Algebraic Aspects of Noncommutative Tori: the Riemann–Hilbert Correspondence

Dissertation

zur

Erlangung des Doktorgrades

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Kalkutta, Indien

Bonn 2007

Angefertigt mit Genehmigung
der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 10. August 2007

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB
Bonn http://hss.ulb.uni-bonn.de/diss online elektronisch publiziert.

Erscheinungsjahr: 2007

For Maa and Baba

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ACKNOWLEDGEMENTS

When I come to think of all people who have been instrumental in shaping my career, the name of my advisor Professor Matilde Marcolli occurs to me first and foremost. Her unflinching support of my wild, possibly naïve, ideas and constant encouragement have been a morale-booster from the very beginning. She gave me enough freedom to carry out my own research. Moreover, she had generously shared her own ideas with me and was always eager to discuss my problems. Indeed, my earnest gratitude goes out to her.

A special word of thanks should be given to Dr. Christian Kaiser, who has been the coordinator of the IMPRS graduate programme and extremely benevolent in sharing his vast knowledge with the students, including me. I would also like to thank the professors at MPI for taking time out of their busy schedule and offering courses which made seemingly impregnable mathematical topics accessible to graduate students - Professors G. Harder, G. Faltings, Yu. I. Manin, D. Zagier and D. Huybrechts, to name only a few.

At various points of time I have had numerous discussions and/or email correspondences with many experts in their respective fields, namely (in alphabetical order of last names) Professors M. van den Bergh, A. Connes, B. Keller, P. J. Smith, B. Toën and V. Vologodsky, to whom I would like to express my gratitude.

I would like to thank Max-Planck-Institut für Mathematik, Bonn for the scholarship and numerous travel grants to attend workshops and conferences, which helped me keep abreast of the recent developments in various areas of mathematics considerably. Moreover, the working conditions were ideal and the non-academic staff were extremely cooperative.

My stay at MPI had given me the opportunity to professionally interact with various visitors to whom I owe a large part of my knowledge. I gratefully acknowledge the useful discussions with the following people (in alphabetical order of last names) - Kürsat Aker, Arend Bayer, Nikolai Durov, Payman Kassaei, Fernando Muro, Behrang Noohi, Jorge Plazas, S. Spallone, Ramesh Sreekantan, Walter D. van Suijlekom, amongst others.

At the non-professional level, I would like to thank all my friends, fellow IMPRS students and postdocs for making my social life a thoroughly enjoyable one. I was truly blessed with a multicultural circle of friends and the numerous cultural, as well as mathematical, interactions with them have helped me grow as a person. Taking the privilege of calling by the first names, I would like to thank the following friends whole-heartedly and wish them all the best with their future pursuits: Anders, Andres, Anton, Bram, Carolina, Ciprian, Clara, Cristina, Emanuele, Eugene, Hermes, Jorge (Pineda), Jorge (Plazas), Ivan, Leonardo, Maryna, Masha, Meng, Obi, Özgür, Sandrine, So, Vasilisa, Volker and Walter.

Finally, I express my gratitude towards to all the referees of my thesis committee, namely Professor Matilde Marcolli, Professor D. Huybrechts, Professor S. Albeverio and Professor M. Drees. A special word of thanks goes out to my advisor and Professor D. Huybrechts for pointing out several errors in the initial draft and making many useful suggestions.

1. Introduction

Noncommutative geometry in its various forms has come to the forefront of mathematical research lately and noncommutative tori constitute perhaps the most extensively studied class of examples of noncommutative differentiable manifolds. They were introduced by Connes during the early eighties [**Con80**] and were systematically studied by Connes [**Con80**], Rieffel [**Rie81, Rie83**] and others. Recently Polishchuk and Schwarz have provided a new perspective on them which is quite amenable to techniques in algebraic geometry [**PS03, Pol04b**]. At the same time Soibelman and Vologodsky have introduced noncommutative elliptic curves as certain equivariant categories of coherent sheaves [**SV03**]. The guiding principle behind both constructions is replacing a mathematical object by its category of appropriately defined representations, *viz.,* vector bundles with connections in the former case, denoted by $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$, and coherent sheaves in the latter, denoted by $\mathcal{B}_q$, where $q = e^{2\pi i\theta}$ and θ is an irrational number.

One of the aims of this thesis is to connect the above two constructions by introducing an intermediate category $\mathcal{B}^\tau_q$. Besides the existence of a forgetful functor from $\mathcal{B}^\tau_q$ to $\mathcal{B}_q$ (as the notation might suggest), we construct a faithful and exact functor from $\mathcal{B}^\tau_q$ to $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$. It turns out to be well-adapted to the Tannakian formalism. In fact, our main result is that it is a Tannakian category and via an equivariant version of the Riemann–Hilbert correspondence we show that it is equivalent to the category of finite dimensional representations of (the algebraic hull of) $\mathbb{Z}^2$ (see Theorem 3.16). This allows us to describe the K-theory of $\mathcal{B}^\tau_q$ as the free abelian group generated by two copies of $\mathbb{C}^*$ (see Corollary 3.18).

In the section entitled *Background material* we briefly review the main results of [**PS03**], including the basic definitions and examples. We also discuss the rudiments of noncommutative tori, which are relevant for our purposes as it is known that there are several ways of looking at them. We also show that there is a certain *modularity* property satisfied by the categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ (see Proposition 2.9). A discussion on categorical Riemann–Hilbert correspondence is also included.

In the section entitled *Equivariant coherent sheaves and* $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$, which is a joint work with Walter D. van Suijlekom, we first provide a motivation for the definition of the categories $\mathcal{B}^\tau_q$ and then construct a faithful and exact functor from $\mathcal{B}^\tau_q$ into $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$. We also give a description of the image of our functor and discuss the induced map on the K-theories of the corresponding categories. There is a canonical forgetful functor from $\mathcal{B}_q^\tau$ to $\mathcal{B}_q$. We briefly recall some preliminaries of Tannakian categories. We explain the structure of a Tannakian category on the category $\mathcal{B}_q^\tau$ and prove an equivariant version of the Riemann–Hilbert correspondence on $\mathbb{C}^*$. Via this correspondence, we find that $\mathcal{B}_q^\tau$ is equivalent to the category of finite dimensional representations of $\mathbb{Z}^2$. As a consequence we are able to compute the K-theory of $\mathcal{B}_q^\tau$.

We conclude with a motivation for a possible notion of the fundamental group of noncommutative tori (see Remark 3.19) and with a discussion on the degeneration of the complex structure on noncommutative tori (see Remark 3.20). We formulate a conjecture, which seems to be a natural consequence of our discussion.

Quite separately noncommutative geometry seems to be the unifying framework of geometry. With the extensive use of the categorical language, which is not only aesthetically pleasing, noncommutative algebraic geometry aims to subsume the highly successful world of noncommutative geometry *à la Connes* and interesting examples from physics like *Landau–Ginzburg models*. Derived categories or abstract triangulated categories, which arose in the context of algebraic topology, were considered to be the right objects in this paradigm, in spite of several unpleasant functoriality problems (*e.g.,* the cone construction is not functorial). One tries to circumvent such problems using suitable models of derived categories like the bounded below homotopy category of injective chain complexes, which look rather *ad hoc*. Fortunately the language of DG categories has flourished over the last decade, which seems to nicely resolve most of the issues. Building on the works of Bondal, Drinfeld, Keller, Kontsevich, Lurie, Orlov, Toën and many others, we explain the basics of noncommutative algebraic geometry in the section entitled *noncommutative geometry in DG framework*.

Some recent and interesting works of Connes [**Con99**], Connes–Consani–Marcolli [**CCM**], Connes–Marcolli–Ramachandran [**CMR05**], Manin [**Man04**] and Manin–Marcolli [**MM02**], amongst others, applying the techniques of noncommutative geometry to number theory have attracted a lot of attention lately. We try to reconcile with these developments in the context of noncommutative algebraic geometry. The main intention is to introduce motivic zeta functions of noncommutative spaces and discuss their properties. We define noncommutative Calabi–Yau varieties generalizing the classical ones and propose motivic zeta functions thereof based on the universal motivic measure on the category of noncommutative spaces in a separate section entitled *Noncommutative Calabi–Yau spaces*. It is a part of an on-going project and the expectation is that our work might shed some light on Manin's *Real Multiplication* programme as set out in [**Man04**].

2. Background Material

For the benefit of the reader we recall some basic facts about coherent sheaves on elliptic curves, t-structures on the derived category of coherent sheaves thereof and some rudiments of noncommutative tori. We also recall the definition along with the basic properties of noncommutative elliptic curves and include a discussion on the categorical Riemann–Hilbert correspondence.

2.1. Coherent sheaves on elliptic curves. We restrict our discussion to elliptic curves, though some of the notions carry over to higher dimensions. Let E be an elliptic curve over $\mathbb{C}$ and we denote by $\mathrm{Coh}(\mathsf{E})$ the abelian category of coherent sheaves on E. Let $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}(\mathsf{E})$. Then it is known that there is a bifunctorial isomorphism which is given by the Serre duality

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Ext}^1(\mathscr{G}, \mathscr{F})^*.$$

One also defines an *Euler form* on E as

$$< \mathscr{F}, \mathscr{G} >:= \dim \mathrm{Hom}(\mathscr{F}, \mathscr{G}) - \dim \mathrm{Ext}^1(\mathscr{F}, \mathscr{G}).$$

The Euler characteristic of $\mathscr{F}$ is by definition $\chi(\mathscr{F}) :=< \mathcal{O}_{\mathsf{E}}, \mathscr{F} >$. It is additive on exact sequences.

By $\mathrm{rk}(\mathscr{F})$ we mean the dimension of $\mathscr{F}_\eta$ over $\mathsf{K} = \mathcal{O}_{\mathsf{E},\eta}$ (*i.e.,* function field) where η is the generic point of E. For any $\mathsf{F}^\bullet \in \mathsf{D}^b(\mathsf{E})$ one extends the definition of degree (Euler characteristic) and rank as follows

$$\chi(\mathsf{F}) = \sum_i (-1)^i \dim_{\mathsf{K}} \mathrm{H}^i(\mathrm{RHom}(\mathcal{O}_{\mathsf{E}}, \mathsf{F}^\bullet)))$$
$$\mathrm{rk}(\mathsf{F}) = \chi(\eta^*(\mathsf{F}^\bullet))$$

The *slope* of a coherent sheaf $\mathscr{F}$ is an element of $\mathbb{Q} \cup \{\infty\}$ defined as

$$\mu(\mathscr{F}) = \frac{\chi(\mathscr{F})}{\mathrm{rk}(\mathscr{F})}$$

The same definition extends the notion of slope to the objects in the derived category as the alternating sum of the slopes of the cohomologies.

Definition 2.1. *A coherent sheaf* $\mathscr{F}$ *is called semi-stable (resp. stable) if for any non-trivial exact sequence* $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ *one has* $\mu(\mathscr{F}') \leqslant \mu(\mathscr{F})$ *(resp.* $\mu(\mathscr{F}') < \mu(\mathscr{F})$*) or equivalently* $\mu(\mathscr{F}) \leqslant \mu(\mathscr{F}'')$ *(resp.* $\mu(\mathscr{F}) < \mu(\mathscr{F}'')$*).*

It is well known that every coherent sheaf on E splits as a direct sum of its torsion and torsion-free part. Since E is smooth, projective and of dimension 1, every torsion-free coherent sheaf is locally free. The following theorem from [**HN75**] gives us a good understanding of the indecomposable objects of $\mathrm{Coh}(E)$, which can be shown to be semi-stable.

Theorem 2.2 (Harder-Narasimhan,Rudakov). *Let* X *be a projective curve. Then for any* $\mathscr{F} \in \mathrm{Coh}(E)$ *there exists a unique filtration:*

$$\mathscr{F}_0 \supset \mathscr{F}_1 \supset \cdots \supset \mathscr{F}_n \supset \mathscr{F}_{n+1} = 0$$

such that

- $\mathscr{A}_i := \mathscr{F}_i / \mathscr{F}_{i+1}$ *for* $0 \leqslant i \leqslant n$ *are semi-stable and*
- $\mu(\mathscr{A}_0) < \mu(\mathscr{A}_1) < \cdots < \mu(\mathscr{A}_n)$.

The graded quotients $\mathscr{A}_i$ of the Harder-Narasimhan Filtration of $\mathscr{F}$ are called the *semi-stable factors* of $\mathscr{F}$.

Remark 2.3. For elliptic curves one has a special property. Every coherent sheaf is isomorphic to the direct sum of its semi-stable factors, which is a consequence of the Calabi–Yau property, which says that Hom's are isomorphic to the duals of the Ext's as vector spaces in the category of coherent sheaves.

2.2. Torsion pairs and t-structures. We now recall the definition of a torsion pair in an abelian category and its associated t-structure, which is nicely written down in *e.g.,* [**HRS96**]. The notations employed here are local and should not be confused with their appearances in different forms elsewhere. Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories of an abelian category $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* in $\mathcal{A}$ if the following conditions are satisfied:

(1) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
(2) For all $X \in \mathcal{A}$ $\exists$ $t(X) \in \mathcal{T}$ and a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow X/t(X) \longrightarrow 0$$

such that $X/t(X) \in \mathcal{F}$.

Let $\mathcal{C}$ be a triangulated category. Following [**BBD82**] a *t-structure* $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ on $\mathcal{C}$ is a pair of full subcategories of $\mathcal{C}$ such that the following conditions are satisfied:

Define $\mathcal{C}^{\leqslant n} := \mathcal{C}^{\leqslant 0}[-n]$ and $\mathcal{C}^{\geqslant n} := \mathcal{C}^{\geqslant 0}[-n]$ for all $n \in \mathbb{N}$.

(1) $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{C}^{\leqslant 0}$ and $Y \in \mathcal{C}^{\geqslant 1}$.
(2) $\mathcal{C}^{\leqslant 0} \subset \mathcal{C}^{\leqslant 1}$ and $\mathcal{C}^{\geqslant 1} \subset \mathcal{C}^{\geqslant 0}$.
(3) For all $X \in \mathcal{C}$ there exist $X' \in \mathcal{C}^{\leqslant 0}$ and $X'' \in \mathcal{C}^{\geqslant 1}$ such that

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

is a distinguished triangle in $\mathcal{C}$.

Given a t-structure $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ on $\mathcal{C}$ we denote by $\mathcal{H}$ the full subcategory $\mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ of $\mathcal{C}$ and call it the *heart* of the t-structure.

The following is proposition 2.1. in [**HRS96**].

Proposition 2.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Define*

$$\begin{aligned} \mathcal{D}^{\leqslant 0} &= \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, i > 0, H^0(X^\bullet) \in \mathcal{T}\} \\ \mathcal{D}^{\geqslant 0} &= \{X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0, i < -1, H^{-1}(X^\bullet) \in \mathcal{F}\} \end{aligned}$$

Then $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is a t-structure on $D^b(\mathcal{A})$.

It is clear that $\mathcal{D}^{\leqslant 0}$ determines the t-structure completely as $\mathcal{D}^{\geqslant 1} := \mathcal{D}^{\geqslant 0}[-1]$ can be defined as its right orthogonal, *i.e.,*

$$\mathcal{D}^{\geqslant 1} := \{X \in D^b(\mathcal{A}) \mid \mathrm{Hom}(Y, X) = 0 \ \text{ for all } Y \in D^{\leqslant 0}\}.$$

In view of Theorem A.1 of [**ATJLSS03**] it is very easy to construct a t-structure on $D^b_{\mathrm{QCoh}}(X)$, *i.e.,* the bounded derived category of $\mathcal{O}_X$-modules with quasicoherent cohomologies, namely, by taking $\mathcal{D}^{\leqslant 0}$ to be the cocomplete full subcategory generated by some set of objects closed under extensions and the shift functor $X \to X[1]$ (but not $X \to X[-1]$). However, such t-structures rarely induce t-structures on $D^b_{\mathrm{Coh}}(X)$.

In order to give a non-trivial example of a t-structure we define a *support datum* as a decreasing sequence $\Phi := \{\Phi^n\}_{n \in \mathbb{Z}}$ of families of supports satisfying the following conditions:

- for $n \ll 0$, Φ^n is the set of all closed subsets of X,
- for $n \gg 0$, Φ^n is $\{\emptyset\}$.

The following interesting example of a t-structure on $D^b_{\mathrm{QCoh}}(\mathscr{D}_X)$, *i.e.,* the bounded derived category of $\mathscr{D}_X$-modules with quasicoherent cohomologies is due to Bezrukavnikov, Deligne and Kashiwara. In view of the discussion above we only describe $\mathcal{D}^{\leqslant 0}$.

Example 1. $\mathcal{D}^{\leqslant 0} = \{M \in D^b_{QCoh}(\mathscr{D}_X) \,|\, \mathrm{Supp}(H^k(M)) \subset \Phi^{k-n} \forall\, k\}$

More examples will appear later on in the text.

2.3. Basic facts about noncommutative tori. The noncommutative torus is a particular case of a transformation group C^*-algebra, with $\mathbb{Z}$ acting continuously on the C^*-algebra $C(\mathbb{S}^1)$ of continuous functions on the circle. Pimsner and Voiculescu [**PV80**] and separately Rieffel [**Rie81**] studied their K-theory, while Connes analysed their differential structure [**Con80**]. We will work with the smooth noncommutative torus, which is a dense Fréchet subalgebra of this transformation group C^*-algebra.

Let θ be an irrational real number. The algebra of smooth functions $\mathcal{A}_\theta$ on the noncommutative torus $\mathbb{T}_\theta$ consists of elements of the form $\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2} U_1^{n_1} U_2^{n_2}$ with $(n_1, n_2) \longrightarrow a_{n_1,n_2}$ rapidly decreasing and U_1, U_2 are unitaries satisfying the commutation relation

$$U_2 U_1 = \exp(2\pi i\theta) U_1 U_2 \tag{1}$$

A less *ad hoc* definition of $\mathcal{A}_\theta$ is given as a *smooth crossed product*. This is the smooth analogue of the aforementioned transformation group C^*-algebra. Let $C^\infty(\mathbb{S}^1)$ be the Fréchet $*$-algebra of smooth functions on the circle with the family of seminorms given by

$$\|f\|_\nu = \sup_{s\in\mathbb{S}^1} |\partial_s^\nu f(s)| \,.$$

We equip this algebra with a smooth action α of $\mathbb{Z}$ by automorphisms given by $\alpha_n(f)(s) = f(s + 2\pi n\theta)$. Take the vector space $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ of sequences on $\mathbb{Z}$ of rapid decay that take values in $C^\infty(\mathbb{S}^1)$. In other words, $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ consists of $C^\infty(\mathbb{S}^1)$-valued sequences $\{f_n\}_{n\in\mathbb{Z}}$ such that

$$\|f\|_{\nu,\mu} = \sup_n (1 + |n|^\mu)\|f_n\|_\nu,$$

is finite for all ν and μ. We introduce the following convolution product and involution on $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$,

$$\begin{aligned} (f * g)_n &= \sum_{m\in\mathbb{Z}} f_m \alpha_m (g_{n-m}) \,, \\ (f^*)_n &= \alpha_n (f^*_{-n}) \,. \end{aligned} \tag{2}$$

The Fréchet $*$-algebra $(\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1)), *, ^*)$ is denoted by $C^\infty(\mathbb{S}^1) \rtimes_\theta \mathbb{Z}$ and is called the smooth crossed product of $C^\infty(\mathbb{S}^1)$ by $\mathbb{Z}$.

It is well-known that the Fourier transform maps an element in $\mathcal{S}(\mathbb{Z})$ isomorphically to an element in $C^\infty(\mathbb{S}^1)$. Under this identification, we have the isomorphism $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1)) \simeq C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ as vector spaces.

The above convolution product on the generating unitaries U_1 and U_2 of $C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ translates to the defining relation of Eqn. (1) of $\mathcal{A}_\theta$. Hence, $\mathcal{A}_\theta \simeq C^\infty(\mathbb{S}^1) \rtimes_\theta \mathbb{Z}$.

2.4. Complex structure on $\mathcal{A}_\theta$, depending on τ. .
The two basic derivations δ_1 and δ_2 acting on $\mathcal{A}_\theta$ are as follows:
For $j = 1, 2$

$$\delta_j\left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2} U_1^{n_1} U_2^{n_2}\right) = 2\pi i \sum_{(n_1,n_2)\in\mathbb{Z}^2} n_j a_{n_1,n_2} U_1^{n_1} U_2^{n_2}.$$

Equivalently, one can define δ_1 and δ_2 by $\delta_j(U_i) = 2\pi i \delta_{ij} U_i$ which is then extended to the whole of $\mathcal{A}_\theta$ by applying the Leibniz rule.

The derivations δ_1 and δ_2 are the infinitesimal generators of the action of a commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$ by automorphisms. Inside the complexified Lie algebra generated by δ_1 and δ_2, we are interested in the vector parametrized by two complex numbers ω_1 and ω_2. We denote

$$\delta_\omega = \omega_1\delta_1 + \omega_2\delta_2. \tag{3}$$

If $\omega = (\tau, 1)$ we also set $\delta_\tau = \delta_\omega$, which is the so-called *complex structure* on $\mathcal{A}_\theta$ already present in [**CR87**].

2.5. The category of holomorphic bundles $Vec(\mathbb{T}^\tau_\theta)$. Vector bundles on $\mathcal{A}_\theta$ are by definition finitely generated projective right $\mathcal{A}_\theta$-modules. For brevity, from now on a projective module would actually mean a finitely generated and projective module. Connes in [**Con80**] has shown that up to isomorphism they are all of the form $E_{c,d}(\theta)$, $(c, d) \in \mathbb{Z}^2$. A module is called **basic** if c and d are relatively prime. Since, for (c, d) relatively prime, one has $E_{cr,dr}(\theta) \simeq E_{c,d}(\theta)^{\oplus r}$, they are the basic building blocks of all projective modules and so most of the constructions will done in term of them.

If $c \neq 0$, $E := E_{c,d}(\theta)$ is defined as the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}/m\mathbb{Z})$ equipped with the following right action of $\mathcal{A}_\theta$ [**Con80**]:

$$fU_1(x, \alpha) = f(x - \frac{d + c\theta}{c}, \alpha - 1) \tag{4}$$

$$fU_2(x, \alpha) = \exp(2\pi i(x - \frac{\alpha d}{c}))f(x, \alpha) \tag{5}$$

where $x \in \mathbb{R}$ and $\alpha \in \mathbb{Z}/c\mathbb{Z}$.

If $c = 0$, we set $E_{0,d}(\theta) = \mathcal{A}_\theta^{|d|}$. One defines the degree, the rank and the slope of E respectively as

$$\deg(E) = c \tag{6}$$

$$\mathrm{rk}(E) = d + c\theta \tag{7}$$

$$\mu(E) = \frac{c}{d + c\theta} \tag{8}$$

We introduce a few more notations. Following [**PS03**], given a basic projective module $E_{c,d}(\theta)$ we complete (c, d) to an $SL(2, \mathbb{Z})$ matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and denote it by $E_g(\theta)$. Such a choice is not unique and the choice of the upper row depends only up to an integral multiple of the lower row $(c\ d)$.

There is a left module structure of $A_{g\theta}$ ($g \in SL(2, \mathbb{Z})$ acts on $\theta \in \mathbb{R}$ by fractional linear transformation *i.e.*, $\theta \mapsto \frac{a\theta+b}{c\theta+d}$) on $E_g(\theta)$ making it into an $A_{g\theta} - \mathcal{A}_\theta$ bimodule as in [**Con80**] defined by

$$U_1 f(x, \alpha) = f(x - \frac{1}{c}, \alpha - a) \tag{9}$$

$$U_2 f(x, \alpha) = \exp(2\pi i(\frac{x}{c\theta + d} - \frac{\alpha}{c}))f(x, \alpha) \tag{10}$$

For different completions one may check from the explicit module structure formulae (4), (5),(9) and (10) that only the left module structure differs.

OBSERVATION 1. *When $g\theta = \theta$, $E_g(\theta)$ becomes an $\mathcal{A}_\theta$ bimodule, which is now called a real multiplication bimodule after Manin.*

A *holomorphic structure* on such an $E = E_{c,d}$, $c, d \in \mathbb{Z}$ is determined by a $\mathbb{C}$-linear connection $\nabla : E \longrightarrow E$ satisfying the Leibniz rule

$$\nabla(ea) = \nabla(e)a + e\delta_\tau(a), \tag{11}$$

for all $e \in E$ and $a \in \mathcal{A}_\theta$.

The objects of the category of holomorphic vector bundles $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ are $E = E_{c,d}(\theta)$ for all $c, d \in \mathbb{Z}$, which should be thought of as vector bundles over $\mathbb{T}_\theta$, each endowed with a holomorphic structure ∇.

A morphism $h : E \longrightarrow E'$ is said to be *holomorphic* if it commutes with the connection *i.e.*, $\nabla_{E'}(he) = h\nabla_E(e)$. These are the morphisms of the category.

One defines the cohomology groups $H^* = H^0$ or H^1 of $\mathcal{A}_\theta$ with respect to a holomorphic bundle E equipped with a connection ∇ as follows

$$H^0(E) = H^0(E, \nabla) = \ker(\nabla : E \longrightarrow E)$$

$$H^1(E) = H^1(E, \nabla) = \mathrm{coker}(\nabla : E \longrightarrow E)$$

There is a distinguished family of holomorphic structures on every *basic* module E, given by the choice of a complex parameter z called the **standard holomorphic structure**. Namely, for $c \neq 0$, one sets

$$\nabla_z(f) = \frac{\partial f}{\partial x} + 2\pi i(\tau\mu(E)x + z)f$$

where $f \in E$. For $c = 0$, the holomorphic structure on the trivial module $\mathcal{A}_\theta$ is defined as

$$\nabla_z(f) = 2\pi i z.f + \delta_\tau(f)$$

Definition 2.5. *A* **standard holomorphic bundle** *is a basic projective module equipped with a standard holomorphic structure, as defined above. It is denoted by* $E_g^z(\theta)$.

Besides, the Hom objects admit the structure of a holomorphic module over a noncommutative torus with a possibly different *Rieffel parameter* θ. We state a part of corollary 2.3. form [**PS03**] to illustrate the matter.

Proposition 2.6. *Let* $E = E_g^z(\theta)$ *and* $E' = E_{g'}^{z'}(\theta)$ *be a pair of basic modules equipped with holomorphic structures* ∇_z *and* $\nabla_{z'}$ *respectively. Then one has*

(1)
$$\mathrm{Hom}_{\mathcal{A}_\theta}(E_g^z(\theta), E{g'}^{z'}(\theta)) \cong E_{g'g^{-1}}^{\mathrm{rk}(g,\theta)(z'-z)}(g\theta)$$
where $\mathrm{rk}(g,\theta) = c\theta + d$ *assuming* $(c\ d)$ *is the lower row of* g.

(2) *The subspace* $H^0(\mathrm{Hom}_{\mathcal{A}_\theta}(E, E')) \subset \mathrm{Hom}_{\mathcal{A}_\theta}(E, E')$ *coincides with the subspace of holomorphic morphisms* $E \longrightarrow E'$.

We just restate propositions 2.4. and 2.5. from [**PS03**] below.

Proposition 2.7. *Let* $E = E_g^z(\theta)$ *and* $E' = E_{g'}^{z'}(\theta)$ *be a pair of basic modules equipped with standard holomorphic structures. Then there is a functorial bijection between the space* $H^1(\mathrm{Hom}_{\mathcal{A}_\theta}(E', E))$ *and* *the isomorphism classes of extensions* $0 \longrightarrow E \longrightarrow F \longrightarrow E' \longrightarrow 0$ *in the category of holomorphic bundles on* $\mathcal{A}_\theta$.

PROPOSITION 2.8. *Assume that* $\mathrm{Im}(\tau) < 0$. *Let* $E = E^z_g(\theta)$ *be a basic module equipped with a standard holomorphic structure* ∇_z.
1. *If* $\mu(E) > 0$ *then* $\mathrm{H}^1(E) = 0$ *and* $\mathrm{H}^0(E)$ *has dimension* $\deg(E)$.
2. *If* $\mu(E) < 0$ *then* $\mathrm{H}^0(E) = 0$ *and* $\mathrm{H}^1(E)$ *has dimension* $\deg(E)$.
3. *If* $z \neq 0$ *then* $\mathrm{H}^*(\mathcal{A}_\theta, \nabla_z) = 0$
4. *The spaces* $\mathrm{H}^0(\mathcal{A}_\theta, \nabla_0)$ *and* $\mathrm{H}^1(\mathcal{A}_\theta, \nabla_0)$ *are* 1*-dimensional.*

Some more properties of the category of holomorphic vector bundles are known *viz.,* tensor products, pull-backs and push-forwards, which we do not discuss here. Rather we prove that $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ has a certain *modularity* property. Let us denote by $\mathsf{Vect}(\mathbb{T}^\omega_\theta)$ the category of holomorphic vector bundles on $\mathbb{T}_\theta$ equipped with the complex structure δ_ω (see Eqn. (3)).

PROPOSITION 2.9. (a) *If* g *is an element in* $\mathsf{SL}(2,\mathbb{Z})$, *then* $\mathsf{Vect}(\mathbb{T}^{g\omega}_\theta) \simeq \mathsf{Vect}(\mathbb{T}^\omega_\theta)$.
(b) *If* $\omega_2 \neq 0$ *and* $\tau = \frac{\omega_1}{\omega_2}$, *then* $\mathsf{Vect}(\mathbb{T}^\omega_\theta) \simeq \mathsf{Vect}(\mathbb{T}^\tau_\theta)$.

PROOF. *(a)* Given a $g \in \mathsf{SL}(2,\mathbb{Z})$, we construct a $*$-automorphism σ of $\mathcal{A}_\theta$ such that $\sigma^{-1}\delta_\omega\sigma = \delta_{g\omega}$. Evidently, it is enough to do this for the generators of $\mathsf{SL}(2,\mathbb{Z})$, *i.e.,* $g_1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. For g_1, $\delta_{g_1\omega} = (\omega_1+\omega_2)\delta_1+\delta_2$. We define $\sigma_1 : \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$ as $\sigma_1(U_1) = U_1U_2$, $\sigma_1(U_2) = U_2$. One may easily check that $\sigma_1(U_1)$ and $\sigma_1(U_2)$ satisfy the commutation relation of $\mathcal{A}_\theta$ as in Eqn. (1) and also that

$$\begin{aligned}
\sigma_1^{-1}\delta_\omega\sigma_1(U_1) &= \sigma_1^{-1}\delta_\omega(U_1U_2)\\
&= \sigma_1^{-1}(\omega_1\delta_1 + \omega_2\delta_2)(U_1U_2)\\
&= \sigma_1^{-1}(2\pi i\omega_1U_1U_2 + 2\pi i\omega_2U_1U_2)\\
&= 2\pi i(\omega_1 + \omega_2)\sigma_1^{-1}(U_1U_2)\\
&= 2\pi i(\omega_1 + \omega_2)U_1\\
&= ((\omega_1 + \omega_2)\delta_1 + \delta_2)U_1\\
&= \delta_{g_1\omega}(U_1).
\end{aligned}$$

Similarly, for U_2 one may check that the actions of δ_ω and $\delta_{g_1\omega}$ agree. For g_2, $\delta_{g_2\omega} = -\omega_2\delta_1 + \omega_1\delta_2$ and we define $\sigma_2(U_1) = U_2^{-1}$, $\sigma_2(U_2) = U_1$. Once again one can easily check that the new generators satisfy Eqn. (1) and that the actions of δ_ω and $\delta_{g_2\omega}$ agree on U_1 and U_2. Explicitly, the functor sends $(\mathcal{A}_\theta, \delta_\omega)$ to $(\mathcal{A}_\theta, \delta_{g_i\omega})$, $i = 1, 2$, and twists the module structure by σ_i, $i = 1, 2$, *i.e.,* $e \cdot a := e\sigma_i(a)$, $i = 1, 2$ and

$e \in E$. One verifies that ∇ on E is compatible with $\delta_{g_i\omega}$, $i = 1, 2$, with respect to the twisted module structure. Indeed,

$$\begin{aligned} \nabla(e \cdot a) &= \nabla(e\sigma_i(a)) \\ &= \nabla(e)\sigma_i(a) + e\delta_\omega(\sigma_i(a)) \\ &= \nabla(e)a + e\sigma_i(\delta_{g_i\omega}(a)) \\ &= \nabla(e) \cdot a + e \cdot \delta_{g_i\omega}(a) \end{aligned}$$

where $e \in E$, $a \in \mathcal{A}_\theta$ and $i = 1, 2$.

(b) In our notation, $\delta_\tau = \frac{\delta_\omega}{\omega_2}$. Sending each ∇ to $\nabla' := \frac{\nabla}{\omega_2}$ makes ∇' automatically compatible with δ_τ. More precisely, the functor sends $(\mathcal{A}_\theta, \delta_\omega)$ to $(\mathcal{A}_\theta, \delta_\tau)$ and (E, ∇) to (E, ∇'). □

2.6. The derived category of holomorphic bundles. The idea is to define the derived category as the zeroth cohomology category of a DG-category (or differential graded category), *i.e.,* a category in which the Hom sets have the structure of a differential complex of $\mathbb{C}$-vector spaces. The objects of the DG-category $\mathcal{C} = \mathcal{C}(\theta, \tau)$ are of the form $E[n]$, where $E \in \mathsf{Vec}(\mathbb{T}_\theta)$, such that $\mathrm{rk}(E) > 0$, [1] equipped with a holomorphic structure (compatible with δ_τ) and n is an integer indicating the formal grading. For brevity, $E[0]$ is denoted simply by E. The morphisms are defined as

(12)
$$\mathrm{Hom}^\bullet_{\mathcal{C}}(E[n], E'[n']) = \Big(\mathrm{Hom}_{\mathcal{A}_\theta}(E, E') \xrightarrow{\partial} \mathrm{Hom}_{\mathcal{A}_\theta}(E, E')\Big)[n' - n]$$

where $\partial(f)$, for any $f \in \mathrm{Hom}(E, E')$ is the map $e \mapsto \nabla(f(e)) - f(\nabla(e))$ (see Proposition 2.6 for the definition of ∇ on $\mathrm{Hom}(E, E')$). The composition of morphisms is defined in the obvious manner.

A holomorphic vector bundle on a complex manifold can be viewed as a smooth vector bundle with an action of the $\overline{\partial}$ operator. Regarding δ_τ as a replacement for $\overline{\partial}$, the category $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ is the category of holomorphic bundles with elements in $\mathsf{Vec}(\mathbb{T}_\theta)$ admitting a δ_τ-equivariance condition. The Hom and the Ext groups in this equivariant category are computed by the cohomologies of the following complex

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{A}_\theta}(E, E') \xrightarrow{\partial} \mathrm{Hom}_{\mathcal{A}_\theta}(E, E') \\ &\qquad \partial f(e) \mapsto \nabla_{E'}(f(e)) - f(\nabla_E(e)). \end{aligned}$$

[1] The assumption on the positivity of rank is made to pick only one of $E_{c,d}$ and $E_{-c,-d}$, which can easily be seen to be isomorphic.

It is clear that the kernel of ∂ (or H^0 of the complex) is precisely the set of all holomorphic morphisms between E and E' (see Proposition 2.6), while the cokernel (or the H^1) computes the extensions in the holomorphic category (see Proposition 2.7).

The reader can find an elaborate discussion on DG categories in subsection 4.2. Nevertheless, for the benefit of the reader let us recall that given a DG-category $\mathcal{C}$, its zeroth cohomology category, denoted by $\mathrm{H}^0\mathcal{C}$, consists of the same objects as in $\mathcal{C}$ and for any two $A, B \in \mathrm{H}^0\mathcal{C}$ one sets $\mathrm{Hom}_{\mathrm{H}^0\mathcal{C}}(A, B) = \mathrm{H}^0\mathrm{Hom}^\bullet_{\mathcal{C}}(A, B)$. Let $\mathcal{C}(X_\tau)$, where $X_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, denote the category whose objects are $F[n]$, where $F \in \mathrm{Coh}(X_\tau), n \in \mathbb{Z}$. The morphisms between $F[n]$ and $F'[n']$ in this category are elements of the shifted Dolbeault complex $F^\vee \otimes F'[n'-n]$.

Making the long story short, one constructs a functor $\mathcal{C}(\theta, \tau) \to \mathcal{C}(X_\tau)$ and shows that the induced functor on the cohomology categories is fully faithful, with the image of $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ lying in the heart of a certain t-structure corresponding to θ described below. Then one shows that this functor actually induces an equivalence between $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ and the heart (see [**Pol04a**]), whose derived category is again equivalent to $D^b(X_\tau)$ (see [**Pol05**]). This implies that $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ is abelian and its derived category is equivalent to $D^b(X_\tau)$. Hence, the derived category of holomorphic bundles actually constitutes a full triangulated subcategory of the derived category and the Polishchuk-Schwarz functor, denoted by $\mathcal{S}_\tau : \mathrm{H}^0\mathcal{C}(\theta, \tau) \longrightarrow D^b(X_\tau)$ is exact.

We denote by $\mathcal{C}^{st} = \mathcal{C}^{st}(\theta, \tau)$ the full subcategory of $\mathcal{C}$ consisting of objects $E[n]$, where E is now a standard holomorphic bundle. The image under $\mathcal{S}_\tau$ of such an object is a *stable* object in $D^b(X_\tau)$, which is by definition an object of the form $V[n]$, where V is either a stable vector bundle or a coherent torsion sheaf supported at a point.

2.7. The Polishchuk-Schwarz functor $\mathcal{S}_\tau$. We briefly sketch in steps the salient features of the construction of the Polishchuk-Schwarz functor $\mathcal{S}_\tau$.

1. For $E \in \mathsf{Vec}(\mathbb{T}^\tau_\theta)$ one constructs a sheaf of E-valued sections on $\mathbb{C}$ by $E_\mathbb{C} = \mathcal{O}(\mathbb{C}) \otimes E$.

2. Define $t_\nu : \mathbb{C} \longrightarrow \mathbb{C}$ by $z \mapsto z + \nu$, where $\nu = 1$ or τ. One defines two sheaf isomorphisms ρ_ν, $\nu = 1$ or τ, as

$$\begin{aligned} \rho_\nu : & t_\nu^* E_\mathbb{C} \xrightarrow{\sim} E_\mathbb{C} \\ & \rho_\nu(f)(z) = f(z+\nu)U^\nu \end{aligned}$$

where $U^1 = U_1$, $U^\tau = U_2$.

3. In order to descend to a sheaf on $X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ one needs the ρ_γ's to satisfy a further compatibility condition (or cocycle condition). Upon replacing ρ_τ by $\rho'_\tau = \exp(-2\pi i\theta z)\rho_\tau$ one obtains

$$\rho_1 \circ t_1^*\rho'_\tau = \rho'_\tau \circ t_\tau^*\rho_1$$

which is precisely the required cocycle condition. Now $E_\mathbb{C}$ becomes a $(\mathbb{Z} + \tau\mathbb{Z})$ equivariant coherent sheaf on $\mathbb{C}$ and therefore, descends to a coherent sheaf on X_τ, which we denote by E_{X_τ}.

4. The holomorphic structure ∇ on E descends to a differential of $\mathcal{O}_{X_\tau}$-modules

$$E_{X_\tau} \xrightarrow{d_\nabla} E_{X_\tau}$$
$$d_\nabla(f)(z) \mapsto \nabla(f(z)) + 2\pi i z f(z)$$

The functor is defined as $\mathcal{S}_\tau(E) = E_{X_\tau} \xrightarrow{d_\nabla} E_{X_\tau} \in D^b(X_\tau)$.

5. If $E = E_g^0(\theta)$ is a standard holomorphic bundle such that $\mu(E) > 0$ (resp. $\mu(E) < 0$) then $\mathcal{S}_\tau(E)$ is quasi-isomorphic to $V_{c,d}$ the kernel (resp. cokernel) of d_∇, which turns out to be a stable vector bundle of degree c and rank d on X_τ. We denote them by $V_{c,d}$ (resp. $V_{c,d}[-1]$ as it sits in degree -1), where (c, d) is the bottom row of g. $\mathcal{S}_\tau(\mathcal{A}_\theta^z)$ is quasi-isomorphic to $\mathcal{O}_{X_\tau,z}[-1]$. For the rest, one uses the following properties:

$$E_g^z(\theta) \simeq E_g^0(\theta) \otimes \mathcal{A}_\theta^z \text{ and } \mathcal{S}_\tau(E \otimes A_\theta^z) \simeq t_z^*\mathcal{S}_\tau(E)$$

6. One extends this functor to the whole DG-category by requiring it to commute with coproducts and translations to obtain the following Theorem:

THEOREM 2.10 (Polishchuk-Schwarz). *Assume that* θ *is irrational. Then the functor* $E \mapsto \mathcal{S}_\tau(E)$ *extends to an equivalence of* $H^0\mathcal{C}^{st}(\theta, \tau)$ *with the full subcategory of* $D^b(X_\tau)$ *consisting of stable objects.*

2.8. t-structures on $\mathsf{D}^b(X_\tau)$ depending on θ. First we construct a torsion pair $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leqslant\theta})$ in $\mathrm{Coh}(X_\tau)$ following [**Pol04b**], which would automatically define a t-structure on $\mathsf{D}^b(X_\tau)$, as described in 2.2.

$$\begin{aligned}\mathrm{Coh}_{>\theta} &= \{F \in \mathrm{Coh}(X_\tau) \mid \text{all semi-stable factors of } F \text{ have slope} > \theta\}\\ \mathrm{Coh}_{\leqslant\theta} &= \{F \in \mathrm{Coh}(X_\tau) \mid \text{all semi-stable factors of } F \text{ have slope} \leqslant \theta\}\end{aligned}$$

Note that *torsion sheaves*, having slope $= \infty$, belong to $\mathrm{Coh}_{>\theta}$. The associated t-structure is given by

$$\begin{aligned}\mathsf{D}^{\theta,\leqslant 0} &= \{K^\bullet \in \mathsf{D}^b(X_\tau) \mid \mathrm{H}^{>0}(K^\bullet) = 0, \mathrm{H}^0(K^\bullet) \in \mathrm{Coh}_{>\theta}\}\\ \mathsf{D}^{\theta,\geqslant 0} &= \{K^\bullet \in \mathsf{D}^b(X_\tau) \mid \mathrm{H}^{<-1}(K^\bullet) = 0, \mathrm{H}^{-1}(K^\bullet) \in \mathrm{Coh}_{\leqslant\theta}\}\end{aligned}$$

Let $\mathcal{C}^{\theta,\tau} := \mathsf{D}^{\theta,\leqslant 0} \cap \mathsf{D}^{\theta,\geqslant 0}$ be the heart of the t-structure. In the language of Polishchuk and Schwarz this whole process is described by

> "..... cut $\mathrm{Coh}(X_\tau)$ into two subcategories generated by stable bundles of slopes $< \theta$ and $> \theta$ respectively (we assume that θ is irrational) and then reassemble these subcategories in a different way into a new abelian category."

It is a bit unfortunate, that we use almost the same notation for the derived category of holomorphic bundles on $\mathbb{T}^\tau_\theta$, denoted by $\mathcal{C}(\theta,\tau)$, and the heart of the t-structure $\mathcal{C}^{\theta,\tau}$.

The family of t-structures is extended to $\theta = \infty$ by putting the standard t-structure on $\mathsf{D}^b(X_\tau)$, whose heart is just $\mathrm{Coh}(X_\tau)$. It is shown in [**Pol04b**] that $\mathcal{C}^{\theta,\tau}$ has cohomological dimension 1 and that it is derived equivalent to $\mathrm{Coh}(X_\tau)$. We provide an alternative argument for the derived equivalence.

Proposition 2.11. *The category $\mathcal{C}^{\theta,\tau}$ is equivalent to $\mathsf{D}^b(X_\tau)$ via $\mathcal{S}_\tau$.*

Proof. It is known that if a torsion pair $(\mathcal{T},\mathcal{F})$ in an abelian category $\mathcal{A}$ is *cotilting*, *i.e.*, every object of $\mathcal{A}$ is a quotient of an object in $\mathcal{F}$, then the heart of the t-structure induced by the torsion pair is derived equivalent to $\mathcal{A}$ (see, for instance, Proposition 5.4.3 and the Remark thereafter in [**Bv03**]). Thus, it is enough to check that the torsion pair $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leqslant\theta})$ is cotilting. Just for the sake of completeness we provide an easy argument below.

Given any $\mathscr{F} \in \mathrm{Coh}(X_\tau)$ we need to produce an object in $\mathrm{Coh}_{\leqslant\theta}$ which surjects onto $\mathscr{F}$. Let $\mathscr{L}$ be an ample line bundle on X_τ, *i.e.*,

$\deg(\mathscr{L}) > 0$. By Serre's theorem one may twist $\mathscr{F}$ by a large enough power of $\mathscr{L}$ such that it becomes generated by global sections, *i.e.,* the quotient of a free sheaf. In other words, there exists $N > 0$ large enough such that for all $n > N$ there is an epimorphism $\oplus_{i\in I}\mathcal{O}_{X_\tau} \longrightarrow \mathscr{F} \otimes \mathscr{L}^n$, I finite. One may twist it back to obtain an epimorphism $\oplus_{i\in I}\check{\mathscr{L}}^n \longrightarrow \mathscr{F}$, where $\check{\mathscr{L}}$ is the dual line bundle. This shows that there exists an epimorphism onto $\mathscr{F}$ from a finite direct sum of copies of $\check{\mathscr{L}}^n$. Since $\deg(\check{\mathscr{L}}^n) = -n.\deg(\mathscr{L}) < 0$ it is possible to make the slope of $\check{\mathscr{L}}^n$, which is equal to $\deg(\check{\mathscr{L}}^n)$, less than θ by choosing a large enough n. Being a line bundle $\check{\mathscr{L}}^n$ is clearly semistable and we observe that the direct sum of copies of $\check{\mathscr{L}}^n$ lies in $\mathrm{Coh}_{\leqslant\theta}$.

□

THEOREM 2.12 ([**Pol04a**]). *The functor $\mathcal{S}_\tau$ induces an equivalence between $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ and $\mathcal{C}^{-\theta^{-1},\tau}$.*

REMARK 2.13. The equivalence defined in [**PS03**] between the derived category of holomorphic bundles on $\mathcal{A}_\theta$ and $D^b(X_\tau)$ sends $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ to $\mathcal{C}^{-\theta^{-1},\tau}$ up to some shift, which we ignore, and the real multiplication on $\mathcal{A}_\theta$ descends to an element $F \in \mathrm{Aut}(D^b(X_\tau))$, which preserves $\mathcal{C}^{-\theta^{-1},\tau}$. This does not distort the picture as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\theta = -\theta^{-1}$, which says that $A_{-\theta^{-1}}$ is Morita equivalent to $\mathcal{A}_\theta$.

Summarizing, one has

$$\mathsf{Vec}(\mathbb{T}^\tau_\theta) \cong \mathsf{Vec}(\mathbb{T}^\tau_{-\theta^{-1}}) \cong \mathcal{C}^{\theta,\tau} \text{ and } D^b(\mathcal{C}^{\theta,\tau}) \cong D^b(X_\tau) \tag{13}$$

.

For $Y = \mathbb{T}^\tau_\theta$ or X_τ let us denote by $\mathsf{Qcoh}(Y)$ and $D^b(\mathsf{Qcoh}(Y))$ the abelian category of quasicoherent sheaves on Y and the bounded derived category thereof respectively. *Polishchuk* defines $\mathsf{Qcoh}(\mathbb{T}^\tau_\theta)$ as the inductive completion of $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$ (*i.e.,* $\mathsf{Ind}(\mathsf{Vec}(\mathbb{T}^\tau_\theta))$) in [**Pol07**] and it is shown that there is an equivalence of derived categories between $D^b(\mathrm{QCoh}(\mathbb{T}^\theta_\tau))$ and $D^b(\mathrm{QCoh}(X_\tau))$ extending the equivalence between $D^b(\mathsf{Vec}(\mathbb{T}^\theta_\tau))$ and $D^b(X_\tau)$ (see section 2.4. in *loc. cit.*). Hence, similar to (13) one has

$$\mathrm{QCoh}(\mathbb{T}^\tau_\theta) \cong \mathrm{QCoh}(\mathbb{T}^\tau_{-\theta^{-1}}) \cong \mathsf{Ind}(\mathcal{C}^{\theta,\tau}) \tag{14}$$

and

$$\mathsf{D}^b(\mathsf{Ind}(\mathcal{C}^{\theta,\tau})) \cong \mathsf{D}^b(\mathrm{QCoh}(\mathsf{X}_\tau)). \tag{15}$$

Remark 2.14. Manin explains the work of Polishchuk and Schwarz as the agreement, in some sense, of a double quotient operation in two different orders (see Section 4 of [**Man06**]). More precisely, consider $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}+\theta\mathbb{Z})$, $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and $\tau \in \mathbb{H}^-$. This space can be regarded as $\mathbb{C}/(\mathbb{Z}+\theta\mathbb{Z})$, which is $\mathbb{T}_\theta$ modulo an "infinitesimal action" of δ_τ on vector bundles over $\mathbb{T}_\theta$, described by $\mathsf{Vec}(\mathbb{T}^\tau_\theta)$. Taking the quotient in the other order, one obtains $\mathsf{X}_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with an "action" of $\theta\mathbb{Z}$ on it which is manifested by putting a t-structure on $\mathsf{D}^b(\mathsf{X}_\tau)$, depending on θ and taking the corresponding heart $\mathcal{C}^{\theta,\tau}$. Manin interprets the equivalence $\mathsf{Vec}(\mathbb{T}^\tau_\theta) \cong \mathcal{C}^{\theta,\tau}$ as the agreement of the two quotient spaces by looking at the category of representations of the two objects.

2.9. Noncommutative elliptic curves. Let X be any complex analytic space and G be a discrete group acting on X, such that the quotient space X/G exists as a compact complex analytic manifold. Let us denote the canonical quotient map by $\pi : \mathsf{X} \longrightarrow \mathsf{X}/\mathsf{G}$. Let us denote by $\mathsf{Sh}(\mathsf{X})$ the category of analytic sheaves on X. Then the following result is well-known.

Proposition 2.15. *Let* $\mathsf{Sh}_\mathsf{G}(\mathsf{X})$ *denote the category of* G*-equivariant sheaves on* X. *Then the functor* $\pi^* : \mathsf{Sh}(\mathsf{X}/\mathsf{G}) \longrightarrow \mathsf{Sh}_\mathsf{G}(\mathsf{X})$ *is an equivalence of categories.*

Since the pull back of an $\mathcal{O}_{\mathsf{X}/\mathsf{G}}$-coherent sheaf is $\mathcal{O}_\mathsf{X}$-coherent in the analytic category, the same functor induces an equivalence between $\mathrm{Coh}(\mathsf{X}/\mathsf{G})$ and $\mathrm{Coh}_\mathsf{G}(\mathsf{X})$.

Let X be a Stein space over $\mathbb{C}$ and G be a group acting freely on X. Assume that X/G is a projective algebraic variety such that the pullback $\pi^*(\mathcal{O}_{\mathsf{X}/\mathsf{G}}(1))$ is a trivial analytic line bundle on X. Let $\mathcal{O}(\mathsf{X},\mathsf{G}) := \mathcal{O}(\mathsf{X}) \rtimes \mathsf{G}$, where $\mathcal{O}(\mathsf{X})$ is the algebra of global analytic functions on X. One introduces another category $\mathcal{B}_\mathsf{G}(\mathsf{X})$, which is the category of right $\mathcal{O}(\mathsf{X},\mathsf{G})$-modules such that they admit a finite presentation over $\mathcal{O}(\mathsf{X})$. One first pulls back via the analytic quotient map to obtain an equivalence between G-equivariant coherent sheaves on X and $\mathrm{Coh}(\mathsf{X}/\mathsf{G})$. Then one tries to describe the image of $\mathrm{Coh}_\mathsf{G}(\mathsf{X})$ under the global sections functor. In other words, let $\Phi = \Gamma_\mathsf{X} \circ \pi^*$.

Theorem 2.16 (Soibelman-Vologodsky). *The functor* Φ *is an equivalence between* $\mathrm{Coh}(\mathsf{X}/\mathsf{G})$ *and* $\mathcal{B}_\mathsf{G}(\mathsf{X})$.

REMARK 2.17. One observes that arbitrary sheaves of modules are described by $\mathcal{O}(X, G)$-modules and the finite presentability condition over $\mathcal{O}(X)$ filters out the coherent ones.

When the quotient X/G does not exist as an honest space due to a bad group action the category of G equivariant sheaves on X still makes sense and can be taken as a definition of "Coh(X/G)". This is the key idea behind the definition of noncommutative elliptic curves denoted by $\mathcal{B}_q$ (explained below) by Soibelman and Vologodsky in [**SV03**]. This definition is in the spirit of noncommutative algebraic geometry, where one typically regards an abelian category as the category of coherent (or quasicoherent) sheaves on the would-be space. The justification being certain reconstruction results due to Gabriel (in the noetherian case [**Gab62**]) and Rosenberg (in general [**Ros98**]). However, their proofs are different.

Set $X = \mathbb{C}^*$ and $G = q^{\mathbb{Z}}$, where the action is given by dilations $z \mapsto q^n z$, $n \in \mathbb{Z}$. Henceforth, q is related to θ as in the previous section by $q = \exp(2\pi i\theta)$. One observes that θ being rational translates to q not being a root of unity. When $|q| \neq 1$ the quotient space $\mathbb{C}^*/q^{\mathbb{Z}}$ is an ordinary elliptic curve over the complex numbers (obtained by the Jacobi uniformization) and the quotient map $\pi : \mathbb{C}^* \longrightarrow \mathbb{C}^*/q^{\mathbb{Z}}$ is a map in the category of complex analytic spaces. When $|q| = 1$ and q is a root of unity one still obtains a quotient space which is isomorphic to $\mathbb{C}^*$ as a complex analytic space. However, when $|q| = 1$ and q is not a root of unity the quotient is not a locally compact Hausdorff (one cannot write down enough functions separating points) space. Then by the discussion above $\mathcal{B}_{q^{\mathbb{Z}}}(\mathbb{C}^*)$ (denoted by $\mathcal{B}_q$ for brevity) defines the noncommutative quotient space in terms of its category of coherent sheaves. We introduce some more notation. Let $\mathcal{O}(\mathbb{C}^*)$ be the algebra of global analytic functions on $\mathbb{C}^*$. We spell out the definition of a noncommutative elliptic curve after Soibelman and Vologodsky. It is defined for all q. When $|q| = 1$ and q is not a root of unity it is truly a noncommutative space, *i.e.,* it has no classical counterpart.

DEFINITION 2.18. *A noncommutative elliptic curve is defined by the category* $\mathcal{B}_q$*, i.e., the category of* $\mathcal{O}(\mathbb{C}^*) \rtimes_q \mathbb{Z}$*-modules, which admit a finite presentation over* $\mathcal{O}(\mathbb{C}^*)$*.*

We recall some known properties of $\mathcal{B}_q$ from [**SV03**].

LEMMA 2.19. (1) *For any* $V \in \mathrm{Ob}(\mathcal{B}_q)$ *the corresponding coherent sheaf on* $\mathbb{C}^*$*,* $\mathcal{O}_{\mathbb{C}^*} \otimes V$ *is free of finite rank.*
(2) *The category* $\mathcal{B}_q$ *is abelian for all* q*, such that* $|q| = 1$ *and* q *not a root of unity.*

(3) *The category $\mathcal{B}_q$ is a $\mathbb{C}$-linear symmetric monoidal category.*

2.10. Categorical Riemann–Hilbert correspondence. Classically the Riemann–Hilbert problem asks whether any finite dimensional $\mathbb{C}$-linear representation of the fundamental group of the punctured sphere $\mathbb{P}^1_{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ is obtained as the monodromy datum of a system of linear differential equations. For noncompact Riemann surfaces the answer is positive and the solution to the problem is the Riemann–Hilbert correspondence. It can be neatly phrased as an equivalence of two categories, one of them being that of finite dimensional $\mathbb{C}$-linear representations of a noncompact Riemann surface.

We are going to reformulate a system of linear differential equations in the language of algebraic geometry in terms of locally free sheaves and integrable connections. Let X be a smooth (connected) curve over $\mathbb{C}$ and $\mathscr{F} \in \mathrm{Coh}(X)$.

Definition 2.20. *A connection on $\mathscr{F}$ is a morphism of sheaves $\nabla : \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \Omega^1_X$ satisfying the Leibniz rule $\nabla(sf) = \nabla(s)f + s \otimes d_X(f)$ for all local sections s of $\mathscr{F}$ and f of $\mathcal{O}_X$.*

Remark 2.21. It can be checked easily that a coherent sheaf admitting a connection must be locally free.

It is known that given any pair $(\mathscr{F}, \nabla)$ there is a unique morphism of sheaves $\nabla^p : \mathscr{F} \otimes \Omega^p_X \longrightarrow \mathscr{F} \otimes \Omega^{p+1}_X$ satisfying $\nabla(s \otimes \omega) = (-1)^p \nabla(s) \wedge \omega + s \otimes d\omega$ for each $p \geqslant 0$ ($\nabla^0 = \nabla$), and one has

$$\nabla^{p+q}(s \otimes (\omega_p \wedge \omega_q)) = (-1)^p \omega_p \wedge \nabla^q(\omega_q \otimes s) + \nabla^p(\omega_p \otimes s) \wedge \omega_q$$

for local sections $\omega_p \in \Omega^p_X$ and $\omega_q \in \Omega^q_X$.
The composites $\nabla^{p+1} \circ \nabla^p$ vanish for all p if and only if $\nabla^1 \circ \nabla^0 = 0$. A connection ∇ is called *integrable* if $\nabla^1 \circ \nabla^0 = 0$ and the resulting complex $(\mathscr{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_X, \nabla^\bullet)$ is the de Rham complex of $(\mathscr{F}, \nabla)$.

The pairs $(\mathscr{F}, \nabla)$ over X naturally form a category with morphisms respecting the connections. It is manifestly additive and has a canonical tensor structure. We now explain the relation between a pair $(\mathscr{F}, \nabla)$ and a system of linear differential equations. Henceforth, for brevity, we write $\omega \otimes s$ as ωs.

We choose an affine open $U \subset X$ such that $\mathscr{F}|_U \simeq \mathcal{O}_X(U)^n$. Then we get

$$\nabla|_U : \mathcal{O}_X(U)^n \longrightarrow \Omega^1_X(U) \otimes_{\mathcal{O}_X} \mathcal{O}_X(U)^n = \mathcal{O}_X(U)^n dz.$$

With respect to the canonical $\mathcal{O}_X(U)$-basis $\{e_i\}$ of $\mathcal{O}_X(U)^n$ we write

$$\nabla(e_i) = -\sum_j \Gamma_{ij} e_j dz,$$

where $\Gamma_{ij} \in \mathcal{O}_X(U)$. Writing an arbitrary section $s \in \mathcal{O}_X(U)^n$ as $s = \sum_i e_i s_i$ we obtain

$$\nabla(s) = \sum_i \nabla(e_i s_i) = \sum_i (\nabla(e_i)s_i + e_i ds_i)$$

. Setting $\nabla s = 0$ and simplifying the above equation a little we obtain

$$\frac{ds_i}{dz} = \sum_j \Gamma_{ij} s_j \quad (i = 1, \ldots, n),$$

i.e., a system of linear differential equations $s' = \Gamma s$, where $\Gamma = (\Gamma_{ij}) \in \mathcal{O}_X(U)^{n \times n}$. The matrix Γ is called the *connection matrix* with respect to the chosen basis.

Given any pair $(\mathscr{F}, \nabla)$ and a fixed base point $x \in X$, $(\ker \nabla)_x$ carries an action of the fundamental group $\pi_1(X, x)$ via monodromy and hence becomes a $\pi_1(X, x)$-module. On the other hand to any representation V of the fundamental group we associate the pair $(\tilde{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X, 1 \otimes d_X)$, where $\tilde{V}$ is the vector bundle associated to the principal $\pi_1(X, x)$-bundle $\tilde{X}$ (universal cover) given by $\tilde{V} = \tilde{X} \times_{\pi_1(X,x)} V$.

Theorem 2.22 (Riemann–Hilbert correspondence). *The functors* $(\mathscr{F}, \nabla) \longmapsto (\ker \nabla)_x$ *and* $V \longmapsto (\tilde{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X, 1 \otimes d_X)$ *are inverse equivalences between the category of vector bundles on* X *equipped with an integrable connection and the category of finite dimensional representations of the fundamental group of* X.

Let X be as before and Y be a divisor, *i.e.,* a finite set of points. Set $X^* = X \setminus Y$ and $j : X^* \to X$. Let $\mathscr{F}$ be a vector bundle on $X \setminus Y$ with a connection ∇. Since for curves integrability of the connection is automatically true, we will not mention it further. For higher dimensions one needs to carry this assumption. One can ask whether $(\mathscr{F}^*, \nabla^*)$ on X^* extends to a pair $(\mathscr{F}, \nabla)$ on X with ∇ possibly being meromorphic along Y. The answer turns out be yes and to explain the details we need to introduce a definition.

Definition 2.23. *Let* $(\mathscr{F}^*, \nabla^*)$ *be as above. This pair has regular singularities along* Y *if there exists a vector bundle* $\mathscr{F} \subset j_*\mathscr{F}^*$ *on* X *such that* $j_*(\nabla^*)$ *carries* $\mathscr{F}$ *into* $\Omega^1(Y) \otimes \mathscr{F}$.

where $\Omega^1(\mathsf{Y})$ stands for the sheaf of meromorphic differentials on X (holomorphic on X^* with logarithmic poles along Y).

Concretely, the definition says that for each $y \in \mathsf{Y}$ there exists a basis of $\mathscr{F}^*$ over a punctured open neighbourhood of y such that the connection matrix with respect to that chosen basis has entries that are meromorphic at y with at worst a simple pole. It is known (see [**Del70**]) that any pair $(\mathscr{F}^*, \nabla^*)$ on X^* admits an extension $(\mathscr{F}, \nabla)$ on X with regular singularities along Y. It is unique in the following *meromorphic* sense.

PROPOSITION 2.24. *Let $\mathcal{O}_{\mathsf{X}}[\mathsf{Y}]$ be the sheaf of meromorphic functions on X, with poles along Y. Then, a coherent $\mathcal{O}_{\mathsf{X}}[\mathsf{Y}]$-module $\mathscr{F}$ with support on Y vanishes.*

PROOF. Indeed, let ϕ be a local equation of Y and $\mathscr{F}^*$ be a coherent $\mathcal{O}_{\mathsf{X}}$-module such that $\mathscr{F} = \mathscr{F}^* \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{O}_{\mathsf{X}}[\mathsf{Y}]$. Then by Nullstellensatz, one has $\phi^p \mathscr{F}^* = 0$ for some p, whence $\mathscr{F} = 0$. □

.

As a consequence one obtains the following:

THEOREM 2.25. *The category of vector bundles on $\mathbb{C}^*$ with regular singular connections (at 0) is equivalent to the category of finite dimensional representations of $\pi_1(\mathbb{C}^*, z) \simeq \mathbb{Z}$.*

REMARK 2.26. A more sophisticated version of the Riemann-Hilbert correspondence, which developed from the works of Beĭlinson, Bernstein, Deligne, Kashiwara, Kawai, amongst others, asserts an equivalence of triangulated categories $\mathscr{R}\mathscr{H}\mathrm{om}_{\mathscr{D}_{\mathsf{X}}}(-, \mathcal{O}_{\mathsf{X}^{an}} : \mathsf{D}^b_{rh}(\mathscr{D}_{\mathsf{X}}) \tilde{\to} \mathsf{D}^b_{con}(\mathbb{C}_{\mathsf{X}^{an}})^{op}$ sending regular holonomic $\mathscr{D}$-modules (elements of the heart of the standard t-structure on $\mathsf{D}^b_{rh}(\mathscr{D}_{\mathsf{X}})$) to perverse sheaves of middle perversity (elements of some special *perverse* t-structures considered in [**BBD82**]). We shall see this formulation once again in the final section.

3. Equivariant coherent sheaves and $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$

This section is an excerpt taken from the authors joint work with W. D. van Suijlekom [**Mv**].

Let us now describe another way of obtaining a quotient, based on the observation that there is an honest action of $\theta\mathbb{Z}$ on X_τ and hence on Coh(X_τ). Indeed, the point $\theta \bmod (\mathbb{Z}+\tau\mathbb{Z})$ on X_τ lies on the real axis of the fundamental domain of the torus and its action is restricted to the circle obtained by folding this axis. In fact, the action given by translations of θ on X_τ transforms to the action of multiplication by powers of $q = e^{2\pi i\theta}$ under the Jacobi uniformization, *i.e.,* $z \longmapsto qz$ on $\mathbb{C}^*/\tilde{q}^{\mathbb{Z}}$, $\tilde{q} = e^{2\pi i\tau}$. Once again we are confronted with a double quotient problem, where the actions commute. Namely, it is the improper action of the group $q^{\mathbb{Z}}$ on X_τ, which is itself obtained by the free and proper action of the group $\tilde{q}^{\mathbb{Z}}$ on $\mathbb{C}^*$ (both actions are by multiplication). The *quotient space* $\mathbb{C}^*/q^{\mathbb{Z}}$ is described by the noncommutative elliptic curves $\mathcal{B}_q$. In the formal case when $|q| < 1$ analogues of such objects have been investigated in [**BG96**]. The category $\mathcal{B}_q$ is nothing but the category of $q^{\mathbb{Z}}$-equivariant (analytic) coherent sheaves on $\mathbb{C}^*$ (or equivalently, the category of modules over the crossed product algebra $\mathcal{O}(\mathbb{C}^*) \rtimes_q \mathbb{Z}$, which are finitely presentable over $\mathcal{O}(\mathbb{C}^*)$). It follows from Lemma 3.2 of [**SV03**] that for any $M \in \mathcal{B}_q$ the underlying $\mathcal{O}(\mathbb{C}^*)$-module is free. However, there are interesting actions of $\theta\mathbb{Z}$ or $q^{\mathbb{Z}}$ on the free modules with respect to which they are equivariant. Let us denote by α the induced action by automorphisms of $\theta\mathbb{Z}$ on $\mathcal{O}(\mathbb{C}^*)$:

$$\alpha(f)(z) = f(qz); \qquad (z \in \mathbb{C}^*, q = e^{2\pi i\theta}).$$

Here, we have understood the notation $\alpha := \alpha(1)$ for the generator of $\mathbb{Z}$, so that $\alpha(n) = \alpha^n$. What is lacking in this picture is an infinitesimal action in terms of δ_τ and compatible connections, which accounts for the remaining $\tau\mathbb{Z}$ quotient operation. To this end, we define a derivation on $\mathcal{O}(\mathbb{C}^*)$ by $\delta = \tau z \frac{d}{dz}$. It is this infinitesimal action by δ that will turn out to be the appropriate replacement for the infinitesimal action of the group $\tau\mathbb{Z}$.

3.1. The category $\mathcal{B}^\tau_q$.

Our goal in this section is to define a category alluded to before, which is somehow 'in between' the categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ introduced by Polishchuk and Schwarz and $\mathcal{B}_q$ by Soibelman and Vologodsky. More precisely, we would like to construct a category $\mathcal{B}^\tau_q$ that is functorially related to both of these categories. At the same time, we would like to stay as close as possible to the setting of the Riemann–Hilbert correspondence. The discussion above motivates us to define the following category as a description of the quotient of $\mathcal{B}_q$ by the infinitesimal action of $\tau\mathbb{Z}$.

DEFINITION 3.1. *The category* $\mathcal{B}_q^\tau$ *consists of triples* (M, σ, ∇), *where*

- M *is a finitely presentable* $\mathcal{O}(\mathbb{C}^*)$*-module, i.e., there is an exact sequence,*
$$\mathcal{O}(\mathbb{C}^*)^m \longrightarrow \mathcal{O}(\mathbb{C}^*)^n \longrightarrow M \longrightarrow 0.$$
- σ *is a representation of* $\theta\mathbb{Z}$ *on* M *covering the action* α *of* $\theta\mathbb{Z}$ *on* $\mathcal{O}(\mathbb{C}^*)$*, i.e.,*
$$\sigma(m \cdot f) = \sigma(m) \cdot \alpha(f); \qquad (m \in M, f \in \mathcal{O}(\mathbb{C}^*)).$$
- ∇ *is a* $\theta\mathbb{Z}$*-equivariant connection on* M *covering the derivation* $\delta = \tau z \frac{d}{dz}$ *on* $\mathcal{O}(\mathbb{C}^*)$*, i.e., it satisfies,*
$$\nabla(m \cdot f) = \nabla(m) \cdot f + m \cdot \delta(f),$$
$$\nabla(\sigma(m)) = \sigma(\nabla(m)),$$
for all $m \in M, f \in \mathcal{O}(\mathbb{C}^*)$.

In addition, we impose that the connection ∇ *is a* regular singular connection *on* M*, that is, there exists a module basis* $\{e_1, \ldots, e_n\}$ *of* M *for which the holomorphic functions (on* $\mathbb{C}^*$*)* $z^{-1}A_{ij}$ *(*$i, j = 1, \ldots, n$*) defined by* $A_{ij}e_j = \nabla(e_i)$ *have simple poles at* 0*. We call* $A = (A_{ij})$ *the* matrix of the connection *with respect to that module basis.*

The morphisms in this category are equivariant $\mathcal{O}(\mathbb{C}^*)$*-module maps that are compatible with the connections. We will also write* $M = (M, \sigma, \nabla)$ *when no confusion can arise. For two objects* M *and* N *we denote by* $\mathrm{Hom}^{\theta\mathbb{Z},\delta}_{\mathcal{O}(\mathbb{C}^*)}(M, N)$ *the* $\mathbb{C}$*-linear vector space of morphisms between them.*

The uniqueness of the matrix $A = A_{ij}$ (after the choice of a module basis $\{e_i\}$ for M) is due to the fact that the modules M in $\mathcal{B}_q^\tau$ turn out to be free as $\mathcal{O}(\mathbb{C}^*)$-modules. This was observed in [**SV03**, Lemma 2] and used the fact that the sheaf $M \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$ must be torsion free due to $\theta\mathbb{Z}$-equivariance. Hence it is locally free on $\mathbb{C}^*$ and thus a trivial vector bundle. It also follows from the fact that a coherent sheaf admitting a connection is automatically locally free. As a consequence the $\mathcal{O}(\mathbb{C}^*)$-module of its global sections is free. This freeness as $\mathcal{O}(\mathbb{C}^*)$-modules can be translated into freeness as $\theta\mathbb{Z}$-equivariant $\mathcal{O}(\mathbb{C}^*)$-modules as follows. Suppose that $M \simeq V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*)$ as $\mathcal{O}(\mathbb{C}^*)$-modules with V a complex vector space. Via this identification there is an induced action of $\theta\mathbb{Z}$ on $V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*)$ making this an isomorphism of $\theta\mathbb{Z}$-equivariant $\mathcal{O}(\mathbb{C}^*)$-modules.

Let $\mathcal{B}^\tau$ denote the category of pairs (V, ∇) with V a vector bundle on $\mathbb{C}^*$ and ∇ a regular singular connection on V associated to $\delta = \tau z \frac{d}{dz}$. By the above remarks on the modules M in $\mathcal{B}^\tau_q$, there is a functor from $\mathcal{B}^\tau_q$ to $\mathcal{B}^\tau$ which forgets the action of $\theta\mathbb{Z}$. Thanks to Deligne [**Del70**] (see also, for instance, Theorem 1.1 and the paragraph after Remark 1.2 of [**Mal87**]), we know that the category $\mathcal{B}^\tau$ is equivalent to the category of finite dimensional representations of the fundamental group $\pi_1(\mathbb{C}^*, z') \simeq \mathbb{Z}$ with a base point z'. This result motivates the regular singularity condition we have imposed on the connections in Definition 3.1.

In Section 3.3 we will enhance this Riemann–Hilbert correspondence to an equivariant version and show that a similar statement holds for $\mathcal{B}^\tau_q$. Let us first proceed to examine some of the properties of $\mathcal{B}^\tau_q$ and its relation with the other two categories, *viz.,* $\mathcal{B}_q$ and $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$.

PROPOSITION 3.2. *The category $\mathcal{B}^\tau_q$ is an abelian category.*

PROOF. It is proven in Proposition 3.3 of [**SV03**] that the category $\mathcal{B}_q$ is abelian. One observes readily that there is a faithful functor (forgetting the connection) from $\mathcal{B}^\tau_q$ to $\mathcal{B}_q$. Suppose that $f : M \to N$ is a morphism in $\mathcal{B}^\tau_q$. Since it is also a morphism in $\mathcal{B}_q$, both $\ker f$ and $\operatorname{coker} f$ are equivariant $\mathcal{O}(\mathbb{C}^*)$-modules. Moreover, the map f intertwines the connections on M and N and hence induces compatible connections on $\ker f$ and $\operatorname{coker} f$ making them objects in $\mathcal{B}^\tau_q$. Other axioms like Coim = Im are standard. □

We now view $\mathcal{A}_\theta$ as a module over $\mathcal{O}(\mathbb{C}^*)$ via the homomorphism

$$\psi : \mathcal{O}(\mathbb{C}^*) \to \mathcal{A}_\theta$$
$$\sum_{n\in\mathbb{Z}} f_n z^n \mapsto \sum_{n\in\mathbb{Z}} f_n U_1^n.$$

This is well-defined since a sequence f_n of exponential decay is certainly a Schwartz sequence.

REMARK 3.3. The map is essentially restricting a holomorphic function on $\mathbb{C}^*$ to the unit circle. In fact, it is injective since, if a holomorphic function vanishes on the unit circle, it must vanish on the whole of $\mathbb{C}^*$. Note that $\mathcal{A}_\theta$ is not finitely generated over $\mathcal{O}(\mathbb{C}^*)$ and hence not an element of $\mathcal{B}_q$ or $\mathcal{B}^\tau_q$.

Proposition 3.4. *The following association defines a right exact functor, denoted* ψ_*, *from* $\mathcal{B}^\tau_q$ *to* $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$. *For an object* (M, σ, ∇) *in* $\mathcal{B}^\tau_q$ *we define an object* $(\tilde{M}, \tilde{\nabla})$ *in* $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ *by*

$$\begin{aligned}\tilde{M} &= M \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta \\ \tilde{\nabla} &= 2\pi i\ \nabla \otimes 1 + 1 \otimes (\tau\delta_1 + \delta_2) \\ &= 2\pi i\ \nabla \otimes 1 + 1 \otimes \delta_\tau.\end{aligned}$$

Proof. Observe that $\psi(2\pi i\delta f) = \tau\delta_1(\psi(f))$ as follows from the definitions of δ and δ_1. Moreover, the image of ψ lies in the kernel of the derivation δ_2 on the noncommutative torus (since $\delta_2(U_1)$ is vanishing). Hence one can add δ_2 to $\tau\delta_1$ making $\tilde{\nabla}$ a connection on $\tilde{M}$ covering δ_τ. □

Note that by a simple adjustification one can actually define a right exact functor from $\mathcal{B}^{\omega_1}_q$ to $\mathsf{Vect}(\mathbb{T}^\omega_\theta)$. We also claim that, in fact,

Proposition 3.5. *The module* $\mathcal{A}_\theta$ *over* $\mathcal{O}(\mathbb{C}^*)$ *via the map* ψ *is flat.*

Proof. The algebra $\mathcal{O}(\mathbb{C}^*)$ is a commutative integral domain, since holomorphic functions cannot have disjoint support. Further, from Corollary 3.2 of [**Pir99**] one concludes that the global Ext dimension of $\mathcal{O}(\mathbb{C}^*)$ is 1. Hence it is a *Prüfer domain*, *i.e.,* a domain in which all finitely generated non-zero ideals are invertible. Indeed, Theorem 6.1 of [**FS85**] says that a (fractional) ideal in a domain is invertible if and only if it is projective and, since $\mathcal{O}(\mathbb{C}^*)$ has Ext dimension 1, given any finitely generated ideal I, applying $\mathrm{Hom}(-, M)$ to the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ for an arbitrary M, one finds that $\mathrm{Ext}^1(I, M) = 0$, *i.e.,* I is projective. It is known that a module over a Prüfer domain is flat if and only if it is torsion free (see, *e.g.,* Theorem 1.4 *ibid.*). So we only need to check torsion freeness. We identify $\mathcal{A}_\theta$ as a module over $\mathcal{O}(\mathbb{C}^*)$ with $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ and represent each element as a sequence $\{g_n\}_{n\in\mathbb{Z}}$, $g_n \in C^\infty(\mathbb{S}^1)$, refer to the discussion before (2). The image of the map ψ clearly lies in $C^\infty(\mathbb{S}^1)$, which is identified with the functions supported at the identity element of $\mathbb{Z}$. In other words, for all $f \in \mathcal{O}(\mathbb{C}^*)$, $\psi(f)$ is of the form $\{f_n\}$, where $f_n = 0$ unless $n = 0$. Now consider any $g = \{g_n\} \in \mathcal{A}_\theta$ and suppose that some non-zero $f \in \mathrm{Ann}(\{g_n\})$, *i.e.,* $g * \psi(f) = \{g_n\alpha_n(f_0)\} = 0$. This implies that $g_n(z)f_0(q^n z) = 0$ for all n, $|z| = 1$. Being the restriction of a holomorphic function on $\mathbb{C}^*$, $f_0(q^n z)$ has a discrete zero set on the unit circle. A smooth function on $\mathbb{S}^1$ cannot have a discrete set of points as support and hence each $g_n(z)$ must be identically zero. Thus, whenever an element in $\mathcal{A}_\theta$ has a non-zero element in its annihilator ideal, the element is itself zero. Hence $\mathcal{A}_\theta$ is torsion free from which the result follows. □

COROLLARY 3.6. *The base change functor* ψ_* *induced by the homomorphism* ψ *is exact and faithful.*

PROOF. From the previous Proposition we conclude that the functor sends an exact sequence of $\mathcal{O}(\mathbb{C}^*)$-modules to an exact sequence of $\mathcal{A}_\theta$-modules. It is clear that the induced morphisms respect the induced connections. For the faithfulness, identify each object $M \in \mathcal{B}_q^\tau$ with $V \otimes \mathcal{O}(\mathbb{C}^*)$ with V a vector space; similarly write $M = V \otimes \mathcal{O}(\mathbb{C}^*)$. A morphism in $\mathcal{B}_q^\tau$ from M to M' is then given by an element in $\mathrm{Hom}_{\mathbb{C}}(V, V') \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*)$, whereas a morphism in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ between $\tilde{M}$ and $\tilde{M}'$ is given by an element in $\mathrm{Hom}_{\mathbb{C}}(V, V') \otimes_{\mathbb{C}} \mathcal{A}_\theta$. The functor ψ_* acts on these element by $1 \otimes \psi$ and since ψ is injective, it follows that ψ_* is injective on morphisms. □

REMARK 3.7. However, the functor is not full. It is certainly not essentially surjective as the underlying $\mathcal{A}_\theta$-modules of the objects in the image are all free, whilst $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ has modules which are not free.

The main Theorem of [**Pol04a**] says that the category generated by succesive extensions of all *standard* holomorphic bundles (see 2.5) over $\mathbb{T}_\theta^\tau$ ($\mathbb{T}_\theta$ equipped with the derivation δ_τ) is already all of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Let us recall the definition of a standard holomorphic bundle (or as standard module) in the special case when the underlying module is just $\mathcal{A}_\theta$. Given any fixed $z' \in \mathbb{C}$, the connection $\nabla_{z'}$ is defined by

$$\nabla_{z'}(a) = \delta_\tau(a) + 2\pi i z' \cdot a, \qquad (a \in \mathcal{A}_\theta).$$

The tuple $E_1^{z'} := (\mathcal{A}_\theta, \nabla_{z'})$ is by definition a standard module. Let us denote the full subcategory of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ generated by successive extensions of standard modules of the form $E_1^{z'}$, $z' \in \mathbb{C}$ by $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$. Since the extension of two free modules is again free, it is clear that the underlying $\mathcal{A}_\theta$-module of all objects of $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ is free.

LEMMA 3.8. *With respect to a suitable basis each object of* $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ *is of the form* $(\mathcal{A}_\theta^n, \delta + A)$*, where* A *is an* $n \times n$ *upper triangular matrix in* $M_n(\mathcal{A}_\theta)$ *with diagonal entries in* $\mathbb{C}$.

PROOF. It is known that given any finitely generated projective module M over $\mathcal{A}_\theta$ and a fixed connection ∇ compatible with δ_τ, all other compatible connections are of the form $\nabla + \phi$, $\phi \in \mathrm{End}_{\mathcal{A}_\theta}(M)$. This follows easily from the Leibniz rule (11). Since M is of the form $\mathcal{A}_\theta^n$, ϕ is determined by a matrix $A \in M_n(\mathcal{A}_\theta)$. Let

$$0 \longrightarrow (\mathcal{A}_\theta, \nabla_{z'}) \xrightarrow{\iota} (\mathcal{A}_\theta^2, \delta + A) \xrightarrow{\pi} (\mathcal{A}_\theta, \nabla_{z''}) \longrightarrow 0 \tag{16}$$

be a holomorphic extension in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries $a, b, c, d \in \mathcal{A}_\theta$ and $\iota(a) = (a, 0)$ and $\pi(a_1, a_2) = a_2$. One checks easily that the holomorphicity of ι and π (the fact that they commute with the connections) forces $c = 0, a = z'$ and $d = z''$. Now by induction it follows that the connections obtained by successive extensions are of the desired form.

Conversely, by induction suppose that every connection of the desired form on $\mathcal{A}_\theta^{n-1}$ can be obtained as an iterated extension of modules of the form $E_1^{z'}$. Let A be an upper triangular matrix in $M_n(\mathcal{A}_\theta)$ whose diagonal entries are in $\mathbb{C}$, *i.e.,* A is of the form

$$\begin{pmatrix} z' & b_2 & \cdots & b_n \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix},$$

where $A' \in M_{n-1}(\mathcal{A}_\theta)$ is also of the prescribed type and $b_2, \ldots, b_n \in \mathcal{A}_\theta$. A routine calculation then shows that

$$0 \longrightarrow (\mathcal{A}_\theta, \nabla_{z'}) \xrightarrow{\iota} (\mathcal{A}_\theta^n, \delta + A) \xrightarrow{\pi} (\mathcal{A}_\theta^{n-1}, \delta + A') \longrightarrow 0$$

with $\iota(a) = (a, 0, \ldots, 0)$ and $\pi(a_1, a_2, \ldots, a_n) = (a_2, \ldots, a_n)$ is a holomorphic extension in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Hence $(\mathcal{A}_\theta^n, \delta + A)$ belongs to $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$. □

Remark 3.9. In the exact sequence in Eqn. (16), if b in $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is non-zero and $a, d \in \mathbb{C}$ such that $a \neq d$, then the matrix can be diagonalized by the change of basis matrix $\left(\begin{smallmatrix} 1 & \frac{-b}{d-a} \\ 0 & 1 \end{smallmatrix}\right)$ and hence the sequence splits.

Remark 3.10. Given any matrix $A \in M_n(\mathbb{C})$, with respect to a suitable basis one can reduce it to its Jordan canonical form (it is also upper triangular with diagonal entries in $\mathbb{C}$). Therefore, $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ contains all objects of the form $(\mathcal{A}_\theta^n, \delta + A)$, where $A \in M_n(\mathbb{C})$ with respect to a basis.

As we will see later (Proposition 3.15), each object (M, σ, ∇) in $\mathcal{B}_q^\tau$ is isomorphic to an object, whose matrix of the connection is a constant matrix. This can be accomplished via a change of basis of M. Combining this with the above remark, we conclude that – at the level of objects – the image of ψ_* lies inside $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$.

3.2. The effect on K-theory. We infer from Eqn. (13) that the K-theory (by that we mean the Grothendieck group, *i.e.,* the free abelian group generated by the isomorphism classes of objects modulo the relations coming from all exact sequences) of $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ is isomorphic to that of $\mathsf{D}^b(X_\tau)$ via the Polishchuk–Schwarz equivalence $\mathcal{S}_\tau$. One knows that $\mathsf{K}_0(\mathsf{D}^b(X_\tau)) \cong \mathsf{K}_0(\mathrm{Coh}(X_\tau)) = \mathsf{K}_0(X_\tau) = \mathrm{Pic}(X_\tau) \oplus \mathbb{Z}$. The composition of the functors ψ_* followed by $\mathcal{S}_\tau$ induces a homomorphism between $\mathsf{K}_0(\mathcal{B}^\tau_q)$ and $\mathsf{K}_0(\mathsf{Vect}(\mathbb{T}^\tau_\theta)) = \mathrm{Pic}(X_\tau) \oplus \mathbb{Z}$. One observes that applying ψ_* one obtains only elements in $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ whose underlying $\mathcal{A}_\theta$-modules are free. It is known that for $E \in \mathsf{Vect}(\mathbb{T}^\tau_\theta)$, $\mathrm{rk}\mathcal{S}_\tau(E) = -\mathrm{deg}(E)$ and $\mathrm{deg}\mathcal{S}_\tau(E) = \mathrm{rk}(E)$. The degree of the modules, which are free, is known to be zero (see (6)). Hence the composition of the two functors sends every element in $\mathcal{B}^\tau_q$ to a torsion sheaf on X_τ. One can check that $\mathcal{O}(\mathbb{C}^*)$ equipped with the connection $\delta + z'$, where $z' \in \mathbb{C}$, gets mapped to the standard holomorphic bundle $E^{z'}_1$ as explained after Remark 3.7. From part (c) of Proposition 3.7 of [**PS03**] we know that $\mathcal{S}_\tau(E^{z'}_1)$ is $\mathcal{O}_{-z'}$ (up to a shift in the derived category), which is the structure sheaf of the point $-z'$ mod $(\mathbb{Z} + \tau\mathbb{Z})$ in X_τ. All modules of the form $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z')$ with $z' \in \mathbb{C}$ are endomorphism simple, *i.e.,* $\mathrm{End}(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z') = \mathbb{C}$. Indeed, ignoring the equivariance condition and the connection, $\mathrm{End}(\mathcal{O}(\mathbb{C}^*)) = \mathcal{O}(\mathbb{C}^*)$ and the equivariance condition says that $\sigma(mf) = \sigma(m)f$. However, by definition $\sigma(mf) = \sigma(m)\alpha(f)$ whence $\alpha(f) = f$ implying $f \in \mathbb{C}$. This module is mapped to $(\mathcal{A}_\theta, \delta_\tau + 2\pi i z') = E^{z'}_1$, which in turn is mapped to the endomorphism simple object $\mathcal{O}_{-z'}$ in $\mathcal{C}^{\theta,\tau}$. It is known that, in fact, the Grothendieck group of any nonsingular curve C is isomorphic to $\mathrm{Pic}(C) \oplus \mathbb{Z}$. In this identification the contribution to $\mathbb{Z}$ comes from the rank of the coherent sheaf, whereas $\mathrm{Pic}(C)$ can be regarded as the contribution from the torsion part (actually from the determinant bundle of the sheaf, which may be identified with a torsion sheaf via a Fourier-Mukai transform). Since we obtain only torsion sheaves, the image of the induced map on K-theory lies inside $\mathrm{Pic}(X_\tau)$.

PROPOSITION 3.11. *The map induced by $\mathcal{S}_\tau \circ \psi_*$ between the K-theories of $\mathcal{B}^\tau_q$ and $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ gives a surjection from $\mathsf{K}_0(\mathcal{B}^\tau_q)$ to $\mathrm{Pic}(X_\tau)$.*

PROOF. The divisor class group of X_τ is the free abelian group generated by the points of X_τ modulo the principal divisors, which is also isomorphic to $\mathrm{Pic}(X_\tau)$. The class of each point $z' \in X_\tau$ of the divisor class group can be identified with the class of the torsion sheaf $\mathcal{O}_{z'}$ corresponding to the line bundle $\mathcal{O}(z') \in \mathrm{Pic}^1(X_\tau)$ and they generate $\mathrm{Pic}(X_\tau)$ as a group. By the above argument $\mathcal{O}_{z'}$ is obtained by applying the functor $\mathcal{S}_\tau \circ \psi_*$ to the element $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta - z')$ of $\mathcal{B}^\tau_q$.

Thus one obtains a surjection onto the generating set of $\mathrm{Pic}(X_\tau)$ from which the assertion follows. □

Remark 3.12. From Proposition 2.1 of [**PS03**] we know that the images of $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z_1')$ and $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z_2')$ under ψ_* are isomorphic if and only if $z_1' \equiv z_2' \bmod (\mathbb{Z} + \tau\mathbb{Z})$. More generally, abbreviating the module $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z')$ by $M_{z'}$, one can also rephrase the linear equivalence relation of the divisor class group to conclude that an element of the form $\sum n_i [M_{-z_i'}]$ maps to zero at the level of K-theory whenever $\sum n_i = 0$ and $\sum n_i z_i' \in (\mathbb{Z} + \tau\mathbb{Z})$. However, some of them actually represent the trivial class in the K-theory of $\mathcal{B}_q^\tau$, as we will see in the next section (see Corollary 3.18).

Although the image of $\mathcal{B}_q^\tau$ gives only the free modules in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$, it has the interesting property of being a Tannakian category, as we will explore in the next section. Let us end this section by summarising the relations between $\mathcal{B}_q^\tau$ and the categories $\mathcal{B}^\tau$, $\mathcal{B}_q$, $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$:

$$\begin{array}{ccccc} & & \mathcal{B}_q^\tau & & \\ & \swarrow & \downarrow{\scriptstyle \psi_*} & \searrow & \\ \mathcal{B}_q & & \mathsf{Vect}(\mathbb{T}_\theta^\tau) & & \mathcal{B}^\tau \end{array}$$

where the two diagonal arrows are the forgetful functors discussed before. All of these functors are faithful and exact (but not injective on objects).

3.3. The equivariant Riemann–Hilbert correspondence. We will now analyse further the structure of $\mathcal{B}_q^\tau$ and define a tensor product on it. Our main result is that this – together with a fibre functor – makes $\mathcal{B}_q^\tau$ a Tannakian category. Via an equivariant version of the Riemann–Hilbert correspondence on $\mathbb{C}^*$, we determine the corresponding affine group scheme.

3.3.1. *Preliminaries on Tannakian categories.* We briefly recall the notion of a Tannakian category. For more details, we refer the reader to the original works [**Saa72, Del90, DM82**] (see also Appendix B of [**vS03**]).

Let $\mathcal{C}$ be an k-linear abelian category, for a field k. Then $\mathcal{C}$ is a neutral Tannakian category over k if

(1) The category $\mathcal{C}$ is a *tensor category.* In other words, there is a tensor product: for every pair of objects X, Y there is an object $X \otimes Y$. The tensor product is commutative ($X \otimes Y \simeq Y \otimes X$) and associative ($X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$) and there is a

unit object 1 (such that $X \otimes 1 \simeq 1 \otimes X \simeq X$). The above isomorphisms are supposed to be functorial.

(2) $\mathcal{C}$ is a *rigid* tensor category: there exists a duality $\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, satisfying
- For any object X in $\mathcal{C}$, the functor $\_ \otimes X^{\vee}$ is left adjoint to $\_ \otimes X$, and the functor $X^{\vee} \otimes \_$ is right adjoint to $X \otimes \_$.
- There is an evaluation morphism $\epsilon : X \otimes X^{\vee} \to 1$ and a unit morphism $\eta : 1 \to X^{\vee} \otimes X$ such that $(\epsilon \otimes 1) \circ (1 \otimes \eta) = 1_X$ and $(1 \otimes \epsilon) \circ (\eta \otimes 1) = 1_{X^{\vee}}$.

(3) An isomorphism between $\mathrm{End}(1)$ and k is given.

(4) There is a fibre functor $\omega : \mathcal{C} \to \mathsf{Vect}_k$ to the category of k-vector spaces: this is a k-linear, faithful, exact functor that commutes with tensor products.

An important result is that every Tannakian category is equivalent to the category of finite dimensional linear representations of an affine group scheme H over k. This equivalence is established by ω and the group scheme H is given as the functor of automorphisms of the fibre functor ω which is defined as follows.

Definition 3.13. *Let $(\mathcal{C}, \omega)$ be a Tannakian category. The affine group scheme of automorphisms $\mathrm{Aut}^{\otimes}(\omega)$ of the fibre functor ω is determined as a functor from the category of k-algebras to the category of groups as follows. If R is a k-algebra, then an element σ of $\mathrm{Aut}^{\otimes}(\omega)(R)$ is given by a collection of elements $\{\sigma(X)\}_X$ with X running over the collection of all objects of $X \in \mathcal{C}$. Each $\sigma(X)$ is an R-linear automorphism of $\omega(X) \otimes_k R$ such that the following hold:*

(1) $\sigma(1) = \mathrm{id}_R$.

(2) *For every morphism* $f : X \to Y$ *we have that* $(\mathrm{id}_R \otimes \omega(f) \circ \sigma(X) = \sigma(Y) \circ (\mathrm{id}_R \otimes \omega(f))$.

(3) $\sigma(X \otimes Y) = \sigma(X) \otimes \sigma(Y)$.

3.3.2. *The Tannakian category structure on $\mathcal{B}^{\tau}_{q}$.* In order not to lose the reader in notational complexities, we generalize a little and let (R, δ) be a differential (commutative) ring that carries an action α of a group G. Let $\mathsf{Mod}^{G,\delta}(R)$ denote the category consisting of free G-equivariant differential R-modules. Recall that a differential R-module is an R-module equipped with a map $\nabla : M \to M$ – a connection – that satisfies the Leibniz rule:

$$\nabla(m \cdot r) = \nabla(m) \cdot r + m \cdot \delta(r).$$

Moreover, G-equivariance means that there is an action σ of G such that

$$\begin{aligned}\sigma_g(m \cdot r) &= \sigma_g(m) \cdot \alpha_g(r),\\ \nabla(\sigma_g(m)) &= \sigma_g(\nabla(m)).\end{aligned}$$

We will group the objects in the category $\mathsf{Mod}^{G,\delta}(R)$ into a triple (M, σ, ∇) and denote the morphisms that respect all the structures by $\mathrm{Hom}_R^{G,\delta}(M, N)$.

Proposition 3.14. *The category* $\mathsf{Mod}^{G,\delta}(R)$ *is a rigid tensor category with the tensor product given by*

$$(M, \sigma, \nabla) \otimes (N, \sigma', \nabla') = (M \otimes_{\mathcal{O}(\mathbb{C}^*)} N,\ \sigma \otimes \sigma',\ \nabla \otimes 1 + 1 \otimes \nabla')$$

for any two objects (M, σ, ∇) *and* (N, σ', ∇') *in* $\mathsf{Mod}^{G,\delta}(R)$.

Proof. We start by checking that the tensor product is commutative. First of all, since R is a commutative ring, the 'tensor flip' that maps $M \otimes_{\mathbb{C}} N \to N \otimes_{\mathbb{C}} M$ factorizes to a bijective map of R-modules from $M \otimes_R N$ to $N \otimes_R M$. One also checks that it intertwines the actions $\sigma \otimes \sigma'$ and $\sigma' \otimes \sigma$ and the two connections.

The duality is given as follows, for an object $(M = V \otimes R, \sigma, \nabla)$, V a vector space, we define its dual object $(M^\vee, \sigma^\vee, \nabla^\vee)$ as follows. Define an R-module by,

$$M^\vee := \mathrm{Hom}_R(M, R),$$

with $r \in R$ acting on $f \in M^\vee$ by $(f \cdot r)(m) = f(m) \cdot r = f(m \cdot r)$. It can be equipped with a dual action $\sigma^\vee$ of G by setting for $f \in M^\vee$,

$$\sigma^\vee(f) = \alpha \circ f \circ \sigma^{-1}.$$

One can check that $\sigma^\vee(f)$ is again R-linear:

$$\begin{aligned}\sigma^\vee(f)(m \cdot r) &= \alpha \circ f\left(\sigma^{-1}(m) \cdot \alpha^{-1}(r)\right)\\ &= \alpha \circ f \circ \sigma^{-1}(m) \cdot r\\ &= \left(\sigma^\vee(f) \cdot r\right)(m)\end{aligned}$$

Moreover, the action of R on $M^\vee$ is equivariant with respect to $\sigma^\vee$:

$$\begin{aligned}\sigma^\vee(f \cdot r)(m) &= \alpha \circ (f \cdot r)\left(\sigma^{-1}(m)\right)\\ &= \alpha\left(f(\sigma^{-1}(m)) \cdot r\right)\\ &= \alpha \circ f \circ \sigma^{-1}(m) \cdot \alpha(r).\end{aligned}$$

A dual connection $\nabla^\vee$ is defined by

$$\nabla^\vee(f) = \delta \circ f - f \circ \nabla,$$

which indeed satisfies the Leibniz rule

$$\begin{aligned}\nabla^{\vee}(f\cdot r)(m) &= \delta(f(m))\cdot r + f(m)\cdot\delta(r) - f(\nabla(m))\cdot r\\ &= \left(\nabla^{\vee}(f)\cdot r\right)(m) + (f\cdot\delta(r))(m),\end{aligned}$$

and is $\sigma^{\vee}$-invariant:

$$\begin{aligned}\sigma^{\vee}\left(\nabla^{\vee}(f)\right) &= \alpha\circ(\delta\circ f)\circ\sigma^{-1} - \alpha\circ(f\circ\nabla)\circ\sigma^{-1}\\ &= \delta\circ(\alpha\circ f\circ\sigma^{-1}) - (\alpha\circ f\circ\sigma^{-1})\circ\nabla,\end{aligned}$$

since α and σ commute with δ and ∇, respectively.

Note that since $M = V\otimes R$, we can identify,

$$M^{\vee}\simeq \mathrm{Hom}_R(V\otimes R, R)\simeq \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})\otimes R\simeq V^{*}\otimes R,$$

from which it follows that $M^{\vee\vee}\simeq M$. Indeed, one checks that the induced map respects the extra (G,δ)-structure:

$$\sigma^{\vee\vee}(m)(f) = \alpha\circ m\circ\left(\sigma^{\vee}\right)^{-1}(f) = \alpha\circ m\circ(\alpha^{-1}\circ f\circ\sigma) = f\left(\sigma(m)\right)$$

$$\begin{aligned}\nabla^{\vee\vee}(m)(f) &= (\delta\circ m)(f) - m\circ\nabla^{\vee}(f)\\ &= \delta\left(f(m)\right) - m\circ(\delta\circ f) + f\left(\nabla(m)\right)\\ &= f(\nabla(m)).\end{aligned}$$

for all $m\in M, f\in M^{\vee}$. In addition, it allows one to prove that the association

$$\phi\in\mathrm{Hom}_R^{G,\delta}(N_1, M^{\vee}\otimes_R N_2)\mapsto\tilde{\phi}\in\mathrm{Hom}_R^{G,\delta}(M\otimes_R N_1, N_2)$$
$$\tilde{\phi}(m\otimes n_1) := \phi(n_1)(m)\in N_2.$$

induces an isomorphism. Again, it is enough to show that this map is both G-equivariant and δ-invariant, which is omitted.
In a similar way, one proves that

$$\mathrm{Hom}_R^{G,\delta}(N_1\otimes_R M^{\vee}, N_2)\simeq\mathrm{Hom}_R^{G,\delta}(N_1, N_2\otimes_R M).$$

Finally, there is an evaluation morphism and a unit morphism given in terms of a basis $\{e_i\}$ of V and its dual $\{\hat{e}_i\}$ of V^{*} by

$$\epsilon(m\otimes f) = f(m),\qquad \eta(1_R) = \hat{e}_i\otimes e_i,$$

that satisfy the required properties. □

Let us now return to the category $\mathcal{B}_q^{\tau}$ of Definition 3.1. It is not difficult to see that the above tensor product respects the regular singularity condition in the definition of $\mathcal{B}_q^{\tau}$. Hence this becomes a rigid tensor category as well. We would like to show that it is in fact a Tannakian category by constructing a fibre functor to $\mathsf{Vect}_{\mathbb{C}}$. The following observations turn out to be essential in what follows.

Via a series of changes of basis, it is possible to bring the matrix A in the form of a constant matrix with all eigenvalues in the same transversal of $\tau\mathbb{Z}$. In other words, its eigenvalues never differ by an integer multiple of τ. Before we explain how this can be achieved, recall that a *transversal* to $\tau\mathbb{Z}$ in $\mathbb{C}$ is the image of a section of the projection map $\mathbb{C} \to \mathbb{C}/\tau\mathbb{Z}$ (*e.g.,* the strip $0 \leq \Re(z/\tau) < 1$). We follow the argument of Section 17 in [**Was76**]. Let $A(z) = A_0 + A_1 z + \cdots$ be a matrix with holomorphic entries. We first bring the constant term A_0 in Jordan canonical form via a constant change of basis matrix. Subsequently, we can bring all the eigenvalues of A_0 in the same transversal of $\tau\mathbb{Z}$ by the so-called *shearing transformations.* Let us consider the case of a 2×2 matrix $A(z)$ and write

$$A(z) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

with $a = a_1 z + a_2 z^2 + \cdots$ and similarly b, c and d. Let us suppose that $\lambda_1 - \lambda_2 = k\tau$ for some positive integer k. The change of basis is given by the matrix $D = \mathrm{diag}\{1, z\}$ and transforms A to

$$A' = D^{-1}AD + D^{-1}\delta D = \begin{pmatrix} \lambda_1 & 0 \\ c_1 & \lambda_2 + \tau \end{pmatrix} + \begin{pmatrix} a(z) & zb(z) \\ c_2 z + c_3 z^2 + \cdots & d(z) \end{pmatrix},$$

and one readily observes that the constant term A_0' of this matrix has eigenvalues that differ by $(k-1)\tau$. Proceeding in this way, one can transform A to a matrix that has constant term with eigenvalues in the same transversal. The generalization to arbitrary dimensions is straightforward and can be found in Section 17.1 of *loc. cit.*

PROPOSITION 3.15. *For each object in* $\mathcal{B}_q^\tau$ *there is an isomorphic object* $(M = V \otimes \mathcal{O}(\mathbb{C}^*), \sigma, \nabla)$ *in* $\mathcal{B}_q^\tau$ *with* V *a vector space and*

(1) $\nabla = \delta + A$ *with* A *a constant matrix with all eigenvalues in the same transversal of* $\tau\mathbb{Z}$*,*
(2) σ *is given by* $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ *for an invertible constant matrix* B*.*

PROOF. Since M is a free $\mathcal{O}(\mathbb{C}^*)$-module, there is a vector space V such that $M \simeq V \otimes \mathcal{O}(\mathbb{C}^*)$. We show *1.* by adopting an argument from Section 5 of [**Was76**]. By the above observations, we can write the matrix of the connection as $A = A_0 + A_1 z + \cdots$, with A_0 having eigenvalues that never differ by an element of $\tau\mathbb{Z}$. We construct a matrix $P = I + P_1 z + \cdots$ (P_k in $M_n(\mathbb{C})$) which solves $PA_0 = AP - \delta P$. Comparing the powers of z, we find

$$A_0 P_k - P_k(A_0 + kI) = -(A_k + A_{k-1}P_1 + \cdots + A_1 P_{k-1})$$

which can be solved recursively by our assumption on the eigenvalues of A_0. This gives a formal power series expansion and we would like to show that the entries of P are in fact holomorphic functions on $\mathbb{C}^*$.

Now by Theorem 5.4 of [**Was76**] one knows that the radius of convergence of the entries of P is the same as that of the entries of A, which is infinity. Hence, $P \in M_n(\mathcal{O}(\mathbb{C}^*))$.

Next, the action of σ can be written as $\sigma(\nu \otimes f) = B\nu \otimes \alpha(f)$ for some invertible matrix $B \in M_n(\mathcal{O}(\mathbb{C}^*))$ with n the dimension of V. Expressed in terms of A and B, the equivariance condition $\sigma \circ \nabla = \nabla \circ \sigma$ reads

$$\delta B + [A, B] = 0, \tag{17}$$

and as observed above, we may assume that A has constant entries and with eigenvalues that are all in the same transversal. We adopt the argument from the proof of Theorem 4.4 in [**Mal87**] to show that B is in fact constant. Writing B as a Laurent series $B = \sum_{k \in \mathbb{Z}} B_k z^k$ we obtain the following relations

$$(A - \tau k I_n) B_k = B_k A, \qquad k = 0, 1, \ldots$$

This implies [**Was76**, Theorem 4.1] (see also Lemma 4.6 in [**Mal87**]) that $(A - \tau k I_n)$ and A have at least one common eigenvalue. But since the eigenvalues of A are all in a transversal of $\tau\mathbb{Z}$ in $\mathbb{C}$, this is impossible unless $k = 0$, and we conclude that $B_k = 0$ for all $k \neq 0$. □

Our next task is to show that $\mathcal{B}^\tau_q$ is in fact a Tannakian category and compute the corresponding affine group scheme. For this, we use an equivariant version of the Riemann–Hilbert correspondence.

THEOREM 3.16. (1) *The category $\mathcal{B}^\tau_q$ is a Tannakian category with the fibre functor given by*

$$\omega : \mathcal{B}^\tau_q \longrightarrow \mathsf{Vect}_\mathbb{C}$$
$$(M, \sigma, \nabla) \longmapsto (\ker\nabla)_z,$$

mapping to the germs at a fixed point $z \in \mathbb{C}^*$ of local solutions to the differential equation $\delta f + A f = 0$, where $\nabla = \delta + A$ with respect to a suitable basis of M.

(2) *The category $\mathcal{B}^\tau_q$ is equivalent to the category $\mathsf{Rep}(\mathbb{Z} + \theta\mathbb{Z})$ of finite dimensional representations of $\mathbb{Z} + \theta\mathbb{Z} \simeq \mathbb{Z}^2$.*

PROOF. By the existence and uniqueness of local solutions of linear differential equations, there are n local solutions to the system of differential equations $\delta U = -AU$ once we have fixed the initial conditions, so that $(\ker\nabla)_z$ is an n-dimensional complex vector space. That the functor ω is faithful can be seen as follows. Suppose ϕ is a morphism between two objects (M, σ, ∇) and (M', σ', ∇') and suppose that these objects are of the form as in Proposition 3.15, with the eigenvalues of A, A' in the same transversal. We claim that ϕ is given by a constant matrix so that $\omega(\phi)$ mapping $(\ker\nabla)_z$ to $(\ker\nabla')_z$ coincides with ϕ. The argument is very similar to that used in the second part of the proof of Proposition 3.15 since compatibility of ϕ with the connections implies

$$(A' - \tau k I_n)\phi_k = \phi_k A, \qquad k = 0, 1, \dots$$

where we have written $\phi = \sum_{k\geq 0} \phi_k z^k$. An application of Theorem 4.1 in [**Was76**] then implies that A and $A' - \tau k I_n$ have a common eigenvalue. This is impossible unless $k = 0$ since by assumption A and A' have eigenvalues in the same transversal. We conclude that $\phi_k = 0$ for all $k > 0$.

The general case follows by observing that Proposition 3.15 implies that a morphism between two objects in $\mathcal{B}_q^\tau$ can always be written as $D_2 \circ \phi \circ D_1^{-1}$ with ϕ constant as above and with D_i certain (invertible) change of basis matrices.

For *2.*, fix a transversal $\mathbf{T}$ to $\tau\mathbb{Z}$ in $\mathbb{C}$. We construct a tensor functor $\mathsf{F}_\mathbf{T} : \mathsf{Rep}(\mathbb{Z}^2) \to \mathcal{B}_q^\tau$ that is full, faithful and essentially surjective. Let ρ_1, ρ_2 be two mutually commuting representations of $\mathbb{Z}$ on a vector space V. Then we define $A \in \mathrm{End}(V)$ via $\rho_1(1) = e^{2\pi i A/\tau}$ and B as $\rho_2(1)$. By Lemma 4.5 in [**Mal87**], there exists a unique matrix A such that $\rho_1(1) = e^{2\pi i A/\tau}$ with its eigenvalues in the transversal $\mathbf{T}$ and a unique matrix B' such that $B = e^{2\pi i B'}$. We set $\mathsf{F}_\mathbf{T}(V) = (M, \sigma, \nabla)$ in $\mathcal{B}_q^\tau$ by setting $M = V \otimes \mathcal{O}(\mathbb{C}^*)$, $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ and finally $\nabla(v \otimes f) = Av \otimes f + v \otimes \delta f$; for a morphism $\phi \in \mathrm{Hom}(V, V')$ we simply set $\mathsf{F}_\mathbf{T}(\phi) = \phi \otimes 1$. Once again by Lemma 4.5 *ibid.* the matrices A and B' commute, whence A and $B = e^{2\pi i B'}$ commute. Thus the compatibility condition between σ and ∇ given by Eqn. (17) is satisfied. Moreover, $\mathsf{F}_\mathbf{T}(\phi)$ is compatible with σ and ∇ and thus a morphism in $\mathcal{B}_q^\tau$.

We infer from Proposition 3.15 that the functor $\mathsf{F}_\mathbf{T}$ is essentially surjective, since any object in $\mathcal{B}_q^\tau$ is isomorphic to an object obtained from an element in $\mathsf{Rep}(\mathbb{Z}^2)$ by the above procedure.

Fullness and faithfulness of this functor can be seen as follows. Let V, V' be two vector spaces with the action of $\mathbb{Z}^2$ given by $e^{2\pi i A/\tau}$, B and $e^{2\pi i A'/\tau}$, B' respectively. We can choose the square matrices A and A' such that their eigenvalues lie in the transversal $\mathbf{T}$. It then follows by the same reasoning as before that an element $\rho \in \mathrm{Hom}^{\theta\mathbb{Z},\delta}_{\mathcal{O}(\mathbb{C}^*)}(M, M')$

is given by a constant matrix that intertwines A, B and A', B', respectively. Hence, it is given by an element in $\mathrm{Hom}(V, V')$ that commutes with ρ_1 and ρ_2 (i.e. a morphism in $\mathsf{Rep}(\mathbb{Z}^2)$).

Finally, we show that $F_{\mathbf{T}}$ is a tensor functor. Suppose that (V, ρ_1, ρ_2) and (V', ρ_1', ρ_2') are two objects in $\mathsf{Rep}(\mathbb{Z}^2)$; we need to show that there are natural isomorphisms $c_{V,V'} : F(V) \otimes F(V') \to F(V \otimes V')$. As before, we define the connection matrix A by setting $e^{2\pi i A/\tau} = \rho_1(1)$ and $B = \rho_2(1)$; in the same manner we define A' and B' from ρ_1' and ρ_2'. We then have

$$F(V, \rho_1, \rho_2) \otimes F(V', \rho_1', \rho_2') = (M\sigma \otimes \sigma', \delta + A \otimes 1 + 1 \otimes A'),$$

where $M = (V \otimes \mathcal{O}(\mathbb{C}^*)) \otimes_{\mathcal{O}(\mathbb{C}^*)} (V' \otimes \mathcal{O}(\mathbb{C}^*))$.

One observes that the eigenvalues of the matrix $A \otimes 1 + 1 \otimes A'$ lie possibly outside the transversal $\mathbf{T}$. However, there is a unique matrix $\tilde{A}$ with all its eigenvalues in $\mathbf{T}$ such that

$$(18) \quad e^{2\pi i \tilde{A}/\tau} := e^{2\pi i (A \otimes 1 + 1 \otimes A')/\tau} = e^{2\pi i A/\tau} \otimes e^{2\pi i A'/\tau} \equiv \rho_1(1) \otimes \rho_1'(1).$$

The procedure of associating to $A \otimes 1 + 1 \otimes A'$ the matrix $\tilde{A}$ defines the required map $c_{V,V'}$ since $\tilde{A}$ is the connection matrix that one would have obtained (via $F_{\mathbf{T}}$) from $\rho_1 \otimes \rho_1'$. In fact, it follows that if $A \otimes 1 + 1 \otimes A'$ commutes with $B \otimes B' \equiv \rho_2(1) \otimes \rho_2'(1)$ then so does $\tilde{A}$. This map is natural in V and V' and the usual diagrams expressing associativity and commutativity (*cf.* for instance [**DM82**, Definition 1.8]) are satisfied. Moreover, it is bijective since an inverse can be constructed from Eqn. (18) by using the identification $\mathrm{End}_{\mathbb{C}}(V \otimes V') = \mathrm{End}_{\mathbb{C}}(V) \otimes \mathrm{End}_{\mathbb{C}}(V')$ to obtain A and A' back from $\tilde{A}$. □

Note that the choice of the transversal $\mathbf{T}$ is irrelevant since two functors $F_{\mathbf{T}}$ and $F_{\mathbf{T}'}$ associated to two different transversals $\mathbf{T}$ and $\mathbf{T}'$ to $\tau\mathbb{Z}$ are related via a natural transformation that is given explicitly by a shearing transformation as discussed before Proposition 3.15.

We observe that it is also possible to prove the above equivalence directly by means of the fibre functor ω. For this we consider the full subcategory of $\mathcal{B}^\tau_q$ such that the connection matrices have all eigenvalues in the same transversal $\mathbf{T}$. It follows from Proposition 3.15 that this category is equivalent to $\mathcal{B}^\tau_q$. By constructing the maps $c_{M,M'}$ very similar to those appearing in the above proof, one can show that this is an equivalence of rigid tensor categories. Moreover, the restriction of the fibre functor gives it the structure of a Tannakian category. The fibre functor induces an equivalence with $\mathsf{Rep}(\mathbb{Z}^2)$ by defining the action of $\mathbb{Z}^2$ on $(\ker\nabla)_z$ to be given by the matrices $e^{2\pi i A/\tau}$ and B. Clearly, the functor $\mathsf{F}_{\mathbf{T}}$ from the proof of Theorem 3.16 is the inverse to this fibre functor.

REMARK 3.17. For any group H the category of its finite dimensional representations over $\mathbb{C}$ forms a neutral Tannakian category, which should be equivalent to the category of representations of some affine group scheme, say $\hat{\mathsf{H}}$. The group scheme $\hat{\mathsf{H}}$ is called the *algebraic hull* of H. Strictly speaking, the affine group scheme underlying $\mathcal{B}_q^\tau$ is the algebraic hull of $\mathbb{Z}^2$. We refer the readers to [**van02**] for an explicit computation of the algebraic hull of $\mathbb{Z}$, which is $\mathrm{Hom}(\mathbb{C}/\mathbb{Z}, \mathbb{C}^*) \times \mathbb{G}_a$.

As a consequence we are able to conclude that the K-theory of $\mathcal{B}_q^\tau$ is the same as that of $\mathsf{Rep}(\mathbb{Z}^2)$. An object of $\mathsf{Rep}(\mathbb{Z}^2)$ is a vector space V equipped with two commuting linear invertible endomorphisms. Using the fact that the two endomorphisms commute, *i.e.,* respect each others eigenspaces, one can always find a common eigenvector w. This gives an exact sequence $0 \longrightarrow \langle w \rangle \longrightarrow \mathsf{V} \longrightarrow \mathsf{V}/\langle w \rangle \longrightarrow 0$ in $\mathsf{Rep}(\mathbb{Z}^2)$. Therefore, the K-theory of $\mathsf{Rep}(\mathbb{Z}^2)$ is the free abelian group generated by the simple objects, which are one dimensional representations with two actions a and b, with $a, b \in \mathbb{C}^*$ (the actions are given by multiplication by a and b respectively). The fibre functor sends the isomorphism class of $(\mathcal{O}(\mathbb{C}^*), b\alpha, \delta + z')$ with $z' \in \mathbb{C}$ to the simple object $(\mathbb{C}, b, e^{2\pi i z'/\tau})$ in $\mathsf{Rep}(\mathbb{Z}^2)$. Note that $(\mathcal{O}(\mathbb{C}^*), b\alpha, \delta + z')$ and $(\mathcal{O}(\mathbb{C}^*), b\alpha, \delta + (z' + n\tau))$ are isomorphic via the shearing transformation by z^n. Indeed,

$$(\delta + z')z^n f = n\tau z^n f + z^n \delta f + z' z^n f = z^n (\delta + (z' + n\tau)) f$$

and their images also get identified via the exponentiation. Summarising, we obtain

COROLLARY 3.18. *The K-theory of $\mathcal{B}_q^\tau$ is the free abelian group generated by the isomorphism classes of the objects $(\mathcal{O}(\mathbb{C}^*), b\alpha, \delta + z')$ with $b \in \mathbb{C}^*$ and $z' \in \mathbb{C}/\tau\mathbb{Z}$. Under this identification, one finds that the map on K-theory induced by the functor $\mathcal{S}_\tau \circ \psi_*$ sends the class of $(\mathcal{O}(\mathbb{C}^*), b\alpha, \delta + z')$ to the divisor class of the point $-z' \in X_\tau$ and their linear combinations accordingly.*

REMARK 3.19. One possible perspective of our work is the notion of a fundamental group of noncommutative tori. Given a noncommutative space described by its category of representations in the appropriate sense, *e.g.,* coherent sheaves, vector bundles with connections, *etc.*, it is conceivable that a description of its fundamental group can be obtained by finding a suitably defined Tannakian subcategory inside it. This philosophy stems from the original work of Nori [**Nor82, Nor76**] in the commutative case. For a classical complex torus $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ the fundamental group is just the lattice $\mathbb{Z}+\tau\mathbb{Z}$. The category $\mathcal{B}^\tau_q$ describes the quotient $\mathbb{C}/(\mathbb{Z}+\theta\mathbb{Z})$ with the infinitesimal $\tau\mathbb{Z}$ action providing the complex structure. Our construction proposes $\mathbb{Z}+\theta\mathbb{Z}$ as a candidate for the fundamental group in a Tannakian setting.

Remark 3.20. Invoking Manin's point of view again, we may disregard the order in which the quotients are performed. Ideally one would like to perform the double quotient operation in two different orders and show that they agree even *at infinity.* Consider first $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the infinitesimal action of $\theta\mathbb{Z}$ on it. It is described by the category $\mathcal{C}^{\theta,\tau}$, which is the heart of the t-structure of Example 2.8 on $D^b(X_\tau)$. If $g \in SL(2,\mathbb{Z})$ and g acts on τ by fractional linear transformation then $\mathcal{C}^{\theta,g\tau} \cong \mathcal{C}^{\theta,\tau}$. There is a unique cusp corresponding to the orbit of the rational numbers with respect to the modular group $SL(2,\mathbb{Z})$. This point corresponds to the nodal Weierstraß cubic E. One may consider a similar infinitesimal action of $\theta\mathbb{Z}$ in terms of t-structures on $D^b(E)$ depending on θ and their hearts as studied in [**BK06**]. On the other hand from Proposition 2.9 one finds that the $SL(2,\mathbb{Z})$ invariance of the categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ can be proven without referring to the equivalence with $\mathcal{C}^{\theta,\tau}$. In fact, it is possible to substitute any real value (in particular rational number) for τ in δ_τ. However, δ_τ does not remain injective for rational values of τ. In fact, one can check that for each rational number p/q, the kernel of $\delta_{p/q}$ is a *-subalgebra of $\mathcal{A}_\theta$ generated by $U_1^{-q}U_2^{p}$.

It is still plausible that the categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$, with $\tau \in \mathbb{Q}$ will be related to the hearts of the t-structures on $D^b(E)$ by functors similar to $\mathcal{S}_\tau$. The following observation might be a useful summary.

The action of $SL(2,\mathbb{Z})$ extends to the whole lower half plane $\mathbb{H}^-$. When adjoined with $\mathbb{P}^1(\mathbb{R})$, the quotient space contains the usual modular curve with an invisible stratum arising from the action of $SL(2,\mathbb{Z})$ on the irrationals, which has been investigated by Connes, Douglas and Schwarz in [**CDS98**] and separately by Manin and Marcolli in [**MM02**]. On the one hand, for a fixed θ, the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}^-$ is encoded in the isomorphism $\mathcal{S}_\tau \cong \mathcal{S}_{g\tau}$ of Polishchuk–Schwarz functors. In particular, $D^b(X_\tau) \cong D^b(X_{g\tau})$ inducing $\mathcal{C}^{\theta,\tau} \cong \mathcal{C}^{\theta,g\tau}$. On the other hand, for a fixed τ, the action of $SL(2,\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ manifests itself by the action induced by the *twist functors* $T_{\mathcal{O}}, T_{k(p_0)} \in \mathrm{Aut}(D^b(X_\tau))$ on the t-structures, $(D^{\theta,\leqslant 0}, D^{\theta,\geqslant 0}) \mapsto (D^{g\theta,\leqslant 0}, D^{g\theta,\geqslant 0})$ up to a shift (see Proposition 2.6 [**Pol04b**]).

A conjecture follows naturally:

CONJECTURE 3.21. *An appropriately defined category* $\mathsf{Vect}(\mathbb{T}_{\theta}^{\tau=\infty})$ *is equivalent to the heart of a t-structure on* $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$, E *being the nodal cubic, via a Polishchuk–Schwarz kind of functor.*

4. Noncommutative Geometry in DG framework

For a long time it was felt that the language of triangulated categories is deficient for many purposes in geometry. The language of DG categories seems to have resolved most of the technical and aesthetic problems. This section is rather speculative in nature and is a part of an on-going project. It is, in fact, a crystallisation of the authors personal communications with Matilde Marcolli, Yu. I. Manin, amongst others, and inspired by the works of Bondal, Drinfeld, Keller, Kontsevich, Lurie, Orlov, Toën, to name only a few. We first prepare the readers for the seemingly abstruse definition of the category of noncommutative spaces. [2] Readers should also refer to the articles of J. Lurie on derived algebraic geometry [**Lura, Lurb**], which seem to develop a geometry taking E_∞ rings as local models of spaces as opposed to honest commutative rings. Noncommutative geometry in their parlance is derived algebraic geometry.

4.1. Motivation. The traditional way of doing geometry with the emphasis on spaces is deficient in many physical situations. Most notably, due to *Heisenberg's Uncertainty Principle* one is forced to consider polynomial algebras with noncommuting variables, like Weyl algebras. One has to do away with the notion of points of a space quite naturally. However, one has perfectly well-defined algebras, albeit noncommutative, with which one can work. One very successful approach from this point of view is that of Connes [**Con94**]. It has many applications and a large part of the classical geometry (differential or spin, to be precise) can be subsumed in this setting. One might want to take a closer look at the key features of classical (algebraic) geometry and try to generalise them. For geometers the first task is to develop a theory over the algebraically closed fields of characteristic zero and by *Lefschetz Principle* there is no harm in assuming our ground field k to be actually $\mathbb{C}$.

From spaces to categories; from functions to sheaves: It quite common in mathematics to study an object via its representations (in an appropriate sense). It is neat to assemble all representations into a category and study it. In this manner from groups one is led to study Tannakian categories, from algebras certain triangulated categories of tilting modules and so on. This process is called categorification or possibly in a more fancy terminology *geometrization.*

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[2] A part of the material presented here can be found in a recent preprint of Kontsevich [**Kon**].

We have already done away with the traditional notion of a space via its set of points. For the time being it is described by its functions. The topology of a space allows us to define functions locally and glue them (if possible to a global one). Every classical space comes hand in hand with its *structure sheaf* of functions (continuous, smooth, holomorphic, algebraic, etc. determining the structure of the underlying space). The representations of the structure sheaf, which are nothing but quasicoherent sheaves, determine the space. In this manner one replaces the notion of a space by its category of quasicoherent sheaves, an idea that goes back to Grothendieck, Manin among others.

The category of quasicoherent sheaves is a *Grothendieck* category when the underlying space is quasi-compact and quasi-separated [**TT90**]. There are many approaches towards developing a theory by treating abelian categories (or some modifications thereof, like Grothendieck categories) as the category of quasicoherent sheaves on *noncommutative spaces* (*cf.* [**AZ94, van01, Ros98**], to name only a few).

REMARK 4.1. There is another point of view inspired by the Geometric Langlands programme and the details can be found, for instance, in [**Fre07**]. The guiding principle here is a generalisation of Grothendieck's *faisceaux-fonctions correspondence.* The faisceaux-fonctions correspondence appears naturally in the context of étale ℓ-adic sheaves. Associated to any complex of étale ℓ-adic sheaves $\mathscr{K}^\bullet$ over a variety V defined over a finite field $\mathbb{F}_q$ is a function $f^{\mathscr{K}^\bullet} : V \longrightarrow \mathbb{C}$ given by

$$f^{\mathscr{K}^\bullet}(x) = \sum (-1)^i \mathrm{Tr}(\mathrm{Fr}_{\bar{x}} \,|\, \mathrm{H}^i(\mathscr{K}^\bullet)_{\bar{x}}).$$

Here $x \in V(\mathbb{F}_q)$ and $\bar{x}$ denotes a geometric point of V over x. Of course, one has to fix an identification $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. According to Grothendieck all *interesting* functions appear in this manner and extrapolating this idea we regard constructible sheaves as the only source of interesting functions even over $\mathbb{C}$.

The lack of Verdier Duality, which is a generalisation of Poincaré Duality and hence an important feature, makes the naïve category of constructible ℓ-adic sheaves undesirable. Instead one works with the category of so-called perverse sheaves.[3] They are objects which live in a bigger derived category. Via a version of the Riemann-Hilbert correspondence over $\mathbb{C}$ the category of perverse sheaves (of middle perversity) is equivalent to the category of regular holonomic $\mathscr{D}$-modules.

[3]It is known that the derived category of coherent sheaves also admits a dualizing complex (see Proposition 1 [**Bez**]).

More precisely, let X be a complex manifold, $D^b_{rh}(\mathscr{D}_X)$ denote the bounded derived category of complexes of $\mathscr{D}_X$-modules with regular holonomic cohomologies and $D^b_c(\mathbb{C}_X)$ denote the bounded derived category of sheaves of complex vector spaces with constructible cohomologies. Then Kashiwara proved in [**Kas84**] $\mathscr{R}\mathscr{H}om_{\mathscr{D}_X}(-, \mathcal{O}_X) : D^b_{rh}(\mathscr{D}_X) \tilde{\to} D^b_c(\mathbb{C}_X)^{op}$ is a equivalence of triangulated categories. Under this equivalence the standard t-structure on $D^b_{rh}(\mathscr{D}_X)$, whose heart is the abelian category of regular holonomic $\mathscr{D}$-modules on X, is mapped to the heart of the t-structure of middle perversity on $D^b_c(\mathbb{C}_X)$. The heart of this t-structure is the category of perverse sheaves (of middle perversity), which can be regarded as another generalisation of functions. As opposed to a quasicoherent sheaf, the model for a function in this setting is a quasicoherent sheaf with a flat connection. Indeed, simplistically a $\mathscr{D}$-module can also be viewed as a quasicoherent sheaf with a flat connection. A quasicoherent sheaf (resp. $\mathscr{D}$-module) corresponds to a polynomial (resp. constructible locally constant) function.

The passage to derived categories: In the category of smooth schemes any morphism $f : X \to Y$ gives rise to two canonical functors on the category of sheaves, *viz.*, pull-back f^* and push-forward f_*. One should naturally expect any generalisation of classical geometry to allow such operations. We see that restricting to abelian categories is not enough as functors like push-forwards are not exact. The natural framework for such functors to exist is that of derived categories or abstract triangulated categories. Besides, if one chooses to work with perverse sheaves as substitutes for functions there is no way around.

Adding finite correspondences to morphisms: Denoting by $\mathcal{V}ar$ the category of complex algebraic varieties, Top that of *nice* topological spaces (here *nice* should imply all properties typical of the complex points of a complex algebraic variety like a complex manifold) one has a tensor functor $\mathcal{V}ar \to$ Top associating to a complex algebraic variety its underlying space with analytic topology. The tensor structure on the two categories is given by direct product. To a topological space in Top one can associate its singular cochain complex which is also a tensor functor to D_{ab}, the category of complexes of abelian groups with bounded cohomology complexes of finitely generated abelian groups. According to Beĭlinson and Vologodsky [**BV**] the basic objective of the theory of motives is to fill in a commutative diagram

$$\begin{array}{ccc} \mathcal{V}\mathfrak{ar} & \longrightarrow & \mathcal{D}_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathrm{Top} & \longrightarrow & \mathsf{D}_{\mathfrak{ab}} \end{array}$$

where $\mathcal{D}_{\mathcal{M}}$ is the rigid tensor triangulated category of motives. The upper horizontal arrow should be faithful and defined purely geometrically and right vertical arrow should respect the tensor structures. In order to construct the upper horizontal arrow one first embeds $\mathcal{V}\mathfrak{ar}$ inside a DG category and then takes a localization. In order to accomplish the first step, *i.e.,* to embed $\mathcal{V}\mathfrak{ar}$ inside some DG category one needs to enlarge the Hom sets to include finite correspondences. This endows $\mathcal{V}\mathfrak{ar}$ with an additive structure.

Triangulated structure is not enough: The hope is to be able to construct a rigid tensor category of *motivic* noncommutative spaces which allows basic operations like pull-back, push-forward and finite correspondences (as morphisms). In the classical setting, we have a construction of $\mathcal{D}_{\mathcal{M}}$ as a triangulated category due to Voevodsky (see *e.g.,* [**FSV00**]). However, one would like to extract the *right* category of motives inside it (possibly as an abelian rigid tensor category). One basic operation in $\mathcal{V}\mathfrak{ar}$ is that of direct product, which defines the tensor structure. It should also survive in $\mathcal{D}_{\mathcal{M}}$. The tensor product of two triangulated categories unfortunately does not carry a natural triangulated structure. Also one runs into trouble in trying to define inner Hom's. The framework of DG (differential graded) categories comes in handy.

4.2. Overview of DG categories. Before we are able to spell out the definition of the category of noncommutative spaces we need some preparation on DG categories, which will be quite concise. For details we refer the readers to *e.g.,* [**Dri04**],[**Kel06b**]. They can be defined over k, where k is not necessarily a field. However, as mentioned before, we set $\mathsf{k} = \mathbb{C}$ and, unless otherwise stated, all our categories are assumed to be k-linear.

A category $\mathcal{C}$ is called a DG category if for all $\mathsf{X}, \mathsf{Y} \in \mathrm{Obj}(\mathcal{C})$ $\mathrm{Hom}(\mathsf{X}, \mathsf{Y})$ has the structure of a complex of k-linear spaces (in other words, a DG vector space) and the composition maps are associative k-linear maps of DG vector spaces (or henceforth DG k-modules). In particular, $\mathrm{Hom}(\mathsf{X}, \mathsf{X})$ is a DG algebra with unit.

Let $\mathcal{DG}\mathfrak{cat}$ stand for the category of all small DG categories. The morphisms in this category are *DG functors* $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ such that for all $\mathsf{X}, \mathsf{Y} \in \mathrm{Obj}(\mathcal{C})$

$$\mathsf{F}(\mathsf{X}, \mathsf{Y}) : \mathrm{Hom}(\mathsf{X}, \mathsf{Y}) \longrightarrow \mathrm{Hom}(\mathsf{FX}, \mathsf{FY})$$

is a morphism of DG k-modules compatible with the compositions and the units.

The tensor structure: The tensor product of two DG categories $\mathcal{C}$ and $\mathcal{D}$ can be defined in the obvious manner, *viz.,* the objects of $\mathcal{C} \otimes \mathcal{D}$ are written as $\mathsf{X} \otimes \mathsf{Y}$, $\mathsf{X} \in \mathrm{Obj}(\mathcal{C})$, $\mathsf{Y} \in \mathrm{Obj}(\mathcal{D})$ and one sets

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}(\mathsf{X} \otimes \mathsf{Y}, \mathsf{X}' \otimes \mathsf{Y}') = \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{X}') \otimes \mathrm{Hom}_{\mathcal{D}}(\mathsf{Y}, \mathsf{Y}')$$

with natural compositions and units.

Happily enough, the category of DG functors $\mathscr{H}\mathsf{om}(\mathcal{C}, \mathcal{D})$ between two DG categories $\mathcal{C}$,$\mathcal{D}$ with appropriately defined morphisms is once again a DG category. With respect to the above-mentioned tensor product $\mathcal{DG}\mathfrak{cat}$ becomes a symmetric tensor category with an internal Hom functor given by $\mathscr{H}\mathsf{om}$, *i.e.,*

$$\mathrm{Hom}(\mathcal{B} \otimes \mathcal{C}, \mathcal{D}) = \mathrm{Hom}(\mathcal{B}, \mathscr{H}\mathsf{om}(\mathcal{C}, \mathcal{D})).$$

However, for our purposes this defintion of the internal Hom functor will prove to be inaccurate later on.

The derived category of a DG category: The standard refence for the construction is [**Kel94**]. We recall some basic facts here. Let $\mathcal{C}$ be a small DG category. A right DG $\mathcal{C}$-module is by definition a DG functor $\mathsf{M} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}_{\mathsf{dg}}(\mathsf{k})$, where $\mathcal{C}_{\mathsf{dg}}(\mathsf{k})$ denotes the DG category of complexes of k-linear spaces. Note that the composition of morphisms in the opposite category is defined by the *Koszul sign rule*: the composition of f and g in $\mathcal{C}^{\mathrm{op}}$ is equal to the morphism $(-1)^{|\mathsf{f}||\mathsf{g}|}\mathsf{gf}$ in $\mathcal{C}$. Every object X of $\mathcal{C}$ defines canonically a *free* right module $\mathsf{X}^{\wedge} := \mathrm{Hom}(-, \mathsf{X})$. A morphism of DG modules $\mathsf{f} : \mathsf{L} \to \mathsf{M}$ is by definition a morphism (natural transform) of DG functors such that $\mathsf{fX} : \mathsf{LX} \to \mathsf{MX}$ is a morphism of complexes for all $\mathsf{X} \in \mathrm{Obj}(\mathcal{C})$. We call such an f a quasi-isomorphism if fX is a quasi-isomorphism for all X, *i.e.,* fX induces isomorphism on cohomologies.

Definition 4.2. *The derived category* $\mathsf{D}(\mathcal{C})$ *of* $\mathcal{C}$ *is defined to be the localization of the category of right DG* $\mathcal{C}$*-modules with respect to the class of quasi-isomorphisms.*

Remark 4.3. With the translation induced by the shift of complexes and triangles coming from short exact sequence of complexes, $\mathsf{D}(\mathcal{C})$ becomes a triangulated category. The Yoneda functor $\mathsf{X} \mapsto \mathsf{X}^\wedge$ induces an embedding of $\mathrm{H}^0(\mathcal{C}) \to \mathsf{D}(\mathcal{C})$. Here $\mathrm{H}^0(\mathcal{C})$ stands for the zeroth-cohomology category[4] with the objects same as $\mathcal{C}$ and morphisms replaced by the zeroth cohomology, *i.e.*, $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{C})}(\mathsf{X},\mathsf{Y}) = \mathrm{H}^0\mathrm{Hom}_{\mathcal{C}}(\mathsf{X},\mathsf{Y})$.

Definition 4.4. *The triangulated subcategory of* $\mathsf{D}\mathcal{C}$ *generated by the free DG* $\mathcal{C}$*-modules* $\mathsf{X}^\wedge$ *under translations in both directions, extensions and passage to direct factors is called the* **perfect** *derived category and denoted by* $\mathrm{per}(\mathcal{C})$. *A DG category* $\mathcal{C}$ *is said to be* **pretriangulated** *if the above-mentioned Yoneda functor induces an equivalence* $\mathrm{H}^0(\mathcal{C}) \to \mathrm{per}(\mathcal{C})$.

Remark 4.5. A *pretriangulated* category does not have a triangulated structure. Rather it is a DG category, which is equivalent to the notion of an *enhanced triangulated category* in the sense of Bondal–Kapranov [**BK89**]. The associated zeroth cohomology category is honestly triangulated and equivalent to $\mathrm{per}(\mathcal{C})$. Let us mention that some authors also call pretriangulated categories as triangulated DG categories.

Definition 4.6. *A DG functor* $\mathsf{F}:\mathcal{C} \to \mathcal{D}$ *is called a Morita morphism if it induces an equivalence* $\mathsf{F}^*:\mathsf{D}(\mathcal{C}) \to \mathsf{D}(\mathcal{D})$.

4.3. The category of noncommutative spaces. The definition provided below is a culmination of the works of several people spanning over two decades including Bondal, Drinfeld, Keller, Kontsevich, Lurie, Orlov and Toën, amongst others. This list of names is certainly not definitive and it only reflects the authors ignorance of the history behind this development.

Definition 4.7. *The category of noncommutative spaces* $\mathcal{NCS}$ *is the localization of* $\mathcal{DG}\mathfrak{cat}$ *with respect to Morita morphisms.*

Thanks to Tabuada [**Tab05b, Tab05a**] we know that $\mathcal{DG}\mathfrak{cat}$ has a Quillen model category structure, where the weak equivalences are the Morita morphisms and the fibrant objects are certain pretriangulated DG categories. As a consequence we deduce that $\mathcal{NCS}$ is the homotopy category of $\mathcal{DG}\mathfrak{cat}$ in the sense of Quillen. This enables us to conclude that each object of $\mathcal{NCS}$ is isomorphic to a pretriangulated

[4]It is also called the homotopy DG category.

DG category. The tensor product of $\mathcal{DG}\mathfrak{cat}$ induces one on $\mathcal{NCS}$ after replacing any object by its cofibrant model since the tensor product by a cofibrant DG module preserves weak equivalences.

REMARK 4.8. We have deliberately included geometric correspondences in the category of noncommutative spaces. These spaces are somewhat *motivic* in nature and it is expected to be a feature of this geometry. We do not want to think of $\mathcal{NCS}$ as a 2-category.

However, the internal Hom functor cannot be derived from $\mathcal{DG}\mathfrak{cat}$. Thanks to Toën [**Toë07**] (also *cf.* [**Kel06a**]) one knows that there does exist an internal Hom functor given by

$$\mathscr{H}\mathsf{om}(\mathcal{C},\mathcal{D}) = \text{cat. of } \mathsf{A}_\infty\text{-functors } \mathcal{C} \to \mathcal{D} \tag{19}$$

here $\mathcal{D}$ needs to be a pretraingulated DG category which is no restriction since we know that in $\mathcal{NCS}$ every object is isomorphic to a pretriangulated DG category. The DG structure of $\mathcal{D}$ endows $\mathscr{H}\mathsf{om}(\mathcal{C},\mathcal{D})$ with a DG structure as well. We will not be able to discuss A_∞-categories and A_∞-functors here. Let us mention that a DG category is a special case of an A_∞-category and we refer the readers to, *e.g.,* [**Kel06a**] for a highly readable survey on the same.

REMARK 4.9. The Hom sets in $\mathcal{NCS}$ are commutative monoids (under a direct sum of kernels see Theorem 4.16 and the Remark thereafter below) and it is possible to talk about exact sequences in $\mathcal{NCS}$ in the sense of Quillen's *admissible exact sequences* (see Definition 4.18 below).

DEFINITION 4.10 (Kontsevich).

- *A noncommutative space (DG category)* $\mathcal{C}$ *is called smooth if the bimodule given by the DG bifunctor* $(\mathsf{X},\mathsf{Y}) \mapsto \mathrm{Hom}(\mathsf{X},\mathsf{Y})$ *is in* $\mathrm{per}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$*.*
- *It is called proper if if is isomorphic in* $\mathcal{NCS}$ *to a DG algebra whose homology is of finite total dimension.*

The motivation behind the definition is the following result (Corollary 3.1.8 [**Bv03**]).

THEOREM 4.11 (Bondal–van den Bergh, Keller). *Assume that* X *is a quasi-compact and quasi-separated scheme. Then* $\mathsf{D}_{\mathsf{Qcoh}}(\mathsf{X})$ *is equivalent to* $\mathsf{D}(\Lambda)$ *for a suitable DG algebra* Λ *with bounded cohomology.*

Note that in this theorem $\mathrm{D}_{\mathrm{QCoh}}(\mathsf{X})$ denotes the honest derived category of complexes of $\mathcal{O}_{\mathsf{X}}$-modules with quasicoherent cohomologies and $\mathsf{D}(\Lambda)$ likewise.

Viewing classical geometry in this setting: Thanks to a quotient construction in the world of DG categories proposed by Drinfeld (see Theorem 3.4 of [**Dri04**]) [5] we are able to define the *DG category of quasicoherent sheaves* on an honest scheme X as

$$\mathcal{D}_{\mathrm{dg}}(\mathsf{X}) = \frac{\text{DG cat. of complexes over QCoh}(\mathsf{X})}{\text{DG subcat. of acyclic complexes}} \quad , \tag{20}$$

which is how we view classical schemes in this framework. It is also known that $\mathrm{H}^0\mathcal{D}_{\mathrm{dg}}(\mathsf{X}) \tilde{\rightarrow} \mathsf{D}_{\mathrm{QCoh}}(\mathsf{X})$. As mentioned above there are reconstruction Theorems available from QCoh(X) and from $\mathsf{D}_{\mathrm{QCoh}}(\mathsf{X})$ only under certain assumptions [**BO01**], which glaringly exclude abelian varieties. For abelian varieties we do have an understanding of the derived category and its autoequivalences [**Orl02**].

Remark 4.12. Those who prefer regular holonomic $\mathscr{D}$-modules as substitutes for functions can perform the above operation after replacing QCoh(X) by the category of regular holonomic $\mathscr{D}$-modules.

One may complain that non-isomorphic classical spaces (*e.g.,* an abelian variety and its dual) become isomorphic in $\mathcal{NCS}$. We would like to argue that it is a feature of the geometry, rather than a drawback. Since we have enhanced the morphisms between our spaces by incorporating certain finite correspondences, we have also increased the chance of objects becoming isomorphic. In fact, due to Mukai [**Muk81**] we know that an abelian variety is derived equivalent to its dual precisely via a *correspondence-like* morphism, which is a Fourier–Mukai transform. Roughly, given any two smooth projective varieties X and Y and an object in $\mathscr{E} \in \mathsf{D}^{\mathsf{b}}(\mathsf{X} \times \mathsf{Y})$ one constructs an exact Fourier–Mukai transform (also sometimes called an integral transform) $\Phi^{\mathscr{E}}_{\mathsf{X}\to\mathsf{Y}} : \mathsf{D}^{\mathsf{b}}(\mathsf{X}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{Y})$ as follows:

$$\Phi^{\mathscr{E}}_{\mathsf{X}\to\mathsf{Y}}(-) = \pi_{\mathsf{Y}*}\left(\mathscr{E} \otimes \pi^*_{\mathsf{X}}(-)\right),$$

where π_{X} (resp. ϕ_{Y}) denotes the projection $\mathsf{X} \times \mathsf{Y} \to \mathsf{X}$ (resp. $\mathsf{X} \times \mathsf{Y} \to \mathsf{Y}$). Here all functors are assumed to be appropriately derived. The object $\mathscr{E}$ is called the *kernel* of the Fourier–Mukai transform. In the case of the equivalence between an abelian variety A and its dual $\hat{\mathsf{A}}$ the kernel is given by the Poincaré sheaf $\mathcal{P}$. Given a *divisorial correspondence* in $\mathsf{X} \times \mathsf{Y}$ one can consider the corresponding line bundle

[5] Drinfeld works over a ground ring k (not necessarily a field) and hence the result is true only under a certain *homotopic flatness* condition over k. However, this is automatically true when k is a field, which is what we have assumed. The proposed quotient is also unique in an appropriate sense.

on $X \times Y$ and use that as the kernel of a Fourier–Mukai transform. Conversely, given a kernel $\mathscr{E} \in D^b(X \times Y)$ of a Fourier–Mukai transform one obtains a cycle (correspondence modulo an equivalence relation) in $X \times Y$ by applying the Chern character to $\mathscr{E}$.

4.3.1. *Very brief interlude on pure motives.* The way one makes $\mathcal{V}ar$ additive is by enriching the Hom sets to geometric correspondences (up to an adequate equivalence relation). Here honest morphisms are seen as correspondences via their *graphs*. Let us explain it in a bit more detail. A *global intersection theory* C is simultaneously a contravariant functor from $\mathcal{V}ar$ to the category of Λ-algebras and a covariant functor from $\mathcal{V}ar$ to Λ-modules, for some fixed commutative ring Λ. For $\phi : X \to Y$ in $\mathcal{V}ar$ the image under C as a contravariant (resp. covariant) functor is denoted by ϕ^* (resp. ϕ_*). For any two varieties $X, Y \in \mathcal{V}ar$ we are given a Λ-algebra homomorphism

$$C(X) \otimes_\Lambda C(X) \longrightarrow C(X \times Y)$$

We denote the image of $x \otimes y$ under this map by $x \times y$. For the rest of the axioms we refer the readers to, *e.g.,* [**Gro58, Man68**]. However, two axioms are worth mentioning.

Multiplication Axiom: Let $X \in \mathcal{V}ar$ and $\delta_X : X \to X \times X$ be the diagonal morphism. Then The composition homomorphism of Λ-algebras

$$C(X) \otimes_\Lambda C(X) \longrightarrow C(X \times X) \xrightarrow{\delta_X^*} C(X)$$

coincides with the homomorphism of multiplication : $\delta_X^*(x \times y) = xy$.

Projection Formula: Let $\phi : X \to Y$ be a morphism in $\mathcal{V}ar$ and let $x \in C(X)$ and $y \in C(Y)$. Then

$$\phi_*(x\phi^*(y)) = \phi_*(x)y$$

Example 2. (1) $C(X) = K(X)$ *and* $\Lambda = \mathbb{Z}$
(2) $C(X) = A(X)$ *the Chow ring (the adequate equivalence relation is the rational equivalence)*
(3) $C(X) = H^{ev}(X, \Lambda)$*, where* Λ *is a* $\mathbb{Q}$*-algebra.*

Any element of the ring $C(X \times Y)$ is called a C-correspondence between X and Y. Let $f \in C(X \times Y)$ and $g \in C(Y \times Z)$. The correspondence

$$g \circ f = p_{13*}(p_{12}^*(f)p_{23}^*(g)) \in C(X \times Z),$$

where p_{12} stands for the projection $X \times Y \times Z \to X \times Y$ and so on.

It is possible to define a graded version of the same by requiring the functor C to take values in graded (by degrees $\geqslant 0$) Λ-algebras such that $C^i(X) = 0$ for $i > \dim X$ with ϕ^* being homogeneous of degree zero and ϕ_* homogeneous of degree $m - n$, where $\phi : X \to Y$ and X, Y are equidimensional of dimensions n and m respectively.

We enhance the morphisms of $\mathcal{V}ar$ by setting $\mathrm{Hom}(X, Y) = C^n(X \times Y)$. Given a morphism $\phi : Y \to X$ in $\mathcal{V}ar$ its *graph* is the morphism $\Gamma_\phi := (\phi \times \mathrm{id}_Y) \circ \delta_Y : Y \to X \times Y$. The morphism ϕ in $\mathcal{V}ar$ is identified with the element $\Gamma_{\phi *}(\mathrm{id}_Y) \in C(X \times Y)$.

Definition 4.13. *An additive category $\mathcal{D}$ is called pseudo-abelian if for any projector (idempotent) $p \in \mathrm{Hom}(X, X)$, $X \in \mathrm{Obj}(\mathcal{D})$ there exists a kernel* $\ker p$ *and the canonical homomorphism* $\ker p \oplus \ker(\mathrm{id}_X - p) \to X$ *is an isomorphism.*

There is a canonical *pseudo-abelian completion* $\overline{\mathcal{D}}$ of any additive category $\mathcal{D}$. The object of $\overline{\mathcal{D}}$ are pairs (X, p), where $X \in \mathrm{Obj}(\mathcal{D})$ and $p \in \mathrm{Hom}_{\mathcal{D}}(X, X)$ is an arbitrary projector. Define Hom sets as

$$\mathrm{Hom}_{\overline{\mathcal{D}}}((X, p), (Y, q)) = \frac{\{f \in \mathrm{Hom}_{\mathcal{D}}(X, Y) \ \text{ such that } fp = qf\}}{\{\text{subgroup of } f \ \text{ such that } fp = qf = 0\}}$$

Roughly, given a graded global intersection theory C we can construct the category of C-motives from $\mathcal{V}ar$ by first enhancing the morphisms to degree zero correspondences and then taking the pseudo-abelian completion of it. In the resulting category the motive of $\mathbb{P}^n$ should decompose as $\mathbb{P}^n = \mathrm{pt} \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2} \oplus \cdots \oplus \mathbb{L}^{\otimes n}$. The object $\mathbb{L}$ is called the *Lefschetz motive* and it should be formally inverted in order to obtain the category of pure motives and morphisms should also be defined appropriately, but we gloss over these details here.

Our ground field has always been k. Now we also assume that $\Lambda = \mathbb{Q}$ and our global intersection theory is that of cycles modulo numerical equivalence. Restricting oneself to the category of connected curves and applying the above machinery one obtains a category of motives of curves (with respect to the our chosen intersection theory). This category admits a better description.

Proposition 4.14 ([**Man68**]). *The category of motives of curves is equivalent to the category of abelian varieties up to isogeny.*

Remark 4.15. The functor associates to a curve its Jacobian variety. It turns out that the category of abelian varieties up to isogeny is abelian and semisimple.

The category of motives is expected to be semisimple and Tannakian (Jannsen showed that the category of motives modulo numerical equivalence is semisimple [**Jan92**]). The category $\mathcal{NCS}$ has some *motivic features* (Remark 4.9 says that $\mathcal{NCS}$ is "close to being additive"). It also has a tensor structure and an internal Hom functor. However, not all objects T are *rigid*, *i.e.,* the canonical morphism $\mathsf{T} \otimes \mathsf{T}^{\vee} \to \mathrm{Hom}(\mathsf{T},\mathsf{T})$ is not an isomorphism. However, the smooth and proper noncommutative spaces are rigid in the above sense.

4.4. Noncommutative motives. The first step of the construction of pure motive entails a linearization of the category $\mathcal{V}\mathsf{ar}$ by including geometric correspondences. We have argued that correspondences induce DG functors (indeed, the kernel of a Fourier–Mukai transform should be thought of as a correspondence). The following Theorem [**Toë07**] says that all DG-functors are described by a Fourier–Mukai *kernel.* The slogan is that DG functors between DG categories is more relevant to geometry than exact functors between honest derived categories. We state the Theorem in its full generality and hence let k be any commutative ring.

THEOREM 4.16 (Toën). *Let* X *and* Y *be quasi-compact and separated schemes over* k *such that* X *is flat over* $\mathrm{Spec}\,\mathsf{k}$*. Then there is a canonical isomorphism in* $\mathcal{NCS}$

$$\mathsf{D}_{\mathsf{dg}}(\mathsf{X} \times_{\mathsf{k}} \mathsf{Y}) \xrightarrow{\sim} \mathscr{H}\mathsf{om}_{\mathsf{c}}(\mathsf{D}_{\mathsf{dg}}(\mathsf{X}), \mathsf{D}_{\mathsf{dg}}(\mathsf{Y})),$$

where $\mathscr{H}\mathsf{om}_{\mathsf{c}}$ *denotes the full subcategory of* $\mathscr{H}\mathsf{om}$ *formed by coproduct preserving quasi-functors (cf. Remark 4.17 below for the definition). Moreover, if* X *and* Y *are smooth and projective over* $\mathrm{Spec}\,\mathsf{k}$*, we have a canonical isomorphism in* $\mathcal{NCS}$

$$\mathrm{Perf}_{\mathsf{dg}}(\mathsf{X} \times_{\mathsf{k}} \mathsf{Y}) \xrightarrow{\sim} \mathscr{H}\mathsf{om}(\mathrm{Perf}_{\mathsf{dg}}(\mathsf{X}), \mathrm{Perf}_{\mathsf{dg}}(\mathsf{Y})),$$

where $\mathrm{Perf}_{\mathsf{dg}}$ *denotes the full subcategory of* D_{dg}*, whose objects are perfect complexes.*

REMARK 4.17. The above Theorem admits a natural generalization to abstract DG categories (not necessarily of the form $\mathsf{D}_{\mathsf{dg}}(\mathsf{X})$ for some scheme X), which can also be found in *ibid.*. The above theorem asserts an equivalence of categories. It can be suitably *decategorified*, in order to have an understanding of the morphisms on the right hand side only. For a DG categories $\mathcal{A}, \mathcal{B}$, let $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ denote the full subcategory of the derived category $\mathsf{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ (see Definition 4.2) of $\mathcal{A}$–$\mathcal{B}$-bimodules formed by the bimodules M such that $- \otimes \mathsf{X} : \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ sends a free $\mathcal{A}$-module to a $\mathcal{B}$-module quasi-isomorphic to a free one (recall an $\mathcal{A}$-module is free if, by definition, it is of the form $\mathrm{Hom}(-, \mathsf{X})$, $\mathsf{X} \in \mathrm{Obj}(\mathcal{A})$. The decategorified statement is that morphism sets (between $\mathcal{A}$ and $\mathcal{B}$) in $\mathcal{NCS}$ are in a natural bijection with isomorphism classes of objects in $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ *ibid.*. Objects of $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ are called *quasi-functors* as they induce honest functors $\mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\mathcal{B})$.

Let us remind the readers that $\mathscr{H}\mathsf{om}$ denotes the internal Hom functor as in (19).

Generalizing this intuition we conclude that the morphisms in $\mathcal{NCS}$ already contain all geometric correspondences. However, $\mathcal{NCS}$ is not a k-linear category, namely, there is no abelian group structure on the set of morphisms. However, there is a semi-additive structure given by the direct sum of kernels of two DG functors. We abelianize them by applying the so-called K_0-decategorification (see, for instance, [**Kon**],[**Tab05a**]).

Recall from Remark 4.9 that it is possible to talk about exact sequences in $\mathcal{NCS}$. We provide one formulation of exact sequences of DG categories (see, *e.g.,* Theorem 4.11 of [**Kel06b**] for other equivalent definitions).

Definition 4.18. *A sequence of DG categories*

$$\mathcal{A} \xrightarrow{\mathrm{P}} \mathcal{B} \xrightarrow{\mathrm{I}} \mathcal{C}$$

such that $\mathrm{I}\mathsf{P} = 0$ *is called exact if and only if* P *induces an equivalence of* $\mathrm{per}(\mathcal{A})$ *onto a thick subcategory of* $\mathrm{per}(\mathcal{B})$ *and* I *induces an equivalence between the idempotent closure of the Verdier quotient* $\mathrm{per}(\mathcal{B})/\mathrm{per}(\mathcal{A})$ *and* $\mathrm{per}(\mathcal{C})$*.*

Remark 4.19. In the classical setting, if X is a quasi-compact quasi-separated scheme, $\mathsf{U} \subset \mathsf{X}$ a quasi-compact open subscheme and $\mathsf{Z} = \mathsf{X} \setminus \mathsf{U}$, then the following sequence

$$\mathrm{Perf}_{\mathsf{dg}}(\mathsf{X} \text{ on } \mathsf{Z}) \longrightarrow \mathrm{Perf}_{\mathsf{dg}}(\mathsf{X}) \longrightarrow \mathrm{Perf}_{\mathsf{dg}}(\mathsf{U})$$

is exact according to the definition, where $\mathrm{Perf}_{\mathsf{dg}}(\mathsf{X} \text{ on } \mathsf{Z})$ denotes the full subcategory of $\mathrm{Perf}_{\mathsf{dg}}(\mathsf{X})$ of perfect complexes supported on Z.

Caution: The object $\mathrm{Perf}_{\mathsf{dg}}$ should not be confused with per as in Definition 4.4.

This property should remind the author of the excision operation (*cf.* Equation (21) below). One knows that there is a well defined K-theory functor on $\mathcal{NCS}$ (or on *spectra*), which agrees with Quillen's

K-theory of an exact category $\mathcal{B}$, when applied to $\mathsf{D}^{\mathsf{b}}_{\mathsf{dg}}(\mathcal{B})$. Now we define the category of noncommutative motives.

Definition 4.20. *The category of noncommutative motives $\mathcal{NCM}$ is the additive category defined as:*

- $\mathrm{Obj}(\mathcal{NCM}) = \mathrm{Obj}(\mathcal{NCS})$
- $\mathrm{Hom}_{\mathcal{NCM}}(\mathcal{C}, \mathcal{D}) = \mathsf{K}_0(\mathrm{rep}(\mathcal{C}, \mathcal{D}))$

As a motivation we provide two Theorems: the first Theorem buys us the *additivization* of $\mathcal{NCS}$, while the second one shows *excision compatibility.*

Theorem 4.21. [**Tab05a**] *A functor F from $\mathcal{NCS}$ to an additive category factors through $\mathcal{NCM}$ if and only if for every exact DG category $\mathcal{B}$ endowed with two full exact DG subcategories $\mathcal{A}, \mathcal{C}$ which give rise to a semiorthogonal decomposition $\mathrm{H}^0(\mathcal{B}) = (\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}))$ in the sense of* [**BO**]*, the inclusions induce an isomorphism $\mathsf{F}(\mathcal{A}) \oplus \mathsf{F}(\mathcal{C}) \xrightarrow{\sim} \mathsf{F}(\mathcal{B})$.*

Such a functor is called an *additive invariant* of noncommutative spaces. The simplest example is $\mathcal{A} \longmapsto \mathsf{K}_0(\mathrm{per}(\mathcal{A}))$.

Theorem 4.22. [**DS04**] *The functor $\mathcal{A} \longmapsto \mathsf{K}(\mathcal{A})$ (Waldhausen K-theory) yields, for each short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathcal{NCS}$, a long exact sequence*

$$\cdots \longrightarrow \mathsf{K}_i(\mathcal{A}) \longrightarrow \mathsf{K}_i(\mathcal{B}) \longrightarrow \mathsf{K}_i(\mathcal{C}) \longrightarrow \cdots \longrightarrow \mathsf{K}_0(\mathcal{B}) \longrightarrow \mathsf{K}_0(\mathcal{C}).$$

Remark 4.23. The category $\mathcal{NCM}$ is additive and the composition is induced by that of $\mathcal{NCS}$. Strictly speaking, we should perform a formal idempotent completion (or pseudo-abelian completion) of $\mathcal{NCM}$ as discussed in Subsection 4.3.1. However, we are mostly going to work with the Grothendieck ring of $\mathcal{NCM}$, which is affected neither by the closure under direct sums nor by the idempotent completion.

Remark 4.24. Certain non-isomorphic objects of $\mathcal{NCS}$ become isomorphic in $\mathcal{NCM}$, *e.g.,* it is shown in [**Kel98**] that, if the ground field k is algebraically closed, each finite dimensional algebra of finite global dimension becomes isomorphic to a product of copies of k in $\mathcal{NCM}$, whereas such a thing is true in $\mathcal{NCS}$ if and only if the algebra is semisimple.

A careful reader should have noticed that we have glossed over the issue of the choice of a (graded) global intersection theory which was central to the construction of the category of pure motives in the classical setting. Let us recall that we made $\mathcal{V}\mathsf{ar}$ into an additive category by setting $\mathrm{Hom}(X, Y) = C^n(X \times Y)$, where C is the chosen graded global intersection theory and X equidimensional of dimension n. We could have imitated the same construction without taking into account the grading of the intersection theory and simply setting $\mathrm{Hom}(X, Y) = C(X \times Y)$. The resulting category is called the category of C-correspondences. Manin mentions in [**Man68**] (end of Section 3) that every *geometric* cohomology theory should be a cohomological functor on the category of C-correspondences, *i.e.,* every correspondence in $C(X \times Y)$ should induce a well-defined morphism $H^*(X) \to H^*(Y)$. Now we turn the argument around. We call a correspondence *geometric* if it induces a morphism between the universal cohomology theories. The existence of the universal cohomology theory is itself unresolved. Our spaces are defined in terms of the (quasicoherent) cohomology theories they support. Indeed, the DG category describing a space contains all cochain complexes whose cohomology complex is a cohomology theory on that space. We pretend that a morphism (a functor) in $\mathcal{NCS}$ is a morphism between the cohomology theories on the two spaces, as if given by some geometric correspondence. Argument in favour of that - if a correspondence induces morphisms between all cohomology theories then, if there were a universal one, it would also induce a morphism between them. Theorem 4.16 should be put in this perspective. The passage from $\mathcal{NCS}$ to $\mathcal{NCM}$ is simply linearizing the category. The category $\mathcal{NCS}$ contains correspondences, in particular those given by codimension 1 subvarieties, and in $\mathcal{NCM}$ we allow linear combinations of them, *i.e.,* all divisors. The equivalence relation can be regarded as a *universal* one, namely, one which identifies two correspondences which induce isomorphic morphisms between the corresponding *universal cohomology theories*. Note that in $\mathcal{NCM}$ we set the Grothendieck group of $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ as morphisms between $\mathcal{A}$ and $\mathcal{B}$. Chow correspondences obtained by taking the rational equivalence relation is geometric. The connection should be an analogue of the *Chern character* map which identifies the K-theory with the Chow group after tensoring with $\mathbb{Q}$. The recent work of Bressler, Gorokhovsky, Nest and Tsygan [**BGNT**] should be useful.

4.5. Motivic measures and motivic zeta functions. We provide a rather simplistic point of view on motivic measures. With respect to a motivic measure it is possible to develop a theory of *motivic integration* (see, *e.g.,* [**DL02**]), which we shall not discuss here. A good reference for most of the intricacies and ramifications is, *e.g.,* [**Loo02**].

Let k be a field (not necessarily k for the time being) and let Sch_k be the category of schemes of finite type over k. Consider the

Grothendieck ring of Sch_k, denoted by $\mathrm{Fun}^{\mathrm{poor}}(k)$ [6], which is defined as the free abelian group generated by isomorphism classes of objects in Sch_k modulo relations

$$[X] = [Z] + [X \setminus Z], \tag{21}$$

where Z is a closed subscheme of X. The multiplication is given by the fibre product over k. There is a unit given by the class of $\mathrm{Spec}\, k$.

Let A be any commutative ring. An A-valued *motivic measure* is a ring homomorphism $\mu : \mathrm{Fun}^{\mathrm{poor}}(k) \to A$. If A has a unit the homomorphism is required to be unital.

Example 3. *Let $k = \mathbb{C}$, $A = \mathbb{Z}$ and $\mu(X) = \chi_c(X)$, i.e., the Euler characteristic with compact supports.*

Example 4. *Let $k = \mathbb{F}_q$, $A = \mathbb{Z}$ and $\mu(X) = \#X(\mathbb{F}_q)$, i.e., the number of $\mathbb{F}_q$-points.*

Remark 4.25. There is a way to naturally enhance the notion of $\mathrm{Fun}^{\mathrm{poor}}(k)$ to the categorical level. The *motivic category of poor man's functions* is obtained by setting $\mathrm{Hom}(X, Y) := \mathrm{Fun}^{\mathrm{poor}}(X \times_k Y)$, objects being the same as that of Sch_k. This category attains a tensor structure under fibre product and sums are given by disjoint unions. One could also consider a more general setting, *i.e.,* working over a noetherian base scheme S instead of k.

Let us fix an A-valued motivic measure μ and, for a smooth $X \in \mathrm{Sch}_k$, let $X^{(n)}$ denote the n-fold symmetric product of X. Set $X^{(0)} := \mathrm{Spec}\, k$. Then associated to μ there is a *motivic zeta function* (possibly due to Kapranov [**Kap**]) of X defined by the formal series

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu\left(X^{(n)}\right) t^n \in A[[t]]. \tag{22}$$

Example 5. *If $k = \mathbb{F}_q$, $A = \mathbb{Z}$ and $\mu(X) = \#X(\mathbb{F}_q)$ as in Example 4 one recovers the usual Hasse–Weil zeta function of X. Indeed, the $\mathbb{F}_q$-valued points of $X^{(n)}$ correspond to the effective divisors of degree n in X.*

Let us denote $\mu(\mathbb{A}^1_k)$ by $\mathbb{L}$. Then we have the following rationality statement for curves (see Theorem 1.1.9 *ibid.*).

[6] It is called *poor man's motivic functions*, a name apparently suggested by Drinfeld.

THEOREM 4.26 (Kapranov). *If* X *is any one dimensional variety (not necessarily non-singular) of genus* g*, then* $\zeta_\mu(X,t)$ *is rational. Furthermore, the rational function* $\zeta_\mu(X,t)(1-t)(1-\mathbb{L}t)$ *is actually a polynomial of degree* $\leqslant 2g$ *and satisfies the functional equation below.*

$$\zeta_\mu(X, 1/\mathbb{L}t) = \mathbb{L}^{1-g}t^{2-2g}\zeta_\mu(X,t) \tag{23}$$

REMARK 4.27. The rationality statement fails to be true in higher dimensions, *e.g.,* if X is a complex projective non-singular surface of geometric genus $\geqslant 2$ [**LL03**]. In fact, a complex surface X has rational motivic zeta function if and only if it has *Kodaira dimension* $-\infty$ [**LL04**].

5. Noncommutative Calabi–Yau spaces

This section grew out of an attempt to realise Matilde Marcolli's vision on motives of noncommutative curves, which was communicated to the author quite a while ago. The main goal is to introduce zeta functions of noncommutative curves *in a motivic framework* and extract arithmetic information out of them. That the zeta functions of varieties contain crucial arithmetic information is a gospel truth by now.

Before we move forward let us mention that such ideas are prevalent in noncommutative geometry, *e.g.,* Connes' spectral realization of the zeros of the Riemann zeta function [**Con00, Con99**]. Some other important works in this direction are [**CMR05**],[**Den01, Den03**],[**Pla**] and [**HP05**], to name only a few. Also the readers should take a look at [**Mar05**] for a more holistic point of view.

Following Theorem 4.14 we argue that the category of noncommutative motives of noncommutative curves should be the full subcategory of $\mathcal{NCM}$ generated by DG categories which resemble those of abelian varieties, *i.e.,* the inclusion of abelian varieties inside $\mathcal{NCM}$ (see Equation (20)). Given an abelian surface the cokernel of the multiplication by 2 map (isogeny) is a Kummer surface with 16 singular points, whose (minimal) resolution of singularities is a $K3$ surface. It is an example of a Calabi–Yau manifold of dimension 2. So even if we look at motives of curves Calabi–Yau varieties show up rather naturally. We propose to treat such varieties as they are, rather than working up to isogenies. Calabi–Yau varieties are interesting from the point of view of physics as well. For us a Calabi–Yau variety is just a variety, whose canonical class is trivial (no assumption on the fundamental group). If X is a smooth projective variety of dimension n, the Serre functor[7] is given by $(- \otimes \omega_X)[n]$, where ω_X is the canonical sheaf of X.

DEFINITION 5.1. *A DG category $\mathcal{C}$ in $\mathcal{NCS}$ is called a noncommutative Calabi–Yau space of dimension n if $H^0(\mathcal{C})$ is triangulated (i.e., $\mathcal{C}$ is pretriangulated as in Definition 4.4) and there exists a natural isomorphism between the Serre functor and $[n]$. In other words, there exists bifunctorial isomorphisms $\mathrm{Hom}(A, B) \tilde{\to} \mathrm{Hom}(B, A[n])^*$ in $H^0(\mathcal{C})$.*

It is clear that if X is a Calabi–Yau variety then $D_{dg}(X)$ is a Calabi–Yau category in the above sense. With this characterization of noncommutative Calabi–Yau varieties we arrive at the definition of the category of noncommutative motives of noncommutative Calabi–Yau varieties.

DEFINITION 5.2. *The category of noncommutative motives of noncommutative Calabi–Yau varieties, denoted by $\mathcal{NCM}_{CY}$, is the full additive subcategory of $\mathcal{NCM}$ consisting of noncommutative Calabi–Yau spaces.*

EXAMPLE 6. *It is expected that via a noncommutative version of the construction of the Jacobian of a curve the category of motives of noncommutative curves can be seen as a full subcategory of $\mathcal{NCM}_{CY}$. The way to view an abelian variety in this setting is not clear to the author yet. The category $\mathcal{NCM}_{CY}$ contains honest elliptic curves (as they are their own Jacobians) as given by the inclusion of classical geometry in this setting (see Equation (20)). The noncommutative torus $\mathbb{T}^{\tau}_{\theta}$ is also included via its DG derived category of holomorphic bundles as described in Subsection 2.6. It is isomorphic to $D_{dg}(X_\tau)$ via the Fourier–Mukai type Polishchuk–Schwarz functor $\mathcal{S}_\tau$ (see Proposition 3.1 [**PS03**]).*

5.0.1. *The universal motivic measure on $\mathcal{NCS}$.* Let us recall from Section 4.5 that an A-valued motivic measure μ is a ring homomorphism from $\mathrm{Fun}^{\mathrm{poor}}(k) \to A$. We have replaced the category of k-schemes by a more sophisticated category of noncommutative spaces $\mathcal{NCS}$. We were lucky that $\mathrm{Fun}^{\mathrm{poor}}(k)$ actually had a ring structure. What was essential was the *excision-friendliness*, *i.e.,* μ respected cutting along closed subschemes. This is captured by an appropriate notion of the Grothendieck group of $\mathcal{NCS}$. We do not want to get into the details of this notion here. The work of Tabuada [**Tab05a**] and the

[7]In an k-linear category $\mathcal{A}$ an additive autoequivalence S is called a Serre functor if there exists a bifunctorial isomorphism $\mathrm{Hom}(A, B) \tilde{\to} \mathrm{Hom}(B, SA)^*$ for any two $A, B \in \mathrm{Obj}(\mathcal{A})$. If it exists it is unique up to isomorphism.

discussion on motivic function spaces in Section 2 of [**Kon**] should be useful. But assuming the existence of an appropriate generalization of $\mathrm{Fun}^{\mathsf{poor}}(\mathsf{k})$ to $\mathcal{NCS}$ we can speculate about the motivic measure on it:

DEFINITION 5.3. *An* A*-valued motivic measure is a ring homomorphism from the conjectural motivic ring of* $\mathcal{NCS}$ *to* A.

REMARK 5.4. We can define a *universal* $\mathbb{M}_{\mathsf{meas}}$-valued motivic measure μ_{mot} by requiring it to admit a unique ring homomorphism f to A for any A-valued motivic measure μ_{A} making the diagram below commute:

$$\begin{array}{ccc} \text{Conjectural motivic ring of } \mathcal{NCS} & \xrightarrow{\mu_{\mathsf{mot}}} & \mathbb{M}_{\mathsf{meas}} \\ & \searrow^{\mu_{\mathsf{A}}} & \Big\downarrow{\scriptstyle \exists ! \mathsf{f}} \\ & & \mathsf{A} \end{array}$$

Now we argue about the existence of the motivic Grothendieck ring of $\mathcal{NCS}$. Since every object in $\mathcal{NCS}$ is quasi-equivalent to a pretriangulated DG category we seek a Grothendieck ring of pretriangulated DG categories.

In [**BLL04**] the authors precisely construct a Grothendieck ring of pretriangulated DG categories. It was pointed out by the authors that it is crucial to work with DG categories (and not honest triangulated ones) as the tensor product of triangulated categories is not triangulated in general. Let us briefly recall their construction.

The Grothendieck ring $\mathcal{G}$ is generated as a free abelian group by the isomorphism classes of pretriangulated DG categories in $\mathcal{NCS}$ (or quasi-equivalence classes of objects in $\mathcal{DG}\mathsf{cat}$) modulo relations analogous to the *cutting and pasting* procedure of $\mathrm{Fun}^{\mathsf{poor}}(\mathsf{k})$. The authors nicely reinterpret the excision relations as those coming from *semiorthogonal decompositions* (see [**BO**] for the details of semiorthogonal decomposition). One writes $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]$ if and only if there exist representatives $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ in $[\mathcal{A}]$, $[\mathcal{B}]$, $[\mathcal{C}]$ respectively such that

(1) $\mathcal{A}'$, $\mathcal{C}'$ are DG subcategories of $\mathcal{B}'$,
(2) $\mathrm{H}^0(\mathcal{A}')$, $\mathrm{H}^0(\mathcal{C}')$ are admissible subcategories of $\mathrm{H}^0(\mathcal{B}')$,
(3) $(\mathrm{H}^0(\mathcal{A}'), \mathrm{H}^0(\mathcal{C}'))$ is a semiorthogonal decomposition of $\mathrm{H}^0(\mathcal{B}')$.

REMARK 5.5. Part (3) implies that $\mathrm{H}^0(\mathcal{A}') = (\mathrm{H}^0(\mathcal{C}'))^{\perp}$, which is Lemma 2.25 in [**BLL04**]. An exact sequence $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$ of pretriangulated DG categories (*cf.* Definition 4.18) induces an *exact* sequence of honest triangulated categories $\mathrm{H}^0(\mathcal{A}) \longrightarrow \mathrm{H}^0(\mathcal{B}) \longrightarrow \mathrm{H}^0(\mathcal{C})$ by definition. However, existence of a semiorthogonal decomposition is a stronger condition. It says that $\mathrm{H}^0(\mathcal{C})$ is a triangulated subcategory of $\mathrm{H}^0(\mathcal{B})$ and $\mathrm{H}^0(\mathcal{A}) = (\mathrm{H}^0(\mathcal{C}))^{\perp}$, *i.e.,* the sequence is *split* (*cf.* Theorem 4.21). It is plausible that one obtains something sensible by allowing all possible exact sequences as relations.

The product $\bullet$ is defined as follows:

$$\mathcal{A}_1 \bullet \mathcal{A}_2 := \mathscr{P}\mathsf{erf}(\mathcal{A}_1 \otimes \mathcal{A}_2),$$

where $\mathscr{P}\mathsf{erf}(\mathcal{A})$ is a pretriangulated DG category defined in *ibid.* and should not be confused with $\mathrm{per}(\mathcal{A})$ as in Definition 4.4 or with perfect complexes. For the benefit of the reader let us elaborate on that. For a DG category $\mathcal{A}$ an $\mathcal{A}^{\mathrm{op}}$-module, *i.e.,* a DG functor from $\mathcal{A}^{\mathrm{op}}$ to the DG category of complexes over k is called *semifree* if it admits a finite filtration such that the successive quotients are free DG modules (up to a shift), *i.e.,* modules of the form $\mathrm{Hom}(-, \mathsf{X})$ for some $\mathsf{X} \in \mathrm{Obj}(\mathcal{A})$. Let us denote the category of semifree modules over $\mathcal{A}$ by $\mathrm{SF}(\mathcal{A})$. The inclusion functor $\mathrm{SF}(\mathcal{A}) \to \mathcal{A}^{\mathrm{op}}$-modules induces an equivalence of triangulated categories between $\mathrm{H}^0(\mathrm{SF}(\mathcal{A}))$ and the derived category of $\mathcal{A}$ [**Dri04**]. The category $\mathscr{P}\mathsf{erf}(\mathcal{A})$ is defined as the full DG subcategory of $\mathrm{SF}(\mathcal{A})$ consisting of objects which become isomorphic to an object in $\mathrm{per}(\mathcal{A})$ after passing on to the zeroth cohomology category. Roughly speaking, $\mathscr{P}\mathsf{erf}(\mathcal{A})$ is a *DG version* of $\mathrm{per}(\mathcal{A})$. The tensor product of two pretriangulated DG categories is made again pretriangulated by means of this construction.
The product $\bullet$ preserves quasi-equivalences of DG categories and hence descends to a product on $\mathcal{G}$. It is proven in [**BLL04**] that the product is associative and commutative. There is a unit given by the class of $\mathsf{DG}(\mathsf{k})$, *i.e.,* the DG category of finite dimensional chain complexes over k. That this product corroborates the geometric picture is justified by Theorem 6.6 *ibid.*.

Proposition 5.6. *The product of two noncommutative Calabi–Yau categories is again a noncommutative Calabi–Yau category.*

Proof. We need to check that $\mathcal{A} \bullet \mathcal{B}$ is a noncommutative Calabi–Yau DG category of dimension $\mathsf{m} + \mathsf{n}$ for $\mathcal{A}, \mathcal{B} \in \mathrm{Obj}(\mathcal{NCM}_{\mathrm{CY}})$ of dimensions m, n respectively. Indeed,

$$\begin{aligned}
& \mathrm{Hom}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(A\otimes B, A'\otimes B'[m+n]) \\
= \; & H^0\mathrm{Hom}_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(A,A')\otimes \mathrm{Hom}_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(B,B')[m+n] \\
= \; & H^{m+n}_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(\mathrm{Hom}(A,A')\otimes \mathrm{Hom}(B,B')) \\
= \; & H^{m+n}\left((\oplus_i \mathrm{Hom}^i_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(A,A')\otimes \mathrm{Hom}^{k-i}_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(B,B'))^\bullet\right) \\
= \; & \oplus_l \mathrm{Hom}^{l+m}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(A,A')\otimes \mathrm{Hom}^{n-l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(B,B') \\
= \; & \oplus_l \mathrm{Hom}^{l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(A,A'[m])\otimes \mathrm{Hom}^{-l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(B,B'[n]) \\
= \; & \oplus_l \mathrm{Hom}^{l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(A',A)^*\otimes \mathrm{Hom}^{-l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(B',B)^* \\
= \; & \left(\oplus_l \mathrm{Hom}^{l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(A',A)\otimes \mathrm{Hom}^{-l}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}(B',B)\right)^* \\
= \; & H^0(\mathrm{Hom}^\bullet_{\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2)}(A'\otimes B', A\otimes B))^* \\
= \; & \mathrm{Hom}_{H^0(\mathscr{P}\mathsf{erf}(\mathcal{A}_1\otimes\mathcal{A}_2))}\mathrm{Hom}(A'\otimes B', A\otimes B)^*
\end{aligned}$$

□

We call a $\mathcal{G}$-valued motivic measure a universal one. Let us denote the image of $X \in \mathcal{NCS}$ inside $\mathcal{G}$ by $[X]$, *i.e.*, $\mu_{mot}(X) = [X]$.

Example 7. *The image inside $\mathcal{G}$ of the noncommutative torus $\mathbb{T}^\tau_\theta$ is isomorphic to that of the complex torus $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, which we denote by $[X_\tau]$. Then the universal $\mathcal{G}$-valued motivic zeta function of the noncommutative torus $\mathbb{T}^\tau_\theta$ is given by*

$$\zeta_{\mu_{mot}}(\mathbb{T}_\theta\tau, t) = \sum_{n=0}^{\infty}[X_\tau]^n t^n \qquad \in \mathcal{G}[[t]],$$

where $[X_\tau]^n := [X_\tau]\bullet\cdots\bullet[X_\tau]$ n times and $[X_\tau]^0 = 1$.

It is perhaps better to replace $[X]^n$ by $[\mathrm{Sym}(X)]^n$ (after defining it appropriately) in the definition of the zeta function to corroborate with the original definition of a motivic zeta function.

It is shown in [**BLL04**] that there is a canonical surjective ring homomorphism $\mathrm{Fun}^{\mathrm{poor}}(k) \to \mathcal{G}_{hon}$ with $(\mathbb{L}-1)$ in the kernel, where $\mathcal{G}_{hon}$ is the subring of $\mathcal{G}$ generated by certain pretriangulated DG categories associated to honest smooth projective varieties over k.

Remark 5.7. The above example is, on the face of it, quite a setback. We didn't gain anything by all the hard work as the motivic zeta function of a noncommutative torus $\mathbb{T}^\tau_\theta$ with a complex structure turns out to be that of the complex torus X_τ. However, in the category of motivic noncommutative spaces $\mathbb{T}_\theta^\tau$ is isomorphic to X_τ via the Polishchuk–Schwarz functor $\mathcal{S}_\tau$. It is an interesting problem in itself to figure out exactly which piece of information can tell the difference. A fantastic outcome could be: complex multiplication and real multiplication are *mirror symmetric* problems, the mirror map being $\mathcal{S}_\tau$. One immediate step is to extend the Polishchuk–Schwarz equivalence to higher dimensions. Noncommutative tori are defined as deformation quantizations of the algebra of functions on the commutative torus (see *e.g.,* [**Rie90**]). The B-model of a conformal field theory associates to a complex torus its derived category of coherent sheaves. The Polishchuk–Schwarz equivalence says that deforming the complex torus to a noncommutative torus does not produce anything new for the B-model.

A more concrete problem, which the author is investigating at the moment, is the construction of a Tannakian category structure on $\mathcal{NCM}_{\mathrm{CY}}$ and the computation of its fundamental group. This would augment a classical result of Milne [**Mil99**].

One nagging point is that certain natural physical constructions do not allow us to define a category in which the composition of morphisms obeys associativity (it is associative only up to homotopy). Hence some mathematicians have resorted to working with A_∞ categories which encode such properties. The world of A_∞ categories subsumes that of DG categories. However, Costello [**Cos07**] has shown that every A_∞ category is quasi-isomorphic to a DG category in a functorial manner. Perhaps it is okay to restrict ourselves to DG categories, where compositions are honestly associative.

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SUMMARY

In this thesis we have tried to figure out some algebraic aspects of noncommutative tori, aiming at generalizing them to arbitrary noncommutative spaces. In the second section all relevant definitions, some examples and motivations have been provided.

In the third section we look at the example of noncommutative tori and see how they can be related to similar objects called noncommutative elliptic curves. We extract a suitably well-behaved subcategory of the category of holomorphic bundles over noncommutative tori. This category turns out to admit a Tannakian structure with $\mathbb{Z} + \theta\mathbb{Z}$ as the fundamental group. The key to this construction is an equivariant version of the classical Riemann–Hilbert correspondence. The aim was to construct homotopy theoretic invariants of noncommutative tori, *e.g.,* fundamental groups and we make a proposal to that end.

The last two sections constitute an attempt to rewrite some parts of noncommutative algebraic geometry in the framework of DG categories. We provide a description of the category of noncommutative spaces and their associated noncommutative motives. We had some arithmetic applications in mind, namely, introducing and studying motivic zeta functions of noncommutative tori. We propose a universal motivic measure on the category of noncommutative spaces. In it lies a subcategory consisting of noncommutative Calabi–Yau spaces containing elliptic curves and noncommutative tori. In this setting we introduce a motivic zeta function of noncommutative tori; more generally that of noncommutative Calabi–Yau spaces. Our work should be put in perspective with the *Real Multiplication* programme of Manin.

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