

# Contents

*There are  $10^{11}$  stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them astronomical numbers. Now we should call them economical numbers —  
Richard Feynman (1918–1988)*

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