# Table of Contents

## About the Author

V

## Introduction

VII

### Chapter I. Points and lines in the plane

1. In which setting and in which plane are we working? And right away an utterly simple problem of Sylvester about the collinearity of points 1

1.2. Another naive problem of Sylvester, this time on the geometric probabilities of four points 6

1.3. The essence of affine geometry and the fundamental theorem 12

1.4. Three configurations of the affine plane and what has happened to them: Pappus, Desargues and Perles 17

1.5. The irresistible necessity of projective geometry and the construction of the projective plane 23

1.6. Intermezzo: the projective line and the cross ratio 28

1.7. Return to the projective plane: continuation and conclusion 31

1.8. The complex case and, better still, Sylvester in the complex case: Serre’s conjecture 40

1.9. Three configurations of space (of three dimensions): Reye, Möbius and Schläfli 43

1.10. Arrangements of hyperplanes 47

1. XYZ 48

Bibliography 57

### Chapter II. Circles and spheres

II. Introduction and Borsuk’s conjecture 61

II.2. A choice of circle configurations and a critical view of them 66

II.3. A solitary inversion and what can be done with it 78

II.4. How do we compose inversions? First solution: the conformal group on the disk and the geometry of the hyperbolic plane 82

II.5. Second solution: the conformal group of the sphere, first seen algebraically, then geometrically, with inversions in dimension 3 (and three-dimensional hyperbolic geometry). Historical appearance of the first fractals 87

II.6. Inversion in space: the sextuple and its generalization thanks to the sphere of dimension 3 91

II.7. Higher up the ladder: the global geometry of circles and spheres 96

II.8. Hexagonal packings of circles and conformal representation 103
II.9.  Circles of Apollonius ................................................. 113
II. XYZ ................................................................. 116
Bibliography ............................................................... 137

Chapter III. The sphere by itself: can we distribute points on it evenly? 141
III.1. The metric of the sphere and spherical trigonometry .................. 141
III.2. The Möbius group: applications .................................... 147
III.3. Mission impossible: to uniformly distribute points on the sphere S^2: ozone, electrons, enemy dictators, golf balls, virology, physics of condensed matter ........................................... 149
III.4. The kissing number of S^2, alias the hard problem of the thirteenth sphere 170
III.5. Four open problems for the sphere S^3 .............................. 172
III.7. A conceptual approach for the kissing number in arbitrary dimension 175
III. XYZ ................................................................. 177
Bibliography ............................................................... 178

Chapter IV. Conics and quadrics 181
IV.1. Motivations, a definition parachuted from the ladder, and why . . . 181
IV.2. Before Descartes: the real Euclidean conics. Definition and some classical properties ........................................ 183
IV.3. The coming of Descartes and the birth of algebraic geometry . . . 198
IV.4. Real projective theory of conics; duality .............................. 200
IV.5. Klein’s philosophy comes quite naturally ................................ 205
IV.6. Playing with two conics, necessitating once again complexification . 208
IV.7. Complex projective conics and the space of all conics ............... 212
IV.8. The most beautiful theorem on conics: the Poncelet polygons . . . 216
IV.9. The most difficult theorem on the conics: the 3264 conics of Chasles 226
IV.10. The quadrics ................................................................ 232
IV. XYZ ................................................................. 242
Bibliography ............................................................... 245

Chapter V. Plane curves 249
V.1. Plain curves and the person in the street: the Jordan curve theorem, the turning tangent theorem and the isoperimetric inequality . . . 249
V.2. What is a curve? Geometric curves and kinematic curves ............ 254
V.3. The classification of geometric curves and the degree of mappings of the circle onto itself ............................................ 257
V.4. The Jordan theorem ..................................................... 259
V.5. The turning tangent theorem and global convexity ..................... 260
V.6. Euclidean invariants: length (theorem of the peripheral boulevard) and curvature (scalar and algebraic): Winding number .......... 263
V.7. The algebraic curvature is a characteristic invariant: manufacture of rulers, control by the curvature 269
V.8. The four vertex theorem and its converse; an application to physics 271
V.9. Generalizations of the four vertex theorem: Arnold I 278
V.10. Toward a classification of closed curves: Whitney and Arnold II 281
V.11. Isoperimetric inequality: Steiner’s attempts 295
V.12. The isoperimetric inequality: proofs on all rungs 298
V.13. Plane algebraic curves: generalities 305
V.14. The cubics, their addition law and abstract elliptic curves 308
V.15. Real and Euclidean algebraic curves 320
V.16. Finite order geometry 328
V. XYZ 331
Bibliography 336

Chapter VI. Smooth surfaces 341
VI.1. Which objects are involved and why? Classification of compact surfaces 341
VI.2. The intrinsic metric and the problem of the shortest path 345
VI.3. The geodesics, the cut locus and the recalcitrant ellipsoids 347
VI.4. An indispensable abstract concept: Riemannian surfaces 357
VI.5. Problems of isometries: abstract surfaces versus surfaces of $\mathbb{E}^3$ 361
VI.6. Local shape of surfaces: the second fundamental form, total curvature and mean curvature, their geometric interpretation, the theorem egregium, the manufacture of precise balls 364
VI.7. What is known about the total curvature (of Gauss) 373
VI.8. What we know how to do with the mean curvature, all about soap bubbles and lead balls 380
VI.9. What we don’t entirely know how to do for surfaces 386
VI.10. Surfaces and genericity 391
VI.11. The isoperimetric inequality for surfaces 397
VI. XYZ 399
Bibliography 403

Chapter VII. Convexity and convex sets 409
VII.1. History and introduction 409
VII.2. Convex functions, examples and first applications 412
VII.3. Convex functions of several variables, an important example 415
VII.4. Examples of convex sets 417
VII.5. Three essential operations on convex sets 420
VII.6. Volume and area of (compacts) convex sets, classical volumes: Can the volume be calculated in polynomial time? 428
VII.7. Volume, area, diameter and symmetrizations: first proof of the isoperimetric inequality and other applications 437
VII.8. Volume and Minkowski addition: the Brunn-Minkowski theorem and a second proof of the isoperimetric inequality .......................................................... 439
VII.9. Volume and polarity ......................................................................................... 444
VII.10. The appearance of convex sets, their degree of badness ............................ 446
VII.11. Volumes of slices of convex sets ................................................................... 459
VII.12. Sections of low dimension: the concentration phenomenon and the Dvoretzky theorem on the existence of almost spherical sections ........................................................................................................... 470
VII.13. Miscellany ....................................................................................................... 477
VII.14. Intermezzo: can we dispose of the isoperimetric inequality? ..................... 493
Bibliography ............................................................................................................. 499

Chapter VIII. Polygons, polyhedra, polytopes .......................................................... 505
VIII.1. Introduction .................................................................................................... 505
VIII.2. Basic notions .................................................................................................. 506
VIII.3. Polygons ........................................................................................................ 508
VIII.4. Polyhedra: combinatorics .............................................................................. 513
VIII.5. Regular Euclidean polyhedra ........................................................................ 518
VIII.6. Euclidean polyhedra: Cauchy rigidity and Alexandrov existence .............. 524
VIII.7. Isoperimetry for Euclidean polyhedra .......................................................... 530
VIII.8. Inscribability properties of Euclidean polyhedra; how to encage a sphere (an egg) and the connection with packings of circles .......................................................... 532
VIII.9. Polyhedra: rationality ..................................................................................... 537
VIII.10. Polytopes ($d \geq 4$): combinatorics I ......................................................... 539
VIII.11. Regular polytopes ($d \geq 4$) ....................................................................... 544
VIII.12. Polytopes ($d \geq 4$): rationality, combinatorics II ...................................... 550
VIII.13. Brief allusions to subjects not really touched on ........................................ 555
Bibliography ............................................................................................................. 558

Chapter IX. Lattices, packings and tilings in the plane ................................................. 563
IX.1. Lattices, a line in the standard lattice $\mathbb{Z}^2$ and the theory of continued fractions, an immensity of applications ................................................................. 563
IX.2. Three ways of counting the points $\mathbb{Z}^2$ in various domains: pick and Ehrhart formulas, circle problem .................................................................................. 567
IX.3. Points of $\mathbb{Z}^2$ and of other lattices in certain convex sets: Minkowski’s theorem and geometric number theory ................................................................. 573
IX.4. Lattices in the Euclidean plane: classification, density, Fourier analysis on lattices, spectra and duality ................................................................. 576
IX.5. Packing circles (disks) of the same radius, finite or infinite in number, in the plane (notion of density). Other criteria ......................................................... 586
IX.6. Packing of squares, (flat) storage boxes, the grid (or beehive) problem ........ 593
IX.7. Tiling the plane with a group (crystallography). Valences, earthquakes ......... 596
IX.8. Tilings in higher dimensions ............................................................................. 603
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>XII.7. Do the mechanics determine the metric?</td>
<td>779</td>
</tr>
<tr>
<td>XII.8. Recapitulation and open questions</td>
<td>781</td>
</tr>
<tr>
<td>XII.9. Higher dimensions</td>
<td>781</td>
</tr>
<tr>
<td>Bibliography</td>
<td>782</td>
</tr>
<tr>
<td>Selected Abbreviations for Journal Titles</td>
<td>785</td>
</tr>
<tr>
<td>Name Index</td>
<td>789</td>
</tr>
<tr>
<td>Subject Index</td>
<td>795</td>
</tr>
<tr>
<td>Symbol Index</td>
<td>827</td>
</tr>
</tbody>
</table>