\}$.

(2) Compute the gain of the error system $\hat{\gamma}_k = \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-G_k u)_T\|}{\|u_T\|}$ and add the maximizing input signals u^*_{k+1} and time horizon T^*_{k+1} to the respective sets, $\mathcal{U}_{k+1} = \mathcal{U}_k \cup \{u^*_{k+1}\}$, $\mathcal{T}_{k+1} = \mathcal{T}_k \cup \{T^*_{k+1}\}$.

(3) Compute $\check{\gamma}_k = \inf_G \max_{u\in\mathcal{U}_k,T\in\mathcal{T}_k} \frac{\|(Nu-G_k u)_T\|}{\|u_T\|}$ by using the constraint convex minimization described above. In [3], the linear model is parametrized by a finite number N of linear terms

$$G = \sum_{i=0}^{N} \frac{d_i}{1 + T_i s}$$

where the time constants are fixed in advance. The resulting constraint convex optimization problem is then solved to get the model candidate G_{k+1}.

(4) Repeat steps 2 and 3 until $\check{\gamma}_k - \hat{\gamma}_k$ is sufficiently small (e.g. below a prescribed tolerance).

An obvious advantage of the procedure is that we obtain a non-decreasing sequence of lower bounds from the values of $\check{\gamma}_k$, and a sequence of upper bounds $\hat{\gamma}_k$, from which we can construct a non-increasing sequence $\tilde{\hat{\gamma}}_l = \sup_{l\le k} \hat{\gamma}_k$. No convergence results of these two sequences are available for a general system N, but in practical cases, the upper and

lower bounds allow us to terminate when a specified accuracy is reached. The application example in Section 4.4 is computed with this procedure.

The computation of gains plays a central role in the above procedure, and we will discuss possible approaches to perform that computation next.

Gain Computation via Optimal Control and Direct Optimization. The problem to compute the quantity

$$\|P\|^{\mathcal{U}} = \sup_{u \in \mathcal{U}, T>0} \frac{\|(Pu)_T\|}{\|u_T\|}$$

can be viewed as the problem to compute a maximizing pair (u, T) of the expression $\|(Pu)_T\| / \|u_T\|$. This formulation constitutes in fact an optimal control problem with free horizon.

Some researchers use necessary conditions from optimal control theory (as found in, e.g., [104, 82, 15, 22]) for the derivation of a solution to this problem. Nikolaou and Manousiouthakis [99] propose to use the identity $\sup_u \frac{\|Nu\|}{\|u\|} = \sup_{\delta>0} \left\{ \frac{1}{\delta} \sup_u \frac{\|Nu\|}{\delta} \right\}$ to compute the gain and the identity $(\|N\| \leq) \|N\|_\Delta = \sup_{u_0} \left\| L_{u_0} \right\|$ with L_{u_0} the linearization around a trajectory u_0 to compute the incremental gain (see also [31]). The related quantity of a stability measure is computed with non-smooth optimal control techniques in [20]. The results given in [124] for discrete-time second order Volterra models show that it is possible to use this approach for special system classes. There are, however, several serious concerns about the suitability of this approach for general systems. Firstly, the conditions are only necessary for a global maximum. The numeric solution can as well represent a local maximum, or even a minimum or saddle-point. Secondly, and even more importantly, existence of an optimal solution is not guaranteed, i.e. it is not known whether an input signal u^* exists such that

$$\frac{\|(Pu^*)_{T^*}\|}{\left\|u^*_{T^*}\right\|} = \sup_{u \in \mathcal{U}, T>0} \frac{\|(Pu)_T\|}{\|u_T\|}.$$

As indirect methods of optimal control are not applicable in general, it is possible to use direct optimization methods instead. In order to do so, the input signal is approximated by a finite number of parameters, and the resulting finite-dimensional optimization problem is solved directly. Because the discretized optimization problem is typically of high order and is non-convex, the use of randomized algorithms is an option in order to compute a good approximation of the gain [121]. A method that has a stronger theoretical foundation, but is restricted to low order systems of special structure, can be based on the theory of dissipative systems, as discussed next.

Gain Computation via Dissipation Inequalities. In Chapter 2, dissipation inequalities have been introduced as a means to determine the gain of a nonlinear system. This is illustrated below by a small example.

E . Consider the system P given by

$$\begin{aligned} \dot{x} &= -x - x^3 + u \\ y &= x \end{aligned}$$

for $X = V = \mathbb{R}$ and $p = 2$. The storage function $S(x) = \gamma x^2$ satisfies

$$\begin{aligned} \dot{S} &= 2\gamma x\left(-x - x^3 + u\right) \\ &= -2\gamma x^2 - 2\gamma x^4 + 2\gamma xu - \gamma^2 u^2 - x^2 + \gamma^2 u^2 + x^2 \\ &\leq -(x - \gamma u)^2 - (2\gamma - 1)\, x^2 + \gamma^2 u^2 \end{aligned}$$

and finally $\dot{S} \leq -x^2 + \gamma^2 u^2$ if $\gamma \geq 1$. On the other hand, suppose $\gamma < 1$. For any S we can choose $u = x + x^3$ to get $x = const.$ and thus $S = const.$, and by choosing a sufficiently small x the expression

$$\int_0^T -x^2 + \gamma^2 u^2 \mathrm{d}t = \int_0^T \left(\gamma^2 - 1\right) x^2 + 2\gamma^2 x^4 + \gamma^2 x^6 \mathrm{d}t$$

can be made strictly negative. Hence, if $\gamma < 1$ there can be no S such that $S(x(T)) - S(x(0)) = 0 \leq \int_0^T -x^2 + \gamma^2 u^2 \mathrm{d}t$ for all x and u, and the L_2-gain of the example system is $\|P\|_2 = 1$. ■

The example demonstrates that Proposition 2.2 allows to tightly characterize the gain of a general nonlinear system P given in state-space form over a given set of input signals $\mathcal{U}(V)$. However, there are three main difficulties in applying this procedure to the gain computation of general nonlinear systems.

(1) In the above example, we took the whole of $\mathbb{R}$ as range for the inputs, and accordingly the whole of $\mathbb{R}$ as invariant region. Considering unbounded inputs often results in unbounded gains, so it is useful to consider only a restricted set of admissible input values. If the inputs are constrained for instance by an amplitude bound, we first have to determine an according invariant region X (for deriving an upper bound on the gain) or the reachability set respectively (to tightly approximate the gain). But the determination of such an invariant region or reachability set is a very demanding task in general.

(2) If for a given V we have determined the region X, we have to find a storage function S for the supply rate $s(u, y) = \gamma^p |u|^p - |y|^p$ on $X \times V$. Therefore, we have to find a common positive semi-definite solution to an inhomogeneous linear partial differential inequality with variable coefficients where the disturbance term can vary within the given bound on u. Also this step can only be hoped to be solved for special system classes.

(3) In order to get a tight approximation (i.e. an error bound by calculating a lower bound) of the gain, the non-existence of a storage function for a given range of values for γ has to be demonstrated as in the example above. Otherwise one has to be satisfied by obtaining an upper bound on the gain.

Regarding problems (1) and (2), we will see that these tasks can be solved to some extend for the special class of polynomial systems. An approach for general nonlinear systems using the notion of input-to-state stability and a gridding technique can be found in [79]. Problem (3) however cannot be solved by constructive methods, and it remains an open problem in general. Fortunately, for the systems analysis and controller design tasks described in Chapters 3 and 4, an upper bound on gains will turn out to be more important

than lower bounds, so we will focus on the solution of problems (1) and (2). If an invariant set X is known, and the dissipation inequality is satisfied on $X \times V$ for a given γ, then the system has an L_2-gain on $\mathcal{U}(V)$ that is smaller or equal to γ.

Gains and Invariant Sets of Linear Systems. In the case of linear systems G, the L_2-gain can be characterized by taking into account sinusoidal inputs only [146]. In this case, any bound on the input that allows for an open ball in $\mathbb{R}^m$ does not have an impact on the L_2-gain, and $\|G\|_2^{\mathcal{U}(V)} = \|G\|_2$. A similar argument is valid for the case of the L_∞-gain [25]. However, for the additive error nonlinearity measure, we need to compute the gain of the difference between a nonlinear system and a linear system. In such a case it will be helpful to calculate the invariant sets of the linear system part separately. In the case of a bound given by $\max_i |u_i(t)| \leq u_{max}$, the L_∞-gain can be used to get bounds on each state separately. If the bound on the input is given as an ellipsoid, invariant sets under norm-bounded disturbances can also be efficiently computed [14, 11]. For a linear system given by

$$\dot{x} = Ax + Bu \tag{3.22}$$

where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^z$ and the input is bounded by $u^T u \leq 1$ a set $\mathcal{P} = \{x \in \mathbb{R}^z : x^T P x \leq 1\}$ is invariant if and only if for some $\alpha > 0$

$$PA + A^T P + \alpha P + \frac{1}{\alpha} PBB^T P \leq 0. \tag{3.23}$$

Moreover, if the pair (A, B) is controllable the smallest such set for a given α is the set $\mathcal{P}^* = \{x \in \mathbb{R}^z : x^T S^{-1} x \leq 1\}$ with S^{-1} the positive definite solution of

$$AS + SA^T + \alpha S + \frac{1}{\alpha} BB^T = 0. \tag{3.24}$$

A minimization of the trace of S (maximization of the trace of P) over the real parameter $\alpha > 0$ delivers the tightest ellipsoid that is invariant under the given perturbation, i.e. the best approximation of the reachable set. If the system matrix A is Hurwitz one only needs to check $0 < \alpha < -2 \max \mathrm{Re}\{\lambda(A)\}$ [14]. In principle, one can mimic the same procedure as above also for nonlinear systems [13]. This approach leads however to invariant sets that are also globally attractive, a property that is far too strong for our purposes and that leads to overly conservative results.

E . Consider the linear system $\dot{x} = -ax + bu$. Then equation (3.24) delivers $-2as + \alpha s + b^2/\alpha = 0$ or $p = \alpha(2a - \alpha)/b^2$. Hence, for any $0 < \alpha < 2a$, the set $\{|x| \leq b/\sqrt{\alpha(2a - \alpha)}\}$ is invariant for all $|u| \leq 1$, and the smallest of those sets is $\{|x| \leq b/a\}$ for $\alpha = a$. For any other input bound $|u| \leq \beta$, the invariant set is accordingly obtained as $\{|x| \leq \beta b/a\}$.

Invariant Sets and Gains of Nonlinear Systems. To determine an invariant region for general nonlinear systems is a very difficult problem. This task can be greatly facilitated by considering the special class of polynomial systems. The inequalities arising in systems analysis are then also often of polynomial type, and they can be checked with limited

conservativeness by sum-of-squared (SOS) methods. Such methods rely on the fact that a polynomial $p(x)$ of given order may be written as a sum of squares, i.e.

$$p(x) = \sum_i (p_i(x))^2 ,$$

if and only if it can be represented as

$$p(x) = m(x)^T M m(x)$$

where $m(x)$ is the vector of monomials of x and M is a positive semi-definite matrix [21]. Thus, the proof of positivity of $p(x)$ can be reduced to the sufficient condition that M be positive definite.

The SOS technique allows to check invariance of a level set of a positive definite function $V(x)$ in the following way. For systems admitting a unique solution, it is possible to characterize positive invariance of a set by the behaviour of the vector field on the boundary of that set [11]. Let the input be bounded by $u(t)^T u(t) \leq r_u^2$ for all t. If

$$\frac{\partial V}{\partial x} f(x(t), u(t)) \leq 0 \text{ for all } V(x) = 1 \text{ and } u(t)^T u(t) \leq r_u^2$$

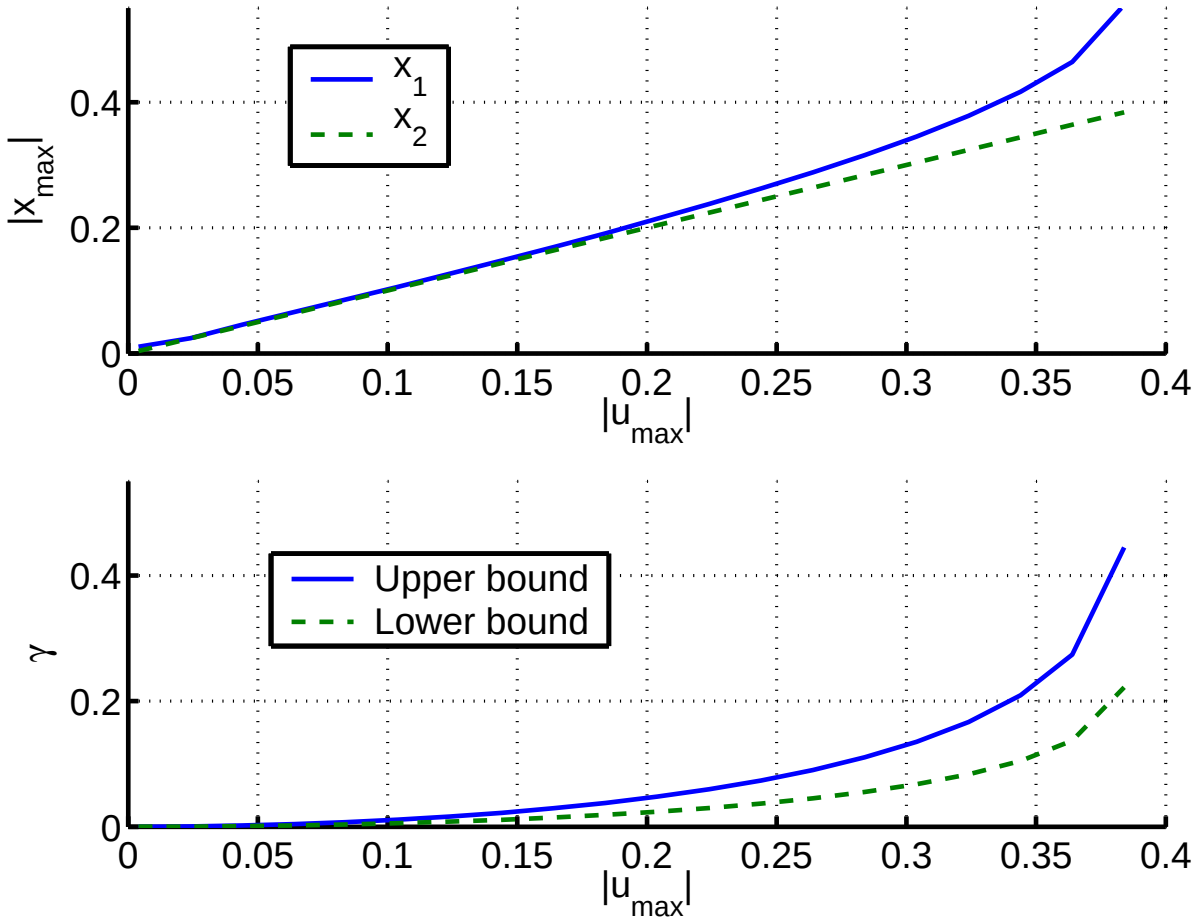
then the level set $\{V(x) \leq 1\}$ is positive invariant under all admissible inputs. With the help of the Positivstellensatz, a generalization of the S-procedure, this equation can be verified with the help of SOS methods [74]. The drawback of the proposed method is that the Lyapunov-function $V(x)$ can not be calculated by semi-definite programming techniques, as V and $\frac{\partial V}{\partial x}$ do not enter in an affine way in the polynomial inequalities. Instead, a candidate function has to be known, and then the set $\{V(x) \leq 1\}$ can be tested for invariance.

Once an invariant set has been computed, there are different ways to compute the L_2-gain over $\mathcal{U}(V)$ with the help of the dissipation inequality $\frac{\partial S}{\partial x} f(x(t), u(t)) \leq \gamma^p |u|^p - |y|^p$. The key problem is to find a storage function S that satisfies this inequality *on S only*, in order to avoid unnecessary conservatism. If the bounds on the states are given for each input and state separately, then the substitution $\tilde{x}_i = \frac{2x_i x_{i,max}}{1+x_i^2}$ and $\tilde{u}_i = \frac{2u_i u_{i,max}}{1+u_i^2}$ in the dissipation inequality leads to a condition that can be tested globally without introducing additional conservatism. If the set X is given in the form $X = \{V(x) \leq 1\}$, then once again results from algebraic geometry lead to a formulation that can be worked with via SOS, and this time the smallest number γ that satisfies (2.2) on $X \times V$ can be computed directly.

E . Consider the system P given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 + u \\ \dot{x}_2 &= -x_2 + u \\ y &= x_1 - x_2 \end{aligned}$$

for $V = [-u_{max}, u_{max}]$ and $p = 2$. The invariant region can be computed with the computational procedure described above, by using the Lyapunov function candidate $V(x) = x^2/x_{max}^2$ and searching for a feasible x_{max} for the subsystem of state x_1. For this simple problem, it is also possible to compute x_{max} directly by solving the equation $-x_{max} + x_{max}^3 + u_{max} = 0$, and both methods lead to the same results. The L_∞-gain from u to x_2 can immediately



F 3.13. Reachable set, characterized by x_{max} and bounds on the gain γ of the system $\dot{x}_1 = -x_1 + x_1^3 + u$, $\dot{x}_2 = -x_2 + u$, $y = x_1 - x_2$ over the operating region $|u| \leq u_{max}$.

seen top be one. The reachable magnitude of the states are displayed in Fig. 3.13, upper part, as a function of u_{max}. These results can in a second step be used to compute an upper bound on the L_2-gain, while a lower bound is obtained by application of Theorem 3.20. Both bounds are shown in Fig. 3.13, lower part. For inputs with a larger amplitude than $u_{max} = \frac{2}{9}\sqrt{3} \approx 0.385$ the system is unstable and the L_2-gain is unbounded. For very small values, the steady-state gain can be seen to tend to zero, as, in fact, the equation of x_2 is the linearization of the equation for x_1. This example shows that for small polynomial systems, the characterization of the gain via dissipation inequalities indeed opens a possibility to compute (an upper bound on) the gain. In fact, the system P corresponds to the additive error system Δ of the nonlinear system $\dot{y} = -y + y^3 + u$ and the state space linearization as linear model, $\dot{\tilde{y}} = -\tilde{y} + u$. Therefore the bounds shown in Fig. 3.13 are nothing else but bounds for the additive error model index over $\mathcal{U}(V)$. This example emphasizes the utility of regional analysis for nonlinear systems, when a global analysis is not possible. ■

Summary. The computation of nonlinearity measures is in principle a complex problem, as it requires the solution of a minimax problem over infinite-dimensional sets. However,

the convex structure of the minimization part allows the efficient approximation of the value of four of the nonlinearity measures and the corresponding best linear models. In order to improve the accuracy of the solution, the computation of gains over sets is necessary. We have pointed out that the necessary conditions for optimal control should not be used for the derivation of a solution, as it cannot be guaranteed that the maximizing input indeed exists. Therefore two alternative approaches have been given. One possibility is the use of (direct) global optimization techniques. Randomized algorithms are an interesting option, because they can also be applied to treat big optimization problems without a priori known structure. They thus can help to improve the results as compared to ad hoc techniques. The disadvantage of using those algorithms is, however, that they do not provide error bounds on the solution. In order to get a rigorous upper bound on the gain of a system, dissipation theory in connection with SOS-techniques can be used, if small scale systems are considered.

3.9. Summary

In this chapter, we introduced a novel framework for the derivation and assessment of linear models for nonlinear systems. The novel framework unifies and extends results known from literature about nonlinearity assessment and linear models for nonlinear systems. As the related nonlinearity measures known from literature, the techniques are applicable to finite gain stable systems only. The key idea is to measure the quality of a linear model G by the gain of the modeling error Δ over a specified region of operation. The modeling error Δ is defined such that the interconnection of G and Δ reproduce the input-output behaviour of the given nonlinear system N. The novel aspect is thereby to consider interconnections other than the parallel connection, that corresponds to an additive modeling error. The setup and the definitions of nonlinearity measures have been justified in a mathematical rigorous way. We have given results that relate the mathematical setup to engineering intuition. It has thus been shown for some of the interconnection structures that the best model for a vanishing operating regime corresponds to the state-space linearization at the operating point, and that the best linear model for a memoryless nonlinear system is also memoryless. With these results at hand, linear models and bounds on the nonlinearity measures have been derived for Hammerstein systems, Wiener systems, Lurie systems and similar system classes. These results gave a first hint on how to choose the best interconnection structure for a given system, based on an a priori known structure. Finally, we discussed the computation of nonlinearity measures for general systems. Due to the convex structure of the related optimization problems, four of the measures can be computed at feasible cost.

The results given in this chapter show for the first time relations between existing nonlinearity measures, and extend them to a more general setup. As compared to previous work on linear modeling and nonlinearity measures in continuous time, at the same time a broader view on this problem and more detailed results on the foundation of nonlinearity measures is obtained. These foundations will prove to be especially useful for the design of linear controllers for nonlinear systems as discussed in the next chapter.

CHAPTER 4

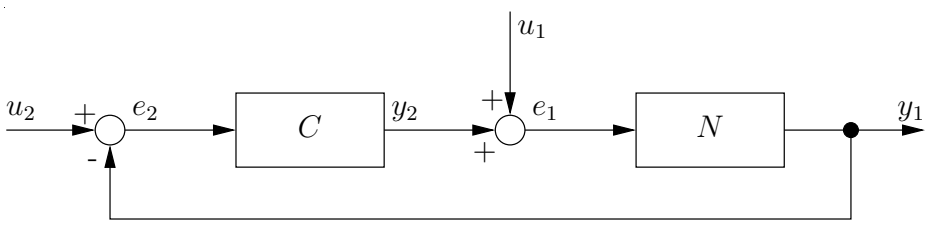
A Small Gain Approach to Linear Control of Nonlinear Systems

The problem considered in this chapter is the design of a linear controller C for a nonlinear plant N. The setup of the control system is given in Fig. 4.1. Two exogenous signals are entering the control loop corresponding to what are usually the disturbance signal at the plant input (u_1) and the reference signal (u_2). The regulated variables are the controller output y_2 and the plant output y_1 or their respective inputs e_i.

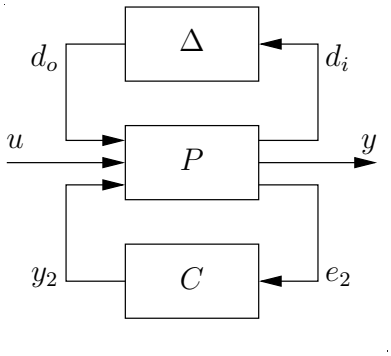
The usual approach to the design of linear controllers for nonlinear systems is to use the linearization at the operating point as a linear model, design a linear controller C and analyze the stability and performance of the closed loop with the nonlinear plant. Using this method, no stability and performance guarantees can be made and the degree of nonlinearity of the plant is not taken into account in the controller design step. In this section, we present a novel approach to linear controller design for nonlinear systems that guarantees stability of the closed loop containing the nonlinear process.

The central idea of the method is to use the framework and the results from nonlinearity analysis in Chapter 3. In fact, all nonlinearity measures give the gain of the error system when a given nonlinear process is represented as the interconnection of a (best) linear model and this error system. This chapter establishes how this information can be used for control-relevant nonlinearity assessment and linear controller design for nonlinear systems.

It is necessary to note that, as the method developed in this chapter makes use of the nonlinearity measure framework, it is restricted to finite-gain stable nonlinear systems. So, in order to leave no room for ambiguity, the goal of the proposed controller design method



F 4.1. General setup for the considered design of a linear controller C for the nonlinear plant N.



F 4.2. Generalized plant and setup for robust control.

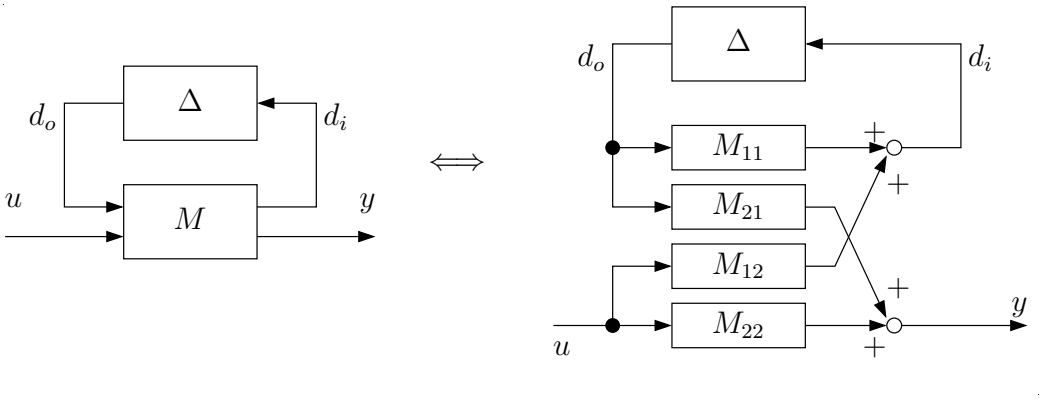
is not to stabilize unstable plants by linear feedback. Instead, the purpose of our efforts is to be able to design a linear controller using a linear model in order to enhance the performance of the nominal control system. Although only stability of the control loop with the nonlinear plant can be guaranteed, also the performance is expected to increase. Examples will show that our effort succeed, and that in particular set-point tracking can be achieved also for the nonlinear system.

Before turning to the main results, we first briefly review the principles of linear robust control. After some preliminary results in Section 4.1 the road is paved for the main results in Section 4.2 that establish stability for nonlinear systems with a linear controller. Results for all interconnection types along with some illustrating examples are given. In Section 4.3, some possible extension to performance synthesis are sketched, and the example of linear control of a practically relevant nonlinear CSTR closes the chapter in Section 4.4.

4.1. Principles of (Non-Global) Robust Control

The goal of linear robust control is to stabilize a given linear plant G despite the presence of a linear modeling error (or *uncertainty*) Δ, only the gain of which is known. The usual setup in linear robust control is the one depicted in Fig. 4.2. Here, Δ is the uncertainty, C the controller and P is the so-called generalized plant, that contains a model of the process G and the information about how G, Δ and C are interconnected. The fundamental principle of linear robust control can then be sketched as follows: The control system is robustly stable, if (a) the control system is nominally stable, i.e. stable for $\Delta = 0$, and if (b) a small gain condition is satisfied that guarantees the stability of the interconnection of P, C and Δ. Denote with M the nominal feedback loop, i.e. the interconnection of P and C. Thus, one obtains the usual $M - \Delta$-structure depicted in Fig. 4.3. As the nominal closed loop system M is linear, it can be split up into four parts,

$$\begin{pmatrix} d_i \\ y \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} d_o \\ u \end{pmatrix}. \tag{4.1}$$



F 4.3. Basic principle of robust control: Robust stability follows from the conjunction of nominal stability and a small gain condition.

The nominal closed loop system M is said to be stable, if all its subsystems have finite gain. Then the inequality[1]

$$\|d_i\| \leq m_{11} \cdot \|d_o\| + m_{12} \cdot \|u\| \tag{4.2}$$

with $m_{11} = \|M_{11}\|$ and $m_{12} = \|M_{12}\|$, and $\|d_o\| \leq \|\Delta\| \cdot \|d_i\|$ leads to the bound

$$\|d_o\| \leq (1 - \|\Delta\| \cdot m_{11})^{-1} \|\Delta\| \cdot m_{12} \cdot \|u\| \tag{4.3}$$

if the small gain condition $\|\Delta\| \cdot m_{11} < 1$ is satisfied. This bound for $\|d_o\|$ in turn gives a bound on $\|y\|$ as

$$\|y\| \leq m_{21} \cdot \|d_o\| + m_{22} \cdot \|u\| \tag{4.4}$$

with $m_{21} = \|M_{21}\|$ and $m_{22} = \|M_{22}\|$, or

$$\|y\| \leq \left(m_{21} \cdot (1 - \|\Delta\| \cdot m_{11})^{-1} \|\Delta\| \cdot m_{12} + m_{22}\right) \|u\| \tag{4.5}$$

and the closed loop is stable also in presence of the error term Δ. For linear systems, it can be shown that if the small gain condition is not satisfied, then there is a linear dynamic system Δ with gain $\|\Delta\|$ that destabilizes the closed loop [146]. In this sense, the condition $\|\Delta\| \cdot \|M_{11}\| < 1$ is *necessary and sufficient* for robust stability in the linear case.

The goal of this chapter shall be to parallel this procedure in order to design linear controllers for nonlinear systems. To this end, the nonlinear process is partitioned in a linear model G and a nonlinear error term Δ. Note that there are two differences to the case of linear robust control. Firstly, the term Δ is not uncertain, but known. However, Δ will be nonlinear (and this is the second difference), rendering the information about Δ void for linear controller design. Note that, also in the linear case, using the small gain condition introduces conservatism if the uncertainty Δ is known, while in the linear case, there is no need to introduce Δ, as the data could be directly used in the controller design. In the nonlinear case, however, this conservatism is the price to pay for the complexity reduction in the controller design step (i.e. designing a linear controller instead of a nonlinear one).

[1] In order to keep the equations readable, we will in the following shortly write $\|x\| \leq \|y\|$ for $\|x_T\|_p \leq \|y_T\|_p$ for all $T > 0$. The following exposition applies to the choice of any $1 \leq p \leq \infty$.

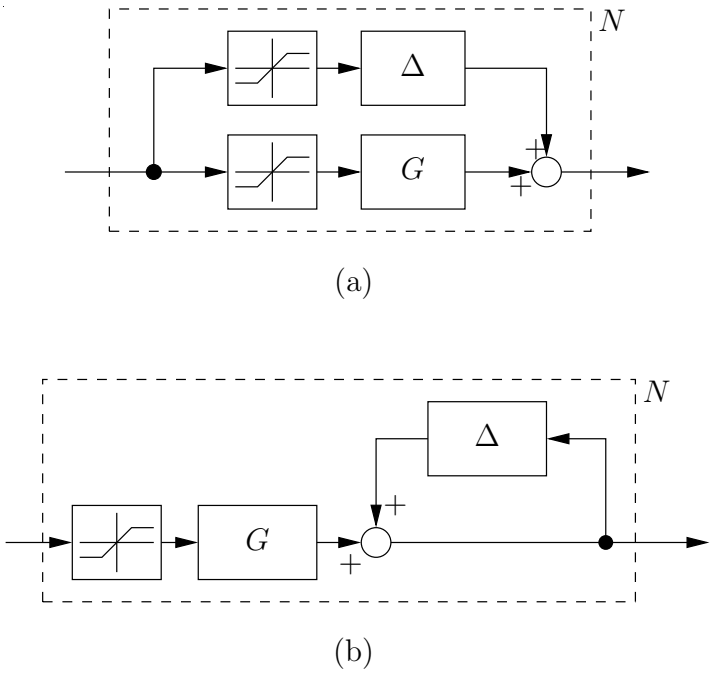
When robust controller design methods are applied, it is sometimes mentioned that this strategy also takes into account possible modeling errors due to nonlinear behaviour. It is indeed true that (4.1)-(4.5) remain valid if Δ is a nonlinear operator. However, as has been mentioned in different places and shown with different examples, global analysis and synthesis for nonlinear systems is not appropriate in many cases, because the difference in behaviour between any linear model and the nonlinear system becomes arbitrarily large for large operating regions. For instance, the nonlinearity measures introduced in Chapter 3 commonly take their maximal value if an unbounded region of operation is considered. But if a bounded region of operation is considered, the reasoning above for robust stability does not longer hold (it only holds if the whole signal spaces L_{pe} are allowed for all signals). Consequently, the main contribution of this chapter is to relax the assumption that the gain of Δ is considered globally to a situation where this is only the case for a given subset of signals.

In virtually all applications, control engineers have to deal with processes whose manipulated variables can take values in a certain range only. Accordingly, only nonlinear behaviour that occurs for these inputs plays a role in controller design. In the sequel, we will therefore consider inputs that are constrained to take values in the set $V = \{v \in \mathbb{R}^m : |v_i| \leq v_{i,max},\ i = 1..m\}$. If no or too large physical bounds are given by the nature of the process, they can be introduced or strengthened artificially in order to represent the desired region of operation. The set of input signals considered for the calculation of the gain of the error system Δ then is $\mathcal{U}(V) = \left\{u \in L_p^m : u(t) \in V,\ t \geq 0\right\}$. Instead of considering restricted signal sets, we can equivalently use the whole signal spaces and think of the system N to incorporate input saturation of the type

$$\Sigma_V : u \in L_{pe}^m \mapsto \tilde{u} \in L_{pe}^m,\ \tilde{u}_i(t) = \begin{cases} -v_{i,max} & \text{if } u_i(t) < -v_{i,max} \\ u_i(t) & \text{if } |u_i(t)| \leq v_{i,max} \\ v_{i,max} & \text{if } u_i(t) > v_{i,max} \end{cases} \tag{4.6}$$

In the sequel we will therefore encounter the system $N \circ \Sigma_V$ where the system N appeared before. Note also that for all nonlinearity measures it holds that $\phi_N^{\mathcal{U}(V)} = \phi_{N\Sigma_V}^{L_{pe}^m}$ and similar relations are true for the gain of error systems corresponding to non-optimal linear models. In Chapter 3 we always represented the plant N as an interconnection of a linear model G and an error term Δ. In the same way we can now represent $N \circ \Sigma_V$ as an interconnection of Σ_V, G and Δ, as exemplary shown in Fig. 4.4 for the additive error and inverse multiplicative output error setups. The nonlinearity measures give the gain of Δ for all signals that may occur in the respective setup. This fact allows to give the inequality $\|d_o\| \leq \|\Delta\| \cdot \|d_i\|$, where $\|\Delta\|$ is equivalent to the nonlinearity measure if G is an optimal model.

Now we need one more ingredient in order to be able to give results that are analogue to (4.3) and (4.5). While we clarified the problems with bounded signals entering the nonlinear systems, we now have a generalized plant P and a nominal closed loop M that contain saturations, so they are not linear any more. However, we do not need exact linearity in the derivation of (4.3) and (4.5). Instead, the validity of the somewhat relaxed conditions (4.2) and (4.4), which express a superposition principle in norm, is sufficient in order to show robust stability. Thus, if we can show that these two inequalities hold for each of the



F 4.4. Representation of a saturated nonlinear system as interconnection of a saturation system Σ_V, a linear model G and an error term Δ, (a) for the additive error setup, and (b) for the inverse multiplicative output error setup.

error model setups, we will obtain for each setup a procedure for linear controller design for nonlinear systems.

If a system has inputs and outputs that consist of stacked vectors, like M does, the existence of a finite gain is equivalent to the existence of separate bounds for the sub-vectors.[2] Therefore, the desired equivalent of (4.2) and (4.4) is given by $\|d_i\| \leq \|P\|\,(\|d_o\| + \|u\|)$, and the same inequality holds for $\|y\|$ replacing $\|d_i\|$. However, this bound is a very rough estimate and we therefore aim at deriving better bounds for the different error model setups in the following.

4.2. Stability of Nonlinear Systems with Linear Controllers

Before turning our attention to the main results, we clarify the kind of stability we want to achieve and we give some helpful lemmas. Consider again the control loop in Fig. 4.1. We assume throughout that the closed loop is well-posed, i.e. that for all considered exogenous inputs, the signals u_i and e_i are uniquely defined and are elements of the respective signal

[2]Consider a general system P with stacked input and output vectors, $y = (y_1\, y_2)^T = P\,(u_1\, u_2)^T = Pu$. The inequalities $\|y_i\| \leq \beta_{i1}\,\|u_1\| + \beta_{i2}\,\|u_2\|$ for $i = 1, 2$ imply $\left\|(y_1\, y_2)^T\right\| \leq \|y_1\| + \|y_2\| \leq (\beta_{11} + \beta_{21})\,\|u_1\| + (\beta_{12} + \beta_{22})\,\|u_2\| \leq (\beta_{11} + \beta_{21} + \beta_{12} + \beta_{22})\left\|(u_1\, u_2)^T\right\|$ where the last inequality is justified by $\|u_i\| \leq \left\|(u_1\, u_2)^T\right\|$ for $i = 1, 2$. Vice versa, from $\|y\| \leq \|P\|\,\|u\|$ follows $\|y_i\| \leq \|P\|\,(\|u_1\| + \|u_2\|)$ for $i = 1, 2$ and the generalization to more than two sub-vectors is straightforward.

spaces. We will call the closed loop stable if the mapping $H_{NC} : (u_1, u_2) \mapsto (y_1, y_2, e_1, e_2)$ has finite gain. The following fact makes our life easier when we have to check for stability.

P 4.1. *[135] The following three statements are equivalent:*

(*i*) $(u_1, u_2) \mapsto (y_1, y_2, e_1, e_2)$ *has finite gain,*

(*ii*) $(u_1, u_2) \mapsto (y_1, y_2)$ *has finite gain,*

(*iii*) $(u_1, u_2) \mapsto (e_1, e_2)$ *has finite gain.*

With the help of the arguments in Section 4.1 it is now possible to give stability conditions for a general nonlinear control system involving a nominal closed loop and an error model.

T 4.2. *Consider a not necessarily linear control system as shown in Fig. 4.3, that satisfies the superposition principle in norm given by (4.2) and (4.4). If*

$$\|\Delta\|^{\mathcal{D}_i} \cdot m_{11} < 1$$

where $\mathcal{D}_i$ denotes the set of signals d_i that may occur as inputs to Δ during closed-loop operation, then the closed loop is stable, i.e. the mapping $u \mapsto y$ has finite gain.

P . Follows directly from the exposition in Section 4.1. □

When we show the superposition principle in norm for the different setups, we will need to make use of the fact that the saturation system Σ_V is incremental gain stable and $\|\Sigma_V\|_\Delta = 1$.

P 4.3. *The saturation operator Σ_V given by (4.6) is finite (incremental) gain stable and $\|\Sigma_V\|_\Delta = 1$.*

P . The result is proven by $\|\Sigma_V u_1 - \Sigma_V u_2\| \leq \|u_1 - u_2\|$ which follows from

$$|(\Sigma_V u_1)_i (t) - (\Sigma_V u_2)_i (t)| \leq \left|u_{1,i}(t) - u_{2,i}(t)\right|$$

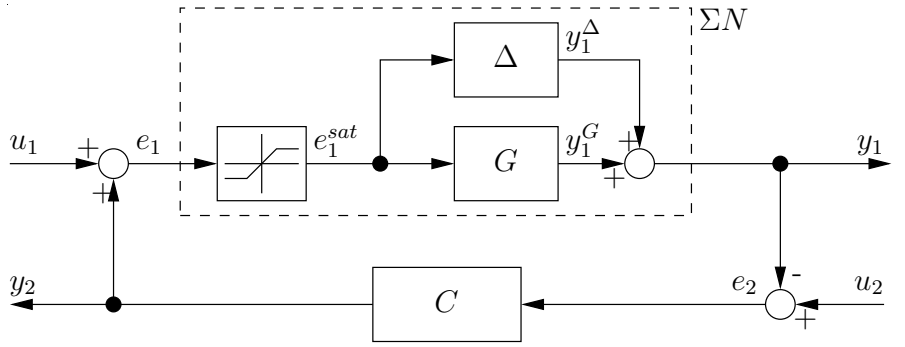
for all components of the vectors $u_1(t)$ and $u_2(t)$. Furthermore, $\Sigma_V 0 = 0$ and the finite incremental gain implies a finite gain. □

Linear stabilization of nonlinear systems in the additive error setup. We are now aiming at tailor-made bounds for the superposition principle in norm of the nominal closed loop M in the additive error setup. Those bounds will allow to show robust stability of the closed loop in presence of the nonlinear perturbation Δ.

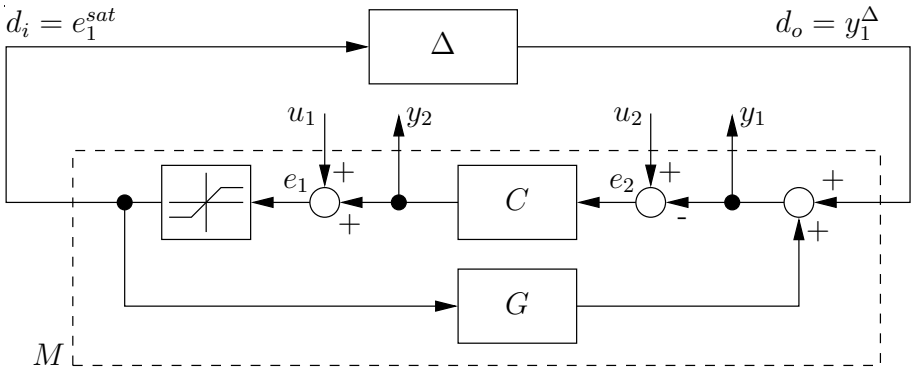
P 4.4. *Assume the control system M in Fig. 4.5 (with $\Delta = 0$) is stable and G is a finite incremental gain stable system. Under these conditions, (4.2) and (4.4) hold and in particular, m_{11} can be taken to be $m_{11} = \left\|(I + CG\Sigma_V)^{-1} C\right\| = \left\|C (I + G\Sigma_V C)^{-1}\right\|$.*

P . By inspection of Fig. 4.5, the equation $e_2 = u_2 - d_0 - G\Sigma_V (u_1 + Ce_2)$ can readily be given. We define $z = G\Sigma_V (u_1 + Ce_2) - G\Sigma_V Ce_2$ and obtain $e_2 = u_2 - d_0 - G\Sigma_V Ce_2 - z$, and hence

$$e_2 = (I + G\Sigma_V C)^{-1} (u_2 - d_0 - z) .$$



(a) Closed loop in standard configuration.



(b) Closed loop in M-Δ configuration.

F 4.5. ”Pulling out the Δ” for the additive error setup.

From $y_2 = Ce_2$ we obtain

$$\|y_2\| \leq \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\| \left(\|d_o\| + \|z\| + \|u_2\|\right)$$

But finite incremental gain stability of G and of Σ_V with $\|\Sigma_V\|_\Delta = 1$ implies $\|z\| \leq \|G\|_\Delta \|u_1\|$ and we can get rid of the dummy variable z to get

$$\|y_2\| \leq \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\| \left(\|d_o\| + \|G\|_\Delta \|u_1\| + \|u_2\|\right).$$

From $d_i = \Sigma_V y_2 + z$ follows

$$\|d_i\| \leq \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\| \left(\|d_o\| + \|G\|_\Delta \|u_1\| + \|u_2\|\right) + \|u_1\|$$

and finally, $y_1 = Gd_i + d_o$ leads to

$$\|y_1\| \leq \|G\| \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\| \left(\|d_o\| + \|G\| \|u_1\| + \|u_2\|\right) + \|G\| \|u_1\| + \|d_o\|$$

and the proof is complete. □

Note that, besides G being an incrementally stable system, we have the weakest possible assumptions. In particular, C is neither required to be stable nor to be linear. This will allow for instance for integral controllers with anti-windup. In principle, we can even

allow for nonlinear models G. For our application, however the class of linear models G is the most important one, and in this case a finite gain implies a finite incremental gain and $\|G\|_\Delta = \|G\|$.

With the above result the road is paved for a sufficient condition on stability of the closed-loop system, when N is given as a nominal linear model and an additive error term.

T 4.5. *For the finite gain stable process N, let a finite incremental gain stable model G and a finite gain stable mismatch term $\Delta = N - G$ be given. Furthermore, assume that a controller C is provided such that the mapping $M_{11} = C(I + G\Sigma_V C)^{-1}$ has finite gain. If the gains satisfy*

$$\|\Delta\|^{\mathcal{U}(V)} \cdot \|M_{11}\| < 1$$

then the control system in Fig. 4.5 is stable, i.e. $H_{(N\Sigma_V)C}$ has finite gain.

P . The result follows immediately from Theorem 4.2 and Proposition 4.4 by considering that only signals in $\mathcal{U}(V)$ may occur as inputs to the modeling error term Δ. □

The stability condition above can be used to set up a controller design procedure consisting of the steps

(1) define the region of operation $\mathcal{U}(V)$ by choosing the $v_{i,max}$,
(2) calculating the nonlinearity measure $\phi_{AE,N}^{\mathcal{U}(V)}$ and the best linear model G^*, or calculate the quality index $\|\Delta\|^{\mathcal{U}(V)}$ for a given linear model G, and
(3) design a linear controller C for the saturated linear system $G \circ \Sigma_V$ such that $\|M\| = \left\|C(I + G\Sigma_V C)^{-1}\right\| < 1/\|\Delta\|^{\mathcal{U}(V)}$.

One might argue that after all, we end up with a difficult design problem, as we have to design a controller that achieves a given L_p-gain despite plant saturation. On the other hand, during any practical controller design input constraints have to be respected. In fact, we do not introduce an additional difficulty here, but we rather benefit from the fact that saturation has to be taken into account anyway, so that the nonlinear behaviour of the process has only to be considered in the respective region.

Performance synthesis for saturated systems is still an area of active research, as can be seen from the many publications in that field in recent years [41, 106, 107, 65, 101, 23, 142, 69, 70, 71]. For the examples considered in this work it turned out to be successful to first do a controller design for the unsaturated system. Afterwards the methods given in [108] or [71] have been used to analyze whether the saturated closed loop also achieves the necessary (incremental) gain. If this was not the case, the performance objectives for the first step have been weakened, and the procedure has been repeated. This procedure proved to be useful and to deliver satisfactory results. The ongoing research in control of saturated systems promises to deliver more effective methods that are easier to use in the future.

In Step 2. above, we do not necessarily have to use the best model and the nonlinearity measure, but any model G along with its associated error model gain $\|\Delta\|^{\mathcal{U}(V)}$ can be used.

One obvious candidate model to be used is for instance the local linearization, as illustrated by the following example.

E . *Linear control of a (globally) unstable system.* Consider the system P given by

$$\dot{y} = -y + y^3 + u$$

and the state space linearization at $y = 0$

$$\dot{\tilde{y}} = -\tilde{y} + u$$

as a linear model. The model error gain $\|\Delta\|^{\mathcal{U}(V)}$ for $V = [-u_{max}, u_{max}]$ and $p = 2$ has actually been computed in the example of Section 3.8 and is depicted in Fig. 3.13 (lower part) as a function of u_{max}. Clearly, a global linear control strategy cannot be successful, and this example once more advocates the application of a regional procedure. In order to design a linear feedback controller, we fix the region of operation to $u_{max} = 0.3$. The modeling error for the local linear model and the chosen operating range is $\|\Delta\|^{\mathcal{U}(V)} = 0.13$. We design an H_∞-controller for the system, solving the mixed-sensitivity problem

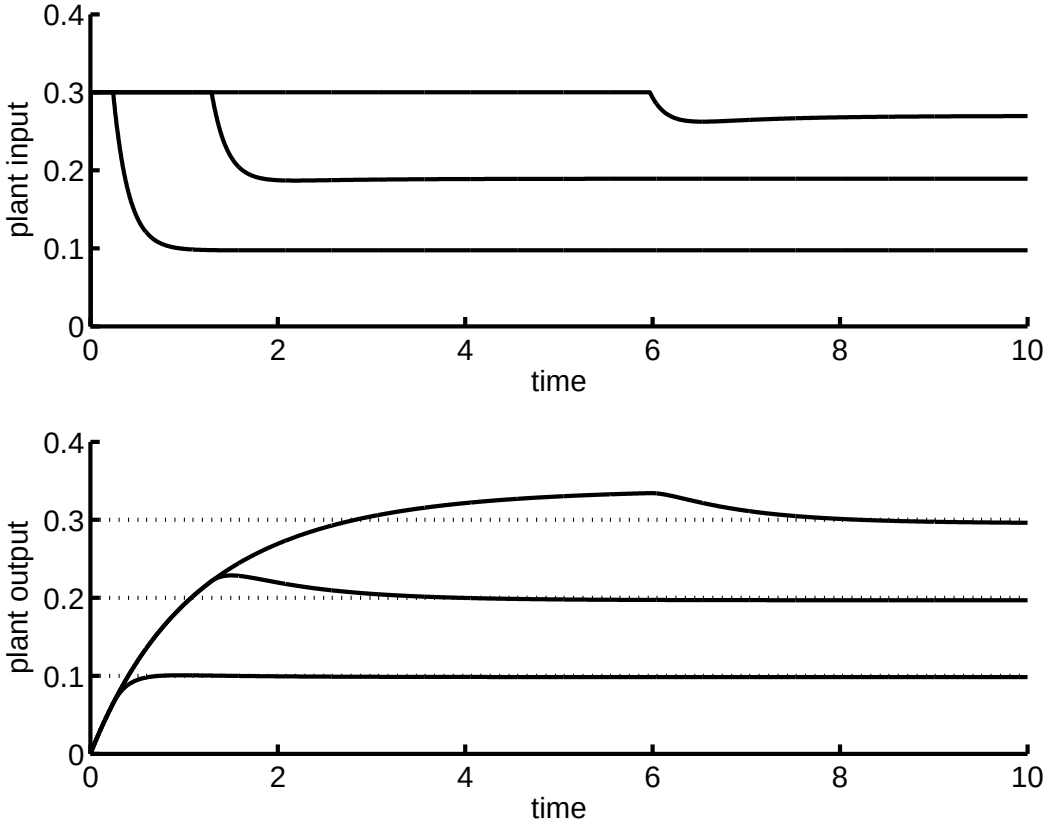
$$\left\| \begin{array}{c} \alpha WS \\ \beta KS \end{array} \right\|_{H_\infty} < 1$$

with the weighting function $W(s) = \frac{s+10}{s+0.1}$ and where a nested $\alpha - \beta-$iteration was done in order to maximize α while still satisfying the condition $\|\Delta\|^{\mathcal{U}(V)} \cdot \|KS\|_{H_\infty} < 1$. An upper bound on the gain of the saturated linear control system was calculated using LMIs by the method given in [71]. The bound could then be used to verify the small gain condition

$$\|\Delta\|^{\mathcal{U}(V)} \cdot \left\| C\left(I + G\Sigma_V C\right)^{-1} \right\| \leq 0.13 \cdot 4.8 = 0.79 < 1$$

proving stability of the closed loop containing the linear controller and the saturated nonlinear system. Plant input and output under closed loop conditions for steps of the reference signal are shown in Fig. 4.6. It can be seen that, due to the high steady state gain, almost asymptotic reference tracking is achieved, although because of the saturation, the system follows higher step more slowly. But the example primarily shows that (i) not necessarily the *best* linear model has to be used, and that (ii) *regional* linear controller design may be a solution for nonlinear systems that are not *globally* stable. ■

The same reasoning that has been used to derive a robust stability result for systems in additive error representation can be redone for the other setups that correspond to a nonlinearity measure. However, in order to assume only the weakest conditions necessary to establish stability, the technical assumptions are different for each setup. In the following, we derive stability conditions for all of the setups sequentially. Then, a generalized procedure is given that possesses the beauty of being applicable to all setups, but for which we require the closed loop to be incremental gain stable instead of just gain stable. This is a stronger requirement, because (gain) stable linear feedback loops incorporating saturation need not be incrementally stable [18]. After the derivation of the various stability results, a short discussion follows.



F 4.6. Input (above) and output (below) of the plant $\dot{y} = -y+y^3+u$ under closed loop conditions for steps (dashed) in the reference signal.

Linear stabilization of nonlinear systems in the multiplicative output error setup. In the case of the multiplicative output error setup, no additional difficulties arise. The whole argument boils again down to deriving a sharp bound for the superposition principle in norm. The interconnection and the nominal closed loop M are shown in Fig. 4.7.

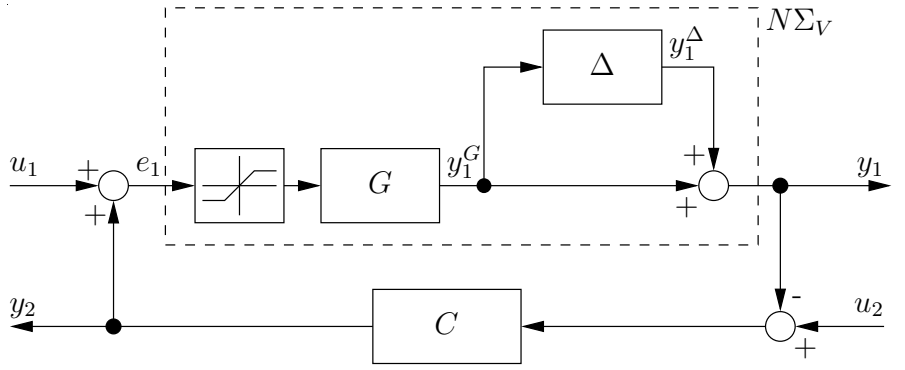
T 4.6. *For the finite gain stable process N, let a finite incremental gain stable model G and a finite gain stable multiplicative output error Δ be given. If a controller C stabilizes the linear saturated system G, i.e. the closed loop $H_{(G\Sigma_V)C}$ is stable, and*

$$\|\Delta\|^{G\mathcal{U}(V)} \cdot m_{11} < 1$$

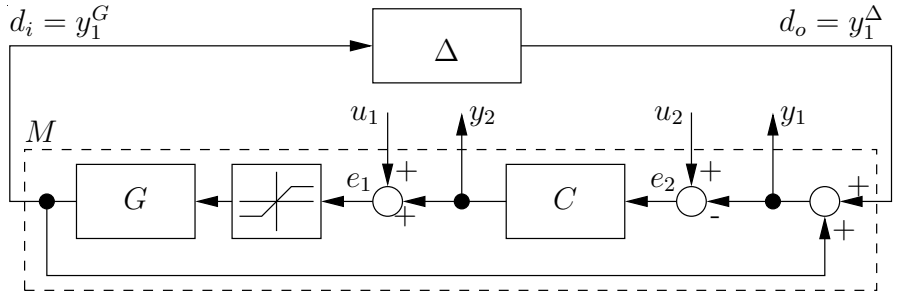
where $m_{11} = \left\|G\Sigma_V C(I + G\Sigma_V C)^{-1}\right\|$, then the controller also stabilizes the nonlinear closed loop, i.e. $H_{(N\Sigma_V)C}$ is stable.

P . The equation of the closed loop for e_2 is $e_2 = u_2 - d_o - G\Sigma_V(u_1 + Ce_2)$. We introduce again artificially the variable $z = G\Sigma_V(u_1 + y_2) - G\Sigma_V y_2$ with $\|z\| \leq \|G\|_\Delta \|u_1\|$. Thus we arrive at $e_2 = (I + G\Sigma_V C)^{-1}(u_2 - d_o - z)$. From $d_i = y_1^G = G\Sigma_V(u_1 + Ce_2) = G\Sigma_V Ce_2 + z$ follows

$$\|d_i\| \leq m_{11}\|u_2 - d_o - z\| + \|G\|_\Delta \|u_1\| \leq m_{11}(\|u_2\| + \|d_o\|) + (1 + m_{11})\|G\|_\Delta \|u_1\|.$$



(a) Closed loop in standard configuration.



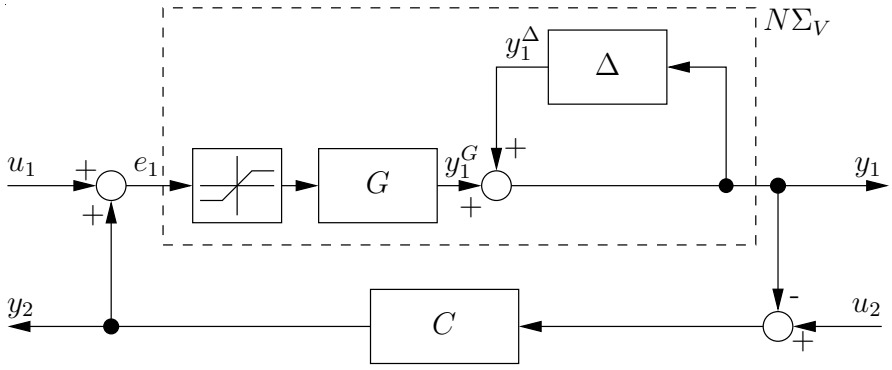
(b) Closed loop in $M - \Delta$-configuration.

F 4.7. ”Pulling out the Δ” in the multiplicative output error setup.

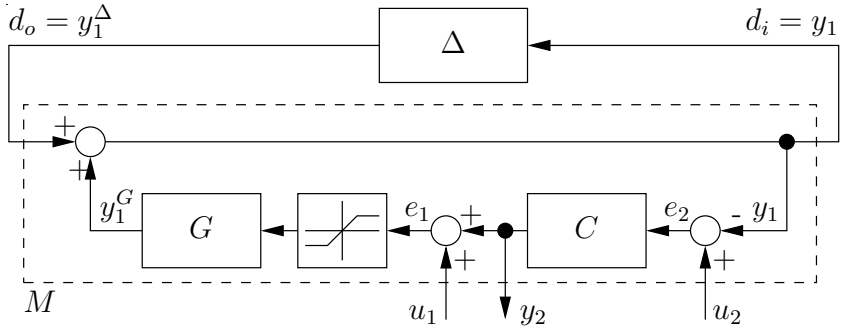
A bound for $y_1$follows immediately from $y_1 = y_1^G + d_o$ and stability of $H_{(G\Sigma_V)C}$ guarantees a bound for $\|y_2\|$. Application of Theorem 4.2 completes the proof as in Corollary 4.5. □

Linear stabilization of nonlinear systems in the inverse multiplicative output error setup. The corresponding result can be obtained for the inverse multiplicative output error setup without difficulty. The closed loop with the nonlinear plant in inverse multiplicative output error representation and the configuration to analyze M are shown in Fig. 4.8. However, in the case of the inverse multiplicative output error setup, the error term Δ may be a relation instead of a mapping. We discussed in Chapter 3 the conditions which assure that the interconnection of the error system Δ and the nominal linear model G result in a mapping that corresponds to N. We assume that these conditions are met, as we implicitly did above for the other setups. Thus, although Δ is allowed to be a relation, we do not need more assumptions than above, i.e. in addition to the fact that we assume G and Δ to be a valid model/error term pair for the nonlinear system N, we only assume that the closed loop with N and C is well-posed.

T 4.7. *For the finite gain stable process N, let a finite incremental gain stable linear model G and a finite gain stable multiplicative output error term Δ be given, where Δ is allowed to be a relation. If a controller C stabilizes the linear saturated system G, i.e. the*



(a) Closed loop in standard configuration.



(b) Closed loop in $M - \Delta$-configuration.

F 4.8. ”Pulling out the Δ” in the inverse multiplicative output error setup.

closed loop $H_{(G\Sigma_V)C}$ is stable, and

$$\|\Delta\|^{N\mathcal{U}(V)} \cdot m_{11} < 1$$

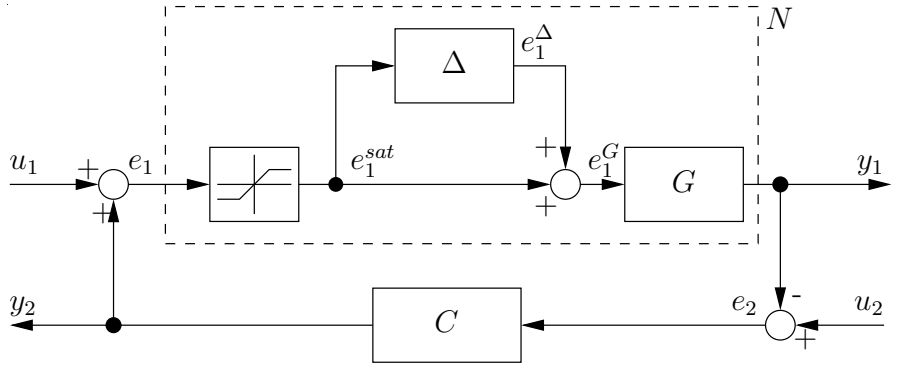
where $m_{11} = \left\|(I + G\Sigma_V C)^{-1}\right\|$, then the controller also stabilizes the nonlinear closed loop, i.e. $H_{(N\Sigma_V)C}$ is stable.

P . Consider the equation of the closed loop $e_2 = u_2 - d_o - G\Sigma_V(u_1 + Ce_2)$ in Fig. 4.8 b). The occurrence of u_1 can be replaced by an artificial input $z = G\Sigma_V(u_1 + Cy_2) - G\Sigma_V Cy_2$ with $\|z\| \le \|G\|_\Delta \|u_1\|$. Then $e_2 = u_2 - d_o - G\Sigma_V Ce_2 - z$ and hence $e_2 = (I + G\Sigma_V C)^{-1}(u_2 - d_o - z)$. Finally, $d_i = y_1 = u_2 - e_2$ leads to

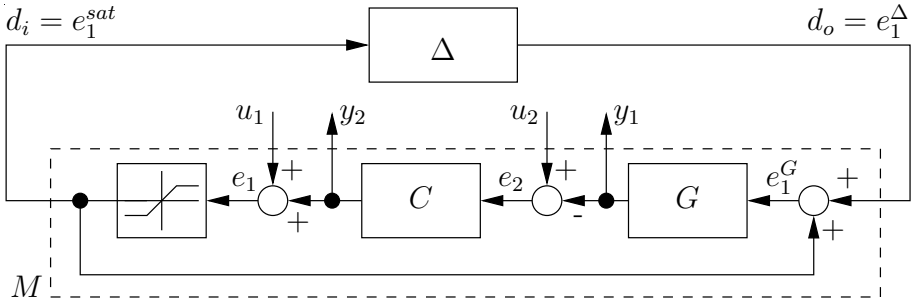
$$\|d_i\| \le \|u_2\| + \left\|(I + G\Sigma_V C)^{-1}\right\| (\|d_o\| + \|u_2\| + \|G\|_\Delta \|u_1\|)$$

and stability of $H_{(G\Sigma_V)C}$ and application of Theorem 4.2 completes the proof. □

Linear stabilization of nonlinear systems in the multiplicative input error setup. In order to keep the influence of u, e apart, we have to assume slightly tighter conditions as



(a) Closed loop in standard configuration.



(b) Closed loop in $M - \Delta$-configuration.

F 4.9. "Pulling out the Δ" in the multiplicative input error setup.

for the results above. The setup is depicted in Fig. 4.9 for consideration of the nominal closed loop M.

T 4.8. *For the finite gain stable process N, let a finite gain stable linear model G and a finite gain stable multiplicative input error model Δ be given. If a controller C is given such that the nominal closed loop $H_{(G\Sigma_V)C}$ is stable and one of the conditions*

- *(i)* *C is linear and finite gain stable,*
- *(ii)* *G has a stable inverse,*
- *(iii)* *$\Sigma_V (I + CG\Sigma_V)^{-1} C$ is incremental gain stable*

holds, and the small gain condition

$$\|\Delta\|^{\mathcal{U}(V)} \cdot m_{11} < 1$$

is satisfied where $m_{11} = \left\|CG (I + \Sigma_V CG)^{-1}\right\|$, then the controller also stabilizes the nonlinear closed loop, i.e. $H_{(N\Sigma_V)C}$ is stable. The conclusion also holds if C stabilizes the nominal closed loop and the small gain condition is satisfied with $m_{11} = \left\|C (I + G\Sigma_V C)^{-1}\right\| \|G\|$.

P . The equation of the closed loop reads $e_2 = u_2 - G (d_o + \Sigma_V (u_1 + Ce_2))$. First consider the cases (ii) and (iii). Introducing $z = \Sigma_V (u_1 + Ce_2) - \Sigma_V Ce_2$, $\|z\| \leq \|u_1\|$ puts the

equation in the form $e_2 = u_2 - Gd_o - G\Sigma_V Ce_2 - Gz$, from which $e_2 = (I + G\Sigma_V C)^{-1}(u_2 - Gd_o - Gz)$. With $d_i = \Sigma_V(u_1 + Ce_2) = \Sigma_V Ce_2 + z$ one obtains $d_i = \Sigma_V C(I + G\Sigma_V C)^{-1}(u_2 - Gd_o - Gz) + z$. If (ii) $\left\|G^{-1}\right\|$ is bounded, then $d_i = \Sigma_V C(I + G\Sigma_V C)^{-1} G\left(G^{-1}u_2 - d_o - z\right) + z$ and

$$\|d_i\| \leq \left\|\Sigma_V C(I + G\Sigma_V C)^{-1} G\right\| \left(\left\|G^{-1}\right\| \|u_2\| + \|d_o\| + \|u_1\|\right) + \|u_1\|.$$

Note that for a finite-gain stable system P we have $\|P(x+y)\| \leq \|P(x+y) - Py\| + \|Py\| \leq \|P\|_\Delta \cdot \|x\| + \|Py\|$ and therefore if (iii) $\Sigma_V C(I + G\Sigma_V C)^{-1}$ is incremental gain stable, then

$$\begin{aligned}\|d_i\| &\leq \left\|\Sigma_V C(I + G\Sigma_V C)^{-1}\right\|_\Delta \|u_2\| + \left\|\Sigma_V C(I + G\Sigma_V C)^{-1} G(-d_o - z) + z\right\| \\ &\leq \left\|C(I + G\Sigma_V C)^{-1}\right\|_\Delta \|u_2\| + \left\|C(I + G\Sigma_V C)^{-1} G\right\| (\|d_o\| + \|u_1\|) + \|u_1\|.\end{aligned}$$

If (i) C is linear and finite gain stable, consider the equation $-y_1 = -G(d_o + \Sigma_V(u_1 + C(u_2 - y_1)))$. Introducing $z = \Sigma_V(u_1 + C(u_2 - y_1)) + \Sigma_V C(-y_1)$, $\|z\| \leq \|u_1\| + \|C\|\|u_2\|$ leads to $-y_1 = -Gd_o - G\Sigma_V C(-y_1) - Gz$, or $-y_1 = (I + G\Sigma_V C)^{-1} G(-d_o - z)$. From $d_i = \Sigma_V C(-y_1) + z$ we get

$$\|d_i\| \leq \left\|C(I + G\Sigma_V C)^{-1} G\right\| (\|d_o\| + \|u_1\| + \|C\|\|u_2\|) + \|u_1\| + \|C\|\|u_2\|.$$

Note that the expressions $C(I + G\Sigma_V C)^{-1} G = CG(I + \Sigma_V CG)^{-1}$ are equal, and if C is linear, then so are $C(I + G\Sigma_V C)^{-1} G = (I + CG\Sigma_V)^{-1} CG$. Finally, if none of (i)-(iii) is true, then from $d_i = \Sigma_V C(I + G\Sigma_V C)^{-1}(u_2 - Gd_o - Gz) + z$ we still have

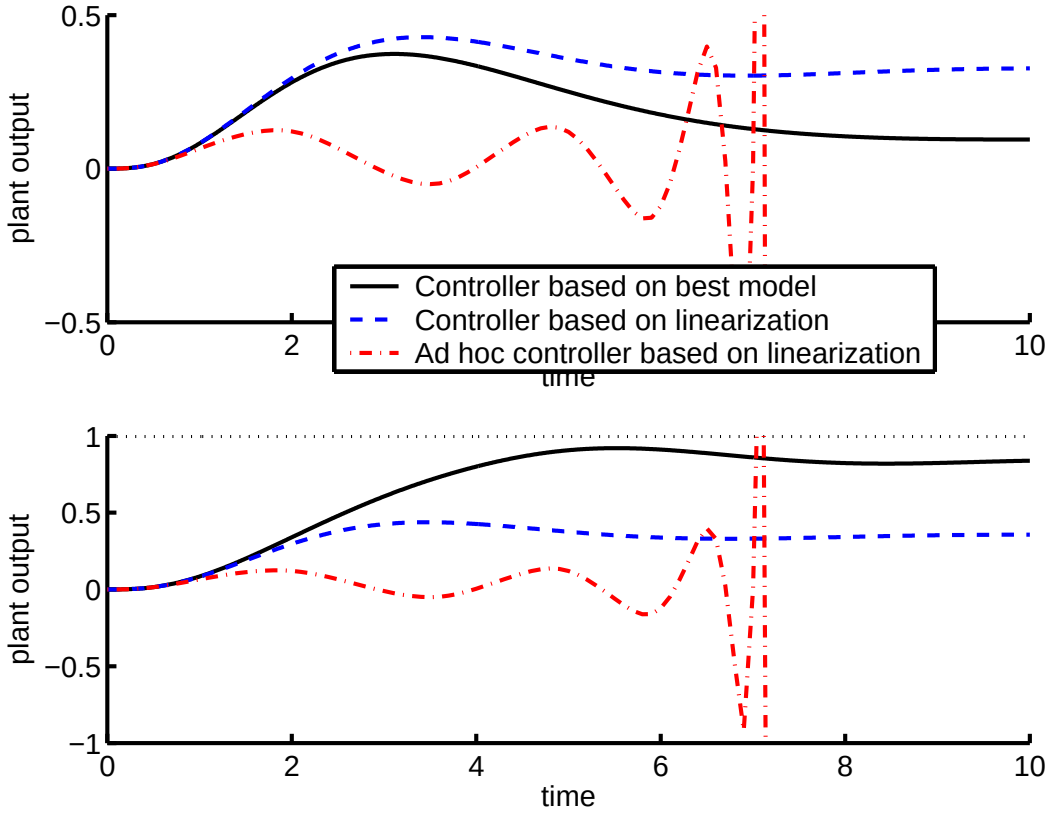
$$\|d_i\| \leq \left\|C(I + G\Sigma_V C)^{-1}\right\| (\|u_2\| + \|G\|\|d_o\| + \|G\|\|u_1\|) + \|u_1\|.$$

The bounds for y_1 and y_2 follow from stability of $H_{(G\Sigma_V)C}$ because by linearity of G the input (u_1, u_2, d_o) to M is equivalent to the input $(u_1, u_2 + Gd_o, 0)$, and by stability of G, $u_2 + Gd_o$ is bounded above, and the proof is complete. □

The example of the system $\dot{y} = -y + y^3 + u$ above showed that any linear model together with its model quality index can in principle be used for linear controller design, like the local (state-space) linearization. However, the next example shows that it may indeed occur that the best model is much better suited for controller design than the local approximation.

E . *Linear control of a Hammerstein system.* Consider the system P given by the series connection of the nonlinearity N_f given by $f(x) = 0.05x + x^3$ and the linear dynamics $L(s) = \frac{1}{(s+1)^3}$ for an operating range of $|u| \leq 1$. The best linear model $G(s) = \frac{0.55}{(s+1)^3}$ and the nonlinearity measure $\phi^{\mathcal{U}}_{MIE,N_f} = 0.91$, as well as the local linearization $G_{loc}(s) = \frac{0.05}{(s+1)^3}$ and the respective model quality index $\|\Delta\|^{\mathcal{U}} = 20$ have been given as examples in Section 3.6. In order to achieve stability, the product of the norm of the complementary input sensitivity, $CG(I + \Sigma_V CG)^{-1}$, and the error gain (or nonlinearity measure) must be strictly lower than one. Moreover, the closed loop must be incremental stable, or the controller must be stable (see Theorem 4.8).

It has not been possible to design an integral controller with anti-windup for the best linear model such that the closed loop was incrementally stable, so a controller structure without integrator has been chosen. The controller design was accomplished using internal model

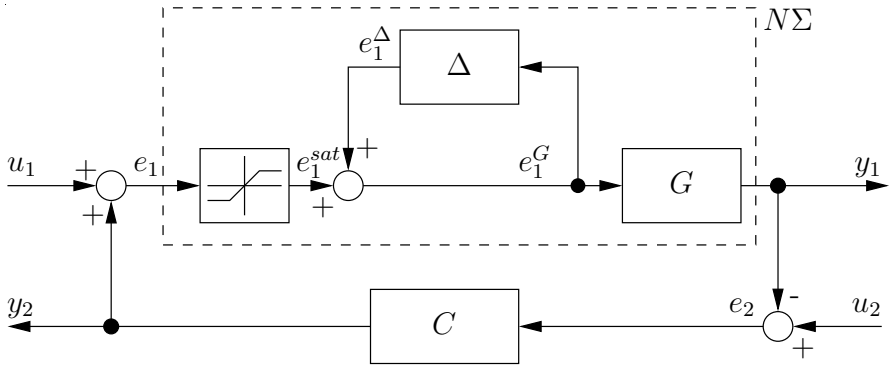


F 4.10. Performance of different controllers for the Hammerstein example system to unity steps in the plant input disturbance (above) and in the reference signal (below).

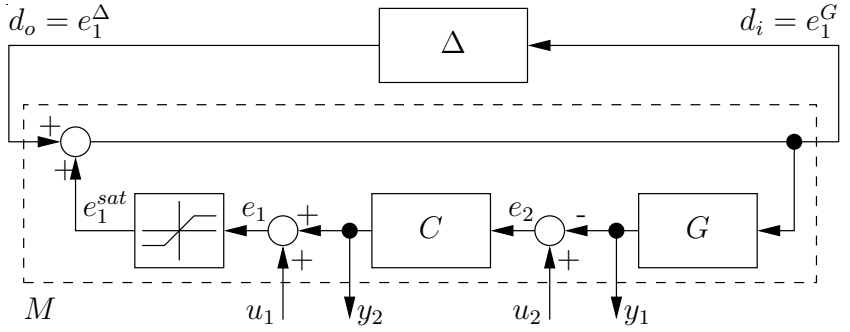
control, but the desired closed loop behaviour $F(s) = \frac{0.75}{(s+1)^3}$ was chosen to have a steady-state error, in order to fulfill the requirement of a non-integral controller. Both, the gain and incremental gain were computed in order to verify the stability conditions. The input and output of the plant under closed loop conditions are shown in Fig. 4.10, where the scenario is a unity step plant input disturbance (above) and a unity step reference input (below). It can be seen that the closed loop is stable. The input disturbance is attenuated significantly, and the reference step can be followed to 83%.

The advantage of using the best model becomes obvious when the above results are compared to the controller based on the local linearization. First note that the model quality index $\|\Delta\|^{\mathcal{U}} = 20$ implies that the complementary sensitivity cannot have a gain larger than or equal to 0.05 in order to guarantee stability using Theorem 4.8. Therefore, an internal model control design was chosen, with a desired closed loop behaviour of $F(s) = \frac{0.049}{(s+1)^3}$. Also in this case, the small gain condition was met for the saturated closed loop. The performance in terms of disturbance attenuation and tracking is considerably worse compared to the first controller, as seen from Fig. 4.10.

Finally, for the sake of comparison, a linear controller was design based on the linearization without considering the nonlinear plant / linear model mismatch. The performance



(a) Closed loop in standard configuration.



(b) Closed loop in $M - \Delta$-configuration.

F 4.11. "Pulling out the Δ" in the inverse multiplicative input error setup.

objective was chosen equal to the first controller. Fig. 4.10 shows the performance of this last controller for the given disturbance and reference signals. Disregarding the nonlinear error term in the controller design process obviously leads to instability. Note that in this case, no saturation has been introduced at the plant input. If this is done, the closed loop still remains unstable, exhibiting persistent oscillations.

This example shows two important facts: firstly, the best linear model can significantly improve the performance of the closed loop. Secondly, disregarding the stability conditions derived above may lead to instability, justifying the approach proposed in this work. ■

Linear stabilization of nonlinear systems in the inverse multiplicative input error setup. In order to be able to show stability also for the nonlinear closed loop in inverse multiplicative input error setup (see Fig. 4.11), we have again to assume the somewhat tighter conditions as for the multiplicative input error setup.

T 4.9. *For the finite gain stable process N, let a finite gain stable linear model G and a finite gain stable inverse multiplicative input error model Δ be given. If a controller C is given such that the nominal closed loop $H_{(G\Sigma_V)C}$ is stable and one of the conditions*

(i) *C is linear and finite gain stable,*
(ii) *G has a stable inverse,*
(iii) $\Sigma_V (I + CG\Sigma_V)^{-1} C$ *is incremental gain stable*

holds, and the small gain condition

$$\|\Delta\|^{NG^{-1}\mathcal{U}(V)} \cdot m_{11} < 1$$

is satisfied where $m_{11} = \left\|(I + \Sigma_V CG)^{-1}\right\|$*, then the controller also stabilizes the nonlinear closed loop, i.e.* $H_{(N\Sigma_V)C}$ *is stable. The conclusion also holds if C stabilizes the nominal closed loop and the small gain condition is satisfied with* $m_{11} = 1 + \left\|C (I + G\Sigma_V C)^{-1}\right\| \|G\|$.

P . The equation of the closed loop reads $e_2 = u_2 - G (d_o + \Sigma_V (u_1 + Ce_2))$. First consider the cases (ii) and (iii). Introducing $z = \Sigma_V (u_1 + Ce_2) - \Sigma_V Ce_2$, $\|z\| \leq \|u_1\|$ puts the equation in the form $e_2 = u_2 - Gd_o - G\Sigma_V Ce_2 - Gz$, from which $e_2 = (I + G\Sigma_V C)^{-1} (u_2 - Gd_o - Gz)$. With $d_i = d_o + \Sigma_V (u_1 + Ce_2) = \Sigma_V Ce_2 + d_o + z$ one obtains $d_i = \Sigma_V C (I + G\Sigma_V C)^{-1} (u_2 - Gd_o - Gz) + d_o + z$. If (ii) $\left\|G^{-1}\right\|$ is bounded, then

$$\begin{aligned} d_i &= \Sigma_V C (I + G\Sigma_V C)^{-1} G \left(G^{-1}u_2 - d_o - z\right) + d_o + z \\ &= \Sigma_V CG (I + \Sigma_V CG)^{-1} \left(G^{-1}u_2 - d_o - z\right) - \left(G^{-1}u_2 - d_o - z\right) + G^{-1}u_2 \\ &= -(I + \Sigma_V CG)^{-1} \left(G^{-1}u_2 - d_o - z\right) + G^{-1}u_2 \end{aligned}$$

and thus

$$\|d_i\| \leq \left\|(I + \Sigma_V CG)^{-1}\right\| \left(\left\|G^{-1}\right\| \|u_2\| + \|d_o\| + \|u_1\|\right) + \left\|G^{-1}\right\| \|u_2\| .$$

Note that for a finite-gain stable system P we have $\|P (x + y)\| \leq \|P (x + y) - Py\| + \|Py\| \leq \|P\|_\Delta \cdot \|x\| + \|Py\|$ and therefore if (iii) $\Sigma_V C (I + G\Sigma_V C)^{-1}$ is incremental gain stable, then

$$\begin{aligned} \|d_i\| &\leq \left\|\Sigma_V C (I + G\Sigma_V C)^{-1}\right\|_\Delta \|u_2\| + \left\|\Sigma_V C (I + G\Sigma_V C)^{-1} G (-d_o - z) + d_o + z\right\| \\ &\leq \left\|C (I + G\Sigma_V C)^{-1}\right\|_\Delta \|u_2\| + \left\|(I + \Sigma_V CG)^{-1}\right\| (\|d_o\| + \|u_1\|) . \end{aligned}$$

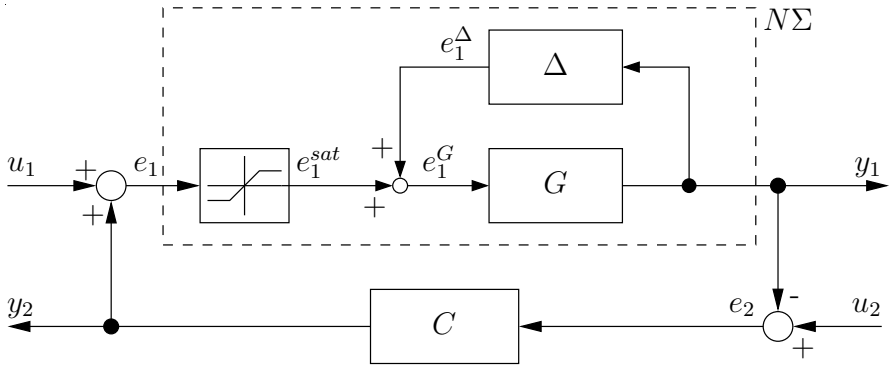
If (i) C is linear and finite gain stable, consider the equation $-y_1 = -G(d_o + \Sigma_V(u_1 + C(u_2 - y_1)))$. Introducing $z = \Sigma_V (u_1 + C (u_2 - y_1)) - \Sigma_V C (-y_1)$, $\|z\| \leq \|u_1\| + \|C\| \|u_2\|$ leads to $-y_1 = -Gd_o - G\Sigma_V C (-y_1) - Gz$, or $-y_1 = (I + G\Sigma_V C)^{-1} G (-d_o - z)$. From $d_i = \Sigma_V (u_1 + Cu_2 + C (-y_1)) + d_o = \Sigma_V C (-y_1) + d_o + z$ we get

$$\begin{aligned} \|d_i\| &= \left\|\Sigma_V C (I + G\Sigma_V C)^{-1} G (-d_o - z) + d_o + z\right\| \\ &\leq \left\|(I + \Sigma_V CG)^{-1}\right\| (\|d_o\| + \|u_1\| + \|C\| \|u_2\|) \end{aligned}$$

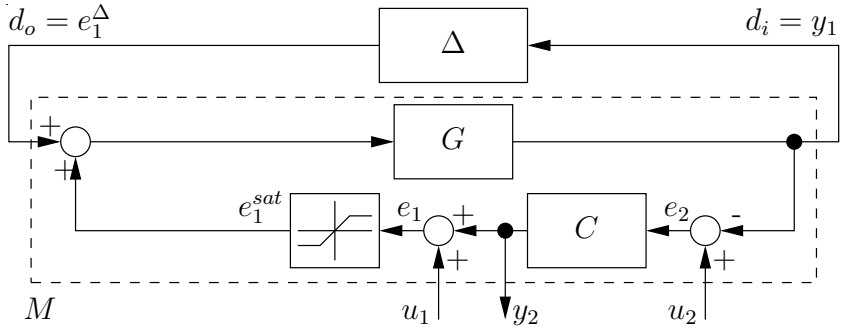
Finally, if none of (i)-(iii) is true, then from $d_i = \Sigma_V C (I + G\Sigma_V C)^{-1} (u_2 - Gd_o - Gz) + d_o + z$ we still have

$$\|d_i\| \leq \left\|C (I + G\Sigma_V C)^{-1}\right\| (\|u_2\| + \|G\| \|d_o\| + \|G\| \|u_1\|) + \|d_o\| + \|u_1\| .$$

The bounds for y_1 and y_2 follow from stability of $H_{(G\Sigma_V)C}$ because by linearity of G the input (u_1, u_2, d_o) to M is equivalent to the input $(u_1, u_2 + Gd_o, 0)$, and by stability of G, $u_2 + Gd_o$ is bounded above, and the proof is complete. □



(a) Closed loop in standard configuration.



(b) Closed loop in $M - \Delta$-configuration.

F 4.12. "Pulling out the Δ" in the feedback error setup.

Linear stabilization of nonlinear systems in the feedback error setup. We have again to assume finite incremental stability for the nominal (linear saturated) closed loop in Fig. 4.12 in order to be able to show stability also for the nonlinear closed loop.

T 4.10. *For the finite gain stable process N, let a finite gain stable linear model G and a finite gain stable inverse multiplicative input error model Δ be given. If a controller C is given such that the nominal closed loop $H_{(G\Sigma_V)C}$ is stable and one of the conditions*

(i) *C is linear and finite gain stable,*
(ii) *G has a stable inverse,*
(iii) *$(I + CG\Sigma_V)^{-1}$ is incremental gain stable*

holds, and the small gain condition

$$\|\Delta\|^{NG^{-1}\mathcal{U}(V)} \cdot m_{11} < 1$$

is satisfied where $m_{11} = \left\|G\left(I + \Sigma_V CG\right)^{-1}\right\|$, then the controller also stabilizes the nonlinear closed loop, i.e. $H_{(N\Sigma_V)C}$ is stable. The conclusion also holds if C stabilizes the nominal closed loop and the small gain condition is satisfied with $m_{11} = \left\|(I + G\Sigma_V C)^{-1}\right\| \|G\|$.

P . The equation of the closed loop reads $e_2 = u_2 - G\left(d_o + \Sigma_V\left(u_1 + Ce_2\right)\right)$. First consider the cases (ii) and (iii). Introducing $z = \Sigma_V\left(u_1 + Ce_2\right) - \Sigma_V Ce_2$, $\|z\| \leq \|u_1\|$ puts the equation in the form $e_2 = u_2 - Gd_o - G\Sigma_V Ce_2 - Gz$, from which $e_2 = \left(I + G\Sigma_V C\right)^{-1}\left(u_2 - Gd_o - Gz\right)$. With $d_i = y_1 = u_2 - e_2$ one obtains $d_i = u_2 - \left(I + G\Sigma_V C\right)^{-1}\left(u_2 - Gd_o - Gz\right)$. If (ii) $\left\|G^{-1}\right\|$ is bounded, then

$$\|d_i\| \leq \left\|G\left(I + \Sigma_V CG\right)^{-1}\right\|\left(\left\|G^{-1}\right\|\|u_2\| + \|d_o\| + \|u_1\|\right) + \|u_2\|.$$

Note that for a finite-gain stable system P we have $\|P\left(x + y\right)\| \leq \|P\left(x + y\right) - Py\| + \|Py\| \leq \|P\|_\Delta \cdot \|x\| + \|Py\|$ and therefore if (iii) $\left(I + G\Sigma_V C\right)^{-1}$ is incremental gain stable, then

$$\begin{aligned}\|d_i\| &\leq \|u_2\| + \left\|\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \|u_2\| + \left\|\left(I + G\Sigma_V C\right)^{-1} G\left(-d_o - z\right)\right\| \\ &\leq \left\|G\left(I + \Sigma_V CG\right)^{-1}\right\|\left(\|d_o\| + \|u_1\|\right) + \left(1 + \left\|\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta\right)\|u_2\|.\end{aligned}$$

If (i) C is linear and finite gain stable, consider the equation $-y_1 = -G(d_o + \Sigma_V(u_1 + C(u_2 - y_1)))$. Introducing $z = \Sigma_V\left(u_1 + C\left(u_2 - y_1\right)\right) - \Sigma_V C\left(-y_1\right)$, $\|z\| \leq \|u_1\| + \|C\|\|u_2\|$ leads to $-y_1 = -Gd_o - G\Sigma_V C\left(-y_1\right) - Gz$, or $-y_1 = \left(I + G\Sigma_V C\right)^{-1} G\left(-d_o - z\right)$. As $d_i = y_1$ we get

$$\|d_i\| \leq \left\|G\left(I + \Sigma_V CG\right)^{-1}\right\|\left(\|d_o\| + \|u_1\| + \|C\|\|u_2\|\right).$$

Finally, if none of (i)-(iii) is true, then from $d_i = u_2 - \left(I + G\Sigma_V C\right)^{-1}\left(u_2 - Gd_o - Gz\right)$ we have

$$\|d_i\| \leq \left\|\left(I + G\Sigma_V C\right)^{-1}\right\|\left(\|u_2\| + \|G\|\|d_o\| + \|G\|\|u_1\|\right) + \|u_2\|.$$

The bound for y_1 is immediate, the bound for y_2 follows from stability of $H_{(G\Sigma_V)C}$ because by linearity of G the input (u_1, u_2, d_o) to M is equivalent to the input $(u_1, u_2 + Gd_o, 0)$, and by stability of G, $u_2 + Gd_o$ is bounded above, and the proof is complete. □

A generalization for incrementally stable nominal closed loop systems. In addition to the results stated above, a very general result can be put up that applies to all of the error model setups. The advantage of the following result can be used to derive stability conditions for all setups in a unified and elegant way. However, in order to provide that general result, we have to make the assumption on finite incremental gain stability of the closed loop transfer operators. This may be a restriction in general [18], so that the following result cannot completely replace the detailed work for the specific setups given above. The generalized plants for all of the model error setups can be deduced from the general nominal closed loop depicted in Fig. 4.13.

Under the assumption that all closed-loop transfer operators are incremental gain stable, we aim at developing a bound

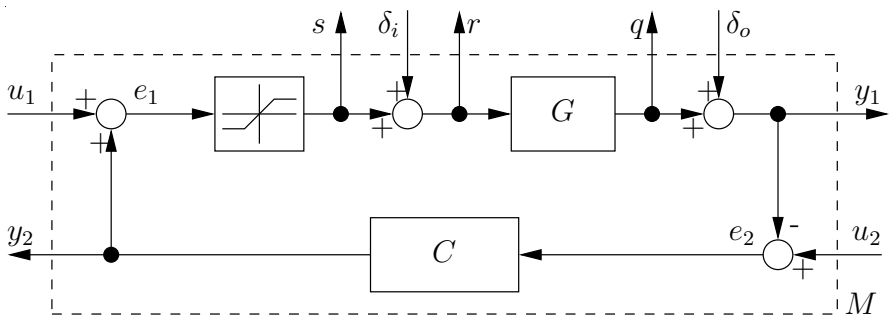
$$\|y\| \leq \beta_1\|\delta_i\| + \beta_2\|\delta_o\| + \beta_3\|u_1\| + \beta_4\|u_2\|$$

for all possible outputs $y = s, r, q, y_1, y_2$. This can indeed be accomplished as follows. First consider the equation of the closed loop

$$e_2 = u_2 - \delta_o - G\delta_i - G\Sigma_V\left(u_1 + Ce_2\right) = u_2 - \delta_o - G\delta_i - G\Sigma_V Ce_2 - Gz$$

with $z = \Sigma_V\left(u_1 + Ce_2\right) - \Sigma_V Ce_2$, $\|z\| \leq \|u_1\|$. We thus get

$$e_2 = \left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right)$$



F 4.13. General nominal closed loop with possible connections to an external error model.

as the basic closed loop transfer equation. Now, for the output y_2 we can write

$$y_2 = Ce_2 = C\left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right)$$

and using the incremental norm property we arrive at

$$\|y_2\| \leq \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \left(\|\delta_o\| + \|u_2\|\right) + \left\|CG\left(I + \Sigma_V CG\right)^{-1}\right\| \left(\|\delta_i\| + \|u_1\|\right). \tag{4.7}$$

For the output s,

$$\begin{aligned} s &= \Sigma_V\left(u_1 + Ce_2\right) = \Sigma_V Ce_2 + z \\ &= \Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right) + z \end{aligned}$$

and by considering that $\Sigma_V C\left(I + G\Sigma_V C\right)^{-1}(-Gz)+z = \left(I + \Sigma_V CG\right)^{-1}(-z)$ one can conclude that

$$\begin{aligned} \|s\| \leq\ & \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \left(\|\delta_o\| + \|u_2\|\right) + \left\|CG\left(I + \Sigma_V CG\right)^{-1}\right\|_\Delta \|\delta_i\| \\ & + \left\|\left(I + \Sigma_V CG\right)^{-1}\right\| \|u_1\|. \end{aligned} \tag{4.8}$$

From

$$\begin{aligned} r &= s + \delta_i \\ &= \Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right) + \delta_i + z \end{aligned}$$

we get in a similar way as before the estimate for r,

$$\|r\| \leq \left\|C\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \left(\|\delta_o\| + \|u_2\|\right) + \left\|\left(I + \Sigma_V CG\right)^{-1}\right\| \left(\|\delta_i\| + \|u_1\|\right). \tag{4.9}$$

The signal q, given by

$$\begin{aligned} q &= Gr \\ &= G\Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right) + G\delta_i + Gz \end{aligned}$$

can then be bounded by

$$\|q\| \leq \left\|G\Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \left(\|\delta_o\| + \|u_2\|\right) + \left\|\left(I + G\Sigma_V C\right)^{-1} G\right\| \left(\|\delta_i\| + \|u_1\|\right), \tag{4.10}$$

and finally, from

$$\begin{aligned} y_1 &= q + \delta_o \\ &= G\Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\left(u_2 - \delta_o - G\delta_i - Gz\right) + \delta_o + G\delta_i + Gz \end{aligned}$$

we can deduce

$$\begin{aligned} \|y_1\| \leq\ & \left\|G\Sigma_V C\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \|u_2\| + \left\|\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \|\delta_o\| \\ & + \left\|\left(I + G\Sigma_V C\right)^{-1} G\right\| \left(\|\delta_i\| + \|u_1\|\right). \end{aligned} \tag{4.11}$$

With the above results, we can derive a condition for robust stability for any of the error setups. For example, consider the inverse multiplicative output error setup. The input to the error term is y_1 and the output of the error term enters the nominal closed loop behind the linear model G, as the signal δ_i. Thus, the last equation (4.11) would give an appropriate bound and following Theorem 4.2 closed loop stability can be guaranteed if the small gain condition

$$\left\|\left(I + G\Sigma_V C\right)^{-1}\right\|_\Delta \cdot \|\Delta\|^{N_{\mathcal{U}}(V)} < 1$$

is satisfied. Clearly, this condition is very similar to the one given in Theorem 4.7, but the assumptions are more restrictive as we required incremental stability of the nominal closed loop in order to arrive at (4.11).

Discussion. For each of the six error setups defined in Chapter 3, we have now stability conditions available that allow to design a controller for a general nonlinear system based on a linear model and the corresponding model quality index (gain of the error system). An immediate question arising is how to choose the right setup for a given plant and control problem, and how the proposed controller design procedure can be related to control-relevant nonlinearity assessment.

On the question which interconnection structure should be chosen, some answers can be given. First of all, it has been shown in Section 3.2 that there are some structural restrictions for the different setups. All setups in which the error term acts on the model input (i.e. the MIE, IMIE and FE setups) are in general only well defined if the nonlinear process does not have more outputs than inputs. Similarly, the setups in which the error term acts on the model output (i.e. the AE, MOE and IMOE setups) are in general only valid if the plant has at least as many outputs as inputs. Moreover, Section 3.6 demonstrated that if the system is known to have a certain structure, like a series or feedback connection of a linear and a nonlinear subsystem, then certain choices for the setups lead to simpler expressions for the nonlinearity measures. The opposite is also true: it has for instance been argued in Section 3.7 that the inverse multiplicative *output* error nonlinearity measure will in many cases be maximal, if a system with a nonlinearity at the *input* (e.g. a Hammerstein system) is considered. Some more advices can be given based on the performance objective, and we postpone them to the first paragraph of the next section.

Concerning control-relevant nonlinearity assessment, it has already been pointed out in the introduction that it is not possible to reduce the question of control-relevant nonlinearity assessment to a single number. The values of the nonlinearity measures defined in this

work already give a hint on whether it may be possible to design a linear controller for a given nonlinear plant or not. However, a final answer to this question can only be given after a linear controller has been designed. Then, the achievable closed loop performance can be compared to the desired performance, or to the performance obtained by nonlinear controller design. Usually the comparison with a nonlinear technique is not possible, or not desirable. In order to nevertheless obtain an indication on the performance loss, the achieved performance can be compared to the performance of a linear controller for the *linear model*, *without* taking the small gain requirement into account. But the evaluation of closed loop performance and the evaluation of a possible performance gain by using nonlinear controller design methods is a difficult question, which still has to await further development (see also the discussion in Section 1.1). Some ideas on analysis and design for performance are given in the next section.

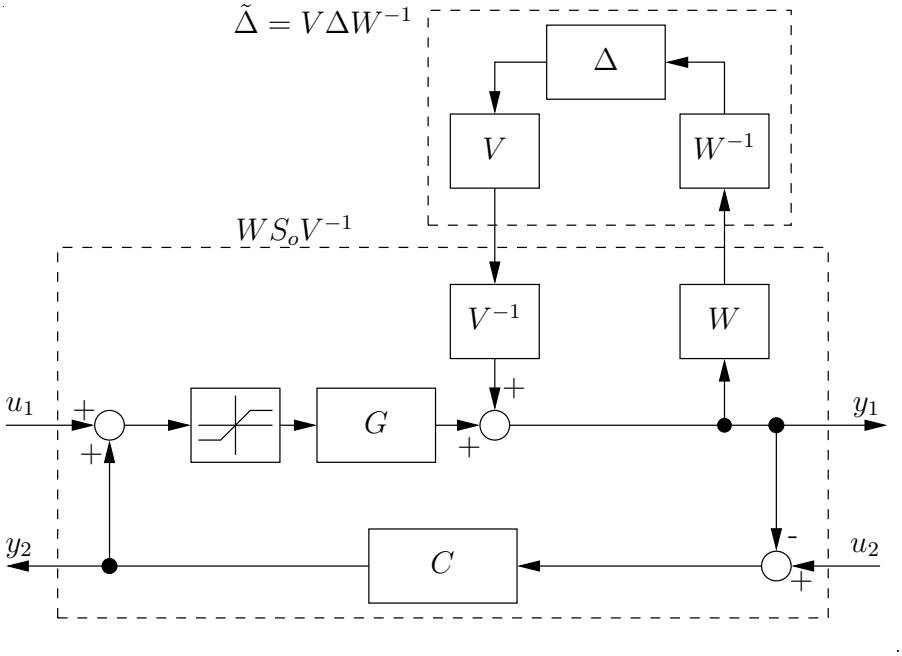
4.3. Some Remarks on Performance

In this section we give possible extensions of the proposed approach to the design of linear controllers for nonlinear systems with guaranteed level of performance. As mentioned in the introduction, an in depth treatment of this topics lies beyond the scope of this work. We nevertheless want to point out directions for future studies that might lead to useful results.

Multipliers and weighted nonlinearity measures. In the last section, stability conditions have been derived for different interconnections of a (saturated) nominal linear model, a controller, and a nonlinear error term. For each of the setups the condition involved the gain of a nominal closed loop and the gain of the error system. For each setup, the involved nominal closed loop operator can be interpreted as a certain type of sensitivity function of the feedback interconnection of the saturated linear model and the controller. For example, the stability condition for the multiplicative output error setup involves the operator $T_o = G\Sigma_V C(I + G\Sigma_V C)^{-1}$, which is exactly the complementary (output) sensitivity function of the nominal closed loop. Similarly, the conditions for the inverse multiplicative output error setup involve the (output) sensitivity function $S_o = (I + G\Sigma_V C)^{-1}$. The respective conditions for the (inverse) multiplicative input error setups involve the (complementary) input sensitivity function, and the condition for the additive error setup the function CS_o.

All these functions play an important role in performance synthesis problems. The function S_o describes the ability of the closed loop to follow reference signals and to attenuate output disturbances. The function S_i signifies the attenuation of input disturbances, the function T_o the attenuation of measurement noise, and the function CS_o characterizes the amount of needed control action. For all functions, a small value indicates small errors in the output variables of the closed loop, which is the goal of controller design. However, it is known that not all of the above quantities can be made small at the same time. Therefore, the typical procedure in linear H_∞ performance synthesis involves dynamic weights in order to describe the frequency ranges in which the above objectives are most important [146].

This observation leads to the suggestion that the setup of nonlinearity assessment and linear modeling should be chosen such that the function involved in the stability condition is also



F 4.14. Closed loop with nonlinear plant represented by a nominal linear model and an inverse multiplicative output error. The multipliers, or weights, V and W are introduced in order to make the performance requirements and the stability requirement match exactly.

the function that is most important for performance synthesis. If, for instance, in a controller design, the attenuation of input disturbances is the most critical aspect, the inverse multiplicative input error setup might be the best choice. This way, the most important performance criterion and the stability condition are not contradicting. Instead, while optimizing the performance of the nominal closed loop, the necessary robustness margin may come for free. In contrast, for optimizing tracking and attenuation of output disturbances, the inverse multiplicative output setup leads to a compatible stability condition.

The idea of choosing nonlinearity assessment and modeling setup according to the desired performance criterion may even be extended. Consider for instance the problem of minimizing the quantity

$$m = \left\| WS_oV^{-1} \right\|.$$

As has been said, the output sensitivity function appears in the conditions for stability in the IMOE setup. But in order to have exactly the same quantity m involved in the stability analysis, the definition of the IMOE nonlinearity measure can be modified to a "weighted" IMOE nonlinearity measure representing the gain of the error model shown in Fig. 4.14. The corresponding weighted nonlinearity measure would then have to be defined as

$$\phi^{\mathcal{U}}_{IMOE,N,V,W} = \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(V(Nu - Gu))_T\|}{\|(W(Nu))_T\|}. \tag{4.12}$$

The weights, or multipliers, V and W have to be chosen to be stable and proper, and to have stable and proper inverses, as usual in H_∞-performance synthesis. The modified

"performance-relevant" nonlinearity measure defined this way is of course not bounded by one any more. But the modified nonlinearity measure has the advantage of not imposing further restrictions on controller design additionally to the performance requirements, as the performance objective is to minimize m and the new stability condition becomes

$$\left\|\tilde{\Delta}\right\|^{\mathcal{U}} \cdot \left\|WS_oV^{-1}\right\| = \phi^{\mathcal{U}}_{IMOE,N,V,W} \cdot \left\|WS_oV^{-1}\right\| < 1.$$

With some abuse of notion, the mechanism by which this method may improve performance can be explained as follows. The use of the weights may flatten out the frequency response of both, the sensitivity function and the error system. In the linear case, this happens if the system gains are achieved for input of different frequency, thus reducing the conservatism of the stability condition. Similar procedures and extensions can of course be made for all of the nonlinearity measures.

Robust Controller Design with Guaranteed L_p-Performance. In the last paragraph we have discussed how the proposed controller design procedure can be modified as to improve the performance of the nonlinear closed loop. Although the outlined approach can be helpful, no guarantees on the actually achieved performance are given. In a linear context, the structured singular value is the right analysis tool for robust performance [146]. Doyle *et al.* [29] give a similar procedure in order to analyze the performance of a nonlinear system. Consider therefore once again the $M-\Delta$-structure given in Fig. 4.3 (left part). Assume that M already incorporated possible performance weights and a controller. If $\|M\| \leq 1$ and $\|\Delta\| \leq 1$ then

$$\|d_i\|_2^2 + \|y\|_2^2 \leq \|d_o\|_2^2 + \|u\|_2^2$$

and

$$\|d_o\|_2^2 \leq \|d_i\|_2^2$$

and consequently the inequality

$$\|y\|_2 \leq \|u\|_2$$

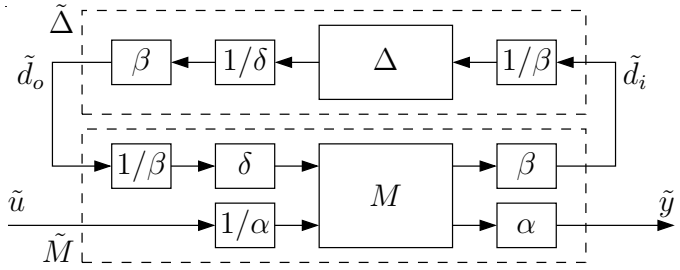
follows, proving that the L_2-gain is bounded by one. The procedure can also be done for other signal spaces L_p, if the vector norm is chosen such that

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_p^p = \|x\|_p^p + \|y\|_p^p .$$

In order to get a more general result, consider Fig. 4.15. In order to deal with a general uncertainty, not necessarily bounded by one, we introduced the scaling factor $\delta = \|\Delta\|$. Moreover, we introduce additional degrees of freedom by introducing the real scalar parameters α and β. By applying the above procedure, if

$$\left\|\tilde{M}\right\| = \left\|\begin{pmatrix} \beta I & 0 \\ 0 & \alpha I \end{pmatrix} M \begin{pmatrix} \delta I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\beta} I & 0 \\ 0 & \frac{1}{\alpha} I \end{pmatrix}\right\| \leq 1 \tag{4.13}$$

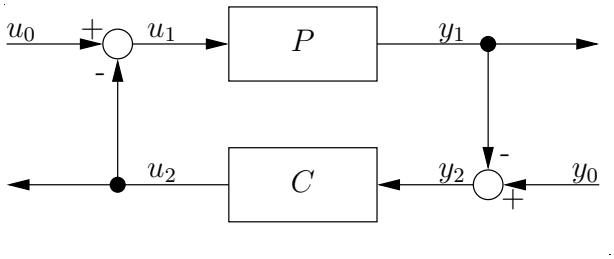
robust performance is guaranteed with $\|\tilde{y}\| \leq \|\tilde{u}\|$, or equivalently, $\|y\| \leq \|u\|$. As α and β are arbitrary, they can be chosen so as to minimize $\left\|\tilde{M}\right\|$, reducing the conservatism of the approach given in [29].



F 4.15. Generalized $M - \Delta$-structure with free real scalar parameters α and β.

A crucial point for the above approach is to respect a certain region of operation of the plant. In the example given in [29] it was shown by physical considerations that the system always stays in a certain part of the state space. Therefore, the technique of introducing an artificial saturation proposed in this work was not necessary. But in order to use this idea of performance synthesis, it is important to lay out a general procedure that incorporates a restriction to a fixed region of operation. Contrary to the case of stabilization, this cannot be done when unbounded external signals are allowed. This can be explained by considering the example of reference tracking. Assume a saturation is introduced at the plant input in order to prescribe the region of operation. As the system is stable, it will not be possible to follow any given desired output. In this case, the tracking behaviour will be arbitrarily bad if large reference values are given, rendering a performance analysis of the closed loop obsolete. The necessity to restrict the admissible external signals for performance analysis and synthesis of the saturated system is widely recognized in the literature [106, 107, 101, 23, 142, 69, 71]. So, a promising direction for future research is to apply the results on performance synthesis for the linear saturated system in order to guarantee that $\left\|\tilde{M}\right\| \leq 1$ for a restricted set of external inputs u.

Relations to Linear Stabilization of Nonlinear Systems using the Gap Metric. The small gain approach to linear control of nonlinear systems is very appealing due to its simplicity and its intuitive nature. Nevertheless it possesses some disadvantages. First of all, it can only be applied to stable plants. Furthermore, the small gain condition is known to be conservative. Even in the linear context, the small gain condition is only sufficient, but not necessary for the stability of the closed loop. The problem at hand is in fact to design a controller for a model A, and to ascertain that the developed controller also stabilizes the real plant B. In the linear context, a necessary and sufficient condition can be derived using the gap metric [32]. The gap metric concept can be extended to nonlinear systems [39], but the necessity of the criterion is lost. Nevertheless, the gap metric approach promises to reduce the conservatism of the small gain approach considerably. Moreover, the gap metric framework also allows to treat unstable plants and to analyze stability and performance for bounded signals.



F 4.16. Control system convention for the gap metric framework.

The gap metric has a very well developed theoretical foundation, and also characterizations of the (nonlinear) gap metric in terms of stable factorizations are available [73]. These results are however not suitable for numerical computations, and it remains an open problem for future research how the gap metric framework can generally be used efficiently for linear control of nonlinear processes. But there are relations of the gap metric to the multiplicative output error nonlinearity measure, which we will point out next.

The general setup for the gap metric framework is as depicted in Fig. 4.16. The signals (u_0, y_0) are exogenous signals and (u_1, y_1) and (u_2, y_2) are the inputs and outputs of the plant and controller respectively. The signals u_i are elements of the signal set $\mathcal{U}$, where $\mathcal{U}$ is a (subset of a) Banach or extended signal space. The signals $y_i \in \mathcal{Y}$ are defined analogously. For ease of presentation, the following exposition concerns the case where $\mathcal{U}$ is an entire signal space. The case of closed balls in L_p can be treated in a similar way [39]. Note the unusual minus sign that is used at the summation point of y_0 and y_1, introduced to make use of parallel projection operators Π [28].

D . [39] For a mapping $P : u \in \mathcal{U} \to y \in \mathcal{Y}$ Let

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \middle| y = Pu, u \in \mathcal{U} \right\} \subseteq \mathcal{U} \times \mathcal{Y}$$

be the graph of a system P. The directed metric between two graphs is defined by

$$\overrightarrow{\delta}(\mathcal{G}_1, \mathcal{G}_2) = \inf \left\{ \sup_{w \in \mathcal{G}_1, T>0} \frac{\|((\Phi - \mathrm{I})w)_T\|}{\|w_T\|} : \begin{array}{l} \Phi \text{ is a causal bijective} \\ \text{map } \mathcal{G}_1 \mapsto \mathcal{G}_2 \text{ with } \Phi 0 = 0 \end{array} \right\}.$$

With some abuse of notation we also write $\overrightarrow{\delta}(P_1, P_2)$ for the metric between the graphs of the two systems P_1 and P_2. If a controller C for a plant P_1 is given, and the gap metric between P_1 and P_2 is known, then the following sufficient condition on the stability of the closed loop with the controller C and the plant P_2 can be given.

Theorem [39] Let $\Pi_{PC} : (u_0, y_0) \mapsto (u_1, y_1)$ be the mapping from the exogenous signals to the input and output of the plant P (see Fig. 4.16). Let Π_{P_1C} be finite-gain stable (i.e. the closed loop with P_1 and C is stable). If a system P_2 is such that

$$\overrightarrow{\delta}(P_1, P_2) < \left\| \Pi_{P_1 C} \right\|^{-1}$$

then Π_{P_2C} is finite-gain stable and

$$\left\|\Pi_{P_2C}\right\| \leq \left\|\Pi_{P_1C}\right\| \frac{1+\overrightarrow{\delta}(P_1,P_2)}{1-\left\|\Pi_{P_1C}\right\|\overrightarrow{\delta}(P_1,P_2)}.$$

This theorem can now be used for the design of linear controllers for nonlinear systems as follows: Let $P_1 = G$ be a linear model for the nonlinear plant $P_2 = N$. If the linear controller C stabilizes the linear model G, it also stabilizes the nonlinear plant N, provided that $\overrightarrow{\delta}(G,N) < \|\Pi_{GC}\|^{-1}$. Moreover, we can define a new nonlinearity measure that measures the gap between the nonlinear system N and its best linear model G^*.

D . The *gap metric nonlinearity measure* of a (not necessarily stable) nonlinear plant $N : \mathcal{U} \to L^n_{pe}$ is defined by

$$\phi^{\mathcal{U}}_{GAP,N} = \inf_{G\in\mathcal{G}_{m,n}} \overrightarrow{\delta}(\mathcal{G}_G, \mathcal{G}_N)$$

and the linear model G^* with $\overrightarrow{\delta}(G^*,N) = \phi^{\mathcal{U}}_{GAP,N}$ is called the best linear model in the gap metric.

Consider the particular map from $\mathcal{G}_P$ to $\mathcal{G}_N$

$$\Phi : \begin{pmatrix} u \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ Nu \end{pmatrix} \text{ with } \Phi^{-1} : \begin{pmatrix} u \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ Gu \end{pmatrix}$$

and consider the L_{2e} signal spaces. Then we obtain from the definition of the gap metric the following upper bound on the gap metric nonlinearity measure

$$\begin{aligned} \phi^{\mathcal{U}}_{GAP,N} = \inf_{G\in\mathcal{G}_{m,n}} \overrightarrow{\delta}(G,N) &\leq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|_2}{\sqrt{\|(Gu)_T\|_2^2 + \|u_T\|_2^2}} \\ &\leq \inf_{G\in\mathcal{G}_{m,n}} \sup_{u\in\mathcal{U},T>0} \frac{\|(Nu-Gu)_T\|_2}{\|(Gu)_T\|_2} = \phi^{\mathcal{U}}_{MOE,N}. \end{aligned}$$

As $\phi^{\mathcal{U}}_{MOE,N} \leq 1$ it can be directly concluded that $\phi^{\mathcal{U}}_{GAP,N} \leq 1$ for stable N. In order to draw conclusions about the control-relevant nonlinearity, the linear control problem has to be solved for the model G subject to the requirement $\|\Pi_{GC}\| < 1/\phi^{\mathcal{U}}_{GAP,N}$. If this condition can be satisfied (which is easy to check with linear methods), then a linear stabilizing controller can be designed for the nonlinear plant. From the parallel projection property follows that $\|\Pi_{GC}\| \geq 1$, and a gap metric nonlinearity measure value of 1 means that linear control for the nonlinear plant will not be possible using this approach (it might still be possible using other approaches as the theorem given above only proves sufficiency, but not necessity). Note that the theorem directly delivers a bound on the gain of the nonlinear closed loop operator Π_{NC}, giving a hint on closed loop performance.

As compared to the approach presented in this thesis, the gap metric framework to describe the distance between input-output operators has the clear advantage of being applicable to unstable systems. Moreover, the stability conditions are assumed to be less conservative than the small gain theorem. But as the condition is also only sufficient, no conclusive

statement can be made at this point. The two main difficulties for developing useful and practical methods based on the gap metric are expected to be the computation of the metric itself and the specification of the region of operation. First attempts to use the gap metric in practical examples for linear controller design for nonlinear systems have been only possible by using a family of local linear models to compute an approximation of the gap metric [132, 133]. Furthermore, the computation of the gap metric can be related to the kernel and image representations of the involved mappings, showing the complexity of the underlying structure [73]. Finally, it is not yet known which choice of signal space is the best one, and whether same way to fix the operating region is possible as is done in this work.

4.4. Application Example: Linear Control of a Nonlinear CSTR

The purpose of this section is to demonstrate the proposed controller design technique on a practically relevant example. We consider a continuous stirred tank reactor (CSTR) in which the product cyclopentenol (substance B) is to be produced from cyclopentadiene (substance A). But in addition to cyclopentenol (B), the reactions

$$A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C$$
$$2A \overset{k_3}{\rightarrow} D.$$

produce the unwanted byproducts dicyclopentadiene (D) and cyclopentanediol (C). The reactions are governed by the nonlinear differential equations

$$\frac{dc_A}{dt} = \frac{q}{V_R}(c_{A0} - c_A) - k_1 c_A - k_3 c_A^2 \tag{4.14}$$
$$\frac{dc_B}{dt} = -\frac{q}{V_R} c_B + k_1 c_A - k_2 c_B \tag{4.15}$$
$$\frac{dT}{dt} = \frac{q}{V_R}(T_0 - T) + \frac{k_W A_R}{\rho C_P V_R}(T_C - T) \tag{4.16}$$
$$-\frac{1}{\rho C_P}(k_1 c_A \Delta H_{R,AB} + k_2 c_B \Delta H_{R,BC} + k_3 c_A^2 \Delta H_{R,AD}) \tag{4.17}$$
$$\frac{dT_c}{dt} = \frac{1}{m_c C_{pc}}(\dot{Q} + k_W A_R (T - T_C)) \tag{4.18}$$

with the reaction rate coefficients described by

$$k_i = k_{i0} e^{(\frac{E_i}{T})}, i = 1, 2, 3. \tag{4.19}$$

The states c_A, c_B are the concentrations of A and B respectively, T is the reactor temperature and T_c is the coolant temperature. The heat flow $\dot{Q}$ that influences the temperature of the coolant is assumed to be constant. The variable to be regulated is the product concentration c_B in the outflow and the manipulated variable is the inlet flow rate q. The example process given above is typically driven at an operating point that achieves about 70% of the maximally possible yield, denoted as suboptimal operating point (SP). For further details on the process and parameter values we refer to [17].

For the nonlinearity assessment and controller design, we have chosen the additive error setup. This setup has been chosen because the system exhibits not special structure that would favor a different setup. Moreover, the controller is designed in order to achieve set point tracking, which to our experience can be achieved most easily in the additive error setup. Especially an integral term in the controller causes difficulties in the other setups, as the available methods to verify the stability criteria are particularly conservative for neutrally stable systems. The nonlinearity measure and best linear model have been computed using the algorithm given in Section 3.8 and in [121] employing genetic algorithms. The terminating criterion $\frac{\check{\gamma}_k - \hat{\gamma}_k}{(\check{\gamma}_k + \hat{\gamma}_k)/2} < 5\%$ has been used to stop the algorithm, but the actually achieved accuracy was even 2.06%. The value of the additive error nonlinearity measure is obtained as

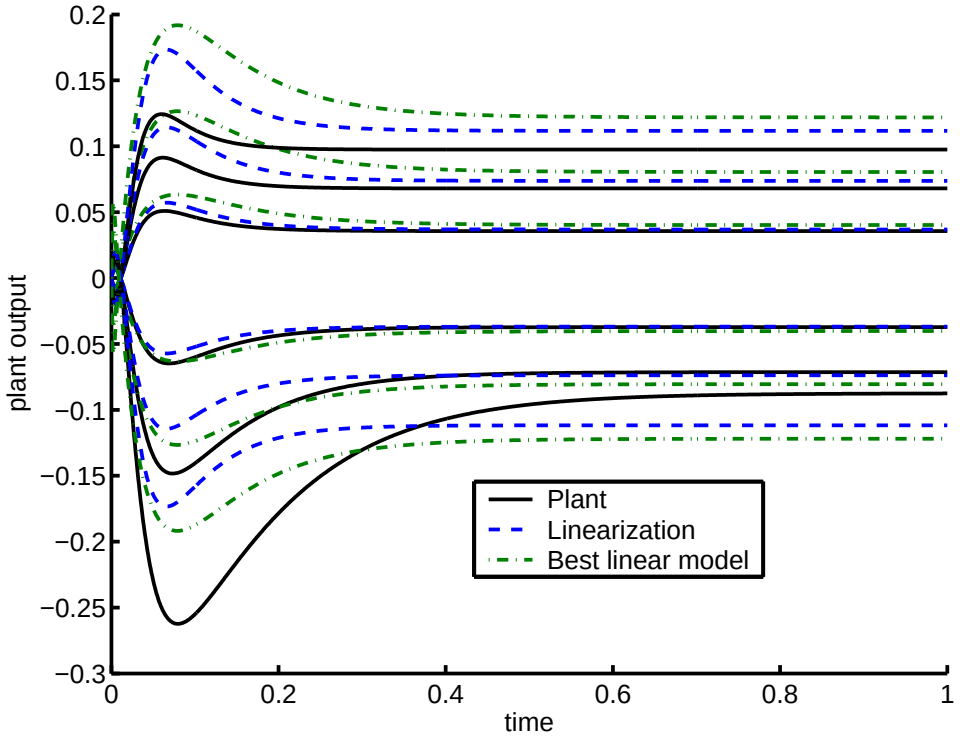
$$\phi_{AE,N}^{\mathcal{U}(\mathcal{V})} = 0.00747$$

in the L_2-setup, for an operating range of $q = 188.3 \pm 100 l/h$. This value seems to be very low, but it has to be compared to the gain of the system for the given operating range, which evaluates to $||N||^{\mathcal{U}(V)} = 0.02359$. The ratio of the nonlinearity measure to the system gain is of $\phi_{AE,N}^{\mathcal{U}(\mathcal{V})} / ||N||^{\mathcal{U}(V)} = 0.317$, and so the system nonlinearity can not be neglected. Moreover, it is worth mentioning that the lower bound of the nonlinearity measure based on the steady-state locus of the plant is $\phi_{AE,N}^{\mathcal{U}(\mathcal{V})} \geq \phi_{AE,N_f}^{\mathcal{U}(\mathcal{V})} = 0.00128$. Comparison with the exact value suggests that the most significant source of the nonlinear behaviour is in the system dynamics of the plant. The gain of the error system, using the local linearization G_{loc} as model is $||\Delta||^{\mathcal{U}(V)} = 0.00876$ and the corresponding ratio to the system gain $||\Delta||^{\mathcal{U}(V)} / ||N||^{\mathcal{U}(V)} = 0.371$. This value is only little higher than the nonlinearity measure and only little difference is expected between using the linearization or the best linear model for controller design. These findings can be supported by studying the step responses of the open-loop nonlinear process, best linear model and linearization, as shown in Fig. 4.17.

Both models seem to approximate the real system behaviour quite well. It can be seen that the best linear model has worse accuracy for the steady-state behaviour, but a better approximation of the dynamic behaviour for negative steps. This compromise is obviously the reason for the best linear model to be slightly superior over the linearization.

It has to be noted that the best model is not unique. This is already true in theory, and there are even more degrees of freedom when the model is computed numerically. For the considered CSTR, computations showed that considering models with real poles only does not imply a loss in model quality. On the other hand, the controller design experience showed that such a model can greatly facilitate the controller synthesis procedure. Therefore, the best linear model was chosen among the linear models with real poles only, and the nonlinearity measure given above is the associated modeling error. This best model

$$G(s) = -7.8 \frac{(s - 106.1)(s + 5.0)}{(s + 937.7)(s + 32.1)(s + 11.4)}$$



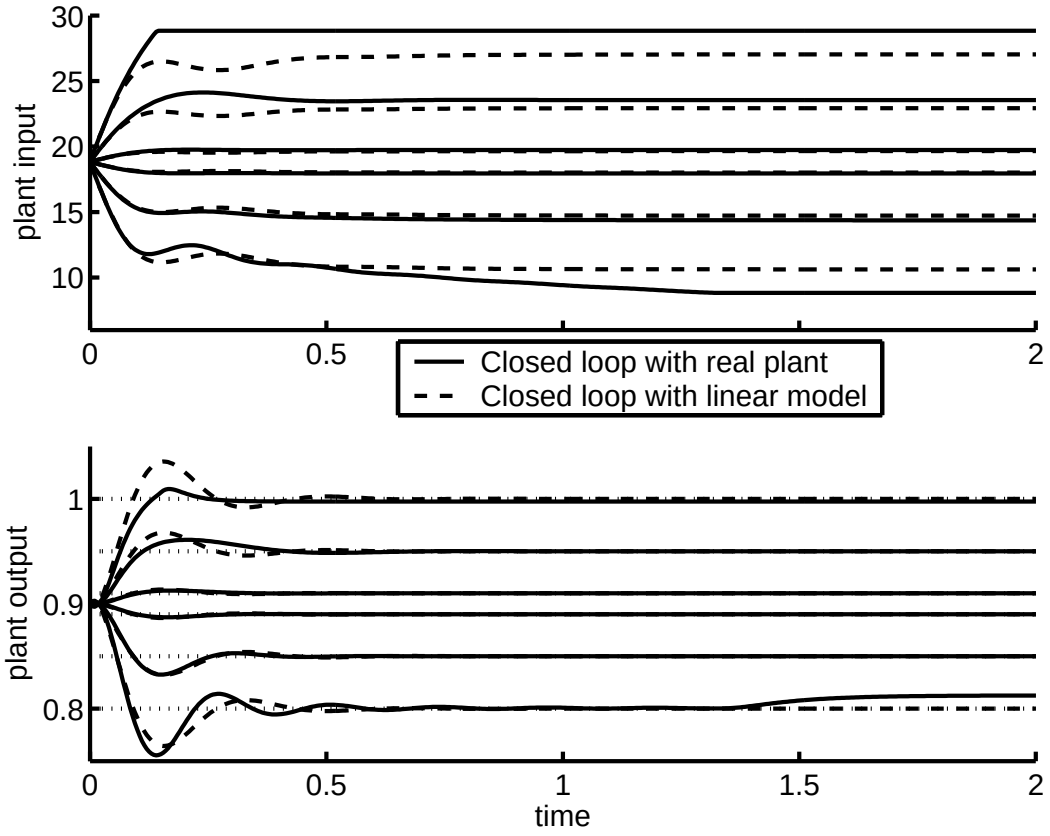
F 4.17. Step responses of the exact nonlinear CSTR model (solid), the best linear approximation (dash-dotted), and the local linearization (dashed). The plant output is the concentration c_B of product B, measured in mol/l as deviation from its steady-state value of $0.9mol/l$. The inputs are equally spaced step inputs in the range of $q = 188.3 \pm n \cdot 33.3l/h$ with $n = 1, 2, 3$.

was also used for controller design. The controller design has been performed using the H_∞ design technique with the mixed-sensitivity problem formulation

$$\left\| \begin{array}{c} \alpha WS \\ \beta KS \end{array} \right\|_{H_\infty} < 1$$

with the weighting function $W(s) = \frac{s+80}{s+0.8}$. A nested $\alpha - \beta-$iteration was done in order to maximize α while still satisfying the condition $\phi_{AE,N}^{\mathcal{U}(\mathcal{V})} \cdot \|KS\|_\infty < 1$. The small gain condition for stability given by Corollary 4.5 was verified by deriving an upper bound via LMIs as described in [108]. Note that the level of saturation does not have any impact on the gain in the single-input case, but is has an impact on the performance (e.g. the size of reference steps that can be followed). In addition to the H_∞ controller design, the controller was augmented with an integral term with anti-windup. The integration time constant was trimmed as high as possible while still respecting the stability condition.

Simulation results of the closed loop with the nonlinear plant and the linear model are given in Fig. 4.18. It can be seen that the set-point tracking behaviour is very satisfactory. It can also be seen that for a reference step of $+0.1mol/l$, the manipulated variable runs into saturation. Comparison with the responses of the closed loop containing the saturated



F 4.18. Responses of the closed loop with nonlinear plant and linear model respectively to steps in the reference signal of different height.

linear model shows that in this case the nonlinear mismatch term does not affect performance, although this could not be guaranteed in advance. This example shows that also for practical systems, the proposed procedure allows to successfully integrate linear model assessment and linear controller design for nonlinear processes.

4.5. Summary

In this chapter, stability conditions and a controller design procedure for linear control of nonlinear systems has been presented. The general procedure is based on the fact that the nonlinear plant is represented as the interconnection of a nominal linear model and a nonlinear error term. Principally any linear model can be used, but the best model, with an error gain given by the corresponding nonlinearity measure, was shown to be superior to other models in some cases. The example of a locally stable, but globally unstable system showed that it may indeed be necessary to do a regional controller design. In our approach, the region of operation is described by constraining the set of allowed input signals to a certain, physically meaningful set. In order to incorporate this restriction in the closed loop, a saturation block has been introduced, that in practice has to be added to either the plant or the controller. Closed loop stability can then be guaranteed by satisfying a small gain

condition of the product of the error gain and the gain of a nominal closed loop system, the exact form of which depends on the interconnection type chosen. We presented results for all interconnection structures that have been defined in Chapter 3. We have also given rules that allow to select the most appropriate setup. Possible approaches to extend the method to performance synthesis have also been given.

The controller design procedure given in this chapter can be applied to nonlinear systems that may be given in any form, as long as they are finite gain stable. The region of operation can be specified very easily, presenting and advantage over previously known results. The only prerequisite for the application of the method is that a linear model is given together with its quality index (modeling error).

CHAPTER 5

Conclusions

It is important to know in which cases the simplification coming along with the use of linear models for nonlinear systems is tolerable in control engineering. The goal of this thesis is to establish methods

(1) to assess the accuracy of a given linear model for a given nonlinear system and
(2) to derive optimal linear models for this system, as well as
(3) to design controllers for a given nonlinear system based on a given linear model.

In order to quantify the quality of a given linear model and to derive an optimal linear model, this work introduced a novel unifying framework. In this framework, the nonlinear plant is represented as the interconnection of a nominal linear model and a nonlinear error term. The model quality can be measured by the gain of the error system, and a best linear model is defined to achieve the smallest possible error gain. Possible interconnection structures include those known from linear robust control theory. The model quality index of the best linear model is a measure of nonlinearity in the input-output behaviour of the nonlinear process, and two nonlinearity measures known from the literature are special cases of the proposed framework. The information about the linear model and the error gain can then subsequently by used to design a linear controller for the nonlinear system.

The main contribution of this thesis for linear modeling of nonlinear systems is to provide a rigorous mathematical foundation of the introduced framework, and to provide characterizations of the nonlinearity measures and best linear models for special system classes. Conditions were given under which an error model can be properly defined, also showing that the given definitions of nonlinearity measures and best linear models are indeed meaningful. We also showed for the additive error nonlinearity measure that the state space linearization gives a unique perfect local linear model, i.e. the unique linear model that achieves zero modeling error in the limiting case of a vanishing operating regime. It was shown that the steady state behaviour establishes a lower bound for three of the nonlinearity measures, and that in those cases, dynamic linear models have no advantage over static linear models for memoryless nonlinear operators. Formulae for the nonlinearity measures of memoryless operators were given, and particularly simple expressions were obtained for memoryless scalar systems based on the notion of a sector. Linear models and the associated modeling errors were derived for Hammerstein and Wiener models, and some further interconnections of a linear and a nonlinear system. It was shown that, if the

approximation is a linear combination of linear models from a given basis, the best linear model is guaranteed to exist for two of the measures. If a strict best linear does not exist, it was shown that an arbitrarily close linear model can be obtained.

The main contribution of this thesis for linear control of nonlinear systems are the stability conditions and controller design procedures for linear control of nonlinear systems. The general procedure is based on the fact that the nonlinear plant is represented as the interconnection of a nominal linear model and a nonlinear error term. The idea is then to use controller design to robustly stabilize the nominal linear system in presence of the nonlinear perturbation term. Principally any linear model can be used, but the best model, with an error gain given by the corresponding nonlinearity measure, was shown to be sometimes significantly superior to other models. This is explained by the fact that the smallest modeling error implies the least conservatism in the controller design. The example of a locally stable, but globally unstable, system showed that it is indeed necessary to do a regional controller design. In our approach, the region of operation is described by constraining the set of allowed input signals to a certain, physically meaningful set. In order to incorporate this restriction in the closed loop, a saturation block has been introduced, that in practice has to be added to either the plant or the controller. Closed loop stability can then be guaranteed by satisfying a small gain condition on the product of the error gain and the gain of a nominal closed loop system. The exact form of the nominal closed loop depends on the interconnection type chosen and we presented the results for all interconnection structures defined previously. Although the proposed method is restricted to finite gain stable processes, the key advantage is that a linear controller can be designed to achieve the desired performance for the linear model, while still guaranteeing closed loop stability for the nonlinear process. It has been shown that this approach can lead to good performance, and that it is also applicable for practically relevant processes.

The results of this thesis build a bridge between nonlinearity measures and robust control theory, allowing a new and better interpretation of known nonlinearity measures, and introducing new ones. As compared to previous research, the power of the proposed framework is to bring together nonlinearity assessment, the development and assessment of linear models for nonlinear systems and the design of linear controllers for nonlinear systems. The framework thus contributes to all related fields of research: to nonlinearity assessment, to linear modeling of nonlinear systems, to control-relevant nonlinearity assessment as well as to linear control of nonlinear systems.

Concerning the areas of nonlinearity assessment and linear modeling for nonlinear systems, the results have shown that two previously known nonlinearity measures are closer related than was known before. Many theoretical results have shown that the definitions of nonlinearity measures are in accordance with engineering intuition, e.g. in the case of local linear approximations, thus strengthening the confidence in the usefulness of the framework. Other results enable the user to more easily evaluate the nonlinearity measure and derive linear models for a system with a known structure, like Hammerstein and Wiener systems, and to decide in advance whether a certain nonlinearity measure (or model quality index) is suited for a given system or not. Both facts contribute to the applicability of the proposed framework. But the given results can be thought to be generalized even

more. Instead of evaluating the gain of the error system for a fixed error setup, one can also think of searching for an optimal generalized plant, thus also optimizing the interconnection structure. In order to be able to apply the analysis and synthesis methods of this thesis in an off-the-shelf manner, advances in the computation of nonlinear system gains are desirable. Furthermore, an analogue approach applicable to unstable systems would significantly enlarge the class of problems that can be treated. One possible starting point for such a method is given by the gap metric framework.

Although the individual nonlinearity measures are computed based on the plant only, the framework also allows to assess the control-relevant nonlinearity. This assessment can on the one hand take into account whether the necessary stability conditions can be satisfied for the best linear model. On the other hand, different nonlinearity measures can be compared, and weighted nonlinearity measures can be used to assess the suitability of linear models with respect to a given control objective. Furthermore, the control-relevant analysis that is done this way does not only refer to an imaginary best linear controller for a nonlinear plant, but a concrete design procedure is known how to actually design a linear controller.

The linear controller design methods for nonlinear systems given in this thesis use the input-output framework and therefore, the presented controller design procedure is not restricted to systems given in state-space form. Moreover, the region of operation of a plant can be specified easier than has been possible before, thus allowing for non-global synthesis. In the future, it will be interesting to see whether the method also proves to be useful in the application to real industrial processes. We have sketched a way how the controller design may be extended to also design linear controllers for nonlinear systems with guaranteed performance. Such a development, as well as the possibility to also treat unstable systems, would contribute to make the ideas given in this thesis a fruitful basis for the application and the further development of linear modeling and linear control of nonlinear systems.

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